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"Meditationis est perscrutari occulta; contemplationis est admirari perspicua....Admiratio generat quæstionem, quæstio investigationem, investigatio inventionem."—*Hugo de S. Victore.*

——"Cur spirent venti, cur terra dehiscat,
Cur mare turgescat, pelago cur tantus amaror,
Cur caput obscura Phœbus furrugine condat,
Quid toties diros cogat flagrare cometas,
Quid pariat nubes, veniant cur fulmina cœlo,
Quo micet igne Iris, superos quis conciat orbes
Tam vario motu."

J. B. Pinelli ad Mazonium.

ALERE FLAMMAM.

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THE

PHILOSOPHICAL MAGAZINE

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[SEVENTH SERIES—VOL. 39]

I. *Reconsideration of the Theory of Free Turbulence* *.

By H. B. Squire, M.A. †

[Received January 25, 1946.]

Summary.

The theories of free turbulence are unsatisfactory as the distinction between dimensional analysis and physical hypotheses is not sharply drawn. Dimensional analysis is applied to the various problems of free turbulence in order to deduce as much as possible, without recourse to physical hypotheses, as to the nature of turbulent flow. The various theories of turbulent flow are then considered in the light of the previous analysis.

It is concluded that the phenomena are too complicated to be explained by the simpler versions of the existing theories. Experiment shows that heat spreads faster than momentum and this is qualitatively in favour of the vorticity transfer theory.

Wall turbulence is considered in an appendix and the logarithmic velocity distribution is deduced from dimensional analysis.

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* This paper is published by permission of the Ministry of Aircraft Production, but the author takes responsibility for the statements made and opinions expressed.

† Communicated by the Author.

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§ 1. INTRODUCTION.

THERE has been a considerable advance in the understanding of turbulent flow since Prandtl's introduction of the momentum transfer theory in 1926 (see Ref. (1), p. 206). The present situation with regard to free turbulence problems * is unsatisfactory as no simple theory has been advanced which is capable of explaining the experimental results. Reichart [2, 3] has made a number of criticisms of the usual method of analysis and these have suggested the present investigation, the object of which is to deduce as much as possible from dimensional analysis, without introducing hypotheses about the nature of turbulent flow. This dimensional analysis has been partly given in a different form by Reichart [3] but is worth repeating, as it is not generally recognized that many of the results commonly associated with theories of turbulent flow can be derived in this way. The distinction between results which can be derived by dimensional analysis and the results which depend on physical hypotheses has not been sharply drawn in ' Modern Developments in Fluid Dynamics ' [1], which is the only book dealing fully with this subject. There is thus a risk that results which really depend on dynamical similarity may be interpreted as confirmation of a physical

---

* Free turbulence problems consist of cases of turbulent flow in which walls play no part and for which the pressure variations are not important ; wakes and jets are the most important of these.

hypothesis. In addition, care is needed before concluding that a theory is confirmed by agreement with experiment in a single case. The measured velocity and temperature distributions in wakes and jets all resemble an error curve in type, and the results are too scattered to permit the slopes of the experimental curves to be determined with confidence. But, since the theories of free turbulent flow link the momentum and thermal exchange with the gradient of the velocity and temperature respectively, comparison should preferably be made with the measured gradients. If the differential equations which are derived theoretically are integrated, disagreement with experiments tends to be masked.

It has been confirmed in some cases of free turbulence that heat spreads more rapidly than momentum (see Appendix III.). This indicates that the momentum transfer theory is in error, as it requires a similarity between these two processes, and the result is qualitatively in favour of the vorticity transfer theory. But the phenomena seem to require a much more complicated theory to explain them adequately than the vorticity transfer theory in its simpler forms provides.

Though this report is mainly concerned with free turbulence, wall turbulence is briefly discussed in Appendix II., in which the logarithmic velocity distribution near a wall is deduced by dimensional analysis.

§ 2. DESCRIPTION OF PROCEDURE.

For the simpler cases of free turbulence the rates of spread and decay of velocity and temperature obey simple rules and the velocity and temperature profiles are similar at all sections. These results are consequencies of the shear stress and heat exchange being dominated by the molar exchange and being effectively independent of viscosity and conductivity. We accept this as true and adopt the following relations, for introduction into the equations of motion :—

(1) The distributions of velocity, temperature, shear stress and heat exchange are similar at all sections.

(2) The shear stress is proportional to the square of the maximum difference in velocity across a section and is independent of viscosity.

(3) The heat exchange is proportional to the product of the maximum difference in velocity and the maximum difference in temperature across a section and is independent of conductivity.

The rates of spread and decay can be determined by assuming power law variations with distance downstream (measured from a convenient origin near the source of the disturbance) for the maximum velocity and temperature differences and for the jet or wake width. Insertion of power laws into the equations enables the indices to be determined by compatibility conditions. This is a tedious process which is omitted here, as it leads only to known results. The indices of the power laws are therefore inserted at the same time as relations (1)–(3) above. When this is done the momentum equation yields a relation between the functions representing the shear stress and velocity distributions, and the energy equation yields an analogous relation between the functions representing the heat exchange and temperature distributions.

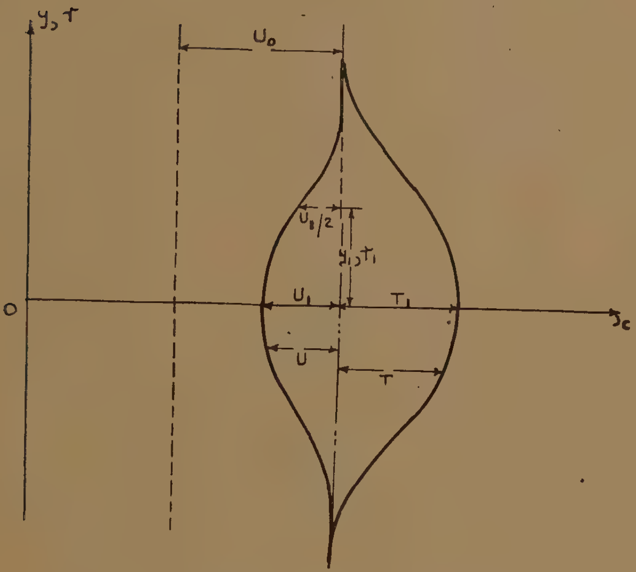
The above procedure is applied to all the standard problems of free turbulence. It is emphasized that the formulæ given in para. 4 are independent of any detailed assumptions as to the nature of turbulent flow (*e. g.*, the momentum transfer theory). If it is desired to obtain explicit expressions for the velocity and temperature distributions further hypotheses must be introduced and the resulting differential equations integrated. But it is possible to draw some useful conclusions immediately; also the relations are in a convenient form for the introduction of these further hypotheses.

§ 3. Notation and Definition.

The notation used is shown in figs. 1–4 for the various cases and is as follows :—

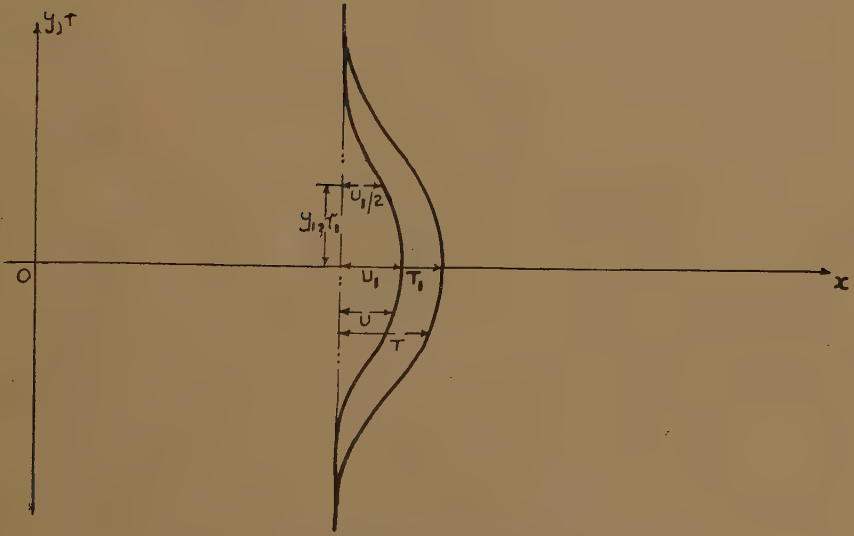
x	Distance along the stream direction measured from an unspecified station close to the introduction of the disturbance.
y, r	Distance normal to the stream from the plane or axis of symmetry respectively.
y_1, r_1	Values of y and r at a station at which the velocity (or velocity defect) is one-half the velocity (or velocity defect) on the plane or axis of symmetry respectively.
η	y/y_1 or r/r_1.
λ	Distance between adjoining rods.
U_0	Main stream velocity (or higher stream velocity for plane mixing region.)
aU_0	Lower stream velocity for plane mixing region.
U	Velocity (or velocity defect) along the stream direction at any point.
V	Velocity normal to the plane or axis of symmetry.
U_1	Velocity (or velocity defect) on the plane or axis of symmetry.
T	Excess temperature above main stream temperature.
T_1	Excess temperature on plane or axis of symmetry.
T_0	Temperature difference between streams for plane mixing region.
$f(\eta)$	Velocity distribution function, equal to U/U_1 for wakes and jets.
$f_1(\eta)$	Temperature distribution function, equal to T/T_1 for wakes and jets.
u	Component of velocity fluctuation in the x direction.
v	Component of velocity fluctuation in the y or r direction.
T'	Magnitude of temperature fluctuation.

Fig. 1.


y, r
U₀
U₁/2
y₁, r₁
O
U₁
T₁
x
U
T

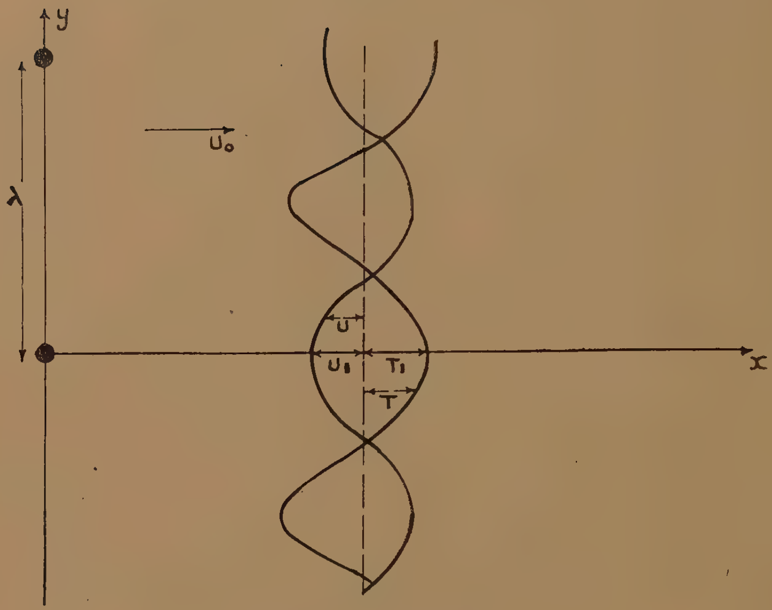
Wake of cylinder or body of revolution.

Fig. 2.


y, r
U₁/2
y₁, r₁
O
U₁
T₁
x
U
T

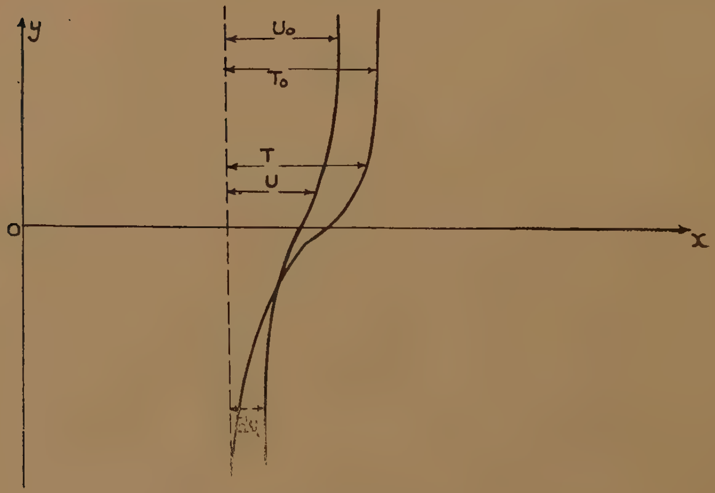
Plane or round jet.

Fig. 3.


y
λ
U₀
U
U₁
T₁
T
x

Wake of row of rods.

Fig. 4.



Plane mixing region.

$\tau=-\rho\overline{uv}$	Shear stress due to mixing.
$\rho c\overline{vT'}$	Rate of heat transfer due to mixing.
ρ	Density.
c	Specific heat (at constant pressure).
$g(\eta)$	Momentum exchange function, equal to $\overline{uv}/U_1^2$ for wakes and jets.
$g_1(\eta)$	Thermal exchange function, equal to $\overline{vT'}/U_1T_1$ for wakes and jets.
ϵ	Eddy viscosity.
ϵ_1	Eddy conductivity.
l	Mixing length.

§ 4. SUMMARY OF RESULTS.

The results of the calculations are collected below. Notes on the method of derivation are given in Appendix I. The governing equations are all given in Ref. (1). For wakes the velocity defects are assumed to be small compared with the stream velocities, and only the first order terms in the equations are retained.

4.1 *Wake of symmetrical cylinder.*

Equations.

$$U_0\frac{\partial U}{\partial x}=\frac{\partial}{\partial y}(\overline{uv})\,. \qquad U_0\frac{\partial T}{\partial x}=-\frac{\partial}{\partial y}(\overline{vT'}). \quad \text{(4.1.1)}$$

Momentum and Energy.

$$U_0\int_0^\infty U\,dy=\text{const.}, \qquad U_0\int_0^\infty T\,dy=\text{const.}$$

Substitutions.

$$\left.\begin{array}{lll} U=U_1f(\eta), & \eta=y/y_1, & T=T_1f_1(\eta), \\ U_1=Ax^{-\frac{1}{2}}, & y_1=Bx^{\frac{1}{2}}, & T_1=A_1x^{-\frac{1}{2}}, \\ \overline{uv}=U_1^2g(\eta), & C=-\dfrac{BU_0}{2A}, & \overline{vT'}=-U_1T_1g_1(\eta). \end{array}\right\} \quad \text{(4.1.2)}$$

Results.

$$g(\eta)=C\eta f(\eta), \qquad g_1(\eta)=C\eta f_1(\eta). \quad . \quad . \quad \text{(4.1.3)}$$

Boundary Conditions.

$$\begin{array}{ll} f(0)=1, & f_1(0)=1, \\ f(1)=0.5, & \text{—} \\ {[f(\eta)]}_{\eta\to\infty}=0, & {[f_1(\eta)]}_{\eta\to\infty}=0. \end{array}$$

Note :—A, A_1, B are dimensional constants ; B is chosen so that $f(\eta)=0{\cdot}5$ for $\eta=1$. C is a number which determines the rate of spread of the wake.

4.2 *Wake of body of revolution.*

Equations.

$$U_0\frac{\partial U}{\partial x}=\frac{1}{r}\frac{\partial}{\partial y}(r\,\overline{uv}), \qquad U_0\frac{\partial T}{\partial x}=-\frac{1}{r}\frac{\partial}{\partial y}(r\,\overline{vT'}). \qquad (4.2.1)$$

Momentum and Energy.

$$U_0\int_0^\infty Ur\,dr=\text{const.}, \qquad U_0\int_0^\infty Tr\,dr=\text{const.}$$

Substitutions.

$$\left.\begin{array}{lll} U=U_1f(\eta), & \eta=r/r_1, & T=T_1f_1(\eta), \\ U_1=Ax^{-\frac{2}{3}}, & r_1=Bx^{\frac{1}{3}}, & T_1=A_1x^{-\frac{2}{3}}, \\ \overline{uv}=U_1{}^2g(\eta), & C=-\dfrac{BU_0}{3A}, & \overline{vT'}=-U_1T_1g_1(\eta). \end{array}\right\}. \qquad (4.2.2)$$

Results.

$$g(\eta)=C\eta f(\eta), \qquad g_1(\eta)=C\eta f_1(\eta). \quad . \quad . \qquad (4.2.3)$$

Boundary Conditions.

As for 4.1.

Note :—As for 4.1.

4.3 *Plane jet.*

Equations.

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=-\frac{\partial}{\partial y}(\overline{uv}), \qquad U\frac{\partial T}{\partial x}+V\frac{\partial T}{\partial y}=-\frac{\partial}{\partial y}(\overline{vT'}). \quad . \qquad (4.3.1)$$

Momentum and Energy.

$$\int_0^\infty U^2\,dy=\text{const.}, \qquad \int_0^\infty UT\,dy=\text{const.}$$

Substitutions.

$$\left.\begin{array}{lll} U=U_1f(\eta), & \eta=y/y_1, & T=T_1f_1(\eta), \\ U_1=Ax^{-\frac{1}{2}}, & y_1=Bx, & T_1=A_1x^{-\frac{1}{2}}, \\ \overline{uv}=-U_1{}^2g(\eta), & & \overline{vT'}=-U_1T_1g_1(\eta). \end{array}\right\}. \qquad (4.3.2)$$

Results.

$$g(\eta)=\tfrac{1}{2}Bf(\eta)\int_0^\eta f(\eta)\,d\eta, \qquad g_1(\eta)=\tfrac{1}{2}Bf_1(\eta)\int_0^\eta f(\eta)\,d\eta. \qquad (4.3.3)$$

Boundary Conditions.

As for 4.1.

Note :—A and A_1 are dimensional constants. B is a number which is chosen so that $f(\eta)=0{\cdot}5$ for $\eta=1$. B determines the rate of spread of the jet.

4.4 *Round jet.*

Equations.

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial r}=-\frac{1}{r}\frac{\partial}{\partial r}(r\,\overline{uv}), \qquad U\frac{\partial T}{\partial x}+V\frac{\partial T}{\partial r}=-\frac{1}{r}\frac{\partial}{\partial r}(r\,\overline{vT'}). \qquad (4.4.1)$$

Momentum and Energy.

$$\int_0^\infty U^2 r\,dr=\text{const.}, \qquad \int_0^\infty UTr\,dr=\text{const.}$$

Substitutions.

$$\left.\begin{aligned} &U=U_1 f(\eta), && \eta=r/r_1, && T=T_1 f_1(\eta),\\ &U_1=Ax^{-1}, && r_1=Bx, && T_1=A_1 x^{-1},\\ &\overline{uv}=-U_1^2 g(\eta), && && \overline{vT'}=-U_1T_1g_1(\eta).\end{aligned}\right\} \qquad (4.4.2)$$

Results.

$$\eta g(\eta)=Bf(\eta)\int_0^\eta \eta f(\eta)\,d\eta, \qquad \eta g_1(\eta)=Bf_1(\eta)\int_0^\eta \eta f(\eta)\,d\eta. \qquad (4.4.3)$$

Boundary Conditions.

As for 4.1.

Note :—As for 4.3.

4.5 *Wake of row of rods.*

Equations.

$$U_0\frac{\partial U}{\partial x}=\frac{\partial}{\partial y}(\overline{uv}), \qquad U_0\frac{\partial T}{\partial x}=-\frac{\partial}{\partial y}(\overline{vT'}). \quad . \quad . \qquad (4.5.1)$$

Substitutions.

$$\left.\begin{aligned} &U=U_1 f(\eta), && \eta=y/\lambda, && T=T_1 f_1(\eta),\\ &U_1=AU_0(\lambda/x), && && T_1=A_1(\lambda/x),\\ &\overline{uv}=U_1^2 g(\eta), && && \overline{vT'}=-U_1T_1g_1(\eta).\end{aligned}\right\} \qquad (4.5.2)$$

Results.

$$g(\eta)=\frac{1}{A}\int_0^\eta f(\eta)\,d\eta, \qquad g_1(\eta)=\frac{1}{A}\int_0^\eta f_1(\eta)\,d\eta. \quad . \qquad (4.5.3)$$

Boundary Conditions.

$$f(0)=1, \qquad f_1(0)=1.$$

Also $f(\eta)$ and $f_1(\eta)$ are periodic functions of η with period unity.

Note :—A and A_1 are constants which determine the rate of decay of the variations of velocity and temperature in the wake. λ is the distance between the rods.

4.6 *Plane mixing region.*

Equations.

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=-\frac{\partial}{\partial y}(\overline{uv}), \qquad U\frac{\partial T}{\partial x}+V\frac{\partial T}{\partial y}=-\frac{\partial}{\partial y}(\overline{vT'}). \quad . \quad (4.6.1)$$

Substitutions.

$$\left.\begin{aligned} &U=U_0f(\eta), && \eta=y/y_0, \quad T=T_0f_1(\eta), \\ &\overline{uv}=-U_0^2(1-a)^2g(\eta), && \eta_0=Bx, \quad \overline{vT'}=-U_0(1-a)T_0g_1(\eta). \end{aligned}\right\} \quad . \quad . \quad . \quad (4.6.2.)$$

Results.

$$g(\eta)=\frac{B}{(1-a)^2}\left[f(\eta)\int_b^\eta f(\eta)\,d\eta-\int_b^\eta[f(\eta)]^2d\eta+D\right],$$

$$g_1(\eta)=\frac{B}{(1-a)}\left[f_1(\eta)\int_b^\eta f(\eta)\,d\eta-\int_b^\eta f_1(\eta)f(\eta)\,d\eta+D_1\right]. \quad . \quad . \quad . \quad (4.6.3)$$

Boundary Conditions.

$$[f(\eta)]_{\eta\rightarrow\infty}=1, \qquad [g(\eta)]_{\eta\rightarrow\pm\infty}=0, \qquad [f_1(\eta)]_{\eta\rightarrow\infty}=1,$$

$$[f(\eta)]_{\eta\rightarrow-\infty}=a, \qquad [g_1(\eta)]_{\eta\rightarrow\pm\infty}=0, \qquad [f_1(\eta)]_{\eta\rightarrow-\infty}=0.$$

Note:—The velocities above and below the mixing region are U_0 and aU_0, respectively. B is a number which is chosen to make the upper edge of the mixing region correspond to $\eta=1$ approximately. b, D and D_1 are constants of integration which are chosen to make g and g_1 vanish for $\eta\rightarrow\pm\infty$.

§ 5. DISCUSSION OF THE VARIOUS THEORIES OF TURBULENT FLOW.

We note first that the constants which appear in the equations relating $g(\eta)$ to $f(\eta)$ must be determined experimentally. These constants measure the rate of spread of the wake or jet, which in turn depends on the strength of the turbulence.

We note next that the relation

$$\frac{g_1(\eta)}{g(\eta)}=\frac{f_1(\eta)}{f(\eta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.1)$$

holds for wakes and jets, both plane and round. This result is surprising but it seems to be a consequence of:

(1) The assumptions introduced at the beginning of paragraph 2.

(2) The similar way in which the velocities U, u and the temperatures T, T′ appear in the equations, and

(3) The similarity of the boundary conditions which U and T are required to satisfy.

All further hypotheses which can be introduced to determine specific expressions for the velocity distribution must satisfy the requirement that $g(\eta)$ vanishes with $f'(\eta)$. This is necessary in order to ensure that the momentum exchange shall vanish when the velocity gradient vanishes. We therefore put

$$g(\eta)=-h(\eta)f'(\eta), \quad \ldots \ldots \quad (5.2)$$

by analogy with the ordinary viscous shear law, where $h(\eta)$ is finite for $f'(\eta)=0$. In the usual dimensional form equation (5.2) is equivalent to Boussinesq's expression

$$\frac{\tau}{\rho}=\epsilon\frac{\partial U}{\partial y}, \quad \ldots \ldots \quad (5.3)$$

where ϵ is a "shear coefficient" or eddy viscosity for turbulent flow, which is finite for $\partial U/\partial y$ equal to zero. Similarly for the heat exchange we may put

$$g_1(\eta)=-h_1(\eta)f_1'(\eta), \quad \ldots \ldots \quad (5.4)$$

corresponding to the dimensional form

$$\overline{vT'}=\epsilon_1\frac{\partial T}{\partial y},$$

where ϵ_1 is an eddy conductivity for turbulent flow.

The different theories of turbulent flow consist of the introduction of different assumptions for the function $h(\eta)$ and its relation to $h_1(\eta)$. These theories are as follows :—

(1) *Constant shear coefficient assumption.*

This may be written

$$h(\eta)=H,$$

where H is a constant. Corresponding to this we should take the eddy viscosity ϵ constant across a section of the wake. Equation (5.2) becomes

$$g(\eta)=-Hf'(\eta).$$

Substitution of this expression into the relations between $g(\eta)$ and $f(\eta)$ deduced in paragraph 4 for the various cases gives differential equations for $f(\eta)$. The velocity distributions deduced by integration of these are similar to the velocity distributions for laminar flow. Some of them have been discussed by Prandtl⁽⁴⁾ and Görtler⁽⁵⁾. They agree satisfactorily with experiment over the central parts but fall off to zero too slowly at the edges of the wake or jet. This suggests that the intensity of turbulence is high and fairly constant in the central region and falls off towards the edges.

The corresponding assumption for heat transfer is

$$h_1(\eta)=H_1,$$

where H_1 is a constant, which, with (5.4), leads to

$$g_1(\eta) = -H_1 f_1'(\eta).$$

A plausible guess would be that H_1 is equal to H, since heat and momentum are both transferred by mixing. But this leads to the result that the rates of spread of heat and momentum should be the same, which is not in accordance with experimental results (see Appendix III.). The difference between $h(\eta)$ and $h_1(\eta)$ is due to the fluctuating pressure gradients which affect the momentum exchange but not the heat exchange (Ref. (1), p. 209).

(2) *Momentum transfer theory.*

The usual dimensional form of this theory is * (Ref. (1), p. 206)

$$\frac{\tau}{\rho} = l^2\left(\frac{\partial U}{\partial y}\right)^2, \quad \epsilon = l^2\frac{\partial U}{\partial y}, \qquad (5.5)$$

where l is the mixing length, which is usually taken to be constant across a section. In our presentation this is equivalent to

$$g(\eta) = -j(\eta)[f'(\eta)]^2, \qquad (5.6)$$

where $j(\eta)$ would be taken to be constant in cases where l is constant across a section. Substitution of (5.6) in the relations between $g(\eta)$ and $f(\eta)$ deduced in paragraph 4 leads to differential equations for $f(\eta)$. The velocity distributions obtained by integration have an infinite curvature at the apex. They are more pointed than the experimental distributions and in some cases fall off to zero too rapidly at the edges.

(3) *Vorticity transfer theory.*

The simplest dimensional form of this theory is

$$\frac{\partial}{\partial y}\left(\frac{\tau}{\rho}\right) = l^2\frac{\partial U}{\partial y}\frac{\partial^2 U}{\partial y^2}, \qquad (5.7)$$

where l is the mixing length. In our presentation this is equivalent to

$$g'(\eta) = -j(\eta)f'(\eta)f''(\eta). \qquad (5.8)$$

With $j(\eta)$ constant (5.8) can be integrated and gives, apart from a factor 1/2, the equation (5.6) of the momentum transfer theory. Similar velocity distributions are obtained as for the momentum transfer theory in some cases.

(4) *Karman's similarity theory.*

For the present investigation Karman's similarity theory can be regarded as an extension of the momentum transfer theory in which the mixing length l is determined from the assumption that, referred to axes moving with the local mean velocity, the eddying motions are similar at all points. This leads to the formula (Ref. (1), p. 349),

---

* For brevity the distinction in sign between $\partial U/\partial y$ and its modulus is ignored.

$$l = \mathrm{K}\frac{\partial \mathrm{U}}{\partial y}\left(\frac{\partial^2 \mathrm{U}}{\partial y^2}\right)^{-1},$$

where K is a constant. The shear stress is given by equation (5.3) and hence

$$\frac{\tau}{\rho} = \mathrm{K}^2\left(\frac{\partial \mathrm{U}}{\partial y}\right)^4\left(\frac{\partial^2 \mathrm{U}}{\partial y^2}\right)^{-2}.$$

With our method of presentation this is equivalent to

$$g(\eta) = \mathrm{K}^2[f'(\eta)]^3[f''(\eta)]^{-2}.$$

Difficulties arise in the application of the theory at points where $f'(\eta)$ or $f''(\eta)$ vanishes.

(5) *Reichardt's theory.*

Reichardt's theory (2, 3) is in a different category from those outlined above. Starting from the experimental results he concluded that the distributions of momentum in wakes and jets fit the probability curve. The consequences of this conclusion are worked out but cannot easily be summarized. The arguments put forward are not wholly convincing.

§ 6. CONCLUDING REMARKS.

The difficulties in the application of Karman's similarity theory are so serious that it need not be discussed further.

The momentum and vorticity transfer theories, as presented above with finite mixing length everywhere, both suffer from the defect that the momentum exchange function is proportional to $[f'(\eta)]^2$ when $f'(\eta)$ is small, which it is near a plane or axis of symmetry. Put another way, they require that the shear function $h(\eta)$ (equation 5.2) and the eddy viscosity (equation 5.3) shall vanish on a plane or axis of symmetry; this can hardly be true unless the turbulence also vanishes there, which is most unlikely. Prandtl appreciated this objection and in his original paper put forward the alternative form for the eddy viscosity :

$$\epsilon = l^2\left[\left(\frac{\partial \mathrm{U}}{\partial y}\right)^2 + l_1^2\left(\frac{\partial^2 \mathrm{U}}{\partial y^2}\right)^2\right]^{\frac{1}{2}},$$

where l_1 is a second mixing length. This assumption seems rather artificial.

The mixing lengths deduced by comparison of these theories with experiment are all large fractions (greater than 0·1) of the width of the jet or wake, and this is an indication that the turbulence and eddy viscosity depend on the overall conditions and not on the local conditions such as the velocity gradient. On the other hand, the turbulence and the eddy viscosity do presumably fall to zero not far beyond the edge of the mixing region, so that the assumption of a constant shear coefficient leads to velocity distribution curves which are incorrect at the edges.

The main conclusion of the above discussion is the negative one that none of the existing theories has been or is likely to be fully confirmed by comparison with experiment *. This is because the variation of the eddy viscosity across a wake or jet cannot be represented by any simple function of the gradient of the velocity.

The experimental data show that heat spreads more rapidly than momentum. This supports in a general way the view that the fluctuating pressure gradients cannot generally be ignored, as the momentum transfer theory implies. On the other hand, no adequate check of the vorticity transfer theory has so far been provided by experiments on free turbulence.

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(5) Görtler, "A new approximation method for the numerical evaluation of free turbulence problems." (Z.A.M.M. vol. xxii. No. 5, Oct. 1942.) (Translated as A.R.C. 7947, FM 718.)
(6) Reichart, "Heat transfer through turbulent friction layers." (Z.A.M.M. vol. xx. No. 6, Dec. 1940, pp. 297–328). (Translated as N.A.C.A. Tech. Memo. No. 1047 and reproduced as A.R.C. 7375, FM 653.)
(7) Fage and Falkner, Proc. Roy. Soc. A 135, p. 702 (1932).

APPENDIX I.

DERIVATION OF THE FORMULÆ IN PARAGRAPH 4.

Paragraph 4.1.—*Wake of symmetrical cylinder.*

Straightforward substitution from (4.1.2) in equations (4.1.1) leads to the formulæ

$$g'(\eta)=C(f+\eta f'),$$
$$g_1'(\eta)=C(f_1+\eta f_1'),$$

which integrate immediately to (4.2.3), since $g(0)=0=g_1(0)$ by symmetry.

Paragraph 4.2.—*Wake of body of revolution.*

Substitution from (4.2.2) in (4.2.1) leads to

$$g+\eta g'=C(2\eta f+\eta^2 f'),$$
$$g_1+\eta g_1'=C(2\eta f_1+\eta^2 f_1'),$$

---

* The apparent confirmation of the vorticity transfer theory by experiments on the wake of a heated cylinder is discussed in Appendix III.

which integrate to (4.2.3), since $g(0)=0=g_1(0)$. It should be noted that the constant C is defined differently in paragraph 4.1 and paragraph 4.2.

Paragraph 4.3.—*Plane jet.*

We write for the stream function

$$\psi=\mathrm{AB}x^{\frac{1}{2}}\mathrm{F}(\eta),$$

so that

$$\mathrm{U}=\frac{\partial\psi}{\partial y}=\mathrm{A}x^{-\frac{1}{2}}\mathrm{F}'(\eta)=\mathrm{U}_1\mathrm{F}'(\eta)=\mathrm{U}_1f(\eta),$$

$$\mathrm{V}=-\frac{\partial\psi}{\partial x}=\mathrm{AB}x^{-\frac{1}{2}}[\eta\mathrm{F}'-\tfrac{1}{2}\mathrm{F}].$$

Substitution in (4.3.1) leads to the result

$$g'=\frac{\mathrm{B}}{2}(\mathrm{F}'^2+\mathrm{FF}'')$$

which integrates to

$$g(\eta)=\frac{\mathrm{B}}{2}\mathrm{F}(\eta)\,\mathrm{F}'(\eta),$$

since $g(0)=0$. Replacing $\mathrm{F}(\eta)$ by $\int_0^\eta f(\eta)\,d\eta$ we obtain the first equation of (4.3.3).

The second equation of (4.3.3) for $g_1(\eta)$ is derived similarly, making use of the above expressions for U and V.

Paragraph 4.4.—*Round jet.*

We write for Stokes's stream function

$$\psi=\mathrm{AB}^2x\mathrm{F}(\eta),$$

so that

$$\mathrm{U}=\frac{1}{r}\frac{\partial\psi}{\partial r}=\frac{\mathrm{A}}{x}\cdot\frac{\mathrm{F}'(\eta)}{\eta}=\mathrm{U}_1f(\eta),$$

$$\mathrm{V}=-\frac{1}{r}\frac{\partial\psi}{\partial x}=\frac{\mathrm{AB}}{x}\left(\mathrm{F}'-\frac{\mathrm{F}}{\eta}\right).$$

Substitution for U, V and $\overline{uv}$ in (4.4.1) leads, after considerable reduction, to

$$\frac{1}{\mathrm{B}\eta}\frac{d}{d\eta}(\eta g)=\frac{\mathrm{F}'^2}{\eta^2}+\frac{\mathrm{FF}''}{\eta^2}-\frac{\mathrm{FF}'}{\eta^3}.$$

Integration gives

$$\eta g=\mathrm{BFF}'/\eta,$$

since $g(0)=0$. Replacing $\mathrm{F}(\eta)$ by $\int_0^\eta \eta f(\eta)\,d\eta$ gives the first equation (4.4.3).

The second equation (4.4.3) for $g_1(\eta)$ is derived as for the plane jet, using the above expressions for U and V.

Paragraph 4.5.—*Wake of a row of rods.*

The results are obtained by direct substitution.

Paragragh 4.6.—*Plane mixing region.*

We put the stream function $\psi=U_0 B x F(\eta)$, where η is defined in (4.6.2), so that

$$U=\frac{\partial\psi}{\partial y}=U_0F'(\eta)=U_0f(\eta),$$

$$V=-\frac{\partial\psi}{\partial x}=BU_0[\eta F'(\eta)-F(\eta)].$$

Also $F(\eta)=\int_b^\eta f(\eta)\,d\eta$ where b is an arbitrary lower limit of integration. Substitution of these expressions for U and V and for $\overline{uv}$ from (4.6.2) in (4.6.1) gives

$$(1-a)^2g'(\eta)=BFF''.$$

Hence
$$(1-a)^2g(\eta)=B\left[FF'-\int_b^\eta F'^2\,d\eta+D\right],$$
$$=B\left[f\int_b^\eta f\,d\eta-\int_b^\eta f^2\,d\eta+D\right],$$

where D is a constant of integration.
Substituting the above expressions for U and V and from (4.6.2) in the second equation of (4.6.1) we obtain

$$(1-a)g_1'(\eta)=BFf_1'.$$

Integration of this equation leads to

$$(1-a)g_1(\eta)=B\left[f_1\int_b^\eta f\,d\eta-\int_b^\eta f_1f\,d\eta+D_1\right],$$

where D_1 is a constant of integration.

APPENDIX II.

NOTE ON WALL TURBULENCE.

The well-established logarithmic velocity distribution for flow near a wall was originally derived by Karman from his similarity theory and concurrently by Prandtl from the momentum transfer theory. It can, however, be deduced from dimensional analysis, as will now be shown.

For turbulent flow near a plane wall with constant shear stress τ_0, we can write

$$\frac{\tau_0}{\rho}=U_\tau^{\,2}=\epsilon\frac{dU}{dy}, \quad \text{. . . . .} \quad \text{(II.1)}$$

where U is the velocity at a distance y from the wall,

U_τ is called the friction velocity,

and ϵ is the eddy viscosity.

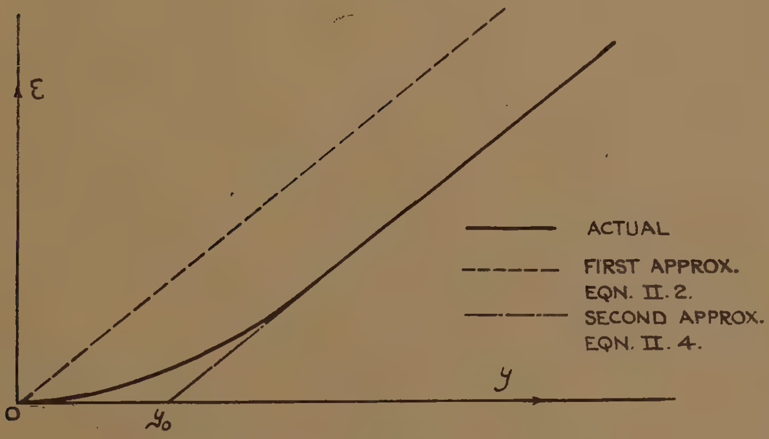
The eddy viscosity ϵ can depend only on U_τ and y, which are the only quantities affecting the motion, the influence of viscosity being negligible by hypothesis. Since ϵ has the dimensions (length)^2/(time) it must be represented by an equation of the form

$$\epsilon = KU_\tau y, \qquad \text{(II.2)}$$

where K is a constant. From (II.1) and (II.2) it follows that

$$U = \frac{U_\tau}{K} \log_e y + \text{const.}, \qquad \text{(II.3)}$$

Fig. 5.



Variation of eddy viscosity near a wall.

which is the standard result. The above treatment has some advantage over the usual ones in that the minimum number of assumptions are made.

The investigation of flow near a plane wall can be carried a stage further without difficulty. We note first that the thickness of the laminar layer near the wall is necessarily a multiple of ν/U_τ.

Dimensional analysis can only be used to show that the eddy viscosity is given by (II.2) for distances from the wall which are large in comparison with the thickness of the laminar layer. The probable variation of the eddy viscosity with distance from the wall is shown diagrammatically in fig. 5. (The variation of the mixing length on the momentum transfer theory is similar.) A second approximation for the eddy viscosity is obviously

$$\epsilon = KU_\tau (y - y_0), \qquad \text{(II.4)}$$

where y_0 is a length related to the laminar layer thickness. Including the effect of viscosity (II.1) is replaced by

$$U_\tau^2=(\nu+\epsilon)\frac{dU}{dy}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (II.5)$$

where $\nu+\epsilon=\nu+KU_\tau(y-y_0)$, when ϵ is given by (II.4).
Integration of (II.5) gives

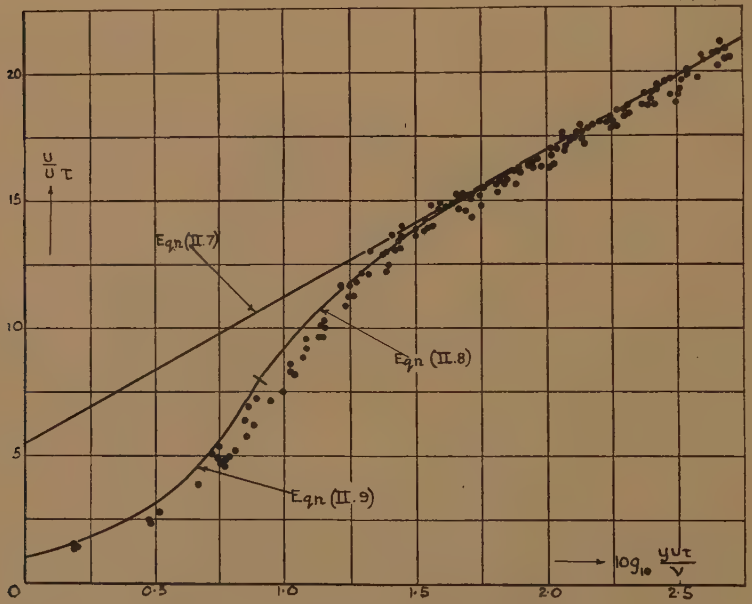
$$\frac{U}{U_\tau}=\frac{1}{K}\log_e\left[\frac{U_\tau(y-y_0)}{\nu}+\frac{1}{K}\right]+A, \quad . \quad . \quad . \quad . \quad . \quad (II.6)$$

where A is a constant.

For values of y such that $\frac{U_\tau y}{\nu}>30$, the experimental results for flow in smooth circular pipes can be represented by the equation (Ref. (1), p. 336)

Fig. 6.

FIG. 6.



Velocity distribution near a wall.

$$\frac{U}{U_\tau}=2{\cdot}50\log_e\left(\frac{U_\tau y}{\nu}\right)+5{\cdot}5, \quad . \quad . \quad . \quad . \quad . \quad . \quad (II.7)$$

and hence we take $K=0{\cdot}40=1/(2{\cdot}50)$ and $A=5{\cdot}5$. An examination of the experimental data for $8<\frac{U_\tau y}{\nu}<30$ shows that eqn. (II.6) represents them fairly well if $\frac{U_\tau y_0}{\nu}$ is taken to be 7·8. With these values (II.6) becomes

$$\frac{U}{U_\tau}=2{\cdot}5\log_e\left(\frac{U_\tau y}{\nu}-5{\cdot}3\right)+5{\cdot}5. \quad . \quad . \quad . \quad . \quad . \quad (II.8)$$

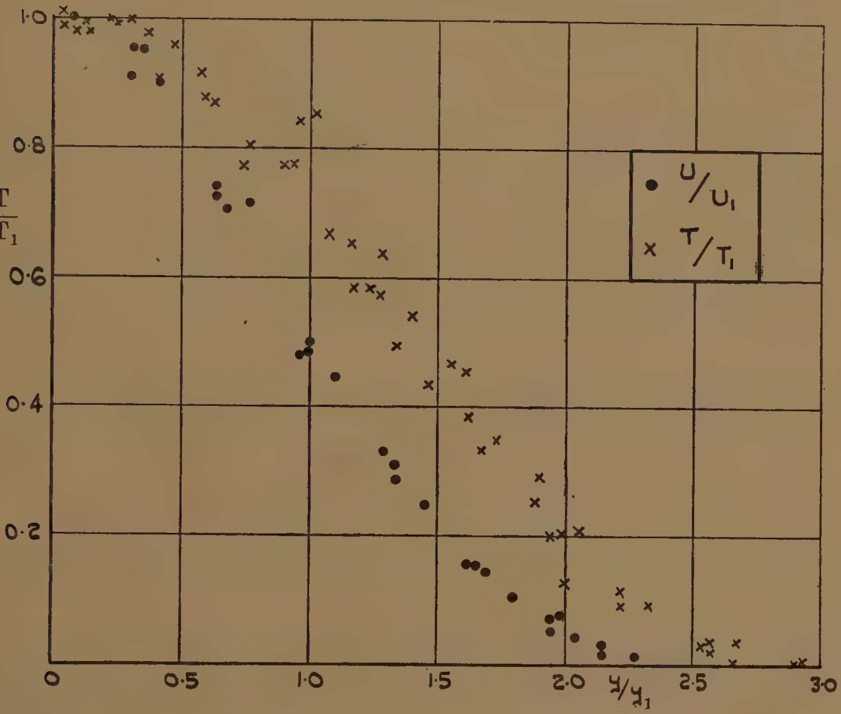
This equation is valid for values of y such that $\frac{U_\tau y}{\nu}>7{\cdot}8$.

For smaller values of y the eddy viscosity is zero, the velocity distribution is roughly linear and is given by the viscous flow formula

$$\frac{U}{U_\tau}=\frac{U_\tau y}{\nu}. \qquad \text{(II.9)}$$

For $\frac{U_\tau y}{\nu}=7\cdot 8$ the two expressions for the velocity join smoothly. The variation of U with y given by (II.8) and (II.9) is compared with the experimental results * in fig. 6; it will be seen that the calculated values are a little too high in the transition layer $6<\frac{U_\tau y}{\nu}<20$; this is due to

Fig. 7.



Fage and Falkner's measurements of velocity and temperature in the wake of a heated cylinder.

the assumption that the eddy viscosity falls off linearly to zero for $\frac{U_\tau y}{\nu}=7\cdot 8$ (fig. 5), whereas in fact it presumably falls off gradually to zero for $y=0$.

APPENDIX III.

Comparison between Theory and Experiment for Cylinder Wakes.

The experimental results obtained by Fage and Falkner (7) for the velocity and temperature in the wake of a heated cylinder are given in fig. 7.

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* The experimental results are taken direct from Reichart (fig. 1 of Ref. (6)).

The first deduction from the results is that the station where the temperature is half the temperature at the centre is farther out than the station where the velocity defect is half the velocity defect at the centre *. In other words, heat spreads more rapidly than momentum. It follows that the momentum transfer theory cannot be valid for these cases, since this theory requires that the velocity and temperature distributions should be similar. The agreement between the measured velocity distribution and the distribution calculated from the momentum transfer theory is therefore fortuitous; this illustrates the difficulty of checking any theory in this field.

The main contention of the vorticity transfer theory, that the effect of the fluctuating pressure gradients on momentum transfer could not generally be ignored, is thus borne out by the experimental results. However, the calculation of velocity and temperature distributions on the vorticity transfer theory requires not only the hypothesis that vorticity is transferable but also some assumption about the variation of the mixing length across the wake or jet. The agreement between theory and experiment for cylinder wakes (Ref. (1), p. 663) is not confirmed for jets (Ref. (1), p. 673), and hence may also be fortuitous.

II. *On Some Focal Properties of an Electron Optical Immersion Objective.*

By P. A. Einstein, M.Sc., and L. Jacob, Ph.D., F.Inst.P.,
Electrotechnics Dept., The University, Manchester †.

[Received July 16, 1947.]

Summary.

(1) The electrolytic tank field plots indicate that a simple exponential law $V = Ae^{kz}$ expresses the early portion of the axial potential distribution of the objective.

(2) The first focal length has a value of the order of the modulator hole diameter; it decreases as the cathode/modulator spacing is increased.

(3) The variation of focal length with initial velocity is surprisingly small—about $\frac{1}{2}$ per cent. from range zero to the most probable velocity (0·087 v.).

(4) The size of the "crossover" is proportional to both the initial radial velocity and to the angle of ejection (within the paraxial limits). It decreases as the cathode is moved away from the modulator.

(5) For constant black-out voltage, the potential of the most probable crossover is independent of geometry.

(6) The cathode image moves further and further from the cathode as the spacing is decreased and magnifications of three and over were recorded.

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* A similar result holds for heated plane jets[3].

† Communicated by the Authors.

1. *Introduction.*

The problem of determining the focal properties of the electron optical immersion objective is complicated by a variety of factors each of which requires special consideration. In the case of any other type of electron lens, it is possible to work with beams of uniform velocity in which the current density is so small as to have a negligible effect on the field. This is not the case with the objective. Not only is the emitting cathode a boundary of the lens field, but the spread in velocities due to thermal emission introduces a chromatic effect, which, like its optical counterpart, results in a whole series of focal lengths, one for each wave-length (or velocity). Superimposed on this is the spherical error. This might be expected to be of some consequence since quite a considerable fraction of the electron beam is emitted from the centre of the cathode at angles of over 70° to the axis. An additional complication results from the space charge of the beam itself. This produces a lowering of the space potential at every point enclosed by the beam, the effect being the more marked the lower the beam velocity, that is, in the region close to the cathode.

In a general survey of the focusing properties of the immersion objective, Johannson (1) examined the quality of the cathode image in its relation to the form of the field, without a detailed assessment of the extent of its specific aberrations. The information yielded was essentially of a qualitative character, and gave a useful overall picture of its performance.

A more detailed investigation of the variation in focal length with electron velocity was undertaken by Maloff and Epstein (2), who indicate a 7/1 variation over a range of speeds from 0·05 to 1·0 volt for their type of system.

The main part of the present work was directed to a detailed examination of the chromatic error, as it affects the character of the crossover and also of the cathode image, when the geometry of the system was varied. A special study was also made of the form of the trajectory with angle of ejection from the cathode, over a range of angle limited only by the paraxial conditions imposed by the equations of motion.

In order to accomplish this, it was necessary to obtain the field distribution along the axis, as a starting point. This was done in the electrolytic tank. A step by step method of approximation on the lines suggested by Maloff and Epstein was then applied to the field plot, and the trajectories traced out through the entire field. In this way the whole course of the image-forming properties of the system was examined.

2. *The Axial Distribution.*

A method of using the electrolytic tank to plot out the field has been described in detail by Jacob (3). To expedite the recording the method was made semi-automatic by amplifying the potential difference between the pick-up probe electrode and one of the main electrodes. This was then applied to a cathode ray oscillograph whose time base was synchronized with the movement of the probe. In this way, the distribution was shown up as a trace on the face of the oscillograph. Fig. 1 shows the elements of the immersion objective lens structure, in which the accelerating electrode was an open-ended cylinder facing the modulating electrode. The axial distributions for the various cathode to modulator (C/M) separations with a fixed (M/A) spacing of 6 mm. throughout are shown in fig. 2. It will be noticed that the curves consist of three distinct parts :—(*a*) a gently rising portion between the cathode and a point on the remote side of the modulator, followed by (*b*) a linear portion having an increased slope and extending close to the accelerating electrode, (*c*) a decrease in slope again, terminating in a flat part indicating a field-free region. This latter is usually the potential of the first focusing electrode.

For the purposes of the subsequent calculations, the region of immediate interest was the curve contained by (*a*). This was specially amplified and additional traces of it obtained for the various geometries, since it includes the position of the crossover and also affects the cathode image. It was found that it could be expressed by a simple exponential function $V=Ae^{kz}$,

TABLE I.

C/M spacing in tank (z)	A (for 17·5 v. R.M.S.)	"k"
0·5 mm.	1·0	0·32
1·0	0·53	0·37
2·0	0·28	0·43
3·0	0·18	0·47
5·0	0·03	0·61

with constants determined exclusively by the geometry and the voltage. These are recorded in Table I. for the various geometries studied in the tank.

This simplification is a great convenience when the first focal lengths corresponding to the various geometries have to be calculated from the usual lens formula. The simple formula, however, only applies to the limits of the initial region explored by the probe. In order to fulfil the condition that the potential takes a value zero at the surface of the cathode, it is necessary to modify the simple law by introducing another exponential term, *i. e.* $V=A(e^{kz}-e^{-kz})$, which thus satisfies the boundary condition. The term e^{-kz} will decrease in value as z increases, so that the simple exponential will then define the distribution. The gradient at the cathode surface now becomes $2Ak$ and thus depends on the value of the contants determined by the log. plots. This requires the field in the immediate vicinity of the cathode to be uniform, the initial paths of the electrons emitted at an angle from the centre of the cathode will take a parabolic form over a very short interval of space ; this will not interfere with the paraxial conditions in the main part of the focusing field. For the purposes of the calculation, the measured values of the potential were used when dealing with the region explored in the tank.

Fig. 1.

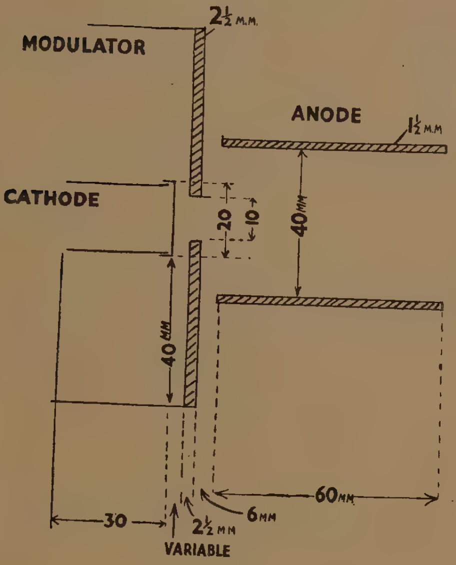
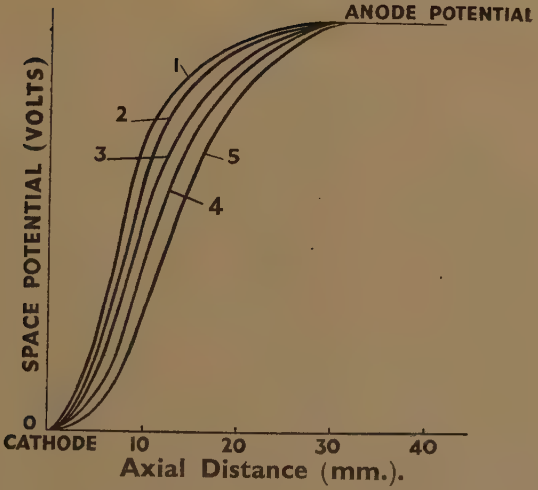

2½ M.M.
MODULATOR
ANODE
1½ M.M
CATHODE
20
10
40MM
40MM
60MM
30
2½MM
6MM
VARIABLE

Fig. 2.



(1) C/M .. .. 0·5 mm.
(2) C/M .. .. 1·0 mm.
(3) C/M .. .. 2·0 mm.
(4) C/M .. .. 3·0 mm.
(5) C/M .. .. 5·0 mm.
Mod. spaced, 6 mm. from anode.

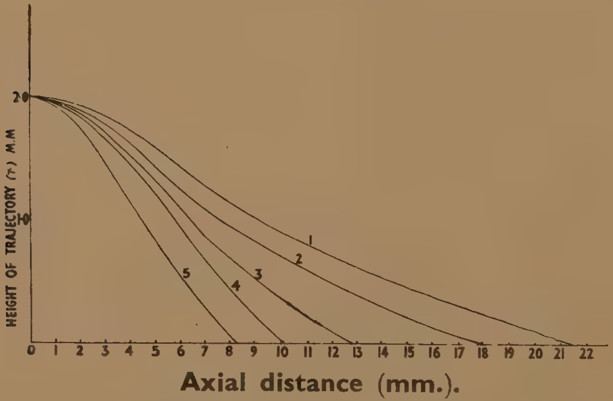
3. *The Plotting of Trajectories : Zero Velocity of Emission.*

Once the potential distribution along the axis is known for any system, it is possible to calculate the refractive effect of the field as the electron passes from one equipotential to another in its passage through the lens. The radial velocity is obtained from the relation established by Maloff and Epstein,

$$v_r = \frac{e}{m}\left(\frac{dV}{dz}\right) \cdot \frac{r}{R} \cdot t + v_{r_0} = C_1 t + v_{r_0} . \qquad \qquad (1)$$

Here $\frac{dV}{dz}$ is the gradient of the field, "r" the height at the first equipotential, R the average radius of curvature (assuming spherical surfaces for the paraxial region), and v_{r_0} the initial radial velocity, and "t" is the time the electron spends in passing between the surfaces.

Fig. 3.



Trajectories : {Parallel emission, zero velocity.} 30-volt black-out.

(1) C/M ..	.. 0·5 mm.	Mod. to anode, 6 mm. (fixed).
(2) C/M ..	.. 1·0 mm.	
(3) C/M ..	.. 2·0 mm.	
(4) C/M ..	.. 3·0 mm.	
(5) C/M ..	.. 5·0 mm.	

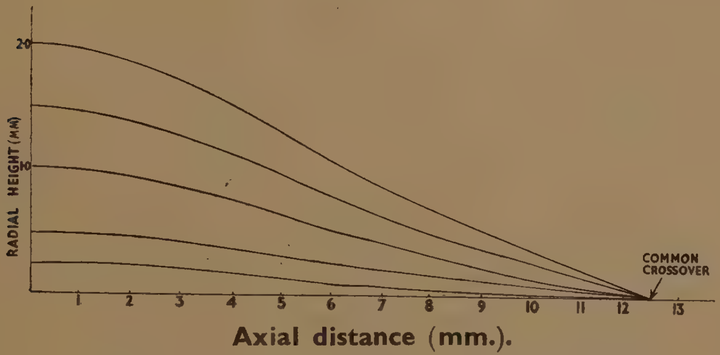
The difference in height in going from one to the other is then

$$\Delta r = \frac{C_1 t^2}{2} + v_{r_0} t . \qquad \qquad (2)$$

By calculating the height corresponding to the known constants of the field plot, the complete step by step process yields a first approximation to the trajectory. This was carried out for all the systems given in Table I., for electrons of zero velocity emitted in a direction parallel to the axis, and starting from a height of 0·2 cm. above the centre of the cathode. The results are shown in fig. 3. The intersections of the paths with the axis yields a first approximation for the focal lengths of the various systems. These will be seen to have a value of the order of the modulator hole diameter. For these particular systems the first focal length decreases the larger the value of the C/M spacing, *i. e.*, with increase in the value of "*k*." This must mean that the longitudinal acceleration becomes of increasing importance as the cathode approaches the modulating electrode; the lens thus becomes weaker. The intersections with the axis not only define the focal lengths, but also the position of the plane of the crossover. This shifts away from the cathode as the latter is brought closer to the modulator. The product of crossover distance and "*k*" has an average value of about 5, which enables the focal length to be determined as an approximation, from the value of "*k*" derived from the field plots.

The effect of varying the initial height of emission for one particular system gives the family of curves of fig. 4, where all meet at the crossover independent of the height of origin. This would be expected to occur if the paraxial conditions were strictly adhered to, as was the case for these calculations; the focusing is then performed by a lens free from aberrations, and the radial height at any point along the trajectory is directly proportional to the initial height. This follows directly from consideration of equations (1) and (2), since C_1 is proportional to "r" and the other terms are constant.

Fig. 4.



C/M .. .. 2·0 mm. } Zero initial velocity,
M/A .. .. 6·0 mm. } parallel emission.

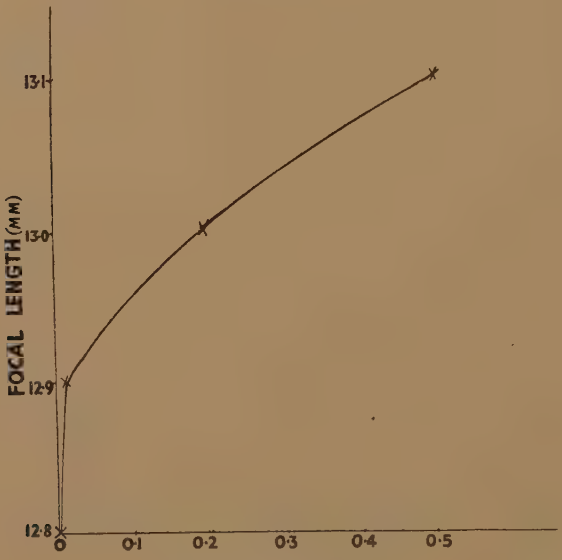
In fact, the common point focus is a necessary condition for all emitting heights where the initial velocity v_{r_0} is constant, when focusing is carried out by such lens fields.

4. *The Longitudinal Chromatic Error.*

The step by step method is particularly helpful when conisdering the effect of initial velocities. Here the velocity acquired by the electron, which has an initial value of V_i volts, is given by $v=5{\cdot}95\times10^7\sqrt{V+V_i}$ cm./sec., where V is the value of the equipotential at the point considered.

The term in v_{r_0} of equations (1) and (2) will be seen to affect the height in the field, hence the effect of initial velocity is to cause the electron to cross the axis at a point determined by the velocity, *i. e.*, to introduce a chromatic error into the focusing system. There will thus be a whole series of focal points on the axis, each corresponding with a specific initial velocity from a single emitting point, over the whole range of velocities.

Fig. 5.



Initial velocity.

The extent of this longitudinal chromatic defect will, for any fixed geometry, depend on the ratio of the initial velocity to that acquired in the field, and may be quite serious for systems which require a low anode potential. To illustrate the extent of the defect, the case of the system C_2M_6A was taken, in which electrons entered the field parallel to the axis at a point 0·2 cm. from the centre of the cathode with the following values of initial velocity :—zero ; 0·01 ; 0·2 ; 0·5 volts respectively. The trajectories were traced through twenty-five surfaces of the field for each case under the fixed condition of a 30 v. black-out, and the end result of the plot is shown in fig. 5 in the form of the variation of crossover point with initial electron velocity. The first three entries of Table II. indicate the data required to enable the calculations to be completed.

The variation of focal length with initial velocity is surprisingly small for the system considered. It amounts at most to about 2 per cent.

over the velocity range chosen: this means an increase of about $\frac{1}{2}$ per cent. for electrons with the most probable velocity of 0·087 volts. The plane of the crossover will thus be shifted to this extent through the variation in initial velocity. If the crossover is to be imaged by a final lens to give the spot on the screen, the plane chosen for the object will be that in which the largest electron concentration occurs within the smallest area. This will normally correspond with that for the most probable velocity of emission from the cathode. Whether or not the focusing will be critical depends further on the paths taken by electron rays from the centre of the cathode which go to form the periphery of the crossover.

5. *The Crossover.*

The existence of a "crossover" or minimum section of the beam was first pointed out by Maloff and Epstein (2). Its position, as already stated, is defined by the first focal plane of the lens field and is usually situated on the side of the modulator remote from the cathode. Its size is in a sense determined by the amount of transverse chromatic aberration, introduced by the spread of initial velocities in a radial direction, *i. e.*, perpendicular to the axis, since, if all the electrons had only normal velocity components, no "crossover" would be formed.

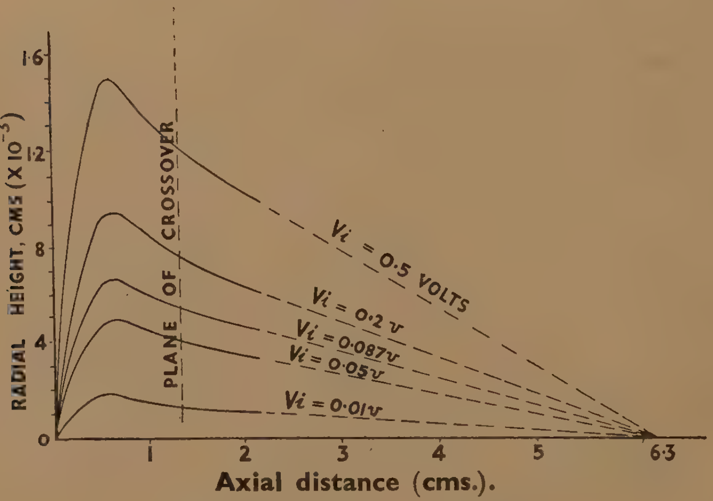
TABLE II.

C_2M_6A : $V_A = 450$ volts ($\equiv -30$ v. black-out). Initial height=0·200 cm.
Initial velocity=0·20 volts (horiz.).

Z cms.	Volts on axis (V)	V_{AV}	(dV/dz) volts/cm.	Rad. of curv. R (cm.)	Time bet. two equip. $\times 10^{-10}$ (sec.)	C_1	Radial velocity $v_r(\times 10^6)$ cm./sec.	Δr $(\times 10^{-4})$ cm.	Radial height cm.
0·03	1·86	0·93	62·7	3·63	4·78	6·06	2·89	6·89	0·1993
0·06	3·79	2·84	63·7	1·85	2·91	12·15	6·42	13·54	0·1980
0·09	5·75	4·77	63·4	1·27	2·28	17·41	10·38	19·12	0·1961

For any constant angle of ejection from the centre of the cathode, and for constant field conditions, the electrons meet the plane of the crossover at different heights determined by their velocities; the largest velocities give the greatest displacement from the axis of the system. In fact, the velocity distribution in the crossover is a pattern of that from the source at the centre of the cathode. The intensity distribution for perfect focusing fields has already been dealt with by Law (4) and Langmuir (5). The ray tracing reveals the variation in size with both velocity and angle within the paraxial range. For the system C_2M_6A which was chosen, a fixed angle of 10° was taken, and the trajectories traced point by point through the field for the ejection velocities in this direction and shown in fig. 6. Examination of the traces shows that the heights which the rays attain in passing from the cathode to the plane of the crossover are, for every point of the axis, proportional to the initial radial velocity, so that it would appear that the crossover size is related in a simple way to the initial conditions. While this is strictly true in theory, in practice the measured size will obviously depend on the sensitivity of the method chosen for measurement. The electrons with the highest speeds will be relatively few in number compared with those having the most probable velocity, so that their contribution to the intensity distribution (possibly a couple of per cent.) may not be appreciably noticed. Their effect in determining crossover size may thus be insignificant. In order to standardize results, it was decided to refer all cases to the most probable crossover (corresponding with the most probable velocity of 0·087 volts determined from the temperature of the emitting cathode).

Fig. 6.

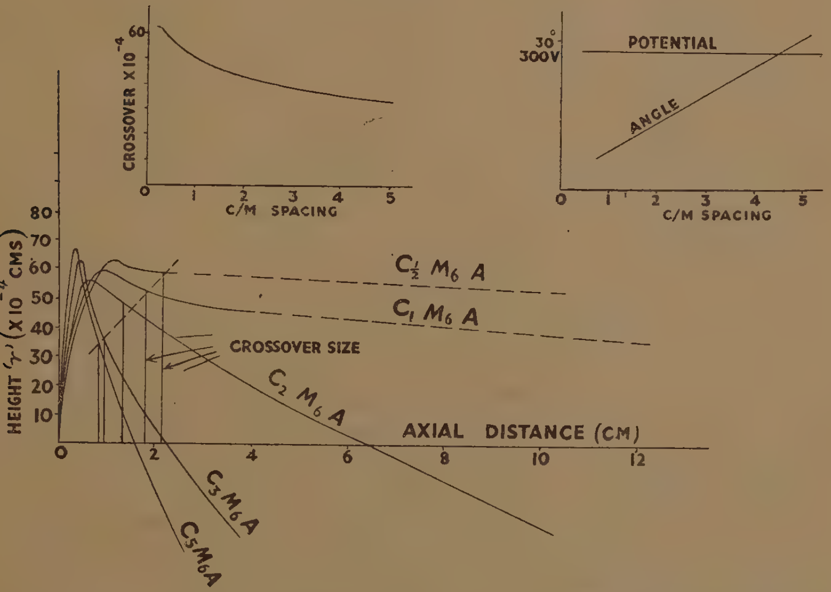


Cath./mod.	2 mm.
Mod./anode	6 mm.
Black-out	30 volts.

V_i= initial velocity.
Most prob. vel.=0·087 volts.
Angle of ejection=10°.

The variation of the crossover size with angle for a constant velocity of ejection would also be expected to show a simple relation. This is found to be so: the height at any point is directly proportional to the angle of emission within the paraxial limits. Using these two general observations, it is possible to deduce the trajectories for any angle or velocity within these limits. When the cathode/modulator spacing is varied, the anode voltage has to be adjusted for each geometry to keep the black-out voltage constant, in this case at −30 volts. The traced size of the most probable crossover under this condition, for the various geometries, is shown in fig. 7. Its value appears to decrease as the cathode is pulled away from the modulator, at first fairly rapidly and then so slowly that there is hardly any further change in size beyond a separation of about 3 mm. The direction of the variation seems to follow that in $1/k$, which defines the power of the lens, so that with such systems and quite apart from the effects of aberration and space charge, the focused spot size will decrease as the C/M separation increases. Such a result can be directly applied, since it enables larger tolerances to be used in design, and is a helpful feature in assembly. The same figure illustrates some further interesting properties connected with the crossover, and revealed by the ray-tracing. If the black-out voltage is kept constant at —30 volts for each geometry, then the value of the potential at the most probable crossover point appears to be independent of the geometry. Its position will, of course, vary from system to system, as pointed out earlier. If, however, the anode voltage is maintained constant for each system, then the potential at the crossover will fall away almost linearly with increase in C/M spacing, *i. e.*, with increase in the value of "*k*."

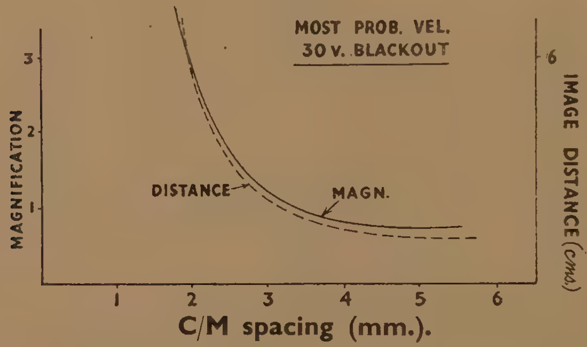
Fig. 7.



The plots also enable the beam angle to be measured directly. This is defined by the intersection with the axis at the crossover. It increases with the "*k*" of the system, indeed, much more rapidly than the increase in "*k*" itself. This is not what would be expected from purely geometrical considerations, since, for example, the angle subtended at the centre of the modulator hole decreases as the cathode is pulled away from the modulator. The shortening of the focus in the latter case must be caused by the powerful effects of the field in the extended low velocity region in this space. It would, however, not be correct to generalize from this particular example, since it is known that with some systems of different electrode configuration, and hence different field forms, the beam angle variation follows the variation in geometry. The increase with the value of "*k*" as found here, however, appears to be general.

The measured value of the beam angle, defined by the final angle of emergence from the accelerating electrode into the field-free space of the first focusing electrode, will, as a rule, be very different from that measured at the crossover, since the beam suffers further refractions all the way till the equipotential corresponding with the accelerator voltage is reached. From here on, it passes rectilinearly through the remainder of the system. There does not appear to be a simple relation between the angle at the crossover, and that measured outside the lens field by the use of a sliding disc engraved with fluorescent material which becomes luminous when the beam strikes it. The former is the more important when applying the Helmholtz–Lagrange law along the beam, since the other data, *i. e.*, potential and size, are already known from the tracings ; the latter is of first importance when measurements of the aberrations are undertaken in connection with subsequent lens systems.

Fig. 8.



6. *The Cathode Image.*

If the ray-tracing is carried out through the entire field of the lens, then the image of the emitting points on the cathode can be constructed step by step to form an image of the emitting source. In reality there is a cathode image for each velocity of emission and its position along the axis is determined by that velocity (as was the case for the crossover), but the calculations have shown that there is no significant variation with velocity. As an example, it was found that with the system C_2M_6A the variation in the image size for a velocity range 0–0·5 volts was only some 5 per cent. To link up with the variation in geometry, the image size was calculated for each of the lens systems previously considered. The magnification and image distance for such systems in vacuum, under the condition of a constant value of black-out voltage, *e. g.* —30 v., in all cases is shown in fig. 8.

The three points relating to the larger spacings were arrived at without difficulty. For the smaller spacings, the rays emitted from the centre of the cathode were found to be so nearly parallel to the axis when they emerged from the field, that some doubt arose as to their points of intersection with the axis to form the cathode image point. Because of this doubt they were not recorded in the figure, which shows generally that the cathode image moves farther and farther from the cathode as the spacing is decreased—at first fairly slowly, and then very rapidly indeed.

In all cases with these systems, the image is found to be inside the first accelerator and at a space potential not far removed from its potential. If the image space is kept field free, the image will be in good focus over a considerable distance on a screen placed to receive it. This remarkable property of large depth of focus is characteristic of these immersion objective systems. They thus make good low power electron microscopes. The cathode image quality, is, however, not perfect, since they generally show some pin-cushion type of distortion which varies with the field conditions.

As regards magnification, this increases as the C/M spacing is reduced, and recorded values of 3 (and probably more) give the order to be expected for the more closely spaced systems.

References.

(1) Johannson, *Ann. d. Phys.* xviii. p. 385 (1933).
(2) Maloff and Epstein, P.I.R.E. xxii. p. 1386 (1934).
(3) Jacob, Phil. Mag. xxv. p. 570 (1938).
(4) Law, P.I.R.E. xxv. p. 954 (1937).
(5) Langmuir, *ibid.* p. 977.

III. *On the Theory of Point Electrons.*

By Professor P. A. M. DIRAC.*

[Received June 7, 1947.]

I WOULD like to answer some criticisms which T. Lewis (1945) has made of my work on the Hamiltonian formulation of the classical equations of motion for point electrons. I shall keep the discussion to the case of a single electron interacting with the electromagnetic field, as Lewis does in most of his paper.

The situation as described by Lewis at the top of p. 536 is not quite correct. Variation of I_1 by itself gives equations of motion which do include the reaction of an electron to its own field, but they apply only to the case when $A^{\mu}_{\text{in}}+A^{\mu}_{\text{out}}=0$. For a single electron this would mean straight-line motion. The addition of I_2 is needed to get the general case of arbitrary A^{μ}_{in}.

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* Communicated by the Author.

In varying the action integral (2.9) of Lewis' paper to get the equation of motion of the electron, one must take into account correctly the variation of the fields. From the physical point of view one would be inclined to look upon A^{μ}_{in} as independent of the electron and to take $\delta A^{\mu}_{\text{in}}=0$. This is what Lewis does. However, to get the correct equation of motion one must take $A^{\mu}_{\text{in}}+A^{\mu}_{\text{out}}$ to be the field independent of the electron, so that at a fixed point in space-time

$$\delta(A^{\mu}_{\text{in}}+A^{\mu}_{\text{out}})=0$$

or

$$\delta(A^{\mu}_{\text{in}}+\tfrac{1}{2}A^{\mu}_{\text{ret}}-\tfrac{1}{2}A^{\mu}_{\text{adv}})=0.$$

One then gets precisely Lewis' equation (2.5). Only by taking $A^{\mu}_{\text{in}}+A^{\mu}_{\text{out}}$ as the independent field can one get the equations of motion in Hamiltonian form.

The main part of Lewis' criticism is concerned with the rôle of the proper-time variables in my theory. Lewis has not understood the main structure of my theory, and as other readers of my work may have similar difficulties I will try to explain it more clearly.

The world-line of the electron is specified in a relativistic manner by giving the four co-ordinates z_{μ} as functions of the proper-time s,—thus it is specified by the four functions $z_{\mu}(s)$. The world-line determines the retarded potentials A^{ν}_{ret} at all points of space-time. Thus the retarded potentials at any point $\mathbf{x}$, namely $A^{\nu}_{\text{ret}}(\mathbf{x})$, are functionals of the four functions $z_{\mu}(s)$, and similarly for the advanced potentials.

For the purposes of the Hamiltonian formulation it is necessary to take a certain point on the world-line, specified by the proper-time s' say. The retarded and advanced potentials at a given point $\mathbf{x}$ are independent of s' (as they were defined already before we introduced s'), so we have

$$\frac{d\,A^{\nu}_{\text{ret}}(\mathbf{x})}{ds'}=0, \qquad \frac{d\,A^{\nu}_{\text{adv}}(\mathbf{x})}{ds'}=0. \quad \ldots\ldots \quad (1)$$

The parameter s' is the independent variable of the Hamiltonian formulation, the Hamiltonian equations of motion being

$$\frac{d\zeta}{ds'}=[\zeta, F],$$

where F is the Hamiltonian function. The dependent variables, typified by ζ, consist of the co-ordinates of the electron $z_{\mu}(s')$ at time s', the momenta $p_{\mu}(s')$ at time s', and certain potential variables $A_{\mu}(\mathbf{x})$ which depend on s' and which therefore differ from all the potentials occurring in the usual formulation of electrodynamics, as the latter do not depend on s'. For the present the $A_{\mu}(\mathbf{x})$ should be considered simply as certain mathematical variables, independent of the electron variables $z_{\mu}(s')$ and $p_{\mu}(s')$, and needed for taking into account the degrees of freedom that the dynamical system has because of the existence of the field. Their physical significance becomes clear only after integration of the equations of motion, when they can be brought into connection with the usual potentials. The Hamiltonian F is a function of the $z_\mu(s')$, $p_\mu(s')$ and $A_\mu(\mathbf{x})$, and leads to well-defined equations of motion as soon as the Poisson Brackets of all these variables are known.

At this stage we may without confusion omit the prime from s' and write it simply s. The resulting notation is then the same as that of Lewis' paper and of my papers to which he refers.

We must fix the P. B.'s of our dynamical variables, the z_μ, p_μ and $A_\mu(\mathbf{x})$. For this purpose one may introduce canonical co-ordinates and momenta, assume the dynamical variables are certain functions of them, and work out the P. B.'s from the usual classical formula. However, it is not necessary to proceed in this roundabout way. It is permissible to postulate any P. B. relations which satisfy the following conditions :—

$$\left.\begin{aligned} [u, v] &= -[v, u], \\ [u_1+u_2, v] &= [u_1, v]+[u_2, v], \\ [u_1u_2, v] &= u_1[u_2, v]+u_2[u_1, v], \\ [u, [v, w]]+[v, [w, u]]+[w, [u, v]] &= 0. \end{aligned}\right\} \quad . \quad (2)$$

Provided these conditions hold, one knows that it must be possible to introduce canonical co-ordinates and momenta leading to the postulated P. B. relations, and one need not bother to work out an example of such canonical co-ordinates and momenta. With this general method of procedure it is not necessary that the Hamiltonian should be expressed as a function of co-ordinates and momenta. It may be given as a function of any dynamical variables (*e. g.* velocities), whose P. B. relations are known.

The P. B. relations I assume are

$$\left.\begin{array}{l} [z_\mu, z_\nu]=0 \qquad [p_\mu, p_\nu]=0 \qquad [p_\mu, z_\nu]=g_{\mu\nu}, \\ [z_\mu, A_\nu(\mathbf{x})]=0 \qquad [p_\mu, A_\nu(\mathbf{x})]=0, \\ [A_\mu(\mathbf{x}), A_\nu(\mathbf{x}')]=\tfrac{1}{2}g_{\mu\nu}\{\Delta(\mathbf{x}-\mathbf{x}'+\boldsymbol{\lambda})+\Delta(\mathbf{x}-\mathbf{x}'-\boldsymbol{\lambda})\}. \end{array}\right\} \quad . \quad . \quad (3)$$

They are easily seen to be consistent with (2), and with the help of (2) one can work out the P. B.'s of any functions of the dynamical variables.

The Hamiltonian is given by Lewis' equation (4.1) and the resulting equations of motion are Lewis' (4.5) and (4.6) and

$$\frac{dA_\mu(\mathbf{x})}{ds}=\tfrac{1}{2}ev_\mu\{\Delta(\mathbf{x}-\mathbf{z}+\boldsymbol{\lambda})+\Delta(\mathbf{x}-\mathbf{z}-\boldsymbol{\lambda})\}. \quad . \quad . \quad . \quad (4)$$

To obtain $dA_\mu(\mathbf{z})/ds$ we must substitute $\mathbf{z}$ for $\mathbf{x}$ in (4) and add on the contribution due to $\mathbf{z}$ depending on s, thus

$$\frac{dA_\mu(\mathbf{z})}{ds}=\tfrac{1}{2}ev_\mu\{\Delta(\boldsymbol{\lambda})+\Delta(-\boldsymbol{\lambda})\}+v_\nu\left.\frac{\partial A_\mu}{\partial x_\nu}\right|_{\mathbf{x}=\mathbf{z}} \quad . \quad . \quad . \quad (5)$$

Now since $\boldsymbol{\lambda}$ is assumed to be a time-like 4-vector, $\Delta(\boldsymbol{\lambda})=\Delta(-\boldsymbol{\lambda})=0$ and (5) reduces to Lewis' (4.8). Lewis' (4.9) then follows. One can thus deduce this equation without meeting the difficulties Lewis mentions.

On integrating (4) we get

$$\mathrm{A}_\mu(\mathbf{x})=\tfrac{1}{2}e\int_{-\infty}^{s} v_\mu(s')\{\varDelta(\mathbf{x}-\mathbf{z}(s')+\boldsymbol{\lambda})+\varDelta(\mathbf{x}-\mathbf{z}(s')-\boldsymbol{\lambda})\}ds'+\alpha_\mu(\mathbf{x}), \quad . \quad (6)$$

where $a_\mu(\mathbf{x})$ is a constant of the motion. A constant here means something independent of s, (or of our earlier s'). We take this constant to be

$$a_\mu(\mathbf{x})=\mathrm{A}_{\mu\,\mathrm{in}}(\mathbf{x}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This is permissible since the right-hand side of (7) is independent of s. The ingoing potentials thus appear as constants of integration of some of the equations of motion.

On putting $\mathbf{x}=\mathbf{z}$ in (6) and using properties of the $\varDelta$ function, we now get

$$\mathrm{A}_\mu(\mathbf{z})=\mathrm{A}_{\mu\,\mathrm{in}}(\mathbf{z})+\tfrac{1}{2}\{\mathrm{A}_{\mu\,\mathrm{ret}}(\mathbf{z}-\boldsymbol{\lambda})-\mathrm{A}_{\mu\,\mathrm{adv}}(\mathbf{z}-\boldsymbol{\lambda})\},$$

which, in the limit $\boldsymbol{\lambda}=0$, gives

$$\mathrm{A}_\mu(\mathbf{z})=\mathrm{A}_{\mu\,\mathrm{in}}(\mathbf{z})+\tfrac{1}{2}\{\mathrm{A}_{\mu\,\mathrm{ret}}(\mathbf{z})-\mathrm{A}_{\mu\,\mathrm{adv}}(\mathbf{z})\}. \quad . \quad . \quad . \quad (8)$$

Thus Lewis' equation (4.9) is the correct equation of motion of the electron with radiation damping. The fact that retarded and advanced potentials appear in (8) does not affect the validity of the deduction of this equation of motion, which follows directly from the P. B. relations (3) and the form of the Hamiltonian F. Lewis' equation (3.4) is quite correct, and follows from (8) and (1) with s for s'.

The P. B. relations connecting $\mathrm{A}_{\mu\,\mathrm{ret}}(\mathbf{x})$ and $\mathrm{A}_{\mu\,\mathrm{adv}}(\mathbf{x})$ with other dynamical variables are not known and cannot be worked out simply, since $\mathrm{A}_{\mu\,\mathrm{ret}}(\mathbf{x})$ and $\mathrm{A}_{\mu\,\mathrm{adv}}(\mathbf{x})$ depend on the electron variables at various times, so that one would have to integrate the electron equations of motion before one could evaluate such P. B.'s. However, if they could be worked out, they would have to be consistent with the other equations of the scheme.

The physical meaning of the potentials $\mathrm{A}_\mu(\mathbf{x})$ is made definite by equations (6) and (7). It may be described roughly as Lewis does at the top of p. 540, and is not confusing if one remembers that the field point $\mathbf{x}$ is completely independent of the electron point $\mathbf{z}(s)$. The latter involves the independent variable s of the Hamiltonian formulation, the former is merely a set of parameters specifying one of the dependent variables $\mathrm{A}_\mu(\mathbf{x})$. One must not restrict them as Lewis suggests by his relation (6.9), because one needs the potentials at field points $\mathbf{x}$ arbitrarily close to the electron point $\mathbf{z}(s)$.

Reference.

(1) Lewis, T., Phil. Mag., vol. xxxvi. pp. 533–541 (1945

IV. *Ill-conditioned Sets of Linear Equations.*

By Professor E. H. Neville *.

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This paper was provoked by the challenge to iterative methods implicit in the application † by Mr. J. Morris of his escalator process to the solution of the set of equations

$$\begin{bmatrix} 5{\cdot}39999 & 5{\cdot}23286 & 4{\cdot}35785 & 3{\cdot}62242 & 2{\cdot}76472 & 1{\cdot}84691 \\ 5{\cdot}23286 & 7{\cdot}87190 & 3{\cdot}62242 & 5{\cdot}25651 & 1{\cdot}84691 & 2{\cdot}80269 \\ 4{\cdot}35785 & 3{\cdot}62242 & 3{\cdot}88141 & 2{\cdot}97304 & 2{\cdot}63974 & 1{\cdot}67936 \\ 3{\cdot}62242 & 5{\cdot}25651 & 2{\cdot}97304 & 4{\cdot}37677 & 1{\cdot}67936 & 2{\cdot}63246 \\ 2{\cdot}76472 & 1{\cdot}84691 & 2{\cdot}63974 & 1{\cdot}67936 & 2{\cdot}01578 & 1{\cdot}14921 \\ 1{\cdot}84691 & 2{\cdot}80269 & 1{\cdot}67936 & 2{\cdot}63246 & 1{\cdot}14921 & 1{\cdot}94065 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \\ v \\ w \end{bmatrix}$$

$$=[1{\cdot}23679 \quad 0{\cdot}48448 \quad 1{\cdot}24950 \quad 0{\cdot}47304 \quad 1{\cdot}06470 \quad 0{\cdot}37831].$$

The escalator process is a successful device for the solution of frequency equations ; to recommend its use in this simpler problem is to ignore the possibility of using iterative processes intelligently. To expose by an example the inadequacy of the routine in which a set of equations

$$\begin{aligned} a_1x+b_1y+c_1z+\ldots&=\alpha, \\ a_2x+b_2y+c_2z+\ldots&=\beta, \\ a_3x+b_3y+c_3z+\ldots&=\gamma, \\ \ldots\ldots\ldots\ldots& \end{aligned}$$

is attacked by the iteration

$$\begin{aligned} a_1x_{n+1}+b_1y_n\quad+c_1z_n\quad+\ldots&=\alpha, \\ a_2x_{n+1}+b_2y_{n+1}+c_2z_n\quad+\ldots&=\beta, \\ a_3x_{n+1}+b_3y_{n+1}+c_3z_{n+1}+\ldots&=\gamma, \\ \ldots\ldots\ldots\ldots\ldots\ldots& \end{aligned}$$

is a criticism only of a particular routine, not of the general principle. Nor is Mr. Morris's process as effective as his paper suggests. To twenty-one places of decimals, the solution of his example is

$$\begin{aligned} x&=+\ \ 5{\cdot}38625\ 24221\ 14004\ 89671\ 5, \\ y&=-\ \ 2{\cdot}81334\ 69056\ 56987\ 06591\ 5, \\ z&=-11{\cdot}59232\ 35480\ 19317\ 71940\ 9, \\ u&=+\ \ 6{\cdot}36482\ 51116\ 16317\ 76363\ 4, \\ v&=+\ \ 7{\cdot}99287\ 21174\ 39987\ 42297\ 9, \\ w&=-\ \ 4{\cdot}20355\ 33598\ 11286\ 99413\ 8. \end{aligned}$$

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* Communicated by the Author.

† Phil. Mag. ser. 7, xxxvii. p. 106 (1946). I have changed the sign on the right, and divided by 10^3 throughout to bring the coefficients into the normal range.

He gives results to eight places of decimals for the first five variables and to seven for w, but it is only to three places that the results are correct, the discrepancy varying from $1{\cdot}2\times 10^{-4}$ for y to $4{\cdot}6\times 10^{-4}$ for z. This is not on account of any inaccuracy in Mr. Morris's arithmetic, but because he has no means of knowing how his errors are accumulating. He substitutes his values in the equations and is satisfied to note that discrepancy begins at the eighth digit. But the very quality of the set of equations implies that discrepancy may begin much sooner in the variables than in the equations; as will appear presently, a loss of three places is to be expected, and seven-figure agreement in the equations has no bearing beyond the fourth figure of the variables: Mr. Morris's solution is slightly more accurate than his check can guarantee, but four of his nine figures are worthless.

What follows is the solution of Mr. Morris's set, which will be called the set M. The set is not, as far as I know, of any importance, but principles are more easily understood from an example than explained in the abstract. We do not assume that the set is known in advance to be ill-conditioned.

The initial step, as I pointed out as long ago as 1915, is to substitute variables X, Y, Z, U, V, W for the numerical values on the right of the equations, and to reverse a transformation in which the coefficients are simple approximations to the coefficients actually given. We cannot foresee the harmless latitude in elements of a determinant of the sixth order, and we bisect the operation. If the set of equations

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=[\xi, \eta, \zeta]$$

is approximately equivalent to

$$[x, y, z]=\begin{bmatrix} \mathrm{A}_1 & \mathrm{A}_2 & \mathrm{A}_3 \\ \mathrm{B}_1 & \mathrm{B}_2 & \mathrm{B}_3 \\ \mathrm{C}_1 & \mathrm{C}_2 & \mathrm{C}_3 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix},$$

three equations,

$$(a_r x+b_r y+c_r z+d_r u+e_r v+f_r w)_{r=1,\,2,\,3}=[\mathrm{X}, \mathrm{Y}, \mathrm{Z}],$$

are reduced approximately to identities by the substitution

$$[x, y, z]=[x_0, y_0, z_0]-\begin{bmatrix} \mathrm{A}_1 & \mathrm{A}_2 & \mathrm{A}_3 \\ \mathrm{B}_1 & \mathrm{B}_2 & \mathrm{B}_3 \\ \mathrm{C}_1 & \mathrm{C}_2 & \mathrm{C}_3 \end{bmatrix}\begin{bmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

where

$$[x_0, y_0, z_0]=\begin{bmatrix} \mathrm{A}_1 & \mathrm{A}_2 & \mathrm{A}_3 \\ \mathrm{B}_1 & \mathrm{B}_2 & \mathrm{B}_3 \\ \mathrm{C}_1 & \mathrm{C}_2 & \mathrm{C}_3 \end{bmatrix}\begin{bmatrix} \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{bmatrix},$$

and the same substitution reduces three equations,

$$(a_rx+b_ry+c_rz+d_ru+e_rv+f_rw)_{r=4,5,6}=[\mathrm{U}, \mathrm{V}, \mathrm{W}],$$

to the form

$$\begin{bmatrix} d_4' & e_4' & f_4' \\ d_5' & e_5' & f_5' \\ d_6' & e_6' & f_6' \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix}=[\mathrm{U}', \mathrm{V}', \mathrm{W}'],$$

where

$$[\mathrm{U}', \mathrm{V}', \mathrm{W}']=[\mathrm{U}, \mathrm{V}, \mathrm{W}]-\begin{bmatrix} a_4 & b_4 & c_4 \\ a_5 & b_5 & c_5 \\ a_6 & b_6 & c_6 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}.$$

We can now replace the coefficients $d_4', e_4', \ldots, f_6'$ by simple approximations, and if we have an approximate reversal

$$[u, v, w]=\begin{bmatrix} \mathrm{D}_4 & \mathrm{D}_5 & \mathrm{D}_6 \\ \mathrm{E}_4 & \mathrm{E}_5 & \mathrm{E}_6 \\ \mathrm{F}_4 & \mathrm{F}_5 & \mathrm{F}_6 \end{bmatrix}\begin{bmatrix} \mathrm{U}' \\ \mathrm{V}' \\ \mathrm{W}' \end{bmatrix},$$

direct substitution gives an approximate reversal of the complete transformation

$$(a_rx+b_ry+c_rz+d_ru+e_rv+f_rw)_{r=1,2,3,4,5,6}=[\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \mathrm{U}, \mathrm{V}, \mathrm{W}].$$

When we attempt to apply this process to the set M, the set at once threatens to be troublesome. The value of the determinant

$$\begin{vmatrix} 5{\cdot}4 & 5{\cdot}2 & 4{\cdot}4 \\ 5{\cdot}2 & 7{\cdot}9 & 3{\cdot}6 \\ 4{\cdot}4 & 3{\cdot}6 & 3{\cdot}9 \end{vmatrix}$$

is about 2·7, and this value is small compared with the values of the first minors ; hence a preliminary solution for x, y, z involves large coefficients, subject to large fluctuations as the coefficients of x, y, z are improved, and such a solution is unsuitable as a basis for successive approximation. Other selections fail in the same way, but the set of equations

$$\begin{bmatrix} 7\frac{5}{6} & 5\frac{1}{4} & 1\frac{5}{6} \\ 5\frac{1}{4} & 4\frac{1}{3} & 1\frac{2}{3} \\ 1\frac{5}{6} & 1\frac{2}{3} & 2 \end{bmatrix}\begin{bmatrix} y \\ u \\ v \end{bmatrix}=[\rho, \sigma, \tau],$$

written as

$$\begin{bmatrix} 94 & 63 & 22 \\ 63 & 52 & 20 \\ 22 & 20 & 24 \end{bmatrix}\begin{bmatrix} y \\ u \\ v \end{bmatrix}=12[\rho, \sigma, \tau],$$

gives

$$3682[y, u, v]=\begin{bmatrix} 2544 & -3216 & 348 \\ -3216 & 5316 & -1482 \\ 348 & -1482 & 2757 \end{bmatrix}\begin{bmatrix} \rho \\ \sigma \\ \tau \end{bmatrix},$$

and we can build a solution on the transformation

$$[y, u, v]=\begin{bmatrix} +0{\cdot}69 & -0{\cdot}87 & +0{\cdot}09 \\ -0{\cdot}87 & +1{\cdot}44 & -0{\cdot}40 \\ +0{\cdot}09 & -0{\cdot}40 & +0{\cdot}75 \end{bmatrix}\begin{bmatrix} \rho \\ \sigma \\ \tau \end{bmatrix}.$$

First we test and improve the transformation. With the suggested values, we have, accurately,

$$\begin{bmatrix} 7{\cdot}87190 & 5{\cdot}25651 & 1{\cdot}84691 \\ 5{\cdot}25651 & 4{\cdot}37677 & 1{\cdot}67936 \\ 1{\cdot}84691 & 1{\cdot}67936 & 2{\cdot}01578 \end{bmatrix}\begin{bmatrix} y \\ u \\ v \end{bmatrix}$$

$$=[\rho, \sigma, \tau]-\begin{bmatrix} -0{\cdot}02466\,92 & +0{\cdot}01794\,26 & +0{\cdot}00895\,05 \\ +0{\cdot}02965\,56 & -0{\cdot}05764\,11 & +0{\cdot}01810\,21 \\ +0{\cdot}00525\,51 & -0{\cdot}00515\,47 & -0{\cdot}00631\,29 \end{bmatrix}\begin{bmatrix} \rho \\ \sigma \\ \tau \end{bmatrix}.$$

It follows that if $\alpha_1, \beta_1, \gamma_1$ are given by

$$[\alpha_1, \beta_1, \gamma_1]=\begin{bmatrix} +0{\cdot}69 & -0{\cdot}87 & +0{\cdot}09 \\ -0{\cdot}87 & +1{\cdot}44 & -0{\cdot}40 \\ +0{\cdot}09 & -0{\cdot}40 & +0{\cdot}75 \end{bmatrix}\begin{bmatrix} -0{\cdot}02466\,92 \\ +0{\cdot}02965\,56 \\ +0{\cdot}00525\,51 \end{bmatrix},$$

then

$$\begin{bmatrix} 7{\cdot}8\dots & 5{\cdot}2\dots & 1{\cdot}8\dots \\ 5{\cdot}2\dots & 4{\cdot}3\dots & 1{\cdot}6\dots \\ 1{\cdot}8\dots & 1{\cdot}6\dots & 2{\cdot}0\dots \end{bmatrix}\begin{bmatrix} \alpha_1\rho \\ \beta_1\rho \\ \gamma_1\rho \end{bmatrix}$$

differs from

$$[-0{\cdot}02466\,92 \quad +0{\cdot}02965\,56 \quad +0{\cdot}00525\,51]\rho$$

by still smaller fractions of ρ. More completely, if

$$\begin{bmatrix} +0{\cdot}69 & -0{\cdot}87 & +0{\cdot}09 \\ -0{\cdot}87 & +1{\cdot}44 & -0{\cdot}40 \\ +0{\cdot}09 & -0{\cdot}40 & +0{\cdot}75 \end{bmatrix}\begin{bmatrix} -0{\cdot}024\dots & +0{\cdot}017\dots & +0{\cdot}008\dots \\ +0{\cdot}029\dots & -0{\cdot}057\dots & +0{\cdot}018\dots \\ +0{\cdot}005\dots & -0{\cdot}005\dots & -0{\cdot}006\dots \end{bmatrix}$$

$$=\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix},$$

then

$$\begin{bmatrix} +0{\cdot}69+\alpha_1 & -0{\cdot}87+\beta_1 & +0{\cdot}09+\gamma_1 \\ -0{\cdot}87+\alpha_2 & +1{\cdot}44+\beta_2 & -0{\cdot}40+\gamma_2 \\ +0{\cdot}09+\alpha_3 & -0{\cdot}40+\beta_3 & +0{\cdot}75+\gamma_3 \end{bmatrix}$$

is the matrix of an improved reversal of the transformation

$$\begin{bmatrix} 7\cdot8\ldots & 5\cdot2\ldots & 1\cdot8\ldots \\ 5\cdot2\ldots & 4\cdot3\ldots & 1\cdot6\ldots \\ 1\cdot8\ldots & 1\cdot6\ldots & 2\cdot0\ldots \end{bmatrix}\begin{bmatrix} y \\ u \\ v \end{bmatrix}=[\rho,\sigma,\tau].$$

This, it need hardly be said, is merely the Newton-Raphson increment in its simplest application*. To five places of decimals,

$$\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix}=\begin{bmatrix} -0\cdot04235 & +0\cdot06206 & -0\cdot01014 \\ +0\cdot06206 & -0\cdot09655 & +0\cdot02081 \\ -0\cdot01014 & +0\cdot02081 & -0\cdot01117 \end{bmatrix},$$

and the transformation is

$$[y,u,v]=\begin{bmatrix} +0\cdot64765 & -0\cdot80794 & +0\cdot07986 \\ -0\cdot80794 & +1\cdot34345 & -0\cdot37919 \\ +0\cdot07986 & -0\cdot37919 & +0\cdot73883 \end{bmatrix}\begin{bmatrix} \rho \\ \sigma \\ \tau \end{bmatrix}.$$

Accurately,

$$\begin{bmatrix} 7\cdot8\ldots & 5\cdot2\ldots & 1\cdot8\ldots \\ 5\cdot2\ldots & 4\cdot3\ldots & 1\cdot6\ldots \\ 1\cdot8\ldots & 1\cdot6\ldots & 2\cdot0\ldots \end{bmatrix}\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}$$

$$=\begin{bmatrix} -0\cdot02588\,36218 & +0\cdot01944\,82706 & +0\cdot00893\,69224 \\ +0\cdot03198\,04373 & -0\cdot06141\,06513 & +0\cdot01902\,11211 \\ +0\cdot00556\,44339 & -0\cdot00557\,45916 & -0\cdot00629\,64484 \end{bmatrix},$$

and therefore, with the second transformation,

$$\begin{bmatrix} 7\cdot8\ldots & 5\cdot2\ldots & 1\cdot8\ldots \\ 5\cdot2\ldots & 4\cdot3\ldots & 1\cdot6\ldots \\ 1\cdot8\ldots & 1\cdot6\ldots & 2\cdot0\ldots \end{bmatrix}\begin{bmatrix} y \\ u \\ v \end{bmatrix}$$

$$=[\rho,\sigma,\tau]-10^{-3}\begin{bmatrix} +1\cdot21\,44218 & -1\cdot50\,56706 & +0\cdot01\,35776 \\ -2\cdot32\,48373 & +3\cdot76\,95513 & -0\cdot91\,90211 \\ -0\cdot30\,93339 & +0\cdot41\,98916 & -0\cdot01\,64516 \end{bmatrix}\begin{bmatrix} \rho \\ \sigma \\ \tau \end{bmatrix}.$$

Applying the second transformation to the defect-matrix in this last identity, we find an increment-matrix

$$10^{-3}\begin{bmatrix} +2\cdot64 & -3\cdot99 & +0\cdot75 \\ -3\cdot99 & +6\cdot12 & -1\cdot24 \\ +0\cdot75 & -1\cdot24 & +0\cdot34 \end{bmatrix},$$

---

* In matrix notation a set of equations is concentrated into $Cx=X$, where x is a column and X is a row. To reverse approximately is to find a matrix E such that CEX differs little from X, that is, such that $CE=1-D$ where 1 is the identity-matrix consisting of positive units in the leading diagonal and zeros elsewhere and D is a matrix all of whose elements are small. If $E_2=E+ED$ and $CE_2=I-D_2$, then $D_2=D^2$ and we continue with $E_3=E_2+E_2D_2$ and so on. After $\log_2 n$ steps, the defect in Cx is D^nX, accurately if the elements are computed in full, and the defect in x is approximately ED^nX.

which is certainly accurate to five places of decimals, and now we have a third transformation

$$[y, u, v] = \begin{bmatrix} +0{\cdot}65029 & -0{\cdot}81193 & +0{\cdot}08061 \\ -0{\cdot}81193 & +1{\cdot}34957 & -0{\cdot}38043 \\ +0{\cdot}08061 & -0{\cdot}38043 & +0{\cdot}73917 \end{bmatrix} \begin{bmatrix} \rho \\ \sigma \\ \tau \end{bmatrix}$$

accurate enough to use for the determination of approximate equations in x, z, w.

Substituting

$$[\rho, \sigma, \tau] = [\mathrm{Y}, \mathrm{U}, \mathrm{V}] - \begin{bmatrix} 5{\cdot}23286 & 3{\cdot}62242 & 2{\cdot}80269 \\ 3{\cdot}62242 & 2{\cdot}97304 & 2{\cdot}63246 \\ 2{\cdot}76472 & 2{\cdot}63974 & 1{\cdot}14921 \end{bmatrix} \begin{bmatrix} x \\ z \\ w \end{bmatrix}$$

we have, to five places of which the last is unreliable,

$$[y, u, v] = [y_1, u_1, v_1] + \begin{bmatrix} -0{\cdot}68459 & -0{\cdot}15451 & +0{\cdot}22217 \\ +0{\cdot}41179 & -0{\cdot}06694 & -0{\cdot}83991 \\ -1{\cdot}08734 & -1{\cdot}11219 & -0{\cdot}07392 \end{bmatrix} \begin{bmatrix} x \\ z \\ w \end{bmatrix},$$

where

$$[y_1, u_1, v_1] = \begin{bmatrix} +0{\cdot}65029 & -0{\cdot}81193 & +0{\cdot}08061 \\ -0{\cdot}81193 & +1{\cdot}34957 & -0{\cdot}38043 \\ +0{\cdot}08061 & -0{\cdot}38043 & +0{\cdot}73917 \end{bmatrix} \begin{bmatrix} \mathrm{Y} \\ \mathrm{U} \\ \mathrm{V} \end{bmatrix},$$

and therefore, from the equations in the set M that have not yet been used,

$$\begin{bmatrix} +0{\cdot}30312 & +0{\cdot}23195 & -0{\cdot}23738 \\ +0{\cdot}23195 & +0{\cdot}18680 & -0{\cdot}20806 \\ -0{\cdot}23738 & -0{\cdot}20806 & +0{\cdot}26734 \end{bmatrix} \begin{bmatrix} x \\ z \\ w \end{bmatrix} = [\mathrm{X}_1, \mathrm{Z}_1, \mathrm{W}_1],$$

where

$$[\mathrm{X}_1, \mathrm{Z}_1, \mathrm{W}_1] = [\mathrm{X}, \mathrm{Z}, \mathrm{W}] - \begin{bmatrix} 5{\cdot}23286 & 3{\cdot}62242 & 2{\cdot}76472 \\ 3{\cdot}62242 & 2{\cdot}97304 & 2{\cdot}63974 \\ 2{\cdot}80269 & 2{\cdot}63246 & 1{\cdot}14921 \end{bmatrix} \begin{bmatrix} y_1 \\ u_1 \\ v_1 \end{bmatrix}.$$

It is now clear that the set of equations M as a whole is sensitive to small changes in the coefficients; a change that is small compared to these coefficients may be comparable with the coefficients in the set of equations in x, z, w, which will be called the set N, and is then compatible with large changes in x, z, w and correspondingly large changes in y, u, v. This is not all; obviously the set N is anything but well-conditioned. This set is approximately

$$\begin{bmatrix} 3 & 2\tfrac{1}{3} & -2\tfrac{1}{3} \\ 2\tfrac{1}{3} & 1\tfrac{5}{6} & -2 \\ -2\tfrac{1}{3} & -2 & 2\tfrac{2}{3} \end{bmatrix} \begin{bmatrix} x \\ z \\ w \end{bmatrix} = 10[\mathrm{X}_1, \mathrm{Z}_1, \mathrm{W}_1],$$

that is,

$$\begin{bmatrix} 9 & 7 & -7 \\ 7 & 5\tfrac{1}{2} & -6 \\ -7 & -6 & 8 \end{bmatrix}\begin{bmatrix} x \\ z \\ w \end{bmatrix}=30[\mathrm{X}_1, \mathrm{Z}_1, \mathrm{W}_1].$$

The inverse of the last coefficient-matrix is

$$\begin{bmatrix} 8 & -14 & -3\tfrac{1}{2} \\ -14 & 23 & 5 \\ -3\tfrac{1}{2} & 5 & \tfrac{1}{2} \end{bmatrix},$$

and the determinant of the coefficient-matrix is therefore $-1\tfrac{1}{2}$, a value fatally small compared with the elements of the inverse matrix. The exact reversal of our approximate transformation is

$$[x, z, w]=\begin{bmatrix} -160 & 280 & 70 \\ 280 & -460 & -100 \\ 70 & -100 & -10 \end{bmatrix}\begin{bmatrix} \mathrm{X}_1 \\ \mathrm{Z}_1 \\ \mathrm{W}_1 \end{bmatrix},$$

but this is not in general an approximation to the solution of the set N or of a set which replaces N if the earlier calculations are carried to a higher order of accuracy.

To prepare the set N for solution we have to express two of the variables in terms of the third by means of two of the equations, choosing variables and equations to secure the most efficient set of coefficients. The inverse matrix of the approximate set directs our choice: if we reverse the transformation

$$\begin{bmatrix} 9 & -7 \\ -7 & 8 \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix}=[\xi, v],$$

the biggest of the four coefficients introduced is 9/23; with any other selection, the biggest coefficient is at least 8/14. We therefore deal next with the pair of equations

$$\begin{bmatrix} +3{\cdot}0312 & -2{\cdot}3738 \\ -2{\cdot}3738 & +2{\cdot}6734 \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix}=[\lambda, \mu]$$

with a view to substituting in due course

$$[\lambda, \mu]=10[\mathrm{X}_1; \mathrm{W}_1]-[+2{\cdot}3195, -2{\cdot}0806]z.$$

It will be noticed that we have multiplied throughout by 10; this brings the set with which we are now concerned into the normal range, and represents a loss of one significant figure in returning from a solution of N to a solution of M.

A direct reversal of the last transformation is perhaps quicker than any iteration. To four places,

$$[x, w]=\begin{bmatrix} +1{\cdot}0829 & +0{\cdot}9616 \\ +0{\cdot}9616 & +1{\cdot}2279 \end{bmatrix}\begin{bmatrix} \lambda \\ \mu \end{bmatrix}.$$

Hence the first and third equations of N are equivalent approximately to

$$[x, w]=[x_2, w_2]+[-0.51108, +0.32434]z,$$

where

$$[x_2, w_2]=\begin{bmatrix} +10.829 & +\ 9.616 \\ +\ 9.616 & +12.279 \end{bmatrix}\begin{bmatrix} X_1 \\ W_1 \end{bmatrix},$$

and this transformation reduces the second equation of N to

$$0.00077z=Z_2,$$

where

$$Z_2=Z_1-0.23195x_2+0.20806w_2.$$

The coefficient of z accounts completely for the character of the set N ; three or four significant figures are necessarily lost in the formation and solution of the final equation.

The structure of the original set of equations is now exposed completely, and we can proceed to a solution as precise as we desire. There are points at which the accuracy is lowered suddenly by a known factor, but there are no wild or unpredictable fluctuations. Roughly, we shall lose five places of decimals in the course of the work ; if we carry a sixth place as a guard, our probable accuracy is greater than our guaranteed accuracy.

Supposing that we require a 10-place solution, the next step is to refine the matrix expressing y, u, v in terms of ρ, σ, τ. The product of the increment-matrix

$$10^{-3}\begin{bmatrix} +2.64 & -3.99 & +0.75 \\ -3.99 & +6.12 & -1.24 \\ +0.75 & -1.24 & +0.34 \end{bmatrix}$$

by the coefficient-matrix

$$\begin{bmatrix} 7.8\ldots & 5.2\ldots & 1.8\ldots \\ 5.2\ldots & 4.3\ldots & 1.6\ldots \\ 1.8\ldots & 1.6\ldots & 2.0\ldots \end{bmatrix}$$

is

$$10^{-3}\begin{bmatrix} +1.19\ 35236 & -1.52\ 92082 & +0.01\ 38020 \\ -2.32\ 66059 & +3.72\ 99511 & -0.91\ 38299 \\ -0.31\ 29690 & +0.40\ 89451 & -0.01\ 18587 \end{bmatrix}.$$

Hence the defect-matrix after the third transformation is

$$10^{-5}\begin{bmatrix} +2.08982 & +2.35376 & -0.02244 \\ +0.17686 & +3.96002 & -0.51912 \\ +0.36351 & +1.09465 & -0.45929 \end{bmatrix},$$

and, multiplying this by the matrix of the third transformation, we have the increment-matrix

$$10^{-5}\begin{bmatrix} +1\cdot 24469 & -1\cdot 59639 & +0\cdot 36983 \\ -1\cdot 59639 & +3\cdot 01680 & -0\cdot 50764 \\ +0\cdot 36983 & -0\cdot 50764 & -0\cdot 14381 \end{bmatrix},$$

which gives for the fourth transformation the matrix

$$\begin{bmatrix} +0\cdot 65030\ 24469 & -0\cdot 81194\ 59639 & +0\cdot 08061\ 36983 \\ -0\cdot 81194\ 59639 & +1\cdot 34960\ 01680 & -0\cdot 38043\ 50764 \\ +0\cdot 08061\ 36983 & -0\cdot 38043\ 50764 & +0\cdot 73916\ 85619 \end{bmatrix}.$$

One more step is necessary. The last increment-matrix provides the defect-matrix

$$10^{-9}\begin{bmatrix} +1\cdot 4\ 20626 & +1\cdot 0\ 84654 & +3\cdot 1\ 40865 \\ +0\cdot 8\ 87196 & +1\cdot 3\ 05733 & +1\cdot 9\ 71911 \\ +0\cdot 9\ 71851 & +0\cdot 3\ 59661 & +0\cdot 6\ 69069 \end{bmatrix},$$

which is exact, and operating with the fourth transformation we have our final increment-matrix

$$10^{-10}\begin{bmatrix} +2\cdot 81825\ 86 & -3\cdot 25837\ 89 & +4\cdot 95363\ 14 \\ -3\cdot 25837\ 89 & +7\cdot 44709\ 38 & -1\cdot 43458\ 56 \\ +4\cdot 95363\ 14 & -1\cdot 43458\ 56 & -0\cdot 02432\ 60 \end{bmatrix},$$

and we can assert that, to 16 places, the second, fourth, and fifth equations in the set M are equivalent to

$$[y, u, v]=[y_1, u_1, v_1] + \begin{bmatrix} -0\cdot 68460\ 66693\ 62446\ 0 & -0\cdot 15451\ 99664\ 96476\ 4 & +0\cdot 22217\ 70385\ 01298\ 4 \\ +0\cdot 41177\ 73799\ 15838\ 7 & -0\cdot 06695\ 62969\ 98981\ 8 & -0\cdot 83993\ 58314\ 23076\ 0 \\ -1\cdot 08733\ 86763\ 55191\ 3 & -1\cdot 11218\ 07944\ 07019\ 9 & -0\cdot 07391\ 49888\ 97486\ 8 \end{bmatrix}\begin{bmatrix} x \\ z \\ w \end{bmatrix},$$

where

$$[y_1, v_1, w_1] = \begin{bmatrix} +0\cdot 65030\ 24471\ 81825\ 9 & -0\cdot 81194\ 59642\ 25837\ 9 & +0\cdot 08061\ 36987\ 95363\ 1 \\ -0\cdot 81194\ 59642\ 25837\ 9 & +1\cdot 34960\ 01687\ 44709\ 4 & -0\cdot 38043\ 50765\ 43458\ 6 \\ +0\cdot 08061\ 36987\ 95363\ 1 & -0\cdot 38043\ 50765\ 43458\ 6 & +0\cdot 73916\ 85618\ 97567\ 4 \end{bmatrix}\begin{bmatrix} \mathrm{Y} \\ \mathrm{U} \\ \mathrm{V} \end{bmatrix}.$$

The expressions just found for y, u, v reduce the first and last equations of the set M to

$$\begin{bmatrix} +3\cdot 02982\ 77542\ 20388 & -2\cdot 37423\ 26487\ 63343 \\ -2\cdot 37423\ 26487\ 63343 & +2\cdot 67302\ 04085\ 83326 \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix}=[\lambda, \mu],$$

where

$$[\lambda, \mu]=10[\mathrm{X}_1, \mathrm{W}_1]-[+2\cdot 31856\ 33283\ 12208 \quad -2\cdot 08100\ 62923\ 84404]z,$$

X_1, W_1 being defined as before :

$$[X_1, W_1]=[X, W]-\begin{bmatrix} 5\cdot23286 & 3\cdot62242 & 2\cdot76472 \\ 2\cdot80269 & 2\cdot63246 & 1\cdot14921 \end{bmatrix}\begin{bmatrix} y_1 \\ u_1 \\ v_1 \end{bmatrix}.$$

Taking for a first approximation the transformation already used, namely

$$[x, w]=\begin{bmatrix} +1\cdot0829 & +0\cdot9616 \\ +0\cdot9616 & +1\cdot2279 \end{bmatrix}\begin{bmatrix} \lambda \\ \mu \end{bmatrix},$$

we complete the reversal by systematic iteration. Defects are computed to the full 16 places, so that they can be corrected from increments alone. There is no latent computation : the elements of the various matrices are transcribed directly from the machine * on which they are calculated.

First defect :

$$10^{-3}\begin{bmatrix} +2\cdot06\ 16400\ 05572\ 5 & +1\cdot83\ 79009\ 58183\ 8 \\ +0\cdot68\ 01104\ 52097\ 9 & +0\cdot86\ 03553\ 51364\ 6 \end{bmatrix},$$

first increment :

$$10^{-3}\begin{bmatrix} +2\cdot88\ 654 & +2\cdot81\ 758 \\ +2\cdot81\ 758 & +2\cdot82\ 376 \end{bmatrix},$$

second transformation :

$$\begin{bmatrix} +1\cdot08578\ 654 & +0\cdot96441\ 758 \\ +0\cdot96441\ 758 & +1\cdot23072\ 376 \end{bmatrix};$$

Second defect :

$$10^{-6}\begin{bmatrix} +5\cdot5114\ 26407\ 8 & +5\cdot3820\ 58719\ 5 \\ +1\cdot9791\ 19243\ 0 & +1\cdot9776\ 68926\ 0 \end{bmatrix},$$

second increment :

$$10^{-6}\begin{bmatrix} +7\cdot8929\ 300 & +7\cdot7510\ 656 \\ +7\cdot7510\ 656 & +7\cdot6245\ 162 \end{bmatrix},$$

third transformation :

$$\begin{bmatrix} +1\cdot08579\ 44329\ 300 & +0\cdot96442\ 53310\ 656 \\ +0\cdot96442\ 53310\ 656 & +1\cdot23073\ 13845\ 162 \end{bmatrix}.$$

Third defect :

$$10^{-11}\begin{bmatrix} +4\cdot1041\ 9 & +4\cdot0332\ 9 \\ +1\cdot4806\ 3 & +1\cdot4528\ 1 \end{bmatrix},$$

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* The one used had capacity 10×12=18. I am grateful to Dr. P. White for checking my operations.

third increment:

$$10^{-11}\begin{bmatrix} +5{\cdot}8842\,6 & +5{\cdot}7804\,4 \\ +5{\cdot}7804\,4 & +5{\cdot}6778\,3 \end{bmatrix}.$$

Hence, to the order proposed,

$$[x, w]=[x_2, w_2]+[-0{\cdot}51050\,79718\,10762 \quad +0{\cdot}32507\,85498\,93254]z,$$

where

$$[x_2, w_2]=\begin{bmatrix} +10{\cdot}85794\,43298\,8843 & +\;9{\cdot}64425\,33112\,3404 \\ +\;9{\cdot}64425\,33112\,3404 & +12{\cdot}30731\,38457\,2978 \end{bmatrix}\begin{bmatrix} X_1 \\ W_1 \end{bmatrix}.$$

Direct substitution now gives

$$\begin{aligned}[y, u, v]&=[y_2, u_2, v_2]\\ &\quad+[+0{\cdot}26720\,21852\,63447 \quad -0{\cdot}55021\,70541\,39764 \quad -0{\cdot}58111\,39094\,75704]z,\end{aligned}$$

where

$$[y_2, u_2, v_2]=[y_1, u_1, v_1]+\begin{bmatrix} -0{\cdot}68460\,66693\,62446\,0 & +0{\cdot}22217\,70385\,01298\,4 \\ +0{\cdot}41177\,73799\,15838\,7 & -0{\cdot}83993\,58314\,23076\,0 \\ -1{\cdot}08733\,86763\,55191\,3 & -0{\cdot}07391\,49888\,97486\,8 \end{bmatrix}\begin{bmatrix} x_2 \\ w_2 \end{bmatrix},$$

and the outstanding equation of the set M becomes

$$0{\cdot}00072\,83464\,96143z=Z_2,$$

where

$$Z_2=Z-4{\cdot}35785x_2-3{\cdot}62242y_2-2{\cdot}97304u_2-2{\cdot}63974v_2-1{\cdot}67936w_2.$$

All that remains is straightforward arithmetic, in which in the long run not more than 12 figures can be significant. If we have only one or two questions to answer, we can determine the intermediate variables numerically in every case. If we have several adjacent sets of equations to solve, or if we want to improve greatly on our first approximation, it may be worth while to eliminate these variables, that is, to express x, y, z, u, v, w explicitly in terms of X, Y, Z, U, V, W.

For the actual set M,

$$\begin{aligned}&[y_1, u_1, v_1]\\ &\quad=[+0{\cdot}01680\,50158\,007 \quad -0{\cdot}16000\,59429\,210 \quad +0{\cdot}64608\,74840\,366],\end{aligned}$$

$$[X_1, W_1]=[-0{\cdot}05779\,05560\,926 \quad +0{\cdot}00992\,97972\,377],$$

$$[x_2, w_2]=[-0{\cdot}53172\,11609\,572 \quad -0{\cdot}43513\,76309\,253],$$

$$\begin{aligned}&[y_2, u_2, v_2]\\ &\quad=[+0{\cdot}28414\,72786\,537 \quad -0{\cdot}01346\,90016\,111 \quad +1{\cdot}25641\,16605\,406],\end{aligned}$$

$$Z_2=-0{\cdot}00844\,32282\,38268\,8,$$

giving to eleven places

$$z=-11{\cdot}59232\,35479\,0, \quad x=+5{\cdot}38625\,24220\,6, \quad w=-4{\cdot}20355\,33597\,7,$$

$$y=-2{\cdot}81334\,69056\,3, \quad u=+6{\cdot}36482\,51115\,5, \quad v=+7{\cdot}99287\,21173\,7,$$

To test these results and to estimate their accuracy, we substitute the values, as far as the tenth place of decimals, in the original equations. We find the defects to be $10^{-10}[\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \mathrm{U}, \mathrm{V}, \mathrm{W}]$, where

$$[\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \mathrm{U}, \mathrm{V}, \mathrm{W}] \\ =[+4\cdot00292 \quad +2\cdot73823 \quad +3\cdot55721 \quad +2\cdot09911 \quad +2\cdot44605 \quad +1\cdot01570],$$

and we solve the set with these values of X, Y, Z, U, V, W. We have now

$$[y_1, u_1, v_1]=[+0\cdot27349\,89149\,18990\,5 \quad -0\cdot32089\,88063\,87536\,1 \quad +1\cdot23020\,70357\,58832\,5],$$

$$[\mathrm{X}_1, \mathrm{W}_1]=[+0\cdot33299\,07264\,08190\,5 \quad -0\cdot31984\,56295\,55780\,1],$$

$$[x_2, w_2]=[+0\cdot53092\,24977\,82173 \quad -0\cdot72499\,36293\,55591],$$

$$[y_2, u_2, v_2]=[-0\cdot25105\,11054\,79787 \quad +0\cdot50667\,11955\,36801 \quad +0\cdot70650\,23658\,37740],$$

$$\mathrm{Z}_2=-0\cdot00086\,90464\,28594,$$

$$z=-1\cdot19317\,71940\,91,$$

$$x=+1\cdot14004\,89671\,49, \qquad w=-1\cdot11286\,99413\,76,$$

$$y=-0\cdot56987\,06591\,47, \qquad u=+1\cdot16317\,76363\,37, \qquad v=+1\cdot39987\,42297\,94.$$

These values are accurate to 11 places, and give the 21-place solution written out at the beginning of the paper. The corresponding defects are

$$10^{-22}[+2\cdot831\,2 \quad +5\cdot123\,7 \quad +2\cdot171\,0 \quad +3\cdot635\,7 \quad +2\cdot196\,3 \quad +3\cdot624\,1].$$

It is particularly to be noticed that discrepancy for us cannot begin sooner in the variables than in the equations, since it is the accuracy of the variables that we have controlled. Ultimately this means that the defects in a set are not independent. We find this useful outcome of our analysis illustrated in the course of reversing to a high degree of accuracy the transformation in six variables. Elimination of the intermediate variables y_1, u_1, v_1 ; $\mathrm{X}_1, \mathrm{W}_1$; x_2, w_2 ; y_2, u_2, v_2 ; Z_2 gives, to the eighth decimal place,

$$[x, y, z, u, v, w] \\ = \begin{bmatrix} +368\cdot67992\,969 & -192\cdot57633\,768 & -700\cdot91360\,981 & +382\cdot02512\,354 & +394\cdot79153\,031 & -218\cdot20772\,656 \\ -192\cdot57633\,768 & +101\cdot43909\,359 & +366\cdot86135\,881 & -201\cdot59493\,776 & -207\cdot06894\,160 & +115\cdot39064\,093 \\ -700\cdot91360\,981 & +366\cdot86135\,881 & +1372\cdot97289\,860 & -755\cdot43310\,368 & -797\cdot85364\,871 & +446\cdot32403\,892 \\ +382\cdot02512\,354 & -201\cdot59493\,776 & -755\cdot43310\,368 & +420\cdot85432\,094 & +443\cdot02929\,065 & -251\cdot94116\,641 \\ +394\cdot79153\,031 & -207\cdot06894\,160 & -797\cdot85364\,871 & +443\cdot02929\,065 & +478\cdot83789\,718 & -270\cdot76137\,174 \\ -218\cdot20772\,656 & +115\cdot39064\,093 & +446\cdot32403\,892 & -251\cdot94116\,641 & -270\cdot76137\,174 & +157\cdot39768\,520 \end{bmatrix} \begin{bmatrix} \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \\ \mathrm{U} \\ \mathrm{V} \\ \mathrm{W} \end{bmatrix}.$$

The coefficients here, ranging from 101 to 1373, show that if the right-hand variables are of order 10^m, the left-hand variables are usually of order 10^{m+3}; for a given set of right-hand values, a set of left-hand values that gives agreement to n places of decimals on substitution is accurate as a rule only to $n-3$ places, and in extreme cases only to $n-4$ places. This is how Mr. Morris was misled; the defect-set of his solution is, to two figures,

$$10^{-7}[+3\cdot1 \quad -0\cdot8 \quad +2\cdot8 \quad -1\cdot3 \quad -3\cdot3 \quad -1\cdot0],$$

and the corresponding increment-set, to this order of accuracy, is

$$10^{-4}[-2{\cdot}2 \quad +1{\cdot}2 \quad +4{\cdot}6 \quad -2{\cdot}5 \quad -2{\cdot}7 \quad +1{\cdot}5].$$

Since we must expect to lose three or four places in an operation, apart from any accumulation of end-figure errors, our eight-place matrix can be relied upon only for six-figure accuracy, and it may well be worth while to use it first for self-improvement. When we multiply by the original coefficient-matrix, every combination is in excess of the required value, 1 or 0, and the defect-matrix is

$$-10^{-8}\begin{bmatrix} 5{\cdot}55\,302 & 4{\cdot}39\,978 & 1{\cdot}63\,251 & 3{\cdot}71\,327 & 3{\cdot}43\,866 & 2{\cdot}65\,544 \\ 7{\cdot}81\,223 & 3{\cdot}23\,045 & 0{\cdot}94\,463 & 5{\cdot}51\,428 & 7{\cdot}02\,331 & 0{\cdot}17\,273 \\ 4{\cdot}22\,282 & 3{\cdot}72\,523 & 1{\cdot}52\,063 & 2{\cdot}92\,120 & 2{\cdot}20\,032 & 2{\cdot}95\,698 \\ 5{\cdot}72\,604 & 2{\cdot}18\,443 & 0{\cdot}83\,609 & 4{\cdot}30\,412 & 5{\cdot}06\,807 & 0{\cdot}82\,058 \\ 2{\cdot}45\,272 & 2{\cdot}48\,504 & 1{\cdot}08\,025 & 1{\cdot}71\,633 & 0{\cdot}86\,308 & 2{\cdot}40\,811 \\ 3{\cdot}38\,766 & 0{\cdot}93\,288 & 0{\cdot}45\,339 & 2{\cdot}66\,746 & 2{\cdot}98\,083 & 0{\cdot}62\,493 \end{bmatrix}.$$

The final increment-matrix, reliable certainly to the fifteenth place, is

$$10^{-8}\begin{bmatrix} +0{\cdot}40\,49956 & -0{\cdot}95\,62054 & -1{\cdot}07\,77587 & +0{\cdot}60\,64222 & +0{\cdot}56\,60209 & -0{\cdot}97\,52491 \\ -0{\cdot}95\,62054 & +0{\cdot}25\,56709 & +0{\cdot}62\,07710 & -0{\cdot}66\,35079 & -0{\cdot}99\,16019 & +1{\cdot}01\,01245 \\ -1{\cdot}07\,77587 & +0{\cdot}62\,07710 & +1{\cdot}04\,90647 & -0{\cdot}71\,98429 & -0{\cdot}57\,38927 & +0{\cdot}30\,40364 \\ +0{\cdot}60\,64222 & -0{\cdot}66\,35079 & -0{\cdot}71\,98429 & +0{\cdot}11\,07069 & +0{\cdot}10\,87063 & -0{\cdot}58\,50322 \\ +0{\cdot}56\,60209 & -0{\cdot}99\,16019 & -0{\cdot}57\,38927 & +0{\cdot}10\,87063 & +0{\cdot}80\,36154 & -0{\cdot}76\,93207 \\ -0{\cdot}97\,52491 & +1{\cdot}01\,01245 & +0{\cdot}30\,40364 & -0{\cdot}58\,50322 & -0{\cdot}76\,93207 & +0{\cdot}13\,33573 \end{bmatrix}.$$

Far from having lost places in comparison with the defect-matrix, this matrix affects the transformation only by as much as a unit in the eighth place.

A single example of the use of the 15-place transformation will suffice. Mr. Morris's proposed solution is

$$[+5{\cdot}38648\,142 \quad -2{\cdot}81346\,708 \quad -11{\cdot}59278\,669 \quad +6{\cdot}36508\,248 \quad +7{\cdot}99315\,004 \quad -4{\cdot}20371\,060]$$

and the set of defects corresponding to this set of values of the variables is, exactly,

$$10^{-7}[+3{\cdot}065\,351 \quad -0{\cdot}806\,266 \quad +2{\cdot}805\,067 \quad -1{\cdot}349\,960 \quad -3{\cdot}323\,370 \quad -1{\cdot}001\,278].$$

The 15-place transformation applied to this set gives the increment-set

$$10^{-4}[-2{\cdot}2\,89978\,85994 \quad +1{\cdot}2\,01743\,43015 \quad +4{\cdot}6\,31419\,80684 \quad +7{\cdot}4\,26316\,16319 \quad +7{\cdot}2\,20774\,39987 \quad +1{\cdot}5\,72401\,88714],$$

and these increments, added to Mr. Morris's values, give a solution agreeing to within two units in the fifteenth place with the 21-place solution.

In conclusion, it is interesting, though irrelevant to the solution of the equations, to notice that we can evaluate with precision the determinant of the original set of coefficients, for this determinant is the product of three factors, one derived from each of the transformations into which our solution is analysed. The order of accuracy is governed by the last factor, 0·00072 83464 96143, which we have found to only 12 figures, The first factor, the determinant of the coefficients of y, u, v in the second, fourth and fifth equations in the set M, is easily found to its full 15 decimals, but we have nothing to gain by taking it beyond the twelfth, to which the value is 9·23009 81597 29. The second factor is the determinant

$$\begin{vmatrix} +0{\cdot}30298\ 27754\ 22039 & -0{\cdot}23742\ 32648\ 76334 \\ -0{\cdot}23742\ 32648\ 76334 & +0{\cdot}26730\ 20408\ 58333 \end{vmatrix}$$

—note that the factor 10^{-1} has been reinserted—which is 0·02461 81075 10695. Multiplying, and dropping the twelfth figure, which is perhaps in error by a unit, we can say that the value of the determinant of the set of equations which we set out to solve is

$$0{\cdot}00016\ 55003\ 8902,$$

correct to the last place.

V. *A Diffusion Problem in which the Amount of Diffusing Substance is Finite.*—I.

By A. H. Wilson, F.R.S.*,
Courtauld's Ltd., Maidenhead.

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Introduction.

1. The present problem arose in connection with attempts to elucidate the effects which occur in the commercial dye-bath. The theoretical formulæ so far available only apply to cases in which the composition of the dye-bath changes infinitesimally during the dyeing process, and they are therefore of very limited application. In order to be able to interpret the results of dyeing experiments it is necessary to have a much more elaborate theory, and the present is the first of a series of papers in which the effects of the various factors are analysed theoretically. Since these factors are of very different natures (for example, the finiteness of the dye-bath, the law of adsorption of the dye in the capillaries of the fibre, the state of aggregation of the dye in the bath), it has been decided to split the theoretical treatment up into several more or less independent parts and to treat the mathematical problems that arise in a more general manner than is required for the particular applications envisaged.

The first paper of the series is concerned essentially with the effect of the finiteness of the dye-bath, and the law of adsorption of the dye is chosen to be as simple as possible and yet give a non-trivial result. This enables us to obtain a complete solution of the problem without having recourse to numerical methods. In the second paper the emphasis is on the law of adsorption of the dye. In this case it is necessary to obtain a numerical solution, so that it is immaterial whether one chooses the

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* Communicated by the Author.

Langmuir or the Freundlich adsorption formula or some more complicated type. A detailed comparison between theory and experiment is left over for the time being since the necessary experimental data are not yet available.

The number of problems to which the analysis given here can be applied, either directly or with trivial modifications, is very large. It has been thought sufficient to solve three problems and to use two different methods depending respectively on an expansion in orthogonal functions and on the Laplace transform.

Diffusion into a Lamina.

2. Suppose that an infinite sheet of uniform material of thickness $2a$ is placed in a solution and that the solute is allowed to diffuse into the sheet and is deposited in the capillaries of the sheet. The sheet occupies the space $-a \leqslant x \leqslant a$, while the solution is of limited extent and occupies the spaces $-l-a \leqslant x \leqslant -a$, $a \leqslant x \leqslant l+a$. The concentration of the solute in the solution is always uniform and is initially c_0, while initially the sheet is free from solute.

If there were no deposition of the solute in the capillaries of the sheet the concentration of the solute in the sheet would be determined by the diffusion equation

$$\frac{\partial c}{\partial t}=\kappa\nabla^2 c, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where κ is the diffusion coefficient. To correct for the adsorption, let s be the amount of solute adsorbed in the sheet per unit volume. Then the simplest assumption that we can make concerning the exchange of free and adsorbed molecules of solute in the capillaries is that, in the adsorption process,

$$\frac{\partial c}{\partial t}=-\lambda c+\mu s, \quad \frac{\partial s}{\partial t}=\lambda c-\mu s, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where λ and μ are constants. This assumption enables us to carry through the whole of the calculation analytically. The use of such adsorption equations as occur in practice leads to non-linear differential equations, which must be solved numerically. This extra complication is dealt with in the second paper of the series.

When we consider simultaneous diffusion and adsorption the differential equations become

$$\frac{\partial c}{\partial t}=\kappa\nabla^2 c-\frac{\partial s}{\partial t}, \quad \frac{\partial s}{\partial t}=\lambda c-\mu s. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the rate of adsorption is very large compared with the rate of diffusion, these equations can be further simplified, since we can replace the second of equations (3) by $s=\mathrm{R}c$, where R is a constant. We then have

$$\frac{\partial c}{\partial t}=\frac{\kappa}{\mathrm{R}+1}\nabla^2 c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in the lamina, the total concentration of solute, free and adsorbed, being $(\mathrm{R}+1)c$.

Equation (4) is the ordinary diffusion equation, which can be solved by the standard methods, but the boundary condition is of an unusual type. The concentration at the surface of the sheet is determined by the condition that the total amount of solute in the system remains constant, and the surface concentration is therefore a function of time. If we attempt to solve the problem directly we are led to a boundary condition which is not of the Sturm-Liouville type. As a result the solutions of equation (4) do not form an orthogonal set, and the usual expansion theorems fail. It is probably this circumstance that led March and Weaver$^{(1)}$, who solved a similar problem, to base their solution upon the use of an integral equation. It turns out, however, that such a complicated method is unnecessary, and that the problem can be solved by elementary means provided one uses the slight artifice of dealing with the amount of solute that has diffused past a given point rather than with the concentration itself.

We define $y(x, t)$, the amount of free solute (per unit area of sheet) contained between the planes at distances 0 and x from the centre of the sheet, by

$$y(x, t)=\int_0^x c(x', t)\,dx'. \qquad \qquad (5)$$

The diffusion equation (4) is now

$$\frac{\partial c}{\partial t}=\frac{\kappa}{\mathrm{R}+1}\frac{\partial^2 c}{\partial x^2}, \qquad \qquad (6)$$

which on integration gives

$$\frac{\partial y(x, t)}{\partial t}=\int_0^x \frac{\partial c(x', t)}{\partial t}\,dx'=\frac{\kappa}{\mathrm{R}+1}\int_0^x \frac{\partial^2 c}{\partial x'^2}\,dx'=\frac{\kappa}{\mathrm{R}+1}\frac{\partial c}{\partial x}, \qquad (7)$$

since, by symmetry, $\partial c/\partial x=0$ at $x=0$. Also, from (5),

$$\frac{\partial y}{\partial x}=c, \quad \frac{\partial^2 y}{\partial x^2}=\frac{\partial c}{\partial x},$$

and hence, on eliminating $\partial c/\partial x$ from (7), we find

$$\frac{\partial y}{\partial t}=\frac{\kappa}{\mathrm{R}+1}\frac{\partial^2 y}{\partial x^2}. \qquad \qquad (8)$$

The boundary condition is obtained by considering the total amount of solute in the system. The solute contained in the sheet is $2(\mathrm{R}+1)y(a, t)$, while the solute in the bath is $2l$ times the (uniform) concentration in the bath. Now the concentration of free solute must be continuous everywhere, and the concentration in the bath must therefore be the same as the concentration of free solute at the surface of the sheet. This shows that the concentration in the bath is $(\partial y/\partial x)_{x=a}$, and the condition for the constancy of the total amount of solute is

$$(\mathrm{R}+1)y+l\,\partial y/\partial x=\text{constant} \qquad \qquad (9)$$

at $x=a$, the value of the constant being lc_0. Except for the presence of the constant, this boundary condition is of exactly the Sturm-Liouville type, whereas this is not so if it is expressed in terms of c.

It is convenient to eliminate the constant by writing

$$y(x,t)=\frac{lc_0x}{(\mathrm{R}+1)a+l}+f(x,t), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and f then satisfies the equations

$$\frac{\partial f}{\partial t}=\frac{\kappa}{\mathrm{R}+1}\frac{\partial^2 f}{\partial x^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$(\mathrm{R}+1)f+l\,\partial f/\partial x=0 \quad (x=a). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

In addition, the initial condition is that $y=0$ for $-a\leqslant x\leqslant a$, that is,

$$\frac{lc_0x}{(\mathrm{R}+1)a+l}+f(x,0)=0 \quad (-a\leqslant x\leqslant a). \quad . \quad . \quad . \quad . \quad (13)$$

2.2. The solution of equations (11) and (12) is quite straightforward. The elementary solutions of (11) which are odd functions of x (by symmetry, c is an even function of x), are

$$f_m=e^{-p_nt}\sin(q_nx/a),$$

where

$$(\mathrm{R}+1)p_n=\kappa q_n^{\,2}/a^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and q_n is, without loss of generality, taken to be strictly positive. The general solution is

$$f=\sum_n \mathrm{A}_n\,e^{-p_nt}\sin(q_nx/a), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and the constants p_n and A_n are to be determined by the boundary condition (12) and the initial condition (13) respectively.

The boundary condition (12) applied to the elementary solution f_n shows that we must have

$$\tan q_n=-\frac{lq_n}{(\mathrm{R}+1)a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

which determines the possible values of q_n, and hence of p_n. The initial condition is

$$\frac{lc_0x}{(\mathrm{R}+1)a+l}+\sum_n \mathrm{A}_n\sin\frac{q_nx}{a}=0 \quad (-a\leqslant x\leqslant a). \quad . \quad . \quad . \quad (17)$$

To solve this equation we need the orthogonality condition

$$\int_0^a \sin\frac{q_mx}{a}\sin\frac{q_nx}{a}\,dx=0 \quad (m\neq n). \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This is merely a particular case of the orthogonality relation for general Sturm-Liouville systems; it can also be proved by direct evaluation of the integral and applying the relation (16). Multiplying (17) by $\sin(q_mx/a)$ and integrating we have

$$\mathrm{A}_m\int_0^a \sin^2\frac{q_mx}{a}\,dx+\frac{lc_0}{(\mathrm{R}+1)a+l}\int_0^a x\sin\frac{q_mx}{a}\,dx=0,$$

which on evaluation gives

$$A_m = -\frac{lc_0}{(R+1)a+l}\frac{4a}{q_m}\frac{\sin q_m - q_m\cos q_m}{2q_m-\sin 2q_m}$$

$$= \frac{2(R+1)alc_0}{(R+1)^2a^2+(R+1)al+q_m^2l^2}\frac{a}{q_m}\frac{1}{\cos q_m}, \quad . \quad . \quad . \quad (19)$$

by (16).

2.3. The most interesting quantity is $M_t = 2(R+1)y(a,\, t)$, the total amount of solute in the sheet at time t. The equilibrium amount of solute is

$$M_\infty = \frac{2(R+1)alc_0}{(R+1)a+l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

In addition to M_∞, there are two constants in the theory which can be taken to be

$$\alpha = \frac{l}{(R+1)a}, \quad \beta = \frac{\kappa}{(R+1)a^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

α is dimensionless, and β has the dimensions of reciprocal time. Now

$$M_t = M_\infty + 2(R+1)\sum_n A_n e^{-p_n t}\sin q_n,$$

so that, with the above notation, we obtain, after substituting for $\tan q_n$ from (16),

$$\frac{M_t}{M_\infty} = 1 - \sum_n \frac{2\alpha(\alpha+1)}{1+\alpha+\alpha^2 q_n{}^2} e^{-\beta q_n{}^2 t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where the q_n's are the non-zero positive roots of

$$\tan q_n = -\alpha q_n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The solution in the particular case of an infinite bath ($l = \infty$) dates back to the earliest days of Fourier series. In this case the roots of (23) are $q_n = (n+\frac{1}{2})\pi$ since $\alpha = \infty$, and we then have

$$\frac{M_t}{M_\infty} = 1 - \sum_{n=0}^{\infty} \frac{8}{(2n+1)^2\pi^2} e^{-\beta(n+\frac{1}{2})^2\pi^2 t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Diffusion into a Circular Cylinder of Radius a.

3.1. In principle this problem is the same as the one discussed in the preceding section except that the trigonometrical functions are replaced by Bessel functions. The reasons for the various transformations have been given in full when considering the simpler case of the sheet, and it will not be necessary to repeat these in this section.

We use cylindrical polar co-ordinates and suppose that the cylinder occupies the space $r \leqslant a$, and the cross-section of the bath in which it is immersed is A, excluding the space occupied by the cylinder. We consider unit length of the cylinder (and of the bath), and only the radial co-ordinate r occurs explicitly. The equations which determine the diffusion are (3) or (4), where we now have

$$\nabla^2 c = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right).$$

We define

$$y(r, t)=2\pi\int_0^r r'c(r', t)dr', \quad z(r, t)=2\pi\int_0^r r's(r', t)dr', \quad . \quad . \quad . \quad (25)$$

which represent the amounts of free and adsorbed solute contained inside the circle of radius r at time t. Multiplying equations (3) by r and integrating, we obtain

$$\frac{\partial y}{\partial t}=\kappa r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial y}{\partial r}\right)-\frac{\partial z}{\partial t}, \quad \frac{\partial z}{\partial t}=\lambda y-\mu z. \quad . \quad . \quad . \quad . \quad (26)$$

In the particular case in which the adsorption takes place much more rapidly than the diffusion, these equations are replaced by

$$\frac{\partial y}{\partial t}=\frac{\kappa}{\mathrm{R}+1}r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial y}{\partial r}\right), \quad z=\mathrm{R}y. \quad . \quad . \quad . \quad . \quad . \quad (27)$$

We first solve equations (27) by the methods of the preceding section, and then solve the more general equations (26).

Diffusion when the Rate of Adsorption is Infinitely Rapid.

3.2. We must first formulate the boundary and initial conditions. The concentration of the free solute at the surface of the cylinder is

$$c(a, t)=\frac{1}{2\pi a}\left(\frac{\partial y}{\partial r}\right)_{r=a}.$$

The condition that the total amount of solute remains constant is therefore

$$\mathrm{A}c(a, t)+y(a, t)+z(a, t)=\mathrm{A}c_0,$$

that is,

$$\frac{\mathrm{A}}{2\pi a}\frac{\partial y}{\partial r}+(\mathrm{R}+1)y=\mathrm{A}c_0 \quad (r=a). \quad . \quad . \quad . \quad . \quad . \quad (28)$$

We also have the initial condition $y=0$ for $0\leqslant r\leqslant a$ at $t=0$. Now change to a new independent variable $f(r, t)$ defined by

$$y(r, t)=\frac{\mathrm{A}c_0r^2}{(\mathrm{R}+1)a^2+\mathrm{A}/\pi}+f(r, t). \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Then $f(r, t)$ satisfies the differential equation

$$\frac{\partial f}{\partial t}=\frac{\kappa}{\mathrm{R}+1}r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial r}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

the boundary condition

$$\frac{\mathrm{A}}{2\pi a}\frac{\partial f}{\partial r}+(\mathrm{R}+1)f=0 \quad (r=a), \quad . \quad . \quad . \quad . \quad . \quad (31)$$

and the initial condition

$$\frac{\mathrm{A}c_0r^2}{(\mathrm{R}+1)a^2+\mathrm{A}/\pi}+f(r, 0)=0 \quad (0\leqslant r\leqslant a). \quad . \quad . \quad . \quad . \quad (32)$$

The elementary solutions of (30) are

$$f_n(r, t)=e^{-p_nt}f_n(r),$$

where

$$\frac{d^2f_n}{dr^2}-\frac{1}{r}\frac{df_n}{dr}+\frac{\mathrm{R}+1}{\kappa}p_nf_n=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (33)$$

The solution of this latter equation that is regular at $r=0$ is $r\mathrm{J}_1(q_nr/a)$, where J_1 is the Bessel function of order 1, and $q_n{}^2/a^2=p_n(\mathrm{R}+1)/\kappa$, q_n, without loss of generality, being taken to be strictly positive. If, corresponding to (21), we define

$$\alpha=\frac{\mathrm{A}}{\pi a^2(\mathrm{R}+1)}, \quad \beta=\frac{\kappa}{(\mathrm{R}+1)a^2}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (34)$$

the general solution of (30) is

$$f=\underset{n}{\Sigma}\mathrm{B}_ne^{-p_nt}\,r\mathrm{J}_1(q_nr/a) \quad (p_n=\beta q_n{}^2). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (35)$$

The boundary condition (31) then gives

$$\frac{\mathrm{A}}{2\pi a}\frac{d}{dr}\left\{r\mathrm{J}_1\left(\frac{q_nr}{a}\right)\right\}_{r=a}+(\mathrm{R}+1)a\mathrm{J}_1(q_n),$$

which, by use of the relation (2, p. 18)

$$\frac{d}{dz}\{z\mathrm{J}_1(z)\}=z\mathrm{J}_0(z),$$

becomes

$$\alpha q_n\mathrm{J}_0(q_n)+2\mathrm{J}_1(q_n)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (36)$$

To determine the constants B_n we use the orthogonality relation, which, from the general theory of self-adjoint differential equations, is

$$\int_0^a\frac{1}{r}f_m(r)f_n(r)dr=\int_0^a r\mathrm{J}_1\left(\frac{q_mr}{a}\right)\mathrm{J}_1\left(\frac{q_nr}{a}\right)dr=0 \quad (m\neq n). \quad . \qquad (37)$$

The initial condition is

$$\frac{\mathrm{A}c_0r^2}{(\mathrm{R}+1)a^2+\mathrm{A}/\pi}+\underset{n}{\Sigma}\mathrm{B}_nr\mathrm{J}_1\left(\frac{q_nr}{a}\right)=0 \quad (0\leqslant r\leqslant a). \quad . \quad . \quad . \qquad (38)$$

Multiplying by $\mathrm{J}_1(q_mr/a)$, integrating and using (37), we obtain

$$\mathrm{B}_m\int_0^a r\left\{\mathrm{J}_1\left(\frac{q_mr}{a}\right)\right\}^2dr+\frac{\mathrm{A}c_0}{(\mathrm{R}+1)a^2+\mathrm{A}/\pi}\int_0^a r^2\mathrm{J}_1\left(\frac{q_mr}{a}\right)dr=0. \quad . \qquad (39)$$

This can be simplified by using the following properties of Bessel functions (2, pp. 132, 135 and 17) :

$$\int_0^x z^2\mathrm{J}_1(z)dz=x^2\mathrm{J}_2(x),$$

$$2\int_0^x z\{\mathrm{J}_1(z)\}^2dz=x^2\{\mathrm{J}_1(x)\}^2-2x\mathrm{J}_1(x)\mathrm{J}_0(x)+x^2\{\mathrm{J}_0(x)\}^2,$$

and

$$x\mathrm{J}_2(x)=2\mathrm{J}_1(x)-x\mathrm{J}_0(x).$$

After substituting for $\mathrm{J}_1(q_n)$ from (36) we find

$$\mathrm{B}_n=\frac{8a\mathrm{A}c_0}{(\mathrm{R}+1)a^2+\mathrm{A}/\pi}\;\frac{1+\alpha}{q_n(4+4\alpha+\alpha^2q_n{}^2)}\;\frac{1}{\mathrm{J}_0(q_n)}. \quad . \quad . \quad . \quad . \qquad (40)$$

The total amount of solute in the cylinder at time t is $\mathrm{M}_t=(\mathrm{R}+1)y(a,\,t)$, and the equilibrium amount is

$$\mathrm{M}_\infty=\frac{(\mathrm{R}+1)a^2\mathrm{A}c_0}{(\mathrm{R}+1)a^2+\mathrm{A}/\pi}=\frac{\mathrm{A}c_0}{1+\alpha}. \qquad (41)$$

Hence

$$\mathrm{M}_t=\mathrm{M}_\infty+(\mathrm{R}+1)\sum_n \mathrm{B}_n a\mathrm{J}_1(q_n)e^{-p_nt},$$

that is, using (36) and (40),

$$\frac{\mathrm{M}_\infty}{\mathrm{M}_t}=1-\sum_n\frac{4\alpha(1+\alpha)}{4+4\alpha+\alpha^2q_n{}^2}\,e^{-\beta q_n{}^2t}, \qquad (42)$$

where the q_n's are the positive roots (other than zero) of (36). For an infinite bath this reduces to

$$\frac{\mathrm{M}_t}{\mathrm{M}_\infty}=1-\sum_n\frac{4}{q_n{}^2}e^{-\beta q_n{}^2t}, \qquad (43)$$

where $\mathrm{J}_0(q_n)=0$ and $q_n>0$.

3·3. *Diffusion when the Rates of Adsorption and Diffusion are comparable.*

The best way of solving this problem is probably by using the Laplace transform.

The formulæ required are (see *e. g.* 3, chapter 4)

$$\bar{y}(s)=\int_0^\infty y(t)\,e^{-st}dt, \qquad \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\bar{y}(s)e^{st}ds=\begin{matrix} y(t) & (t>0) \\ 0 & (t<0)\end{matrix}.$$

In the first formula the real part of s must be sufficiently large for the integral to converge; in the second formula γ must be such that the path of integration lies to the right of all the singularities of $\bar{y}(s)$.

Now, if $y=0$ when $t=0$, we have

$$\int_0^\infty\frac{\partial y}{\partial t}e^{-st}dt=\Big[ye^{-st}\Big]_0^\infty+s\int_0^\infty ye^{-st}dt=s\bar{y}(s).$$

Hence, if we multiply equation (26) by e^{-st} and integrate with respect to t and define $\bar{z}(s)$ as the transform of $z(t)$, we obtain

$$s\bar{y}(r,\,s)=\kappa r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\bar{y}(r,\,s)}{\partial r}\right)-s\bar{z}(r,\,s), \quad s\bar{z}(r,\,s)=\lambda\bar{y}(r,\,s)-\mu\bar{z}(r,\,s). \quad (44)$$

Similarly, the condition for the conservation of the solute, namely

$$\frac{\mathrm{A}}{2\pi}\frac{1}{r}\frac{\partial y}{\partial r}+y+z=\mathrm{A}c_0 \quad (r=a), \qquad (45\,a)$$

becomes

$$\frac{\mathrm{A}}{2\pi a}\frac{\partial\bar{y}}{\partial r}+\bar{y}+\bar{z}=\frac{\mathrm{A}c_0}{s} \quad (r=a). \qquad (45\,b)$$

Since differentiations with respect to s do not occur we may replace the partial differentiation signs by ordinary ones.

Eliminating $\bar{z}$ from equations (44) we find

$$r\frac{d}{dr}\left(\frac{1}{r}\frac{d\bar{y}}{dr}\right)+k^2\bar{y}=0, \quad k^2=-\frac{s(s+\lambda+\mu)}{\kappa(s+\mu)}, \qquad (46)$$

of which the solution is $\bar{y}=\mathrm{F}(s)r\mathrm{J}_1(kr)$. The function $\mathrm{F}(s)$ is determined by the boundary condition (45), which gives at once

$$s\mathrm{F}(s)\left[\frac{\mathrm{A}}{2\pi}k\mathrm{J}_0(ka)+\frac{s+\lambda+\mu}{s+\mu}a\mathrm{J}_1(ka)\right]=\mathrm{A}c_0. \quad . \quad . \quad . \quad (47)$$

Applying the inverse transform, we see that $y(r,\,t)$ is given by

$$y(r,\,t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\mathrm{F}(s)r\mathrm{J}_1(kr)e^{st}\,ds$$

$$=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\mathrm{A}c_0r\mathrm{J}_1(kr)e^{st}ds}{k[(\mathrm{A}/2\pi)s\mathrm{J}_0(ka)-\kappa ka\mathrm{J}_1(ka)]}. \quad . \quad . \quad . \quad . \quad (48)$$

Now $\mathrm{J}_1(kr)/k$, $\mathrm{J}_0(ka)$ and $ka\mathrm{J}_1(ka)$ are all single-valued functions of k^2 and hence of s. The complex integral can therefore be evaluated by the calculus of residues, by deforming the contour into a large semicircle lying in the part of the complex plane for which the real part of s is less than γ. The integral taken along such a circle (provided it does not pass through a pole of the integrand) tends to zero as the radius of the circle tends to infinity, and the expression (48) is equal to the sum of the residues. The poles are all simple and are at $s=0$ and at $s=s_n$, where s_n is a root of

$$\frac{\mathrm{A}}{2\pi}k_n\mathrm{J}_0(k_na)+\frac{s_n+\lambda+\mu}{s_n+\mu}a\mathrm{J}_1(k_na)=0 \quad \left(k_n{}^2=-\frac{s_n(s_n+\lambda+\mu)}{\kappa(s_n+\mu)}\right). \quad (49)$$

The s_n's are all negative.

For small values of ka we have

$$\mathrm{J}_0(ka)=1+\mathrm{O}(k^2a^2), \quad \mathrm{J}_1(ka)=\tfrac{1}{2}ka+\mathrm{O}(k^3a^3).$$

Hence the residue at $s=0$ is

$$\mathrm{A}c_0r^2\Big/\left[\frac{\mathrm{A}}{\pi}+\left(1+\frac{\lambda}{\mu}\right)a^2\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

To find the residue at $s=s_n$, denote by $f(s)$ the expression in square brackets in the denominator of (48). Then, since $s-s_n$ is a simple factor of $f(s)$, the residue is

$$\mathrm{A}c_0r\mathrm{J}_1(k_nr)e^{s_nt}\Big/\left[k_n\left(\frac{df(s)}{ds}\right)_{s=s_n}\right]. \quad . \quad . \quad . \quad . \quad . \quad (51)$$

Now we have

$$\frac{df(s)}{ds}=\frac{\mathrm{A}}{2\pi}\mathrm{J}_0(ka)-\frac{dk}{ds}\left\{\frac{\mathrm{A}}{2\pi}sa\mathrm{J}_1(ka)+\kappa ka^2\mathrm{J}_0(ka)\right\}$$

on using the relations

$$\frac{d\mathrm{J}_0(z)}{dz}=-\mathrm{J}_1(z), \quad \frac{d}{dz}\left\{z\mathrm{J}_1(z)\right\}=z\mathrm{J}_0(z),$$

and so

$$y(r,\,t)=\frac{\mathrm{A}c_0}{(\mathrm{R}+1)(1+a)}\frac{r^2}{a^2}$$
$$+\sum_n\frac{\mathrm{A}c_0 r\mathrm{J}_1(k_nr)e^{s_nt}}{k_n\left[\frac{\mathrm{A}}{2\pi}\mathrm{J}_0(ka)-\frac{dk_n}{ds_n}\left\{\frac{\mathrm{A}}{2\pi}s_na\mathrm{J}_1(k_na)+\kappa k_na^2\mathrm{J}_0(k_na)\right\}\right]}\quad . \quad . \quad (52)$$

where

$$\mathrm{R}=\frac{\lambda}{\mu},\quad \alpha=\frac{\mathrm{A}}{\pi a^2(\mathrm{R}+1)}.\quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

The expression for $z(r,\,t)$ differs from that for $y(r,\,t)$ by having an extra factor $\lambda/(s_n+\mu)$ multiplying the nth term (including the term $s_n=0$). The corresponding multiplying factor for $y+z$ is

$$(s_n+\lambda+\mu)/(s_n+\mu)=-\kappa k_n^2/s_n.$$

The total amount of solute in the cylinder is

$$\mathrm{M}_t=y(a,\,t)+z(a,\,t)$$
$$=\frac{\mathrm{A}c_0}{1+\alpha}-\sum_n\frac{\kappa k_n}{s_n}\frac{\mathrm{A}c_0a\mathrm{J}_1(k_na)e^{s_nt}}{\frac{\mathrm{A}}{2\pi}\mathrm{J}_0(k_na)-\frac{dk_n}{ds_n}\left\{\frac{\mathrm{A}}{2\pi}s_na\mathrm{J}_1(k_na)+\kappa k_na^2\mathrm{J}_0(k_na)\right\}}.$$

The Bessel functions can be eliminated by using the relation (49), and we find

$$\frac{\mathrm{M}_t}{\mathrm{M}_\infty}=1-\sum_n\frac{(1+\alpha)e^{s_nt}}{1-\frac{\mathrm{A}}{2\pi}\kappa k_na^2\frac{dk_n}{ds_n}-\frac{\mathrm{A}}{2\pi}\frac{s_n^2}{\kappa k_n}\frac{dk_n}{ds_n}}\quad . \quad . \quad . \quad . \quad (54)$$

where k_n and s_n are given by (49). This can be further simplified by substituting for dk_n/ds_n, the final result being

$$\frac{\mathrm{M}_t}{\mathrm{M}_\infty}=1-\sum_n\frac{(1+\alpha)e^{s_nt}}{1+\left(1+\frac{\lambda\mu}{(s_n+\mu)^2}\right)\left(\frac{1}{(\mathrm{R}+1)\alpha}+\frac{\alpha}{4\kappa\beta}\frac{s_n^2}{k_n^2}\right)},\quad . \quad . \quad (55)$$

where $\beta=\kappa/\{(\mathrm{R}+1)a^2\}$ as before.

In the particular case of infinitely rapid adsorption, we have

$$k^2=-(\mathrm{R}+1)s/\kappa,\quad 2kdk/ds=-(\mathrm{R}+1)/\kappa,\quad s_n=-\beta q_n^2,$$

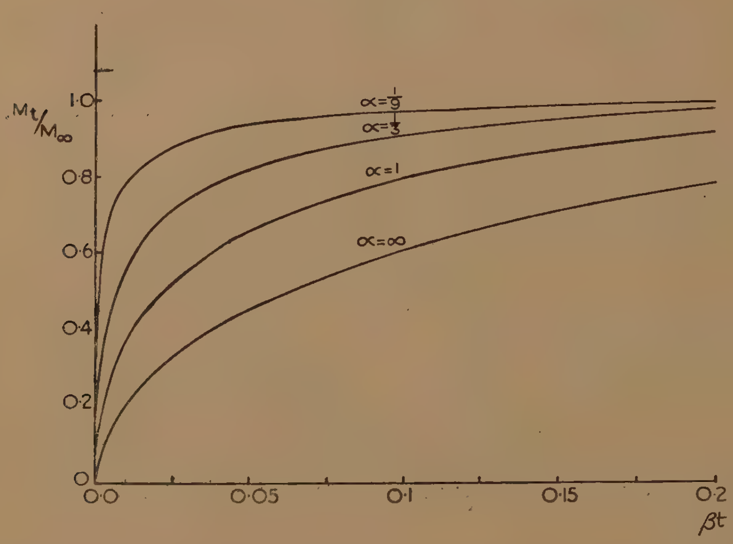
and we regain (42) at once.

Numerical Values.

4. Case 2 is the one of greatest physical importance, and we therefore confine the present discussion to formula (42), which gives M_t/M_∞ as a function of βt and of the parameter α. The latter is a measure of the finiteness of the bath, α being infinite for an infinite bath and decreasing as the bath grows smaller.

Curves giving $\mathrm{M}_t/\mathrm{M}_\infty$ as a function of βt are shown in the figure for four values of α, namely $\alpha=\frac{1}{9}, \frac{1}{3}$, 1 and ∞, corresponding to an exhaustion of the dye-bath of $\frac{9}{10}, \frac{3}{4}, \frac{1}{2}$ and 0 respectively. The main features of the curves, can, of course, be obtained without calculation, and their value lies in the fact that discrepancies between the elementary theory and experiment can now be determined quantitatively. Investigation of the discrepancies will reveal what extra refinements must be introduced into the theory in order to explain the observed facts. It is as yet too early to attempt this programme since a sufficient accumulation of reliable data is still lacking.

Fig. 1.



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(3) Carslaw and Jaeger, "Operational methods in applied mathematics." (Oxford, 1941).

VI. *The Critical Load of a Battened Column.*

By Professor A. J. S. Pippard, M.B.E., D.Sc.,
Imperial College, London *.

[Received February 13, 1947.]

The most satisfactory formula for the determination of the strength of struts is based upon an analysis by the late Prof. John Perry of a member having initial curvature †. The degree of curvature which must be assumed to obtain consistency with test figures was the object of researches made by Prof. Andrew Robertson * and, as a result of these, the Perry-Robertson formula is now the basis of column design in this country.

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* Communicated by the Author.

† See 'Analysis of Engineering Structures,' Pippard and Baker (Arnold). 2nd ed. pp. 132 *et seq.*

The formula involves the determination of the theoretical load which would produce a state of neutral equilibrium for the member if it were straight, *i. e.* the Euler critical load. With solid members this is simple, but a type of strut known as the battened column presents more difficulty.

Fig. 1.

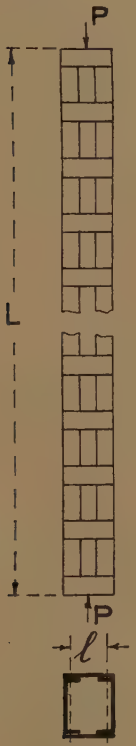
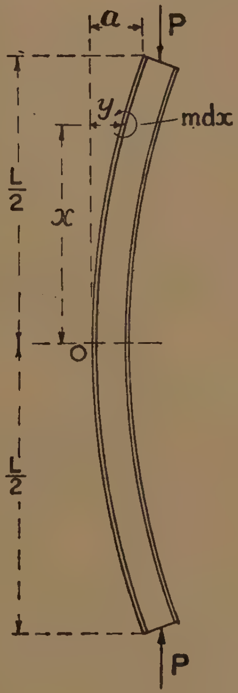


Fig. 2.



In this paper, a theoretical treatment of the problem is given which, although approximate, is based upon a device which has in other problems yielded excellent results when tested experimentally †. There is reason to believe therefore that the present analysis constitutes a satisfactory solution of the problem, although direct tests are not available. The construction of the column is shown diagrammatically in fig. 1. Two flange members, considered equal as is usual in this type of structure, are connected at equal intervals by flat batten plates. The joints between flanges and batten plates are assumed to be rigid so that there is no relative movement of these members at their junctions. It is also assumed that there are a sufficient number of batten plates to prevent premature failure of a section of the flange, *i. e.* the column reaches its critical load before any component fails.

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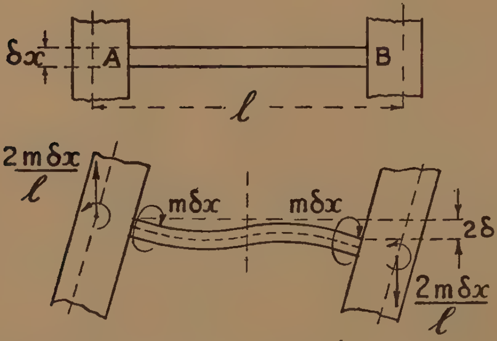
* "The Strength of Struts," Andrew Robertson, Inst. C. E. Selected Papers No. 28.

† "The Stresses in a Radially Spoked Wire Wheel," Pippard and Francis, Phil. Mag. vol. xi. (Feb. 1931).

"The Loading of Inter-connected Bridge-girders," Pippard and de Waele. Journal Inst. C. E. pp. 97–114 (1938).

Let P be the critical axial compression on the pin-ended column,
A the area of each flange,
I the second moment of area of the flange section about the appropriate neutral axis,
k the radius of gyration of the flange section about the same axis,
I_c the second moment of area of each batten plate about the axis of bending,
L the length of the column between pins,

Fig. 3.



l the distance between the lines of centroids of the flanges,
n the number of batten plates on each face of the column,
E Young's modulus for the material of the column
and Q the critical load for one flange $=\dfrac{\pi^2 EI}{L^2}$.

It will be supposed that the flanges, instead of being joined by a finite number of batten plates, are connected by a web which can apply flexural restraint to the flanges, but cannot transmit stress along the axis of the column. The flexural rigidity of this web will be taken as EI′ per unit length of the column.

Fig. 3 shows an element of this web of length δx. When the column is deflected from its initial position the element is distorted as shown and applies couples of magnitude $m\delta x$ to each flange and also reactive shearing forces of magnitude $\dfrac{2m\delta x}{l}$. These forces are resisted by a tension in flange A and a compression in flange B. The integral of the changes in flange-length from these elementary strains causes a displacement of B relative to A of 2δ. By considering the element of the web as a beam it is easily shown that the slope at the flange connections is

$$i_x = \frac{ml}{6\mathrm{EI}'} + \frac{2\delta}{l}, \qquad (1)$$

where m is the intensity of couple applied to the flanges. Since the web and flanges are rigidly connected, i_x is also the slope of the flanges at the point.

If the column is displaced from its normal position when carrying a load less than the critical value, it will recover its straightness when the displacing force is removed. When it carries the critical load it will remain in the displaced position. Fig. 2 shows the column under the critical load having been displaced an arbitrary amount a at the centre. The load is equally shared between the flanges, since they are assumed to be similar, and each carries $\frac{\mathrm{P}}{2}$. Taking an origin at the centre of the displaced column and measuring x and y positive as shown, the total force in each flange at x, due to the web reactions, is

$$\int_x^{\frac{\mathrm{L}}{2}} \frac{2m\,dx}{l}.$$

The increase in length of the element δx of the flange in tension and the decrease of that in compression is

$$\frac{2\delta x}{l\mathrm{AE}} \int_x^{\frac{\mathrm{L}}{2}} m\,dx.$$

The total movement of a point on either flange is

$$\frac{2}{l\mathrm{AE}} \int_0^x \int_x^{\frac{\mathrm{L}}{2}} m\,dx\,dx,$$

and so

$$2\delta = \frac{4}{l\mathrm{AE}} \int_0^x \int_x^{\frac{\mathrm{L}}{2}} m\,dx\,dx. \qquad (2)$$

For either flange,

$$\mathrm{EI}\frac{d^2y}{dx^2} = \frac{\mathrm{P}}{2}(a-y) - \int_x^{\frac{\mathrm{L}}{2}} m\,dx,$$

so,

$$\mathrm{EI}\frac{d^3y}{dx^3} = -\frac{\mathrm{P}}{2}\frac{dy}{dx} + m,$$

and

$$m = \mathrm{EI}\frac{d^3y}{dx^3} + \frac{\mathrm{P}}{2}\frac{dy}{dx}. \qquad (3)$$

From (1) and (2),

$$\frac{dy}{dx} = \frac{ml}{6\mathrm{EI}'} + \frac{4}{l^2\mathrm{AE}} \int_0^x \int_x^{\frac{\mathrm{L}}{2}} m\,dx\,dx, \qquad (4)$$

or

$$\frac{d^3y}{dx^3} = \frac{l}{6\mathrm{EI}'}\frac{d^2m}{dx^2} - \frac{4m}{l^2\mathrm{AE}}.$$

Then, using (3),

$$\frac{d^5y}{dx^5}+c_1\frac{d^3y}{dx^3}-c_2\frac{dy}{dx}=0,$$

where $$c_1=\frac{P}{2EI}-\frac{6I'}{l}\left(\frac{1}{I}+\frac{4}{Al^2}\right),$$

and $$c_2=\frac{12PI'}{l^3AEI}.$$

The appropriate solution of the equation, taking account of symmetry, is

$$y=A\cosh\alpha x+B\cos\beta x+D, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where A, B and D are constants to be determined.

In this equation

$$\alpha=\left(\frac{-c_1+\sqrt{c_1^{\,2}+4c_2}}{2}\right)^{\frac{1}{2}},$$

$$\beta=\left(\frac{c_1+\sqrt{c_1^{\,2}+4c_2}}{2}\right)^{\frac{1}{2}}.$$

When $x=0$, $y=0$; and when $x=\frac{L}{2}$, $\frac{d^2y}{dx^2}=0$ and $y=a$; and the constants are

$$A=-a\left\{\frac{\beta^2\cos\frac{\beta L}{2}}{\alpha^2\left(1-\cos\frac{\beta L}{2}\right)\cosh\frac{\alpha L}{2}-\beta^2\left(\cosh\frac{\alpha L}{2}-1\right)\cos\frac{\beta L}{2}}\right\},$$

$$B=-a\left\{\frac{\alpha^2\cosh\frac{\alpha L}{2}}{\alpha^2\left(1-\cos\frac{\beta L}{2}\right)\cosh\frac{\alpha L}{2}-\beta^2\left(\cosh\frac{\alpha L}{2}-1\right)\cos\frac{\beta L}{2}}\right\},$$

$$D=a\left\{\frac{\alpha^2\cosh\frac{\alpha L}{2}+\beta^2\cos\frac{\beta L}{2}}{\alpha^2\left(1-\cos\frac{\beta L}{2}\right)\cosh\frac{\alpha L}{2}-\beta^2\left(\cosh\frac{\alpha L}{2}-1\right)\cos\frac{\beta L}{2}}\right\},$$

From equations (4) and (3) it is seen that when $x=\frac{L}{2}$, $\frac{d^4y}{dx^4}=0$ and this condition, using equation (5), gives

$$-a\alpha^2\beta^2\cosh\frac{\alpha L}{2}\cos\frac{\beta L}{2}$$

$$\times\left\{\frac{\alpha^2+\beta^2}{\alpha^2\left(1-\cos\frac{\beta L}{2}\right)\cosh\frac{\alpha L}{2}-\beta^2\left(\cosh\frac{\alpha L}{2}-1\right)\cos\frac{\beta L}{2}}\right\}=0.$$

Since this must be satisfied for any value of a,

$$\cos\frac{\beta L}{2}=0,$$

and the condition which determines P, the critical load is

$$\beta L=\pi. \qquad (6)$$

With this value, the constants are

$$A=0\,;\quad B=-a \text{ and } D=a.$$

The deflected form of the column is then

$$y=a\left(1-\cos\frac{\pi x}{L}\right), \qquad (7)$$

and

$$m=\frac{\pi a}{L}\left(\frac{P}{2}-Q\right)\sin\frac{\pi x}{L}, \qquad (8)$$

while

$$2\delta=\frac{4La}{\pi l AE}\left(\frac{P}{2}-Q\right)\sin\frac{\pi x}{L}. \qquad (9)$$

It will be assumed that at the attachments of the batten plates the slopes and displacements in the actual column are the same as at corresponding points on the hypothetical column braced by a continuous web.

Then, if i_p and $2\delta_p$ are the slope and relative vertical displacement respectively at the pth batten plate, the bending moment applied to the flange by the two plates (one on each face) is

$$M_p=\frac{12EI_c}{l}\left(i_p-\frac{2\delta_p}{l}\right),$$

and the total moment applied by all the batten plates on one half of the column from $x=0$ to $x=\frac{L}{2}$ is

$$\frac{12EI_c}{l}\Sigma\left(i_p-\frac{2\delta_p}{l}\right),$$

the summation including all batten-plate attachments on that length.

If it be assumed that the same total moment is applied by the web of the hypothetical column,

$$\frac{12EI_c}{l}\Sigma\left(i_p-\frac{2\delta_p}{l}\right)=\int_0^{\frac{L}{2}}m\,dx=\frac{6EI'}{l}\int_0^{\frac{L}{2}}\left(\frac{dy}{dx}-\frac{2\delta}{l}\right)dx,$$

using equation (1).

From equations (7) and (9), after reduction, this gives

$$I'=\frac{I_c\pi Z}{L}, \qquad (10)$$

where Z is written for $2\Sigma\sin\frac{\pi x}{L}$.

$$\left.\begin{array}{ll}\text{When } n \text{ is even,} & Z=1+\operatorname{cosec}\theta \\ \text{When } n \text{ is odd,} & Z=1+\cot\theta \\ \quad\text{where} & \theta=\pi/2(n-1)\end{array}\right\} \quad \ldots \quad (11)$$

From the value of β we have in the general case

$$\beta^2(\beta^2-c_1)-c_2=0.$$

Substituting the particular value of $\beta=\frac{\pi}{L}$ at the critical load, for c_1 and c_2 and for I' from (10), we obtain

$$\frac{P}{Q}=\frac{2\left\{\frac{\pi l}{L}+\frac{6I_c}{I}\left(1+\frac{4k^2}{l^2}\right)Z\right\}}{\frac{\pi l}{L}+\left(\frac{24I_c}{I}\frac{k^2}{l^2}\right)Z} \quad \ldots \quad (12)$$

If I_e and k_e are the effective second moment of area of the column and the effective radius of gyration respectively,

$$\left.\begin{array}{l} I_e=\frac{PI}{Q}, \\ k_e=\sqrt{\frac{PI}{2AQ}}=k\sqrt{\frac{P}{2Q}} \end{array}\right\} \quad \ldots \quad (13)$$

When n is large, equation (10) becomes

$$I'=\frac{2nI_c}{L}, \quad \ldots \quad (14)$$

and (12) becomes

$$\frac{P}{Q}=\frac{2\left\{\frac{\pi^2 l}{L}+\frac{12nI_c}{I}\left(1+\frac{4k^2}{l^2}\right)\right\}}{\frac{\pi^2 l}{L}+\frac{48nI_ck^2}{Il^2}} \quad \ldots \quad (15)$$

To realize the implications of these formulas it is convenient to consider the case when the batten plates are so closely spaced as to touch one another. The depth of each is then $\frac{L}{n}$ and $I_c=\frac{tL^3}{12n^3}$, where t is the thickness of plate. Equation (12) is then

$$\frac{P}{Q}=\frac{2\left\{\frac{\pi l}{L}+\frac{tL^3}{2n^3I}\left(1+\frac{4k^2}{l^2}\right)Z\right\}}{\frac{\pi l}{L}+\left(\frac{2tL^3k^2}{n^3Il^2}\right)Z}.$$

When n is large, the value of Z approaches $\frac{2n}{\pi}$ and the second term in both the numerator and denominator decreases rapidly with n. When n is very large, $P\rightarrow 2Q$, *i. e.* the batten plates have no effect and the critical load for the column is simply the sum of the critical loads for the two flanges separately. When n is small, however, the second term has a preponderant effect and

$$\frac{P}{Q} \to \frac{l^2+4k^2}{2k^2}.$$

This expression can be written in the form

$$\frac{P}{Q} \to \frac{2\left(\frac{Al^2}{4}+I\right)}{I}.$$

The numerator, which from equation (13) is seen to be I_e, is the second moment of area of the flanges alone about the neutral axis of the column. If the batten plates were joined so that they could transmit stresses along the axis of the column, the webs would be fully effective and would be included in the second moment of area. Both the cases considered are artificial, but they show that P will lie between the crippling load for the two flanges considered as separate struts and the crippling load for a column in which the web plates do not contribute to I_e.

In a practical case, equation (12) can be used to calculate P; $\frac{P}{2A}$ is then the value of p_e in the Perry formula. The value to be assigned to the curvature coefficient can only be determined empirically as for any other type of strut, but certain available evidence indicates that a value of $0{\cdot}0015\ L/k_e$ should be reasonable.

ADDENDUM.

After the above was written, Prof. J. F. Baker of Cambridge University kindly supplied the author with the results of a test made by Mr. W. H. Ng, a research worker in his department. The column tested was formed of two steel channels $2\frac{1}{2}$ in. × $\frac{1}{2}$ in. × $\frac{1}{8}$ in. thick, connected by 9 batten plates on each face. The plates were 3 in. wide, $2\frac{1}{2}$ in. deep and $\frac{3}{16}$ in. thick. The overall test length was 6 ft. and the actual failing load was 10·55 tons.

The data for calculation were as follows :—

$L=72$ in.

$l=1{\cdot}75$ in.

$n=9$.

$A=0{\cdot}406$ sq. in.

$I=0{\cdot}0059$ in. units.

$I_c=0{\cdot}244$ in. units.

$k^2=0{\cdot}0145$.

Substituting these in equation (12), it is found that $P/Q=107$ and, assuming a value for Young's modulus of 13,000 tons per sq. in., Q is 0·146 tons. Hence $P=15{\cdot}6$ tons.

In the Perry-Robertson formula

$$p_c=\frac{p_y+(\eta+1)p_e}{2}-\sqrt{\left\{\frac{p_y+(\eta+1)p_e}{2}\right\}^2-p_yp_e},$$

we have p_c=intensity of axial loading to produce yielding of column,

p_y=yield stress of the material=18 tons per sq. in.,

p_e=intensity of critical loading=15·6/0·812=19·2 tons per sq. in.,

$\eta=0{\cdot}0015\,\mathrm{L}/k=0{\cdot}122$,

and so $p_c=12{\cdot}9$ tons per sq. in.

Therefore, the calculated strength of the column is 12·9×0·812=10·48 tons, which agrees almost exactly with the experimental value.

Little importance attaches to the result of a single test, however, and experimental evidence for columns of widely varying proportions is needed before the validity of the theoretical result can be established.

VII. *A Problem on Random Vectors.*

By R. D. Lord, Heriot-Watt College, Edinburgh *.

[Received April 1, 1947.]

1. *The Problem.*

The following problem has recently been proposed and solved by Horner †: to find the probability distribution of the component along the x-axis of the sum of n equal coplanar vectors, whose directions are random. Taking the vectors of unit length and $\theta_1, \theta_2, \ldots, \theta_n$ as the angles they make with the x-axis, we require the probability function of $X_n=\cos\theta_1+\cos\theta_2+\ldots+\cos\theta_n$, *i. e.*, we require $P_n(x)$, where $P_n(x)dx$ is the probability that $x \leqslant X_n \leqslant x+dx$. Since the directions are random, the θ's are independent random variables with identical probability functions

$$\left.\begin{aligned} p(\theta)&=1/2\pi, \qquad 0\leqslant\theta\leqslant 2\pi \\ &=0, \qquad \theta<0, \quad \theta>2\pi \end{aligned}\right\}. \qquad (1)$$

The required distribution is obviously symmetrical with zero mean. Also $P_n(x)$ will be zero if $|x|>n$. When $n=1$, we merely change the variable in (1) to $x=\cos\theta$, obtaining

$$P_1(x)=1/\pi\sqrt{1-x^2}, \qquad (2)$$

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* Communicated by the Author.

† F. Horner, "A Problem on the Summation of Simple Harmonic Functions of the same Amplitude and Frequency but of Random Phase," Phil. Mag. series (7), vol. xxxvii. pp. 145–162 (1946).

if $|x| \leqslant 1$, and zero otherwise. For $n > 1$, Horner used a geometrical argument leading to the equation

$$P_n(x) = \frac{1}{\pi} \int_{x-1}^{z} \frac{P_{n-1}(u)}{\sqrt{1-(u-x)^2}}\, du, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3$$

where z is the smaller of $(x+1)$ and $(n-1)$. For $n=2$, this gives, when $|x| \leqslant 2$,

$$P_2(x) = K(\sin v)/\pi^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where K is the complete elliptic integral of the first kind with modular angle $v = \text{arc} \sin \frac{1}{2}x$. If $n > 2$, (3) cannot be evaluated in terms of tabulated functions, and numerical integration is necessary. Complete reliance on (3) would mean tabulating $P_n(x)$ for all smaller values of n, and evaluating integrals whose integrands always have at least one infinity. Horner avoids this by relating the problem to the "random walk" problem, thus deriving a series solution (which he does not explicitly state) from Pearson's solution of the latter problem.

The present problem, however, is actually easier than the random walk problem and standard methods for the addition of random variables are applicable. In what follows, I first obtain Horner's solution and a modification of it; then a solution in which $P_n(x)$ is expressed as a single infinite integral involving Bessel functions. This second solution is analoguous to Kluyver's solution of the random walk problem *. Next, I obtain the cumulants of the distribution and use them to show that the distribution tends to normality as n increases. Then I show how Kluyver's solution of the random walk problem can be derived from the Bessel function solution, and finally, I indicate briefly the generalization to more than two dimensions. I should say here that I do not always adhere to Horner's notation and terminology.

2. *Derivation of Horner's Solution.*

Each of the random variables $\cos\theta$ has $P_1(x)$ for probability function, while the sum of any m of them has $P_m(x)$ for probability function. We can therefore regard X_n as the sum of the two random variables

$$\xi_1 = \cos\theta_1 + \cos\theta_2 + \ldots + \cos\theta_m,$$

with probability function $P_m(x)$, and

$$\xi_2 = \cos\theta_{m+1} + \cos\theta_{m+2} + \ldots + \cos\theta_n,$$

with probability function $P_{n-m}(x)$. Hence, by a standard theorem (Kendall, pp. 246–7),

$$P_n(x) = \int_{-\infty}^{\infty} P_m(u) P_{n-m}(x-u)\, du. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

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* See G. N. Watson, 'A Treatise on the Theory of Bessel Functions,' 2nd ed., p. 419, Cambridge (1944). I shall refer to this book as "Watson." For references on the theory of probability I shall quote, as "Kendall," the book 'The Advanced Theory of Statistics,' vol. i. London 1943, by M. G. Kendall.

The range of integration is actually finite since the integral is zero outside a certain finite interval.

When $m=n-1$, the integrand is $P_{n-1}(u)P_1(x-u)$, and on using (2) we get Horner's solution (3). Taking other values of m we obtain variants of this solution.

3. *The Solution in Terms of Bessel Functions.*

To obtain a different type of solution I use the characteristic function (Kendall, p. 54). By definition the characteristic function of $\cos\theta$ is

$$\int_{-\infty}^{\infty} e^{it\cos\theta}p(\theta)d\theta,$$

which, on substituting for $p(\theta)$, is seen to be $J_0(t)$. This is the characteristic function of each of $\cos\theta_1$, $\cos\theta_2$, . . ., $\cos\theta_n$. Since these random variables are independent, the characteristic function $\phi_n(t)$ of their sum is the product of individual characteristic functions (Kendall, p. 180), *i. e.*,

$$\phi_n(t)=J_0{}^n(t).$$

Applying the inversion theorem (Kendall, p. 91)

$$P_n(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}\phi_n(t)dt$$

$$=\frac{1}{\pi}\int_0^{\infty} \cos xt\, J_0{}^n(t)dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

When $n=1$, the integral is "Weber's discontinuous factor" (Watson, p. 405), and gives equation (2). When $n=2$, we make the integral into a repeated integral by using Neumann's integral for $J_0{}^2(t)$ (Watson, p. 31) and invert the order of integration; using Weber's discontinuous factor again and a standard substitution, we obtain the elliptic integral (4).

When $n>2$, it seems impossible to express $P_n(x)$ by a closed formula involving only tabulated functions, but numerical integration is available for any particular n and x. Actually it would be easier and more useful to evaluate the distribution function

$$F_n(x)=\int_{-\infty}^{x} P_n(u)\,du,$$

which is given by an integral more rapidly convergent than (6), namely by

$$F_n(x)-F_n(0)=\frac{1}{\pi}\int_0^{\infty}\frac{\sin xt}{t}J_0{}^n(t)dt.$$

Since the integrand in (6) is $O(t^{-\frac{1}{2}n})$ for large t, the integral is uniformly convergent for $n>2$, and hence $P_n(x)$ is then continuous. For $n=1$ the integrand is $O(t^{-\frac{1}{2}})$, and for $n=2$ it is $O(t^{-1})$, thus explaining the discontinuities in $P_1(x)$ and $P_2(x)$ (for the latter see Horner's fig. 4).

In the case of unequal vectors of lengths $a_1, a_2, \ldots a_n$, it is easy to see that $J_0{}^n(t)$ is to be replaced by $J_0(a_1t)J_0(a_2t)\ldots J_0(a_nt)$.

4. *Cumulants of the Distribution* : *Normal Approximation.*

The cumulant generating function is the logarithm of the characteristic function (Kendall, p. 60), and on expanding up to the term in t^6 is found to be

$$n\left(-\frac{1}{4}t^2-\frac{1}{64}t^4-\frac{1}{576}t^6\ldots\right).$$

The cumulant κ_r is the coefficient of $(it)^r/r\,!$ in this expansion, and hence the odd cumulants are zero and the even cumulants are

$$\kappa_2=\frac{1}{2}n,\quad \kappa_4=-\frac{3}{8}n,\quad \kappa_6=\frac{5}{4}n,\ \ldots.$$

κ_2 is the variance, *i. e.*, the square σ^2 of the standard deviation. Thus

$$\sigma^2=\frac{1}{2}n. \qquad \ldots\ldots\ldots\ldots \qquad (7)$$

In standard measure (*i. e.*, with σ as unit for x) the non-zero cumulants are $\kappa_2=1$, $\kappa_4=-3/2n$, $\kappa_6=5/n^2$, ..., and in general, $\kappa_{2r}=O(n^{-r+1})$. Thus, all cumulants after the second are either zero or tend to zero as n tends to infinity. Hence as n increases the distribution approximate to the normal distribution with standard deviation $\sqrt{n/2}$ and zero mean (Kendall, p. 110).

The normal approximation takes only the mean and standard deviation into account and better approximations can be obtained using the higher cumulants. The Pearson type II approximation appears to be better than the Gram Charlier series of type A, and is substantially better than the normal approximation, which is itself adequate for ordinary purposes when $n=7$ or even less.

5. *Relation to the Problem of the Random Walk.*

The random walk problem is this : to find the probability distribution of the magnitude of the sum of n random vectors. Let $Q_n(r)$ be the required probability function ; by this, we mean, let $Q_n(r)dA$ be the probability that the end point, B say, of the sum-vector OB be situated in an element of area dA distant r from the origin O. We know that $Q_n(r)$ is zero if $r>n$.

If we take $dA=dx\,dy$, keep x constant and integrate the probability with respect to y, we shall obtain the probability that the point B lies in a strip of width dx parallel to the y-axis, *i. e.*, we shall obtain $P_n(x)dx$.

Hence $$P_n(x)=\int_{-\infty}^{\infty}Q_n(r)dy=2\int_0^{\infty}Q_n(r)dy,$$

where the integrand is zero if $y^2>n^2-x^2$. Changing to r as the variable of integration

$$P_n(x)=2\int_0^{\infty}Q_n(r)\left(1-\frac{x^2}{r^2}\right)^{-\frac{1}{2}}dr,$$

where $\left(1-\frac{x^2}{r^2}\right)^{-\frac{1}{2}}$ is to be taken as zero if $r^2<x^2$, and the actual range of integration is x to n. Forming the characteristic function, and then inverting the order of integration in the resulting repeated integral,

$$\begin{aligned}J_0^n(t)&=2\int_{-\infty}^{\infty} e^{ixt}\,dx\int_0^{\infty} Q_n(r)\left(1-\frac{x^2}{r^2}\right)^{-\frac{1}{2}}dr\\ &=2\int_0^{\infty} Q_n(r)\,dr\int_{-\infty}^{\infty} e^{ixt}\left(1-\frac{x^2}{r^2}\right)^{-\frac{1}{2}}dx\\ &=2\int_0^{\infty} Q_n(r)dr\int_0^{\pi} e^{irt\cos u}\,r\,du, \text{ putting } x=r\cos u,\\ &=2\pi\int_0^{\infty} rJ_0(rt)Q_n(r)dr.\end{aligned}$$

But the final integral is a Hankel transform of $Q_n(r)$, and by the inversion theorem (Watson, p. 456)

$$Q_n(r)=\frac{1}{2\pi}\int_0^{\infty} tJ_0(rt)J_0^n(t)dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which is Kluyver's formula. For unequal vectors this equation is modified in the same way as (6) is.

6. *Generalization to more than Two Dimensions.*

In the case of p dimensions let us use generalized polar coordinates with θ as a co-latitude, varying from 0 to π. We have n equal random vectors for which all directions in space are equally likely, and we want the probability function $P_n(x)$ for the sum of the components along $\theta=0$. The probability $p(\theta)d\theta$ that the θ co-latitude of any one of the vectors lies between θ and $\theta+d\theta$ is proportional to $\sin^{p-2}\theta d\theta$, where the constant of proportionality is found from the fact that the integral of this probability over all possible values of θ is unity, *i. e.*, the constant is the reciprocal of $\int_0^{\pi}\sin^{p-2}\theta d\theta$. Hence

$$p(\theta)=\frac{\Gamma(\frac{1}{2}p)}{\Gamma(\frac{1}{2}p-\frac{1}{2})\Gamma(\frac{1}{2})}\sin^{p-2}\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

From this the characteristic function of $\cos\theta$ is found to be

$$\phi(t)=\Gamma(\tfrac{1}{2}p)(\tfrac{1}{2}t)^{-q}J_q(t),$$

where $q=\frac{1}{2}p-1$ (using Watson, p.25), whence $P_n(x)$ is found as in the two-dimensional case. It may be noted that when $p=3$, the component of a single vector has a uniform distribution; this may be seen from (9), or more simply, from Archimede's theorem on the area of a sphere.

The cumulant generating function is

$$n\left(-\frac{t^2}{2p}-\frac{(p+4)}{4p^2(p+2)}t^4\ldots\right),$$

in which the coefficients decrease in numerical value as p increases.

Thus for fixed n the closeness to normality increases with the number of dimensions. The variance is n/p and also decreases, so that the distribution becomes more concentrated near the mean. Also the integrand in the integral for $P_n(x)$ corresponding to equation (6) becomes smaller for large t, so that $P_n(x)$ is never discontinuous if $p>2$. The case $p=1$ fits into this picture of a distribution becoming more normal and " better behaved " (as regards continuity) as p increases ; for in this extreme case, where we are adding steps of $+1$ or -1 along the x-axis, the distribution is completely discontinuous and n must be quite large for the normal approximation to be good.

VIII. *The Effective Stress Concentration at the End of a Crack in Materials having Atomic Constitution.*

By C. Gurney *.

[Received November 22, 1946.]

Summary.

Classical theory of elasticity makes use of the concept of stress at a point, and the forces between atoms in a material of atomic constitution are only proportional to point stresses, if the variation in stress over the volume occupied by an atom is negligible. This is usually the case, but an exception occurs in all cases where theory of elasticity predicts infinite stress, and in particular in the case of the stress distribution at the end of sharp ended cracks, such as are postulated to account for the weakness of glass. To treat them mathematically, cracks are assumed to have elliptic form, and, in the present paper, the average stress over a small area at the end of the major axis of an elliptic crack is computed in terms of the ratio of the lengths of the major and minor axes of the ellipse, and of the ratio of the length averaged over, to the radius of curvature at the end of the crack. Particular interest is centred on the case in which the latter ratio is unity because the smallest radius of curvature in practice may be expected to be of the order of atomic dimensions, as is also the length over which the average stress must be computed to obtain the effective stress concentration for brittle materials. For this case, the effective stress concentration is about half the maximum stress concentration occurring at the line boundary of the hole. In the case of a crack of length $2a$ having zero radius of curvature at its end, the average stress over a length $\varDelta$, is approximately equal to $1{\cdot}8\sqrt{\frac{a}{\varDelta}}$, whereas the peak stress concentration is infinite. For given $\varDelta$, reducing the minimum radius from being numerically equal to $\varDelta$ to zero only increases the effective stress concentration by about 70 per cent.

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* Communicated by the Author.

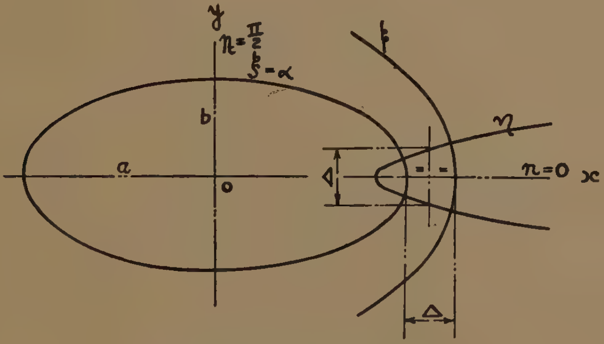
1. *Introduction.*

Classical theory of elasticity applies to a continuous elastic medium and makes use of the concept of stress at a point. In material having atomic constitution, stress at a point is of no significance, the smallest force of practical interest being that between individual atoms. This may be estimated from the theory of elasticity by multiplying the average stress over the volume occupied by an atom by the effective cross-sectional area of the atom. Usually, stress gradients are not so rapid that stresses vary appreciably over the effective diameter of an atom, and stresses computed from the theory of elasticity are then proportional to the forces between atoms. Nevertheless, very rapid stress gradients do sometimes occur ; for example, in all cases where the theory of elasticity predicts infinite stress for finite external forces. In such cases the infinite stress acts on an infinitesimal area. In particular, very rapid stress gradients occur at the ends of the sharp-ended cracks such as are postulated to occur in glass. In his original treatment, Griffith (1920) suggested a criterion of failure, in which the stress distribution in material near the ends of the crack was but indirectly involved—through its contribution to the strain energy of the material, which would be correctly estimated by classical elasticity theory. The very rapid stress gradient at the end of the crack was therefore in itself of no special consequence. Recently, however, the present author (1946) has shown that Griffith's criterion when applied to material of atomic constitution gives the stress at which thermal motions may be expected to cause the crack to start to spread by splitting, and that the rate of spread of the crack by this means may be very slow. Catastrophic failure will not occur, until the applied stress, combined with the disrupting effect of thermal motion, is sufficient to overcome the maximum possible inter-atomic attraction between the atoms at the end of the crack. In connection with this later criterion of failure, the rapid stress gradient at the end of the crack causes the average stress concentration over the cross-sectional area of an atom situated at the end of the crack to be appreciably less than the peak stress concentration at the line boundary of the crack. Calculation of this average stress is therefore of interest. To treat them mathematically, cracks are usually assumed to be elliptical. As the case of an ellipsoidal hole presents difficulty, the calculation is confined to that of an elliptical hole in a flat plate acted on by edge forces. The simplest solution is for radially symmetrical stress at infinity. The stresses at the end of the major axis of a narrow ellipse for this case are approximately the same as those caused by an unidirectional tension at right angles to the major axes of the ellipse. The calculation has therefore been made for the case of symmetrical stress.

2. *Calculation of average stress concentration.*

The calculation of the stress distribution around an elliptic hole is due to Inglis (1913). It is given in elliptic coordinates (see fig. 1). When calculating the average stress over a small area at the ends of the crack it is therefore convenient to treat an area bounded by ellipses and hyperbolas rather than one bounded by cartesian ordinates and abscissæ. Instead, therefore, of averaging the stress over a rectangular area of dimensions $\Delta\times\Delta$ it has been averaged over the area bounded by ellipses passing through the points a, 0 and $(a+\Delta)$, 0 and by a hyperbola passing through the points $\left(a+\frac{\Delta}{2}\right)$, $\frac{\Delta}{2}$ and $\left(a+\frac{\Delta}{2}\right)$, $-\frac{\Delta}{2}$, where a is the semi-major axis of the ellipse; b is the semi-minor axis. It is convenient to consider the half of this area lying above the x-axis. The transformation from cartesian to elliptic co-ordinates is given by

Fig. 1.



Elliptic coordinates.

$$\left.\begin{aligned} x&=c\cosh\xi\cos\eta \\ y&=c\sinh\xi\sin\eta \end{aligned}\right\}, \quad \ldots \ldots \quad (1)$$

where $c^2=a^2-b^2$.

The values of ξ and η bounding the area at the end of the crack are then given by

$$\left.\begin{aligned} \cosh\alpha&=\frac{a}{c}\ \cosh\xi=\frac{a+\Delta}{c} \\ \sinh\alpha&=\frac{c}{b}\ \sinh\xi=\frac{\sqrt{b^2+2a\Delta}}{c}=\frac{b}{c}\sqrt{1+2k} \\ \sin\eta&=\frac{\Delta}{2\sqrt{b^2+a\Delta}}=\frac{\Delta}{2b\sqrt{1+k}} \end{aligned}\right\}, \quad \ldots \quad (2)$$

where $\frac{a\Delta}{b^2}=\frac{\Delta}{r}=k$, where r=radius of curvature at the end of the ellipse.

In expressions (2) Δ^2 has been neglected compared with $b^2+2a\Delta$. The formula derived will therefore imply

$$\frac{\Delta}{c}<\frac{1}{50}.$$

We shall be interested only in ellipses in which the breadth is much less than the lengths, say

$$\frac{b}{c}<\frac{1}{10}.$$

For these values of $\frac{\Delta}{c}$ and $\frac{b}{c}$, $\frac{a}{c}$ is approximately unity and $\sinh\xi$ and $\sin\eta$ are small, so that they may be replaced by ξ and η. The limitation $\frac{b}{c}<\frac{1}{10}$ implies $\frac{r}{c}<\frac{1}{100}$, and for $\frac{r}{c}=\frac{1}{100}$, the biggest permissible value of k is 2. For smaller values of $\frac{b}{c}$ the largest permissible value of k increases as $\left(\frac{c}{b}\right)^2$.

The circumferential stress caused by unit principal stresses acting at infinity in two directions at right angles is

$$\overset{\smile}{\eta\eta}=\frac{\sinh 2\xi\,(\cosh 2\xi+\cosh 2\alpha-2\cos 2\eta)}{(\cosh 2\xi-\cos 2\eta)^2} \quad . \quad . \quad . \quad . \qquad (3)$$

Integrating with respect to ξ between the limits ξ and α gives

$$\int_\alpha^\xi \overset{\smile}{\eta\eta}\,d\xi=\tfrac{1}{2}\left(1-\frac{\cosh 2\alpha-\cos 2\eta}{\cosh 2\xi-\cos 2\eta}\right)+\tfrac{1}{2}\log\frac{\cosh 2\xi-\cos 2\eta}{\cosh 2\alpha-\cos 2\eta} \quad . \quad . \qquad (4)$$

The first expression may be integrated directly with respect to η, giving

$$\int_0^\eta\int_\alpha^\xi \overset{\smile}{\eta\eta}_1 d\xi d\eta=\tfrac{1}{2}\,\frac{\cosh^2\xi-\cosh^2\alpha}{\sinh\xi\cosh\xi}\tan^{-1}\frac{\tan\eta}{\tanh\xi}. \quad . \quad . \quad . \qquad (5)$$

The second expression is not integrable in terms of a finite number of recognised functions, but $\cosh 2\xi-\cos 2\eta$ can be written as the infinite product

$$\cosh 2\xi-\cos 2\eta=2\sin^2\eta\left(1+\frac{\xi^2}{\eta^2}\right)\overset{\infty}{\underset{1}{\pi}}\left(1+\frac{\xi^2}{(n\pi+\eta)^2}\right)\left(1+\frac{\xi^2}{(n\pi-\eta)^2}\right).$$

For small η and ξ the second integral then becomes

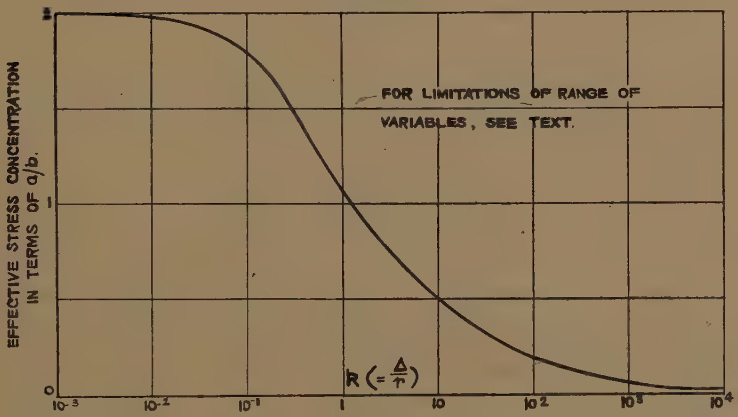
$$\begin{aligned}\int_0^\eta\int_\alpha^\xi \overset{\smile}{\eta\eta}_2 d\xi d\eta&=\tfrac{1}{2}\int_0^\eta\left(\log\frac{\xi^2+\eta^2}{\alpha^2+\eta^2}+\frac{2}{\pi^2}(\xi^2-\alpha^2)\sum_1^\infty\frac{1}{n^2}\right)d\eta\\&=\tfrac{1}{2}\int_0^\eta\left(\log\frac{\xi^2+\eta^2}{\alpha^2+\eta^2}+\frac{\xi^2-\alpha^2}{3}\right)d\eta\\&=\tfrac{1}{2}\left[\eta\log\frac{\xi^2+\eta^2}{\alpha^2+\eta^2}+2\xi\tan^{-1}\frac{\eta}{\xi}-2\alpha\tan^{-1}\frac{\eta}{\alpha}\right], \quad . \qquad (6)\end{aligned}$$

if $\int(\xi^2-\alpha^2)d\eta$ is negligible.

The average stress is obtained by dividing the sum of expressions (5) and (6) by $(\xi-\alpha)\eta$. Performing this division and substituting for ξ and η from expressions (2) gives, for the average stress,

$$\frac{1}{(\xi-\alpha)\eta}\int_0^\eta\int_\alpha^\xi \widetilde{\eta\eta}\,d\xi\,d\eta=\frac{c}{b(\sqrt{1+2k}-1)}\left[\tfrac{1}{2}\log_e\frac{4(1+2k)(1+k)+k^2}{4(1+k)+k^2}\right.$$
$$+2\sqrt{1+k}\left(\frac{1}{\sqrt{1+2k}}+\frac{\sqrt{1+2k}}{k}\right)\tan^{-1}\frac{k}{2\sqrt{1+k}\sqrt{1+2k}}$$
$$\left.-2\frac{\sqrt{1+k}}{k}\tan^{-1}\frac{k}{2\sqrt{1+k}}\right]. \qquad (7)$$

Fig. 2.



Effective stress concentration in terms of k and a/b.

Values of expression (7) are plotted in fig. 2, the ordinate being in terms of $\frac{c}{b}\approx\frac{a}{b}$ and the abscissa in terms of k. The sharpest cracks expected to be obtained in practice are those having radius of curvature at their ends equal to the order of atomic dimensions. Particular interest is therefore centred on values of k of the order of 1. In this region it is seen that the average stress is rapidly changing. For $k=1$, the average stress is $1{\cdot}1\frac{a}{b}$ compared with a peak stress of $2\frac{a}{b}$. The effect of atomic size in this case is therefore roughly to halve the stress concentration.

A case of some theoretical, if little practical interest, is that in which the curvature at the end of the crack is infinite. This leads to infinite point stress concentration, but when averaged over an area of $\Delta\times\Delta$, putting $k=\infty$ and $b=0$ in expression (7) gives an average stress of about

$$1{\cdot}8\sqrt{\frac{c}{\Delta}}.$$

For $k=1$, the corresponding expression is $1{\cdot}1\sqrt{\frac{c}{\Delta}}$, so that reducing the radius of curvature at the end of the crack from Δ to zero, only increases the average stress over a length of Δ by about 70 per cent.

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(1) A. A. Griffith, Phil. Trans. A, ccxxi. p. 163 (1920).
(2) C. Gurney, Proc. Phys. Soc. lix. p. 169 (1947).
(3) C. E. Inglis, Proc. Inst. Naval Architects, lv. p. 219 (1913).

IX. *The Compression of a Gas by a Rigid Piston Moving in a Closed Rigid Tube.*

By Stewart Paterson, M.A., B.Sc.
(Research Department, Imperial Chemical Industries, Ltd., Explosives Division) *.

[Received May 4, 1947.]

We suppose a column of ideal gas with constant ratio (γ) of specific heats to be confined at rest within a rigid parallel tube. A rigid piston commences instantaneously to move inward with constant velocity U relative to the closed end of the tube. A shock wave S_1 will then pass through the gas, to be reflected in turn from the closed end and the piston as successive shock waves $S_2, S_3, \ldots, S_n, \ldots$ When U is small compared with the sound velocity in the uncompressed gas, this process can be represented accurately by the ordinary adiabatic law; otherwise this will not be so. It is of interest to examine the general case, where U is not assumed to be small.

Let D_n, U_n be the numerical velocities of S_n and of the gas behind it relative to the tube; d_n, u_n the corresponding velocities relative to the gas ahead of S_n; p_n, v_n, T_n and a_n the pressure, specific volume, temperature and sound velocity in the gas behind S_n; p_0, v_0, T_0, a_0 being their initial values. Let, further,

$$\pi_n \equiv p_n/p_{n-1}, \quad \phi_n \equiv v_n/v_{n-1}, \quad \tau_n \equiv T_n/T_{n-1} = a_n^2/a_{n-1}^2,$$
$$\delta_n \equiv d_n{}^2/a_{n-1}^2, \quad \psi_n \equiv u_n{}^2/a_{n-1}^2, \quad \lambda \equiv (\gamma+1)/(\gamma-1).$$

Then, an application of the laws of conservation of mass, momentum and energy in the usual way (1, 2) across S_n gives

$$\phi_n \equiv \frac{\pi_n+\lambda}{\lambda\pi_n+1}, \qquad \ldots \ldots \ldots \quad (1)$$

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* Communicated by the Author.

$$\psi_n=\frac{(\lambda-1)^2(\pi_n-1)^2}{(\lambda+1)(\lambda\pi_n+1)}, \qquad (2)$$

$$\delta_n=\frac{\lambda\pi_n+1}{\lambda+1}, \qquad (3)$$

and, of course,

$$\tau_n=\phi_n\pi_n=\pi_n\,.\,\frac{\pi_n+\lambda}{\lambda\pi_n+1}. \qquad (4)$$

These equations would apply to any steady shock wave. The special conditions of our problem give further :—

$$D_{2n+1}=d_{2n+1}, \quad n=0, 1, \ldots, \qquad (5)$$

$$D_{2n}=d_{2n}-U, \quad n=1, 2, \ldots, \qquad (6)$$

$$U_{2n+1}=U, \quad n=0, 1, \ldots, \qquad (7)$$

$$U_{2n}=0, \quad n=1, 2, \ldots, \qquad (8)$$

$$u_n=U, \quad n=1, 2, \ldots \qquad (9)$$

From (9), $\psi_{n-1}=\psi_n\tau_{n-1}$, and so by (2) and (4)

$$\frac{(\pi_{n-1}-1)^2}{\pi_{n-1}(\pi_{n-1}+\lambda)}=\frac{(\pi_n-1)^2}{\lambda\pi_n+1},$$

whence

$$\pi_n=\frac{(\lambda+2)\pi_{n-1}-1}{\pi_{n-1}+\lambda}. \qquad (10)$$

From (1) and (10) we then deduce

$$\phi_n=\frac{(\lambda-2)\phi_{n-1}+1}{\lambda-\phi_{n-1}}, \qquad (11)$$

and from (3) and (11)

$$\delta_n=\frac{(\lambda+1)\delta_{n-1}-1}{\delta_{n-1}+\lambda-1}. \qquad (12)$$

The bilinear recurrence formulæ (10), (11) and (12) can be written in the forms

$$\frac{1}{\pi'_n}=\frac{1}{\pi'_{n-1}}+\frac{1}{\lambda+1},$$

$$\frac{1}{\phi'_n}=\frac{1}{\phi'_{n-1}}+\frac{1}{\lambda-1},$$

$$\frac{1}{\delta'_n}=\frac{1}{\delta'_{n-1}}+\frac{1}{\lambda},$$

where

$$\pi'_n\equiv\pi_n-1, \quad \phi_n'\equiv1-\phi_n, \quad \delta'_n\equiv\delta_n-1.$$

Hence,

$$\pi'_{n+1}=\frac{(\lambda+1)\pi_1'}{\lambda+1+n\pi_1'}, \qquad (13)$$

$$\phi'_{n+1}=\frac{(\lambda-1)\phi_1'}{\lambda-1+n\phi_1'}=\frac{(\lambda-1)\pi_1'}{\lambda+1+n\pi_1'}, \qquad (14)$$

$$\delta'_{n+1}=\frac{\lambda\delta_1'}{\lambda+n\delta_1'}=\frac{\lambda\pi_1'}{\lambda+1+n\pi_1'}, \qquad (15)$$

using (1) and (3).

Again, from (2) and (13),

$$\frac{a_n{}^2}{\mathrm{U}^2}=\frac{1}{\psi_{n+1}}=\frac{[\lambda+1+n\pi_1{}'][\lambda+1+(n+\lambda)\pi_1{}']}{(\lambda-1^2)(\pi_1{}')^2}, \quad . \quad . \quad . \quad . \quad (16)$$

and so

$$\tau_{n+1}=\psi_{n+1}/\psi_{n+2}=\frac{[\lambda+1+(n+1)\pi_1{}'][\lambda+1+(\lambda+n+1)\pi_1{}']}{[\lambda+1+n\pi_1{}'][\lambda+1+(\lambda+n)\pi_1{}']} \quad . \quad (17)$$

and $$\frac{\mathrm{T}_n}{\mathrm{T}_0}=\frac{a_n{}^2}{a_0{}^2}=\frac{\psi_1}{\psi_{n+1}}=\frac{[\lambda+1+n\pi_1{}'][\lambda+1+(\lambda+n)\pi_1{}']}{(\lambda+1)(\lambda\pi_1{}'+\lambda+1)} \quad . \quad . \quad . \quad . \quad (18)$$

Also

$$\frac{d_{n+1}}{\mathrm{U}}=\sqrt{(\delta_{n+1}/\psi_{n+1})}=\frac{\lambda+1+(\lambda+n)\pi_1{}'}{(\lambda-1)\pi_1{}'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

whence $$\frac{\mathrm{D}_{2n+1}}{\mathrm{U}}=\frac{\lambda+1+(\lambda+2n)\pi_1{}'}{(\lambda-1)\pi_1{}'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$\frac{\mathrm{D}_{2n}}{\mathrm{U}}=\frac{\lambda+1+2n\pi_1{}'}{(\lambda-1)\pi_1{}'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

It is easy to show that π_n, δ_n and τ_n tend monotonically to 1 from above, and ϕ_n from below, as $n\rightarrow\infty$; and that $d_{n+1}>d_n>a_0$.

Also $\mathrm{D}_{2n+1}-\mathrm{D}_{2n}=\dfrac{\lambda\mathrm{U}}{\lambda-1}>0$ and $\mathrm{D}_{2n}-\mathrm{D}_{2n-1}=\dfrac{2-\lambda}{\lambda-1}\mathrm{U}$,

which is <0, unless indeed $\gamma>3$.

From (13) we have

$$\frac{p_n}{p_0}=\frac{\Gamma\left(\dfrac{\lambda+1}{\pi_1{}'}\right)\Gamma\left(\dfrac{\lambda+1}{\pi_1{}'}+\lambda+1+n\right)}{\Gamma\left(\dfrac{\lambda+1}{\pi_1{}'}+\lambda+1\right)\Gamma\left(\dfrac{\lambda+1}{\pi_1{}'}+n\right)}, \quad . \quad . \quad . \quad . \quad (22)$$

and from (14) $$\frac{v_n}{v_0}=\frac{\Gamma\left(\dfrac{\lambda+1}{\pi_1{}'}+\lambda\right)\Gamma\left(\dfrac{\lambda+1}{\pi_1{}'}+n+1\right)}{\Gamma\left(\dfrac{\lambda+1}{\pi_1{}'}+1\right)\Gamma\left(\dfrac{\lambda+1}{\pi_1{}'}+\lambda+n\right)}. \quad . \quad . \quad . \quad (23)$$

The n-eliminant of (22) and (23) replaces $p_nv_n{}^\gamma=p_0v_0{}^\gamma$ and might be called the " dynamic adiabatic " for the compression. If $\dfrac{\mathrm{U}}{a_0}$ (and so $\pi_1{}'$) is small, then $\dfrac{\lambda+1}{\pi_1{}'}$ and $\dfrac{\lambda+1}{\pi_1{}'}+n$ are both large, irrespective of n.

Hence $$\Gamma\left(\frac{\lambda+1}{\pi_1{}'}+\lambda+1+n\right)\sim\left(\frac{\lambda+1}{\pi_1{}'}+n\right)^{\lambda+1}\Gamma\left(\frac{\lambda+1}{\pi_1{}'}+n\right),$$

and $$\Gamma\left(\frac{\lambda+1}{\pi_1{}'}+\lambda+1\right)\sim\left(\frac{\lambda+1}{\pi_1{}'}\right)^{\lambda+1}\Gamma\left(\frac{\lambda+1}{\pi_1{}'}\right),$$

so that $$\frac{p_n}{p_0}\sim\left(\frac{n\pi_1{}'}{\lambda+1}+1\right)^{\lambda+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Similarly,
$$\frac{v_n}{v_0} \sim \left(\frac{n\pi_1'}{\lambda+1}+1\right)^{1-\lambda} \qquad (25)$$

Hence
$$p_n v_n^{\gamma} \sim p_0 v_0^{\gamma}.$$

That is, the ordinary adiabatic is closely followed.

If n is also small, from (24) and (25),
$$\frac{p_n}{p_0} \sim 1+n\pi_1',$$
$$\frac{v_n}{v_0} \sim 1-\frac{\lambda-1}{\lambda+1}\,n\pi_1' \sim 1-n\phi_1'.$$

Hence
$$p_n-p_0 \sim n(p_1-p_0), \qquad (26)$$
$$v_0-v_n \sim n(v_0-v_1), \qquad (27)$$

and so
$$\mathrm{T}_n-\mathrm{T}_0 \sim n(\mathrm{T}_1-\mathrm{T}_0). \qquad (28)$$

That is, the pressure, volume and temperature changes caused by the first wave are approximately doubled in the second, trebled in the third and so on.

When π_1' is not small, p_n and v_n are not connected by the normal adiabatic relationship, at any rate for n small. The large changes in pressure and volume and smaller changes in temperature can be deduced at once from (13), (14) and (17). Thus,
$$\pi_n \sim \frac{\lambda+n}{n-1};\quad \phi_n \sim \frac{n}{n-1+\lambda},\quad \text{and}\quad \tau_n \sim \frac{n(\lambda+n)}{(n-1)(\lambda+n-1)}.$$

Moreover,
$$\frac{\mathrm{T}_n}{\mathrm{T}_1} \sim \frac{n(\lambda+n)}{\lambda(\lambda+1)}.$$

When n is large, however,
$$\frac{p_n}{p_0} \sim \frac{n^{\lambda+1}\,\Gamma\left(\frac{\lambda+1}{\pi_1'}\right)}{\Gamma\left(\frac{\lambda+1}{\pi_1'}+\lambda+1\right)}$$

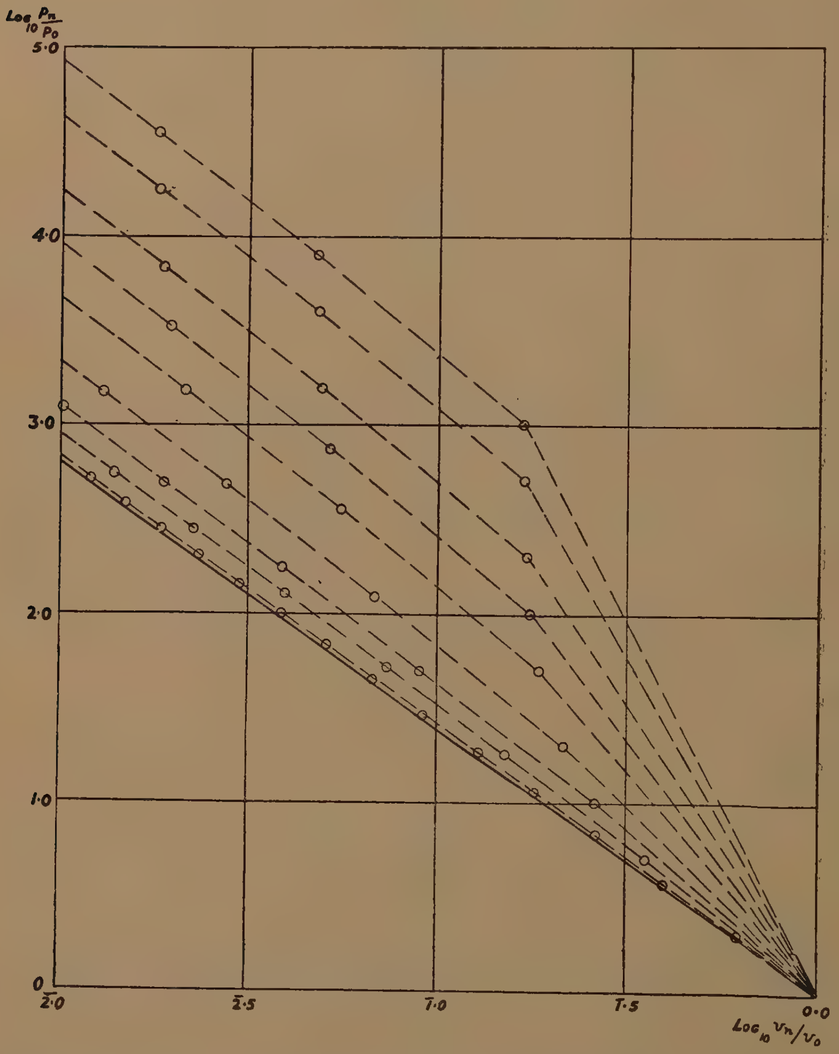
and
$$\frac{v_n}{v_0} \sim \frac{n^{1-\lambda}\,\Gamma\left(\frac{\lambda+1}{\pi_1'}+\lambda\right)}{\Gamma\left(\frac{\lambda+1}{\pi_1'}+1\right)}.$$

Hence,
$$p_n v_n^{\gamma} \sim \mathrm{A} p_0 v_0^{\gamma}, \qquad (29)$$

where
$$\mathrm{A} \equiv \frac{(\pi_1')^2}{(\lambda+1)(\lambda\pi_1'+\lambda+1)}\left[\frac{\Gamma\left(\frac{\lambda+1}{\pi_1'}+\lambda\right)}{\Gamma\left(\frac{\lambda+1}{\pi_1'}+1\right)}\right]^{\gamma-1}. \qquad (30)$$

(v_n, p_n) thus ultimately follows an ordinary adiabatic, though not that through (v_0, p_0). A few values of A for $\gamma=1{\cdot}4$ are given in the table.

Fig. 1.

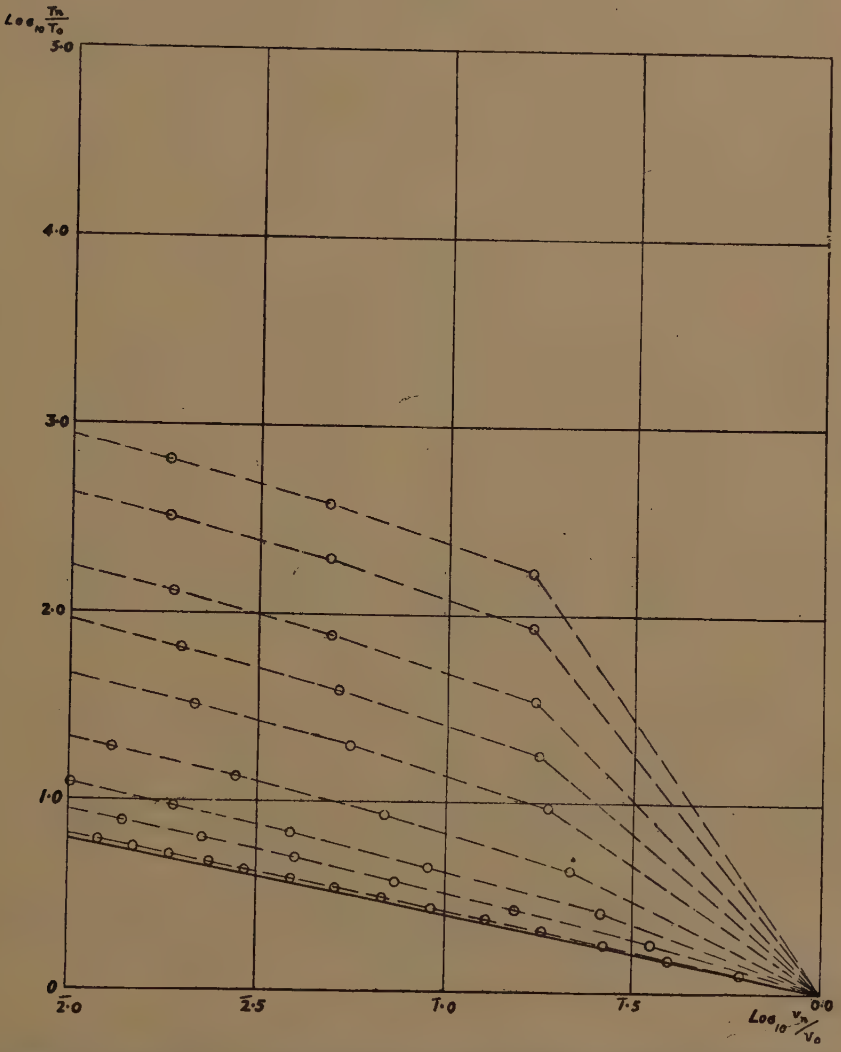


Repeated reflection of shock-waves in an ideal gas with γ constant and equal to 1·4.

The broken curves, reading down, correspond to $\pi_1=1000$, 500, 200, 100, 50, 20, 10, 5, 2. The points on each, reading from right to left, represent conditions behind the first, second, etc., waves. The full curve is the ordinary adiabatic.

As π_1' becomes larger, A approaches $[\Gamma(\lambda)]^{\gamma-1}\pi_1'/\lambda(\lambda+1)$. The relation between p_n, v_n and T_n is shown for the same values of π_1 in figs. 1 and 2.

Fig. 2.



Repeated reflection of shock-waves in an ideal gas with γ constant and equal to 1·4.

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TABLE.

U/a_0	π_1	A	U/a_0	π_1	A
0·0	1	1·0	5·34	50	8·73
0·524	2	1·08	7·63	100	16·85
1·36	5	1·59	10·75	200	33·0
2·18	10	2·31	17·2	500	81·3
3·27	20	3·91	24·4	1000	162·5

The above note may be considered as extending to multiple reflections the results given by Herpin (3) for a single reflection.

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(1) Rankine, Phil. Trans. clx. p. 277 (1870).
(2) Hugoniot, *J. de l'École Polyt.* lvii. p. 3 (1887).
(3) Herpin, *Comptes Rendus*, ccxxiii. p. 276 (1946).

X. *The Most Probable Colorimetric Tensor Derived from "n" Colour Matchings**.

By LUDWIK SILBERSTEIN, Ph.D.

THE components $c_{\iota\kappa}=c_{\kappa\iota}$ of the most probable tensor derivable from n free trichromatic matchings of a given colour are determined by the six equations

$$\frac{\partial}{\partial c_{\iota\kappa}} \sum_{i=1}^{n} \log f_i=0,$$

or

$$\frac{\partial}{\partial c_{\iota\kappa}} \{\tfrac{1}{2}\log D-ac_{11}-bc_{22}-cc_{33}-2dc_{23}-2ec_{31}-2fc_{12}\}=0, \quad . \quad . \quad (A)$$

where D is the determinant $|c_{\iota\kappa}|$ and $a=[x^2]$, etc., $d=[yz]$, etc.

Mr. Anthony Marriage, of Kodak Ltd., has just drawn my attention to an incorrectness of my solutions (16), *loc. cit.*, of the equations (A). His own, much simpler, form of the solutions is

$$c_{11}=\frac{1}{2\mathscr{H}}(bc-d^2), \text{ etc.,} \qquad c_{23}=\frac{1}{2\mathscr{H}}(ef-ad), \text{ etc.,}$$

where

$$\mathscr{H}=\begin{vmatrix} a & f & e \\ f & b & d \\ e & d & c \end{vmatrix}.$$

The following simple reasoning will show that Mr. Marriage's result is correct.

Let $c^{\iota\kappa}$ be the contravariant tensor associated with the covariant tensor $c_{\iota\kappa}$, briefly, *the supplement* of $c_{\iota\kappa}$. It is defined by $c^{\iota\kappa}=D_{\iota\kappa}/D$, where $D_{\iota\kappa}$ is the minor of D. It is well known that the supplement of the supplement is the *original* tensor, that is to say, $c_{\iota\kappa}$ is the supplement of $c^{\iota\kappa}$.

Now, since

$$\frac{\partial D}{\partial c_{11}}=D_{11} \qquad \text{and} \qquad \frac{\partial D}{\partial c_{23}}=D_{23}+D_{32}=2D_{23}, \text{ etc.,}$$

---

* *Corrigendum* to Phil. Mag. Ser. 7, vol. xxxvii. p. 126.

our equations (A) are

$$\left.\begin{array}{l} c^{11}=2a,\ c^{22}=2b,\ c^{33}=2c \\ c^{23}=2d,\ c^{31}=2e,\ c^{12}=2f \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

This, then, is the most probable *contravariant* colorimetric tensor, which itself may be useful. The required *covariant* tensor $c_{\iota\kappa}$ is simply the supplement of (B), *i. e.*,

$$\left.\begin{array}{l} c_{11}=\dfrac{bc-d^2}{2\mathscr{H}},\ \text{etc.} \\[2ex] c_{23}=\dfrac{ef-ad}{2\mathscr{H}},\ \text{etc.} \end{array}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (C)$$

where

$$\mathscr{H}=\tfrac{1}{8}\left|c^{\iota\kappa}\right|=\begin{vmatrix} a & f & e \\ f & b & d \\ e & d & c \end{vmatrix},$$

in perfect agreement with Mr. Marriage's finding.

We may mention that, in the particular case of $d=e=f=0$, the original formulæ (16), *loc. cit.*, agree with (C), both giving $c_{11}=1/2a$, etc., $c_{23}=c_{31}=c_{12}=0$. In other cases, however, the formulæ (16) disagree in some relevant details from the correct ones, (C).

129 Seneca Parkway,
Rochester, New York.
September 20th, 1947.

XI. *Theory of High Pressure Mercury Vapour Discharge.*

To the Editors of the Philosophical Magazine.

Gentlemen,—

In Mr. V. J. Francis's paper in the July 1946 issue of the 'Philosophical Magazine,' the following integral occurred and was evaluated numerically :—

$$S(x)=\int_0^1 \frac{t}{\sqrt{1-x\ln t}}\,.\,dt.$$

However, by a suitable change of variable it is readily shown that the integral may be expressed in terms of tabulated functions :—

$$S(x)=\sqrt{\frac{\pi}{2x}}\exp\left(\frac{2}{x}\right)\left[1-\operatorname{erf}\left(\sqrt{\frac{2}{x}}\right)\right].$$

The values given by the author in fig. 6 are found to be appreciably in error. The errors are presumably due to slips in computation, for even the simple Simpson rule gives S(2)=0·378, compared with 0·379 from the expression given above, whereas fig. 6 shows a value of about 0·29.

Yours faithfully,
R. E. Burgess.

Radio Division,
National Physical Laboratory,
Teddington, Middlesex.
July 24, 1947.

XII. *On the Uniqueness of the Acceleration Transformation.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

THE transformation coefficients connecting the two systems in relative acceleration motion have been given in letters $^{(1)}$ to 'Nature.' Here we show that the transformation is the only possible one that gives the electromagnetic field, a contravariant tensor, * the observed values, *i. e.* to satisfy

$$F^{\mu\nu'} = \frac{\partial x_\mu'}{\partial x_\alpha}\frac{\partial x_\nu'}{\partial x_\beta}F^{\alpha\beta}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

$$\left.\begin{aligned}
&\text{with } F^{\alpha\beta} = \\
&\begin{vmatrix}
0 & 0 & 0 & -i\dfrac{ex}{r^3} \\[2ex]
0 & 0 & 0 & -i\dfrac{ey}{r^3} \\[2ex]
0 & 0 & 0 & -i\dfrac{ez}{r^3} \\[2ex]
i\dfrac{ex}{r^3} & i\dfrac{ey}{r^3} & i\dfrac{ez}{r^3} & 0
\end{vmatrix}, \\
&\text{and } F^{\mu\nu'} = \\
&\begin{vmatrix}
0 & \dfrac{ey'\Gamma}{c^2r'^2} & \dfrac{ez'\Gamma}{c^2r'^2} & -i\left(\dfrac{ex'}{r'^3}-\dfrac{e\Gamma}{c^2r'}+\dfrac{ex'^2\Gamma}{c^2r'^3}\right) \\[2ex]
-\dfrac{ey'\Gamma}{c^2r'^2} & 0 & 0 & -i\left(\dfrac{ey'}{r'^3}+\dfrac{ex'y'\Gamma}{c^2r'^3}\right) \\[2ex]
\dfrac{ez'\Gamma}{c^2r'^2} & 0 & 0 & -i\left(\dfrac{ez'}{r'^3}+\dfrac{ex'z'\Gamma}{c^2r'^3}\right) \\[2ex]
i\left(\dfrac{ex'}{r'^3}-\dfrac{e\Gamma}{c^2r'}+\dfrac{ex'^2\Gamma}{c^2r'^3}\right) & i\left(\dfrac{ey'}{r'^3}+\dfrac{ex'y'\Gamma}{c^2r'^3}\right) & i\left(\dfrac{ez'}{r'^3}+\dfrac{ex'z'\Gamma}{c^2r'^3}\right) & 0
\end{vmatrix},
\end{aligned}\right\} \quad . \; . \; . \quad (2)$$

as solved from experimentally verified Maxwell's equations for an accelerating charge in x-direction.

From pure relativity considerations, it is easy to show that, for motions in the x-direction, $y'=y$ and $z'=z$. Then

$$F^{24'} = \frac{\partial t'}{\partial t}F^{24}, \quad \text{hence} \quad \left(\frac{\partial t'}{\partial t}\right) = \left(1+\frac{\Gamma x'}{c^2}\right)\alpha^3;$$

---

* This really amounts to the arbitrary way of the correspondence between F and E and H.

similarly $F^{31'} = \frac{\partial x'}{\partial t} \frac{F^{34}}{ic}$, hence $\frac{\partial x'}{\partial t} = \frac{\Gamma r'}{c} \alpha^3$; where $\alpha \equiv \frac{r}{r'}$.

Put

$$\begin{aligned}
\frac{\partial x'}{\partial x} &= 1 - \frac{x'\Gamma}{c^2} + \sum_{\nu=1} \alpha_\nu \Gamma^\nu, & \frac{\partial t'}{\partial x} &= -\frac{x'^2\Gamma}{c^3 r'} + \sum_{\nu=1} d_\nu \Gamma^\nu, \\
\frac{\partial x'}{\partial y} &= -\frac{y'\Gamma}{c^2} + \sum_{\nu=1} b_\nu \Gamma^\nu, & \frac{\partial t'}{\partial y} &= -\frac{x'y'\Gamma}{c^3 r'} + \sum_{\nu=1} e_\nu \Gamma^\nu, \\
\frac{\partial x'}{\partial z} &= -\frac{z'\Gamma}{c^2} + \sum_{\nu=1} c_\nu \Gamma^\nu, & \frac{\partial t'}{\partial z} &= -\frac{x'z'\Gamma}{c^3 r'} + \sum_{\nu=1} f_\nu \Gamma^\nu,
\end{aligned} \quad \ldots \quad (3)$$

for on putting $\Gamma=0$, the transformation must be reduced to unity. Now, when the origins of the two systems instantaneously coincide, the relative accelerations must be equal and opposite, *i. e.* $\Gamma_1 = -\Gamma$, as is evident even from purely dimensional considerations. That is $T^{-1}_{(\Gamma)} = T(-\Gamma)$, or

$$S = T(\Gamma, x')T(-\Gamma, x) = 1. \quad \ldots\ldots\ldots \quad (4)$$

From consideration of S_{44}, it follows that $\sum_{\nu=1} d_\nu \Gamma^\nu = 0$ and $x' = x$ and consequently $\alpha = 1$. Other coefficients follow from formula for $F^{14'}$ and the condition (4); namely

$$T(\Gamma) = \begin{vmatrix}
1 - \frac{x\Gamma}{c^2} & -\frac{y\Gamma}{c^2} & -\frac{z\Gamma}{c^2} & \frac{\Gamma r}{c} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{x^2\Gamma}{c^3 r} & -\frac{xy\Gamma}{c^3 r} & -\frac{xz\Gamma}{c^3 r} & 1 + \frac{\Gamma x}{c^2}
\end{vmatrix}. \quad \ldots \quad (5)$$

To prove the integrability of this transformation, we have to restore it into its original form with $v \neq 0$. To this end, we may apply a finite Lorentz transformation to (5) and on putting $\Gamma' = \frac{\Gamma}{(1-\beta^2)^{3/2}}$ as obvious from special relativity, we have

$T(\beta_1 I'') =$

$$\begin{array}{cccc}
\frac{1}{\sqrt{1-\beta^2}} - \frac{x\left(1+\beta\frac{x}{r}\right)}{c^2(1-\beta^2)^2}\Gamma' & -\frac{y\left(1+\beta\frac{x}{r}\right)}{c^2(1-\beta^2)^2}\Gamma' & -\frac{z\left(1+\beta\frac{x}{r}\right)}{c^2(1-\beta^2)^2}\Gamma' & \frac{v}{\sqrt{1-\beta^2}} + \frac{r\left(1+\beta\frac{x}{r}\right)}{c^2(1-\beta^2)^2}\Gamma' \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\frac{v}{{}^2\sqrt{1-\beta^2}} - \frac{x(x+\beta r)}{c^3 r(1-\beta^2)^2}\Gamma' & -\frac{y(x+\beta r)}{c^3 r(1-\beta^2)^2}\Gamma' & -\frac{z(x+\beta r)}{c^3 r(1-\beta^2)^2}\Gamma' & \frac{1}{\sqrt{1-\beta^2}} + \frac{x+\beta r}{c^2(1-\beta^2)^2}\Gamma'
\end{array}\Bigg|.$$

We can then verify that

$$\frac{\partial^2 x_\mu'}{\partial x_\alpha \partial x_\beta} = \frac{\partial^2 x_\mu'}{\partial x_\beta \partial x_\alpha},$$

and hence x', y', z', and t' can all be integrated in terms of x, y, z and t.

The author wishes to thank Prof. H. P. Soh for the suggestion of the problem.

Reference.

(1) 'Nature,' vol. clvii, p. 809, June 15, 1946 and vol. clviii, p. 99, July 20, 1946.

Department of Physics,
National University of Chekiang,
Hangchow, Chekiang, China.
January 11th. 1947.

Yours faithfully,
Wang Mu-Hsien.

XIII. *Notices respecting New Books.*

Annual Report of the Board of Regents of The Smithsonian Institution for 1945. (United States Government Printing Office, Washington, 1946.) For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington 25, D.C. Price $1.75.

The Smithsonian Institution was founded in 1846 by James Smithson and intended to be an institution for the "increase and diffusion of knowledge among men," and one of its interesting and pleasing features (very refreshing in these days when a vociferous minority insists that knowledge, including scientific knowledge, is only of value in so far as it can be directed to some practical or economic end) is the basic principle underlying its scientific work "that researches shall be prosecuted for the new knowledge to be gained alone, without regard to its possible economic usefulness."

The present volume contains the Annual Report submitted to Congress by the Board of Regents of the Institution. There is in it a detailed summary of its activities, including, naturally, special wartime activities. They are very multifarious and not by any means confined to science. Its interests range from art galleries and museums to a national zoological park and an astronomical observatory; but the greater portion of the volume consists of a General Appendix consisting of "a miscellaneous selection of papers (some of them original) embracing a considerable range of scientific investigation and discussion." They deal with all sorts of subjects and include among others :—

Blood and blood derivatives,
Thinking about race,
Conserving wild life,
Plastics and metals,
Conquest of the Northwest Passage,
Medical uses of the cyclotron,

and last (to be mentioned), but not least, a delightful and informative account of our galaxy and the extra-galactic nebulæ by Professor William T. Skilling of San Diego, with the title :—'Our Revolving "Island Universe" and its Spiralling Counterparts.'

This alone is worth the price demanded for the whole volume.

Wm. Wilson.

Les mesures physiques. By ALBERT PÉRARD. [Pp. 136.] (Paris: Presses universitaires de France, 1947.) No price given.

THIS paper-covered book of some 30,000 words is a recent addition to the comprehensive *Que sais-je*? series, which, in its 257 volumes to date, covers a remarkable range of subjects, literary and historical as well as scientific and technological. The author, distinguished director of the *Bureau International des Poids et Mesures*, refers to himself as "un métrologiste blanchi sous le harnais."

A brief introduction, which stresses the fundamental importance of measurement in science, closes in a way which illustrates well the author's meticulous objectivity: "Des mesures précises de masse ont conduit . . . à la réalisation de la bombe atomique. Est-ce là le dernier 'bienfait' de la métrologie? Les avis sont partagés." The first long chapter discusses general principles and methods, and errors of observation. There follows five chapters dealing in turn with the measurement of length, mass, and time; with electrical measurements; and with the measurement of temperature. In the final chapter a short survey is given of the principles of measurement of a variety of other physical magnitudes.

Though physics is not measurement, measurement is a *sine qua non* of physics. In the teaching of physics, however, the basic principles and methods of measurement are often inadequately discussed. This book provides an excellent corrective. It is by no means a treatise on the subject. Hardly any references are given, and it does not demand of the reader an extensive prior knowledge. It does, however, achieve its modest aim admirably, and the clear and straightforward presentation of essentials will be appreciated by many of the learned in physics no less than by the learners. E. C. S.

Physical Science in Art and Industry. By E. G. RICHARDSON. [Pp. xii+299.] English Universities Press, 1947.) Price 15*s*.

THE first edition of this book, published in 1940, is widely appreciated as an account, in manageable compass, of the applications of physics in many diverse fields. The book has now been brought up to date, and a new chapter on the physics of "detection" by acoustic and radar methods replaces the earlier more general chapter on physics and the art of war. Each of the fifteen chapters is a more or less independent essay of some 8,000 words, though the appearance of essentially the same physical problem (*e. g.* the measurement of particle sizes) or of the same physical device (*e. g.* the photoelectric cell) in a variety of different contexts helps to give unity to the book. The fields of application dealt with include locomotion (by land, water and air) and communication; pottery, building materials and textiles; the culinary arts (themes include the physical properties of dough and of jellies); farms, rivers and mines; fine art, architecture and music.

Almost inevitably, in covering so extensive a range, a catalogue effect cannot be entirely avoided, and there is little opportunity to bring the particular physical problems discussed into proper perspective in relation to the wider problems of the group of activities under consideration. Moreover, in a book primarily addressed to professional physicists, the author has perhaps unnecessarily hampered himself by avoiding formulæ or mathematics, a modest use of which in a number of places could have made vague and confusing descriptions much more precise and intelligible. It is, however, ungracious to complain on these grounds, for the author has obviously been at immense pains not to give way to temptations to depart from his self-imposed plan. Those who wish to know how physics can possibly "come into" this or that, a question about which students of physics are often extremely hazy, can be confident of finding an interesting and sometimes quite unexpected answer, or series of answers, for the many fields dealt with in this book. E. C. S.

Table of the Bessel Functions $\mathrm{J}_0(z)$ *and* $\mathrm{J}_1(z)$ *for Complex Arguments.* Prepared by the Mathematical Tables Project, National Bureau of Standards. [Pp. xliv+403; $10\frac{1}{2}'' \times 8''$.] 2nd edition. (New York, Columbia University Press, 1947.) Price $7.50.

The first edition of these tables was reviewed in the issue of this Magazine for May 1944. Since there have been no reports of errors in the tabular material of the first edition, the negatives of this first edition were used to reproduce the tables of the second edition. Changes from the first edition comprise:

(*a*) correction of the labelling of the graphs on pages xv and xvii; and

(*b*) minor revisions in the introduction, *e.g.* the addition of an explicit statement of the relations between the tabulated functions and the ber and bei functions.

J. M. Hammersley.

Tables of Spherical Bessel Functions, Vol. II. Prepared by the Mathematical Tables Project, National Bureau of Standards. [Pp. xx+328; $10\frac{1}{2}'' \times 8''$.] 1st edition. (New York, Columbia University Press, 1947.) Price $7.50.

The first volume of these tables was reviewed in the issue of this Magazine for April 1947. This second volume extends the tabulation of the function $\sqrt{\pi/2x}\,\mathrm{J}_\nu(x)$ for the range $\pm\nu=14\frac{1}{2}(1)30\frac{1}{2}$. The range of x is $0(0{\cdot}01)10(0{\cdot}1)25$ for $\pm\nu=14\frac{1}{2}(1)21\frac{1}{2}$ and $10(0{\cdot}1)25$ for $\pm\nu=22\frac{1}{2}(1)30\frac{1}{2}$. As in the first volume the function is tabulated to 7 significant figures for $x>10$, and to 8, 9, or 10 significant figures for $x \leqslant 10$ (except for a few entries near the zeros of the functions). Ancillary tables give $\Lambda_\nu(x)$ for the following ranges:—

(*a*) $+\nu=\frac{1}{2}(\frac{1}{2})20\frac{1}{2}(1)30\frac{1}{2},\ x=0(0{\cdot}1)10$;
(*b*) $+\nu=\frac{1}{2}(1)30\frac{1}{2},\ x=10(0{\cdot}1)25$;
(*c*) $-\nu=14\frac{1}{2}(1)16\frac{1}{2},\ x=0(0{\cdot}1)9{\cdot}5(0{\cdot}05)10(0{\cdot}1)25$;
(*d*) $-\nu=17\frac{1}{2}(1)30\frac{1}{2},\ x=0(0{\cdot}1)25$;

where $$\sqrt{\pi/2x}\,\mathrm{J}_\nu(x)=\mathrm{C}_\nu x^{\nu-\frac{1}{2}}\Lambda_\nu(x) \quad \text{and} \quad \mathrm{C}_\nu=\sqrt{\pi}/\{2^{\nu+\frac{1}{2}}\Gamma(\nu+1)\}.$$

The values of $\Lambda_\nu(x)$ are given to 9 decimal places for (*a*), and to 7 significant figures in most cases for (*b*), (*c*), and (*d*). Since C_ν is also tabulated to 10 significant figures for $\pm\nu=\frac{1}{2}(1)30\frac{1}{2}$, the ancillary tables permit the calculation of $\sqrt{\pi/2x}\,\mathrm{J}_\nu(x)$ for large ν and $x \leqslant 10$, where interpolation for $\Lambda_\nu(x)$ is easier than for $\sqrt{\pi/2x}\,J_\nu(x)$. For x-way interpolation in $\sqrt{\pi/2x}\,\mathrm{J}_\nu(x)$ and $\Lambda_\nu(x)$, the tables quote second differences (modified when the fourth differences exceed 1000 units of the eighth significant figure), and in a few cases fourth differences as well. Subsidiary tables of Everett coefficients are given at the end of the book. There is a schedule in the introduction showing the accuracy of ν-way Lagrangian interpolation of first, second, and third orders.

Finally the volume gives the first few (real) zeros of the Bessel function of the first kind and its derivative, together with the value of J' when $\mathrm{J}=0$ and the value of J when $\mathrm{J}'=0$. This tabulation is for all (real positive) zeros for which $0<x\leqslant 25$, and extends to all ν such that 2ν is a non-zero positive or negative integer. Here the accuracy of tabulation varies between 6 and 10 decimal places inclusive.

The standard of reproduction is similar to that of the first volume.

J. M. Hammersley

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

XIV. *The Effect of Manganese, Iron and Nickel on the α/β Brass Equilibrium.*

By William Hume-Rothery, F.R.S.*

[Received September 22, 1947.]

Synopsis.

The forms of the $\alpha/(\alpha+\beta)$ and $(\alpha+\beta)/\beta$ isothermal phase boundaries are discussed for the ternary system Cu-Al-Zn, and for ternary systems of the type Cu-Al-X, and Cu-Zn-X, where X is a transition element. It is shown that in the system Cu-Al-Mn and Cu-Zn-Mn, the first additions of manganese affect the $\alpha/(\alpha+\beta)$ and $(\alpha+\beta)/\beta$ boundaries in the way that would be expected if the α/β equilibrium were determined by electron concentration, and manganese acted as a divalent element. On passing from manganese→iron→nickel there is a continual decrease in the apparent valency of the transition element if it is again assumed that the equilibrium is determined by electron concentration (data are not available for the effect of cobalt). The valencies which have to be ascribed to the transition elements from the above point of view are quite different trom the numbers of electrons per atom which Pauling's theories regard as concerned in the bonding process in the crystals of the transition elements.

1. *Introduction.*

It is well known that when copper and silver are alloyed with some elements of higher valency, alloys of the $(\alpha+\beta)$ type are obtained in which, under conditions of true equilibrium, a face-centred cubic (α) phase exists in equilibrium with a body-centred cubic (β) phase. Hume-Rothery, Mabbott and Channel-Evans (1934) showed that in systems where the size factor was favourable the $\alpha/(\alpha+\beta)$ solubility curves were roughly superimposed if the equilibrium diagrams were drawn in terms of electron concentration, which was therefore regarded as the controlling factor. This idea was developed quantitatively by H. Jones (1937) who was able to calculate the approximate positions of the $\alpha/(\alpha+\beta)$ and $(\alpha+\beta)/\beta$ phase boundaries in copper-zinc and copper-aluminium alloys. This simple electron theory assumes that when, *e. g.*, zinc is dissolved in solid copper, the only effect is to increase the electron concentration, and the complications resulting from the differences in size or electro-chemical characteristics of the solvent and solute are ignored. Later work by Zener (1947) has shown that Jones's calculations were in error as regards the effect of temperature on the phase boundaries, although the theory is correct in predicting the approximate electron concentration at which the boundaries occur. Andrews and Hume-Rothery (1941) examined the effect of lattice distortion on the $\alpha/(\alpha+\beta)$ and $(\alpha+\beta)/\beta$ solubility limits in a number of copper alloys at a standard temperature of 671° C. This work showed that if the diagrams were drawn in terms of electron concentration, the general effect of increasing lattice distortion * was to displace the phase boundaries in the direction of lower electron concentration, and to increase the extent (in terms of electron concentration) of the $(\alpha+\beta)$ field. A quantitative relation involving the volume per electron was also discovered for the binary systems Cu-Zn, Cu-Ga, Cu-In and Cu-Sn but not for the system Cu-Al, and this discrepancy was thought to be due to the effect of the electrochemical factor which is greatest in the last system.

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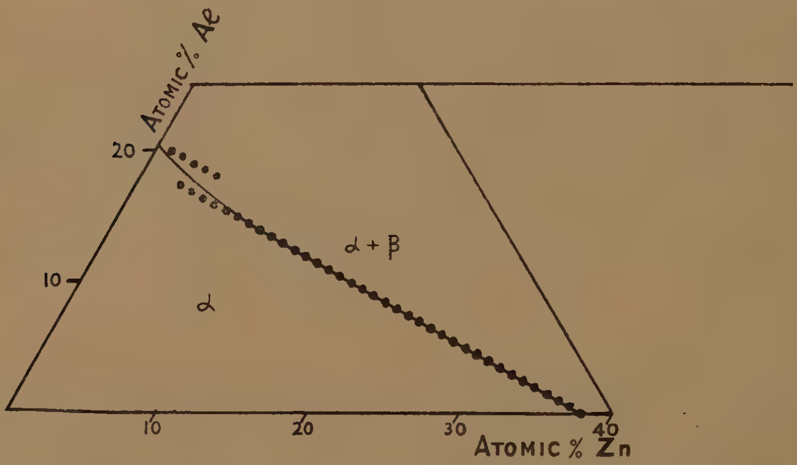
* Communicated by the Author.

Fig. 1.



The 550° C. isothermal $a/(a+\beta)$ boundary in the system Cu–Al–Zn. The dotted lines show the boundaries to be expected if aluminium affected the solubility of zinc in such a way as to maintain a constant electron concentration, and similarly for the effect of zinc on the solubility of aluminium.

It is clear that a complete theory of the α/β equilibrium in ternary alloys of copper will have to include the effects of lattice distortion and electrochemical factor, but in systems where these effects are relatively small the solubility isothermals are often controlled mainly by electron concentration. This effect is shown in the diagrams of Hansen and Bauer (1932) for the ternary system Cu-Al-Zn, which has recently been examined in greater detail and the 550° C. isothermal resulting from this study is shown in fig. 1. In this diagram † the $\alpha/(\alpha+\beta)$ solubility

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* In the copper alloys examined, the lattice distortion was always an expansion.

† We have to thank Dr. Maurice Cook for permission to reproduce this diagram.

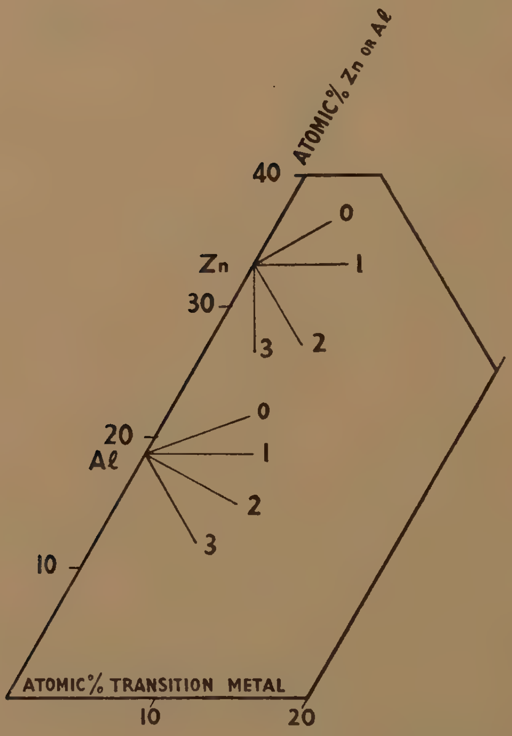
isothermal is approximately a straight line joining the $\alpha/(\alpha+\beta)$ points. In the system Cu-Zn the α-solubility limit at 550° C. is 38 atomic per cent. zinc, and the electron concentration is 1·38, whilst in the system Cu-Al the corresponding values are 20·4 and 1·408. This difference between the electron concentrations in the two systems is due to the differences between the lattice distortions and electrochemical factors. In fig. 1 the dotted lines show the way in which small additions of zinc would be expected to affect the solubility of aluminium in copper, and vice versa, on the assumption that the effect is purely one of electron concentration. It will be seen that when aluminium is added to the copper-zinc alloys, the $\alpha/(\alpha+\beta)$ solubility boundary remains at constant electron concentration over a wide range of composition, whereas when zinc is added to the copper-aluminium alloys there is an immediate departure from the simple electron concentration principle.

2. *The Effect of Transition Elements on the α/β Brass Type of Diagram.*

In considering the influence of transition elements on the α/β equilibrium, it is only possible to predict very general principles. It is known that iron, nickel and cobalt give rise to phases with a γ-brass structure, the compositions of which are variable, and which include those corresponding to the formulæ Fe_5Zn_{21}, Co_5Zn_{21}, Ni_5Zn_{21}. This is often taken to indicate that the transition elements act as though they possessed a zero valency, since the γ-structure is known to occur at an electron concentration of 21/13. If the transition elements did act as metals of zero valency, it would imply that any electrons they contributed to the lattice were counterbalanced by an equivalent absorption of electrons into the incomplete *d*-shells. Actually the compositions of these γ-phases vary over considerable ranges, and the transition elements could be allotted valencies somewhat greater than zero without creating any conflict with the facts. We cannot, of course, predict that exactly the same effect will be shown when these elements are introduced into α/β Cu-Zn or Cu-Al alloys, but we can conclude reasonably that if, in the presence of a transition element, the α/β equilibrium is still determined by electron concentration, a transition element such as nickel will act as though it possessed a relatively low valency. We may therefore expect the addition of nickel to increase the solubility of zinc or aluminium in copper over that portion of the equilibrium diagram where the saturated α-solid solution is in equilibrium with a random β phase. On passing back from nickel→cobalt→iron→manganese . . . we may expect this effect to become less marked, because the tendency to complete the *d*-shell is greatest when the latter is nearly filled. At some stage we shall therefore expect the transition element to act as though it has a valency comparable with that of zinc or aluminium.

At 800° C. the $\alpha/(\alpha+\beta)$ boundary occurs at approximately 33·4 atomic per cent. zinc in the system Cu-Zn; the corresponding electron concentration is 1·33. In the system Cu-Al the corresponding point is 18·7 atomic per cent. aluminium, the electron concentration being 1·37. If we now consider ternary alloys of the type Cu-Zn-X and Cu-Al-X, fig. 2 shows the directions expected for the $\alpha/(\alpha+\beta)$ boundaries on the assumption that these occur at a constant electron concentration, and that the third element X acts as though it possessed the valencies indicated. It will be appreciated that if X acts as a univalent element, then copper and X are equivalent, and the addition of X should leave the atomic percentage of zinc or aluminium unaltered. If X acts as a divalent element, then zinc and X are equivalent, and in the system Cu-Zn-X the $\alpha/(\alpha+\beta)$ boundary should be at a constant atomic percentage of copper, whilst in the system Cu-Al-X it must slope as shown because aluminium has a valency of three. In the same way, if X acts as a trivalent element, the phase boundary should lie at a constant atomic percentage of copper in the system Cu-Al-X, where aluminium and X have the same valency, and should slope as shown in the system Cu-Zn-X, where X has a higher valency than zinc. Similar considerations apply to the $(\alpha+\beta)/\beta$ boundaries if these are determined by electron concentration alone.

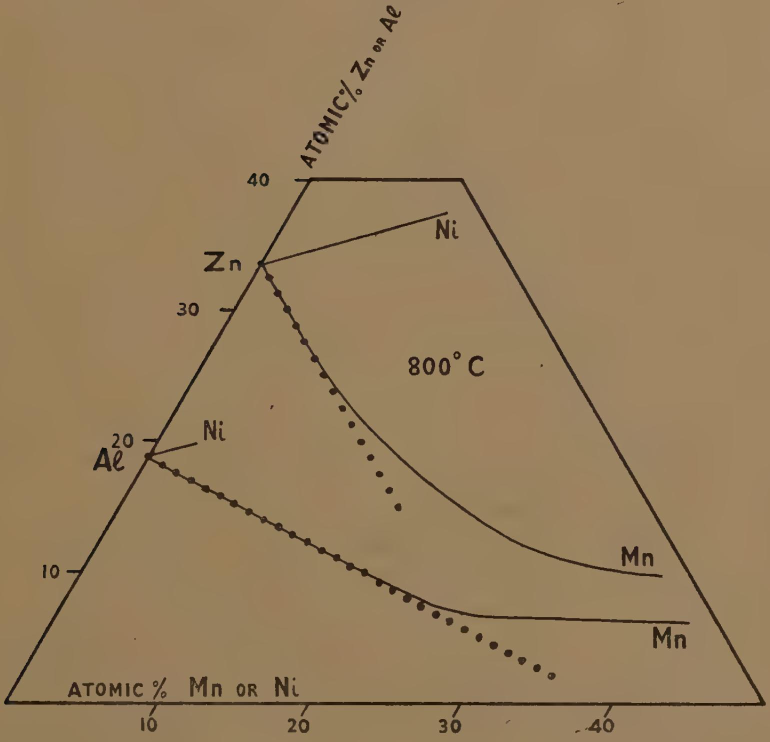
Fig. 2.



The lines marked 0, 1, 2, 3 show the directions expected for the $\alpha/(\alpha+\beta)$ isothermals at 800° C. in systems of the type Cu–Al–X and Cu–Zn–X, if the α/β equilibrium were determined by electron concentration, and the element X had the valencies 0, 1, 2 and 3 respectively.

Data for both the α/(α+β) and (α+β)/β boundaries are available for the systems Cu-Ni-Zn, Cu-Mn-Zn, Cu-Ni-Al and Cu-Mn-Al; the original sources are discussed at the end of this paper. In the system Cu-Fe-Zn, the solubility of iron in the α and β phases is very small (*ca.* 1 per cent.), but the work of Bauer and Hansen (1934) gives one point on the (α+β)/β curve at 800° C., although a corresponding point cannot be obtained for the α/(α+β) curve. When the results for the above systems are compared, it is found that the different investigators do not give exactly the same values for the solubility limits in the binary systems Cu-Zn and Cu-Al. In order to compare the effects in the ternary systems, it has therefore been necessary to adjust some of the curves slightly so that they start from a common point on the coordinate representing the binary system Cu-Zn or Cu-Al.

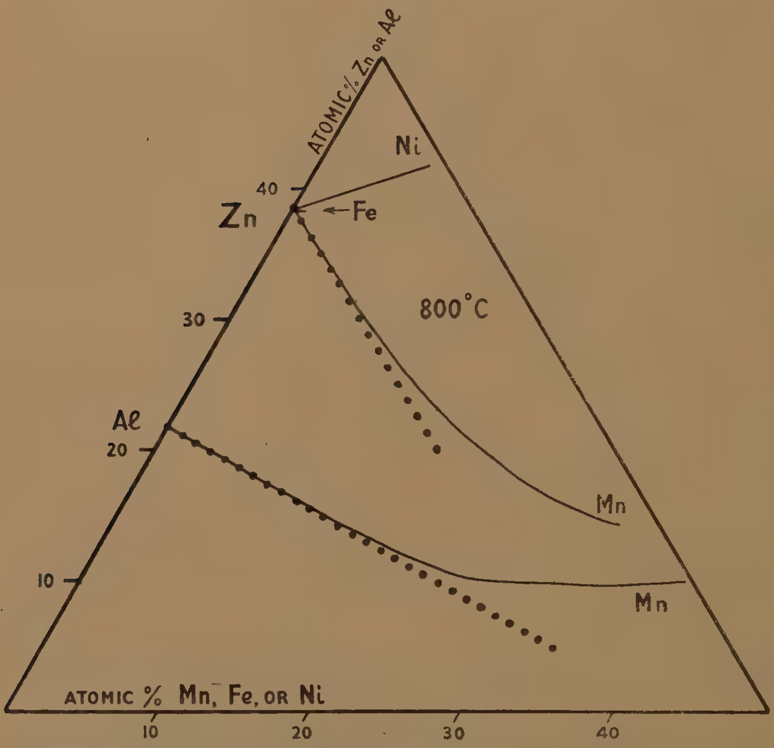
Fig. 3.



The full lines show the experimentally determined α/(α+β) isothermals at 800° C. in the system Cu–Mn–Al, Cu–Ni–Al, Cu–Mn–Zn and Cu–Ni–Zn. The dotted lines show the effects expected if the equilibrium were determined by electron concentration alone, and the transition element possessed a valency of 2.

Fig. 3 shows the α/(α+β) curves at 800° C. for the systems where they are known, whilst fig. 4 shows the corresponding (α+β)/β curves. Although some of these are only approximate, it is clear that regular effects are shown which agree with those outlined above. In the systems Cu-Ni-Zn and Cu-Ni-Al, the solubilities of zinc and aluminium are increased by the first addition of nickel, whilst the corresponding $(\alpha+\beta)/\beta$ curves slope in the same direction; this is in agreement with the view that nickel acts as an element of low valency. On passing from Ni→Fe→Mn the slopes of the $\alpha/(\alpha+\beta)$ and $(\alpha+\beta)/\beta$ curves indicate a regular increase in the apparent valency of the transition metal in agreement with expectation (page 91).

Fig. 4.



The full lines show the experimentally determined $(\alpha+\beta)/\beta$ isothermals in the systems Cu–Mn–Al, Cu–Mn–Zn, Cu–Fe–Zn and Cu–Ni–Zn. The dotted lines show the effects expected if the equilibrium were determined by electron concentration alone, and the transition element possessed a valency of 2.

3. *The Apparent Valencies of Manganese, Iron and Nickel in the α/β Brass Type of Diagram.*

In fig. 3 the lowest curve represents the $\alpha/(\alpha+\beta)$ boundary at 800° C. in the system Cu-Mn-Al, whilst the dotted line shows the boundary which would be expected if the equilibrium were controlled by electron concentration alone and manganese acted as a divalent element. It will be seen that the agreement is astonishingly precise for the first addition of manganese, but that a well marked bend in the curve occurs when the manganese content exceeds about 25 atomic per cent. At this point the total content of manganese plus aluminium is about 33 atomic per cent., so that the bend in the curve occurs at the point where there are roughly 2 atoms of solvent (copper) to 1 atom of solute (manganese + aluminium). The same general characteristics are shown by the $\alpha/(\alpha+\beta)$ isothermals at 850° C., 750° C., 700° C., 650° C. and 600° C. At each of these temperatures the first additions of manganese affect the $\alpha/(\alpha+\beta)$ boundary as though manganese possessed a valency of approximately 2, and as the manganese content is increased there is a bend in the curve at compositions corresponding to roughly 2 atoms of solvent (copper) to 1 atom of solute (manganese + aluminium), although the change in direction is not always so marked as at 800° C.

Fig. 3 also shows the corresponding $\alpha/(\alpha+\beta)$ boundary at 800° C. for the system Cu-Mn-Zn. It will be seen that the boundary curve is a curve which at low manganese content approaches tangentially to the direction to be expected if equilibrium were determined by electron concentration and manganese acted as a divalent element, but which bends away from this direction at a much lower manganese content than does the corresponding curve in the system Cu-Mn-Al. Now in the binary system Cu-Al the $\alpha/(\alpha+\beta)$ boundary at 800° C. is at roughly 18·5 atomic per cent. aluminium, whereas in the binary system Cu-Zn, owing to the lower valency of zinc, the $\alpha/(\alpha+\beta)$ boundary is at roughly 33·5 atomic per cent. zinc. It follows, therefore, that in the system Cu-Zn there is already 1 atom of solute (zinc) to 2 of solvent (copper). The form of the curves therefore suggests strongly that in this class of alloy manganese acts as a divalent element until there is roughly 1 atom of solute to 2 atoms of solvent (copper), and that with higher concentrations of the solutes some new factor becomes increasingly important. The apparently different forms of the $\alpha/(\alpha+\beta)$ curves in the systems Cu-Mn-Zn and Cu-Mn-Al may thus be the result of the fact that in the former system the ratio of solute to solvent has already reached the critical value, whereas in the binary system Cu-Al the boundary is at a much lower proportion of solute, and only reaches the critical value after the addition of 25 atomic per cent. of manganese.

Fig. 4 shows the corresponding isothermals for the $(\alpha+\beta)/\beta$ boundaries, and it is clear that the same general effects are shown, and that manganese in small amounts affects the α/β equilibrium as though it were a divalent element, whilst the departure from this simple relation occurs at a lower percentage of manganese in the system Cu-Mn-Zn than in the system Cu-Mn-Al, as was the case for the $\alpha/(\alpha+\beta)$ boundaries. In both ternary systems, the $(\alpha+\beta)/\beta$ isothermals are of the same general form at other temperatures so long as these are above the temperature at which the $\beta \rightleftharpoons \beta'$ disorder $\rightleftharpoons$ order change takes place.

In the system Cu-Fe-Zn the $(\alpha+\beta)/\beta$ curve extends over a very short range, which is shown in fig. 4. If the equilibrium were determined by electron concentration alone, the direction of this $(\alpha+\beta)/\beta$ boundary is roughly that expected for a univalent metal, but the accuracy of the data is not sufficiently great for this to be examined further.

The system Cu-Ni-Zn has been examined in detail by Schramm (1936), and in fig. 3 the curve marked " Ni " shows the effect of the first additions of nickel on the $\alpha/(\alpha+\beta)$ boundary at 800° C. The direction of this curve is that which would be expected for an element exerting a valency of approximately 0·6 if the equilibrium were determined by electron concentration alone. At higher percentages of nickel the equilibrium changes from $\alpha/(\alpha+\beta)$ to $\alpha/(\alpha+\beta')$, where β' is the body-centred cubic phase with an ordered structure. The position of the change depends on the temperature, and the forms of the $\alpha/(\alpha+\beta)$ and $\alpha/(\alpha+\beta')$ boundaries are different. The same figure shows the corresponding effect of small amounts of nickel on the $\alpha/(\alpha+\beta)$ boundary in the system Cu-Ni-Al, and the direction is that to be expected for an element of valency about 0·4. This direction persists for only a small range, and with higher percentages of nickel the curve has a quite different form owing presumably * to the development of order in the β phase.

The above results are of interest in showing that whatever may be the validity of Pauling's views (1938) regarding the numbers of electrons responsible for cohesion in the crystals of the transition elements, these numbers are not the effective valencies of the elements as regards their influence on the α/β brass equilibrium. According to Pauling, on passing along the series K→Ca→Sc→Ti . . . →Ni→Cu, the number of bonding electrons in the crystals of the elements increases regularly from one to five per atom on passing from potassium to vanadium, and then remains roughly constant at between 5 and 6 electrons per atom on passing to nickel. It is clear that the directions of the curves in figs. 3 and 4 are quite different from those to be expected for elements of valency 5 or 6. It is also clear that although the first additions of nickel increase the solubility of zinc and aluminium in copper, the rate of increase is less than would be expected for an element of zero valency, so that as regards the α/β equilibrium the nickel seems to contribute some electrons to the lattice. According to the older theory of Mott and Jones (1936) solid nickel contains 0·6 electrons per atom in *s*-states, and the direction of the curves in fig. 4 suggests that nickel contributes roughly this number of electrons per atom in the alloys concerned.

APPENDIX.

The $\alpha/(\alpha+\beta)$ and $(\alpha+\beta)/\beta$ isothermals in the system Cu-Al-Mn have been determined by Dean, Long, Graham and Armantrout (1946), whose results have been converted to atomic percentages in the present paper. It is to be noted that the diagram of D., L., G., and A. are not all reproduced on exactly the same scale. The corresponding curves for the system Cu-Zn-Mn have been determined by Dean, Lang, Graham, and Roberson (1946). The $\alpha/(\alpha+\beta)$ boundary in the system Cu-Ni-Al is due to Alexander (1938), and the sources of the remaining data have been given in the course of this paper.

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* See Hume-Rothery (1938) in the discussion of the paper by W. O. Alexander 1938).

ACKNOWLEDGMENTS.

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The Inorganic Chemistry, Laboratory,
The University Museum, Oxford.
September, 1947.

XV. *Semi-rigid Representation of Elastic Cantilevers for Torsion Problems in Aerodynamics.*

By A. R. COLLAR, M.A., D.Sc., F.R.Ae.S.,
Sir George White Professor of Aeronautical Engineering
in the University of Bristol *.

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SUMMARY.

By means of a simple example, the semi-rigid representation of an elastic cantilever, as commonly employed for the solution of torsion problems in aerodynamics, is illustrated. The differences in the approximate and true solutions are exhibited; the conditions under which the approximate method yields the true values for a critical parameter (the critical dynamic head) are noted. It is shown that this critical parameter is stationary with respect to variations in the true mode, provided only that the end condition at the root is satisfied.

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* Communicated by the Author.

1. *Introduction.*

In many aeroelastic problems we are concerned only with the twist of a component under air loads. Since, however, these air loads are functions of the twist, analytical treatment of the problems leads to integral equations which require special artifices for solution, if a solution is possible at all. The conventional method of treatment of such aeroelastic problems has for many years been to replace the actual elastic structure by another having the same external geometry, but which possesses only a small finite number of possible modes of deformation—often only one; and by this means simple and satisfactory practical solutions of the problems concerned have been reached.

The new structure is said to be "semi-rigid" (see for example Duncan (1)). The present note considers some aspects of this particular form of semi-rigid representation; by consideration of a simple example the differences in the elastic and semi-rigid approaches are exhibited, and the conditions are noted under which exact or nearly exact values for critical parameters result from the semi-rigid approach.

It is thought that the treatment adopted may prove a useful guide for other enquiries into the accuracy of such approximate representations.

2. *The Problem Considered.*

We consider what is probably the simplest of all aeroelastic problems, namely, the divergence of a wing fixed at the root and subject to aerodynamic actions: this problem has been considered also, from a slightly different viewpoint, by Duncan (2). Let the wing-span be s, the spanwise variable being y. The chord is c, and it is assumed that the flexural axis lies a distance ec aft of the aerodynamic centre (quarter-chord point) and that a normal load applied on the flexural axis produces no twist at any wing section. The slope of the lift curve is a_1.

Then, with the usual assumptions of strip theory, if the wing twists at the section y through an angle θ, the increment in lift dL at the quarter-chord point on a strip of width dy is

$$dL = qca_1\theta\, dy,$$

where

$$q = \tfrac{1}{2}\rho V^2.$$

Accordingly, the moment about the flexural axis is

$$dM = qec^2a_1\theta\, dy. \qquad \qquad (1)$$

The aerodynamic moments obtained by integration of (1) are balanced by the elastic actions in the wing. The torsional stiffness per unit length is C (this quantity, like c, e, a_1, is assumed to be a known function of y). With regard to the stiffness C, it is assumed that the wing is such that a concentrated torque applied at any section produces twist between this loaded section and the root, but produces no twist outboard of the loaded section. Though not exactly true, this assumption is quite close to reality for modern wings in which the stiffness is due to a torsion box; it would, however, obviously be inaccurate where the torsional stiffness is due to differential bending of spars.

With the definition adopted, the rate of twist at any section due to a concentrated torque M is given by

$$\frac{d\theta}{dy}=\frac{\mathrm{M}}{\mathrm{C}}. \qquad (2)$$

2.1. *The Formal Solution of the Elastic Problem.*

The total moment applied to the section distant y from the root is the sum of all the elementary moments from y to the tip, *i. e.* from (1),

$$\mathrm{M}=q\int_y^s ec^2a_1\theta dy.$$

The local rate of twist is then, from (2),

$$\frac{d\theta}{dy}=\frac{q}{\mathrm{C}}\int_y^s ec^2a_1\theta dy. \qquad (3)$$

The total twist at the section y is the sum of all the elements of twist from the root to y, *i. e.* from (3),

$$\theta=q\int_0^y\frac{1}{\mathrm{C}}\left\{\int_y^s ec^2a_1\theta dy\right\}dy. \qquad (4)$$

In the formulation of this equation it is assumed that only aerodynamic and elastic actions are present, and the balance of these defines formally the critical value of q for which the equilibrium is neutral; the fact that θ is arbitrary to a scalar multiplier expresses the condition of neutral equilibrium. Nevertheless, θ (unlike C, e, c, and a_1) is an unknown function of y; and hence, unless a practical method of solution of the integral equation (4) can be found, a numerical value for q cannot be determined. Even if a variation of θ with y were assumed, so that the integrals in (4) could be evaluated, an unambiguous value for q could not be found unless the correct mode had fortuitously been chosen, since (4) is a functional equation.

2.2. *The Semi-rigid Approach.*

Suppose we choose a single arbitrary twist function $f(y)$, subject to the condition that, at an arbitrarily chosen reference station y_r, $f(y_r)=1$. Then if the twist at this station is θ_r, the twist elsewhere is given by

$$\theta=\theta_r f(y), \qquad (5)$$

which is assumed to be true no matter what the loading distribution may be. In equation (5), θ_r is a generalized co-ordinate which has been deliberately chosen to have a physical significance; when it is possible, such a choice is obviously helpful.

To find the generalized force corresponding to the generalized coordinate θ_r, we give θ_r a small increment $\delta\theta_r$. The corresponding increment $\delta\theta$ in θ is $\delta\theta_r f$; hence the total work done by the aerodynamic actions is

$$\begin{aligned}\delta W&=\int_0^s \delta\theta\, dM\\ &=\int_0^s \delta\theta\,.\,qec^2a_1\theta\,dy\\ &=q\theta_r\delta\theta_r\int_0^s ec^2a_1f^2dy.\end{aligned}$$

Accordingly the generalized force arising from the air loads is

$$\frac{\partial W}{\partial\theta_r}=q\theta_r\int_0^s ec^2a_1f^2\,dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The generalized force obtained from the elastic actions is found from the strain energy; this is

$$\begin{aligned}V&=\tfrac{1}{2}\int_0^s C\left(\frac{d\theta}{dy}\right)^2dy\\ &=\tfrac{1}{2}\theta_r^2\int_0^s C\left(\frac{df}{dy}\right)^2dy,\end{aligned}$$

whence

$$\frac{\partial V}{\partial\theta_r}=\theta_r\int_0^s C\left(\frac{df}{dy}\right)^2dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In the condition of neutral equilibrium, (6) and (7) are equal and we have, therefore,

$$q\int_0^s ec^2a_1f^2dy=\int_0^s C\left(\frac{df}{dy}\right)^2dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Equation (8) provides the final solution of the problem, and it gives the critical value of q explicitly.

3. *Comparison of the Methods.*

The treatment of the elastic beam led to equation (4); the semi-rigid approach led to equation (8). These two equations are formally so different that it is not easy to see at first that they can give the same result. We now proceed to demonstrate that if the correct function f is used in the semi-rigid approach, it gives the correct value for q as found from equation (4); let us denote the value of q given by (4) as q_t and that by (8) as q_s.

In equation (4) put $\theta=\theta_r f(y)$, where f is now the true mode of twist, and differentiate twice; then

$$-\frac{d}{dy}\left(C\frac{df}{dy}\right)=q_tec^2a_1f. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

We now write equation (8) in the form

$$\int_0^s \mathrm{C}\left(\frac{df}{dy}\right)\left(\frac{df}{dy}\right)dy = q_s\int_0^s ec^2a_1f^2\,dy,$$

and integration by parts of the left-hand side yields

$$\left[f\mathrm{C}\left(\frac{df}{dy}\right)\right]_0^s - \int_0^s \frac{d}{dy}\left(\mathrm{C}\frac{df}{dy}\right)f\,dy = q_s\int_0^s ec^2a_1f^2\,dy. \quad . \quad . \quad (10)$$

Now, f vanishes at the root ($y=0$) and $\mathrm{C}\,\frac{df}{dy}$ vanishes at the tip ($y=s$) since it represents the airload at the tip (see equation (2)). Hence the first term in (10) vanishes; and on substitution from (9) in the second term, we deduce

$$q_t\int_0^s ec^2a_1f^2\,dy = q_s\int_0^s ec^2a_1f^2\,dy,$$

so that the values of q given by the two methods are identical.

4. *The Stationary Property of* q *with respect to* f.

The result proved in section 3 is not of great practical help since, in fact, the correct mode f is not known. However, we now proceed to show that, in the application of the semi-rigid method, first order errors in f lead at most to second order errors in the critical value of q.

Suppose the mode employed in the semi-rigid approach is $f+\epsilon$, where f is the correct mode; and let the resultant value of the dynamic head be $q+\delta q$, where q is the correct value. Then

$$\int_0^s \mathrm{C}\left(\frac{df}{dy}+\frac{d\epsilon}{dy}\right)^2 dy = (q+\delta q)\int_0^s ec^2a_1(f+\epsilon)^2\,dy. \quad . \quad . \quad . \quad (11)$$

Retaining only the first order terms in the difference between (8) and (11), we obtain

$$2\int_0^s \mathrm{C}\left(\frac{df}{dy}\right)\left(\frac{d\epsilon}{dy}\right)dy = \delta q\int_0^s ec^2a_1f^2\,dy + 2q\int_0^s ec^2a_1f\epsilon\,dy.$$

Once again we integrate by parts and find

$$2\left[\mathrm{C}\frac{df}{dy}\,\epsilon\right]_0^s - 2\int_0^s \frac{d}{dy}\left(\mathrm{C}\frac{df}{dy}\right)\epsilon\,dy = \delta q\int_0^s ec^2a_1f^2\,dy + 2q\int_0^s ec^2a_1f\epsilon\,dy.$$

Substitution from (9) now leads to

$$2\left[\mathrm{C}\frac{df}{dy}\,\epsilon\right]_0^s = \delta q\int_0^s ec^2a_1f^2\,dy. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Now $\mathrm{C}\,\frac{df}{dy}$ vanishes at $y=s$; hence, provided ϵ vanishes at $y=0$—*i. e.*, provided the approximate mode chosen satisfies the root condition—it follows from (12) that $\delta q=0$. Hence the error in q is at most of the second order of smallness.

5. *Choice of a Mode.*

It is of particular interest to note that the approximate mode chosen does not require to satisfy the tip condition. This fact greatly widens the choice of functions suitable for use in semi-rigid representations. In fact, it is a matter of experience that with the tapers common to modern aircraft wings, tailplanes, etc., the stiffness C is greater near the root than near the tip. At the same time the airloads increase towards the root; and the result is that the rate of twist does not vary greatly over the larger part of the span. As the tip is approached $\frac{d\theta}{dy}$ tends to fall to zero; but, as we have seen, it is not necessary for the mode used in a semi-rigid representation to follow this trend.

The foregoing remarks lead to the very simple conclusion that the great majority of problems of this kind can be treated by assuming a linear mode for the semi-rigid representation; *i. e.*,

$$f(y)=\frac{y}{y_r}. \qquad \qquad (13)$$

Alternatively, if there is good reason for supposing that the true mode departs considerably from linearity, a serviceable representation is often provided by

$$f(y)=\left(\frac{y}{y_r}\right)^a, \qquad \qquad (14)$$

where a is an index which is arbitrarily chosen to suit the problem.

6. *The Expression for the Stiffness.*

The full expression deduced from (11), provided that ϵ vanishes at the root, is

$$\delta q\int_0^s ec^2a_1(f+\epsilon)^2\,dy=\int_0^s \mathrm{C}\left(\frac{d\epsilon}{dy}\right)^2 dy-q\int_0^s ec^2a_1\epsilon^2\,dy. \qquad (15)$$

In the derivation of the stationary property of q with respect to the mode it was tacitly assumed that not only is ϵ of the first order of smallness, but so also is $\frac{d\epsilon}{dy}$ at all points along the span; so that the right-hand side of (15) is of the second order of smallness. It must be observed, therefore, that if a mode is chosen for which $\frac{d\epsilon}{dy}$ is not small, the result will no longer be true. For example, if in (14) the index a is taken to be less than unity, then the approximate mode has an infinite slope at the root and the value of $\frac{d\epsilon}{dy}$ is therefore also infinite. In fact, in this case the overall stiffness also becomes infinite. Accordingly, a further restriction is placed on the choice of the mode, if an approximate mode is used in (8) on the right-hand side.

In practice this difficulty is often avoided by a direct measurement of the stiffness. For, by (7), the right-hand side of (8) is the overall effective stiffness of the wing for the mode chosen; and it is often assumed that this stiffness is not very different from the actual stiffness found in an ordinary stiffness test. This question is dealt with elsewhere (3) more fully; it is sufficient to remark here that, no matter where the reference section is taken, the stiffness found in a stiffness test is always in defect of that appropriate to a distributed load. However, for the normal reference section, *i. e.* at about 0·7 span, the difference is not usually great, being normally of the order 5 to 10 per cent.

Thus equation (8) may be used by replacement of the right-hand side by the measured stiffness and by the use of an approximate mode on the left; and in this case it is possible to use the mode given by (14) without the restriction that a must exceed unity; it is, of course, obvious that a cannot be negative or zero, since the displacement must vanish at the root.

7. *Two Degrees of Freedom.*

The foregoing paragraphs are principally intended to show how the use of a single approximate mode will give a closely correct value for the critical dynamic head. For completeness, however, some remarks on the use of more than one degree of freedom are added here.

If we re-write equation (5) in the form

$$\theta=\theta_{r1}f_1(y)+\theta_{r2}f_2(y)+\ldots+\theta_{rn}f_n(y), \quad . \quad . \quad . \quad . \quad . \quad (16)$$

we may proceed, as before, by giving each generalized coordinate in turn an increment, to determine a set of n equations of moments. The condition of compatibility yields n values of the critical parameter, of which the lowest is the significant one. To illustrate this, we use two generalized coordinates; the equations are then

$$\left\{q\int_0^s \phi f_1^2\,dy-\int_0^s C\left(\frac{df_1}{dy}\right)^2 dy\right\}\theta_{r1} + \left\{q\int_0^s \phi f_1 f_2\,dy-\int_0^s C\left(\frac{df_1}{dy}\right)\left(\frac{df_2}{dy}\right)dy\right\}\theta_{r2}=0,$$

$$\left\{q\int_0^s \phi f_1 f_2\,dy-\int_0^s C\left(\frac{df_1}{dy}\right)\left(\frac{df_2}{dy}\right)dy\right\}\theta_{r1} + \left\{q\int_0^s \phi f_2^2\,dy-\int_0^s C\left(\frac{df_2}{dy}\right)^2 dy\right\}\theta_{r2}=0, \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where $\phi=ec^2a_1.$

If we write

$$f_1=f+\epsilon_1,$$
$$f_2=f+\epsilon_2,$$

where f is the true mode, and if the approximate value of the critical parameter is $q+\delta q$, where q is the correct value, then δq is found from (17), after some reduction, to be given by

$$\begin{aligned}
&\left\{\delta q\int_0^s \phi(f+\epsilon_1)^2\,dy+q\int_0^s \phi\epsilon_1^2\,dy-\int_0^s \mathrm{C}\left(\frac{d\epsilon_1}{dy}\right)^2 dy\right\}\\
&\times\left\{\delta q\int_0^s \phi(f+\epsilon_2)^2\,dy+q\int_0^s \phi\epsilon_2^2\,dy-\int_0^s \mathrm{C}\left(\frac{d\epsilon_2}{dy}\right)^2 dy\right\}\\
&-\left\{\delta q\int_0^s \phi(f+\epsilon_1)(f+\epsilon_2)dy+q\int_0^s \phi\epsilon_1\epsilon_2\,dy-\int_0^s \mathrm{C}\left(\frac{d\epsilon_1}{dy}\right)\left(\frac{d\epsilon_2}{dy}\right)dy\right\}^2=0.
\end{aligned} \quad . \quad . \quad . \quad (18)$$

The constant term in this quadratic for δq is

$$\begin{aligned}
&\left\{q\int_0^s \phi\epsilon_1^2\,dy-\int_0^s \mathrm{C}\left(\frac{d\epsilon_1}{dy}\right)^2 dy\right\}\left\{q\int_0^s \phi\epsilon_2^2\,dy-\int_0^s \mathrm{C}\left(\frac{d\epsilon_2}{dy}\right)^2 dy\right\}\\
&\qquad-\left\{q\int_0^s \phi\epsilon_1\epsilon_2\,dy-\int_0^s \mathrm{C}\left(\frac{d\epsilon_1}{dy}\right)\left(\frac{d\epsilon_2}{dy}\right)dy\right\}^2.
\end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

This is of the fourth order of smallness; the coefficients of δq^2 and δq are each of the second order. Accordingly the smaller value of δq (which alone is significant) is of the second order of small quantities. It may also be remarked, however, that the expression (19) is similar to that leading to the Schwartz inequality, and in certain circumstances it vanishes, whether or not ϵ_1 and ϵ_2 are small functions. Thus if, for example, $\epsilon_2=r\epsilon_1$, where r is an arbitrary constant, (19) vanishes, and then $\delta q=0$ accurately. The significance of this is that a particular linear combination of $f+\epsilon_1$ and $f+r\epsilon_1$ yields f alone; and the method, in fact, yields this particular combination of the two approximate modes, and therefore gives the exact result. In any case, (19) may be expected to be very small, and the use of two approximately correct modes will therefore yield a very closely correct mode and dynamic head.

References.

A general reference which is of considerable help in problems of this nature is 'Rayleigh's Principle,' by G. Temple and W. G. Bickley. The particular references cited in the text are:—

(1) W. J. Duncan, "The representation of aircraft wings, tails and fuselages by semi-rigid structures in dynamic and static problems." Reports and Memoranda of the Aeronautical Research Council, No. 1904.

(2) W. J. Duncan, "The critical conditions of dynamical systems and elastic structures," Phil. Mag. xx. p. 789 (1935).

(3) A. R. Collar and E. G. Broadbent, "The rolling power of an elastic wing, Part I." Reports and Memoranda of the Aeronautical Research Council, No. 2186.

XVI. *Metal Optics at Centimetre Wave-lengths*—I.

By L. Speirs, M.A., M.Sc., D. Phil.,
Lecturer in Physics, University College, Exeter *.

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§ 1. *Introduction.*

The optical properties of metals in the form of thin films have been the subject of a number of investigations in the optical and infra-red regions of the spectrum. Recent developments in the technique of producing and measuring centimetre waves permit this region of the spectrum to be used for the same purpose. The advantage to be gained in using these long wave-lengths is that Maxwell's theory is valid for metals in bulk, whereas it is known to fail in the infra-red, the discrepancy between theoretical predictions and experimental results increasing as the visible region is approached. Thus any discrepancies which may exist between theory and experiment at long wave-lengths must be attributable solely to differences between the properties of these films and those of the metal in bulk. Owing, however, to the fact that diffraction effects are encountered when the dimensions of an obstacle are comparable with the wave-length of the incident radiation, experiments based on the use of plane unguided waves are not suitable at centimetre wave-lengths. The experiments appropriate to this spectral region are consequently those in which the waves are confined to the interior of concentric lines or wave-guides, and this leads to a technique of metal optics which is in some respects different from that employed at shorter wave-lengths. The writer has used two methods to investigate the surface resistance of metal films at a wave-length of 1·25 cm. In one of these the guided wave is allowed to traverse the film and its support once; the reflection and transmission coefficients are found by direct measurement of the reflected and transmitted waves. The second method involves a resonance experiment which can be employed with advantage when dealing with films which are not too thick. This experiment consists in loading a cylindrical cavity resonator with the film supported on a flat disc of "low loss" material, and noting the resonance frequency before and after its insertion. The experimental details of these methods will be described in a later paper, and at present only the theory of the second method, which includes that of the first, will be developed.

The effect of loading a resonator with a film, which may be regarded as a "lossy" dielectric, is twofold: in general both the resonant frequency and the response (Q) of the resonator will be affected. Usually measurements of both these quantities would be required in order to evaluate the phase and attenuation constants of the material. In the case of a metal, however, the real and imaginary parts of the propagation constant can be shown on the basis of Maxwell's theory to be equal to within 1 part in 10^4; consequently a knowledge of the change either in Q, or in the resonant frequency, suffices to yield the phase and attenuation constants of the metal, and hence its surface resistance. The obvious expedient of measuring the change in the Q value of the resonator when it is loaded with the film has not been employed for this purpose, since the connection between the theoretical Q, calculated from energy considerations, and the experimental Q, derived from a frequency response curve, is not apparent. On the other hand, since a connection can be established between the surface resistance of the film and the resonance frequencies of the loaded and unloaded resonator, this method has been adopted in practice.

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* Communicated by Professor F. H. Newman.

The theory of the loaded resonator can be developed in two ways which are fundamentally equivalent but not equally convenient, or instructive. In one method general solutions of the wave equation are assigned to each of the three regions into which the resonator is divided in the presence of the film and its support; the solution for each region consists of two waves travelling in opposite directions and having unknown complex coefficients. The application of boundary conditions at the two interfaces and the metal walls then permits the elimination of the arbitrary coefficients and yields the required resonance condition. The relation so obtained is, however, too complicated to be of practical value, and the effect of approximations is not as clear as in the method to be described. In the latter method boundary matching is applied to evaluate the reflection and transmission ratios of each interface for waves travelling in either direction, and the solution is built up by superposing the plane waves arising from multiple reflections in each region. This method has the advantage of showing very clearly the effect of approximations which are made at an early stage in the calculations, and at the same time yields a number of other useful results. The theory is developed for plane waves and is shown to apply with minor modifications to certain guided waves.

§ 2. *Reflection and Transmission Coefficients of a Metal Film separating two Dielectric Media.*

The three media are designated by subscripts. The subscripts 1 and 3 refer to "lossless" dielectric media of which 3 is the solid film support and 1 may be air; 2 refers to the metal film. The propagation constants of the three media are thus $\gamma_1=j\beta_1$; $\gamma_2=\beta_2(1+j)$; $\gamma_3=j\beta_3$. The plane waves are designated by their electric vectors which are of the form $ae^{j\omega t\pm\gamma z}$ according to the direction of motion. The positive direction of z is taken to be from medium 1 to medium 3 and is normal to the interfaces. The magnetic vector in each case can be obtained from

$$H=jc/\omega\mu \,.\, \partial E/\partial z.$$

If each pair of adjoining media is made semi-infinite in turn, the effect of multiple reflections is avoided and the amplitudes (a', a'') of the reflected and transmitted waves can be obtained in terms of the incident amplitude (a) by boundary matching for each interface and each direction of incidence. Writing the amplitude reflection ratio—$a'/a=re^{j\pi}$, the amplitude transmission ratio $a'/a=t$, and using the notation r_{12} to signify that the reflection takes place in medium 1 on medium 2, and t_{12} that transmission is from medium 1 to medium 2, we have

$$r_{12}=\frac{\mu_1\gamma_2-\mu_2\gamma_1}{\mu_1\gamma_2+\mu_2\gamma_1};\quad t_{12}=1-r_{12}=\frac{2\mu_2\gamma_1}{\mu_1\gamma_2+\mu_2\gamma_1},$$

$$r_{21}=\frac{\mu_2\gamma_1-\mu_1\gamma_2}{\mu_1\gamma_2+\mu_2\gamma_1};\quad t_{21}=1-r_{21}=\frac{2\mu_1\gamma_2}{\mu_1\gamma_2+\mu_2\gamma_1},$$

$$r_{23}=\frac{\mu_2\gamma_3-\mu_3\gamma_2}{\mu_2\gamma_3+\mu_3\gamma_2};\quad t_{23}=1-r_{23}=\frac{2\mu_3\gamma_2}{\mu_2\gamma_3+\mu_3\gamma_2},$$

$$r_{32}=\frac{\mu_3\gamma_2-\mu_2\gamma_3}{\mu_2\gamma_3+\mu_3\gamma_2};\quad t_{32}=1-r_{32}=\frac{2\mu_2\gamma_3}{\mu_2\gamma_3+\mu_3\gamma_2},$$

and from these the following subsidiary relations exist :—

$$r_{12}=-r_{21}\,;\quad t_{12}\times t_{21}=1+r_{12}r_{21}=1-r_{12}{}^2$$

$$r_{23}=-r_{32}\,;\quad t_{23}\times t_{32}=1+r_{23}r_{32}=1-r_{23}{}^2.$$

We now consider the case when a metal film of thickness l is interposed between two semi-infinite dielectric media 1 and 3. On taking into account the multiple reflections which occur in the film, we find that a wave $ae^{jwt-\gamma_1 z}$ in medium 1 which is incident on the film gives rise to a reflected wave $=a'e^{jwt+\gamma_1 z}$ in medium 1, which is given by

$$ae^{jwt+\gamma_1 z}\left[r_{12}e^{j\pi}+t_{12}t_{21}\{r_{23}e^{j\pi-2\gamma_2 l}+(r_{23})^2(r_{21})e^{j3\pi-4\gamma_2 l}+\ldots\}\right],$$

and a transmitted wave $a''e^{jwt-\gamma_3 z}$ in medium 3 given by

$$ae^{jwt-\gamma_3 z}t_{12}t_{23}e^{-\gamma_2 l}\left[1+r_{23}r_{21}e^{j2\pi-2\gamma_2 l}+(r_{23}r_{21})^2e^{j4\pi-4\gamma_2 l}+\ \ldots\right].$$

Introducing an amplitude reflection ratio $r_{1(23)}\,e^{j\pi}=-a'/a$ and an amplitude transmission ratio $t_{1(23)}=a''/a$, we find

$$r_{1(23)}\ =\ \frac{r_{12}+r_{23}(t_{12}t_{21}-r_{12}r_{21})e^{-2\gamma_2 l}}{1-r_{21}r_{23}e^{-2\gamma_2 l}}\ =\ \frac{r_{12}e^{\gamma_2 l}+r_{23}e^{-\gamma_2 l}}{e^{\gamma_2 l}+r_{12}r_{23}e^{-\gamma_2 l}},$$

$$t_{1(23)}\ =\ \frac{t_{12}t_{23}e^{-\gamma_2 l}}{1-r_{21}r_{23}e^{-2\gamma_2 l}}\ =\ \frac{t_{12}t_{23}}{e^{\gamma_2 l}+r_{12}r_{23}e^{-\gamma_2 l}}.$$

Similarly a wave $ae^{jwt+\gamma_3 z}$ in medium 3, which is incident on the film, gives rise to a reflected wave $-a'e^{jwt-\gamma_3 z}$ in medium 3, which is given by

$$ae^{jwt-\gamma_3 z}$$
$$\times\left[r_{32}e^{j\pi}+t_{32}t_{23}\{r_{21}e^{j\pi-2\gamma_2 l}+(r_{21})^2(r_{23})e^{j3\pi-4\gamma_2 l}+(r_{21})^3(r_{23})^2e^{j5\pi-6\gamma_2 l}+\ \ldots\}\right]$$

and a transmitted wave $a''e^{jwt+\gamma_1 z}$ in medium 1 given by

$$ae^{jwt+\gamma_1 z}t_{32}t_{21}e^{-\gamma_2 l}\left[1+(r_{21}r_{23})e^{j2\pi-2\gamma_2 l}+(r_{21}r_{23})^2e^{j4\pi-4\gamma_2 l}+\ \ldots\ \right].$$

Putting $r_{3(21)}e^{j\pi}=-a'/a$ and $t_{3(21)}=a''/a$, we find

$$r_{3(21)}=\frac{r_{32}+r_{21}(t_{32}t_{23}-r_{32}r_{23})e^{-2\gamma_2 l}}{1-r_{21}r_{23}e^{-2\gamma_2 l}}=\frac{r_{32}e^{\gamma_2 l}+r_{21}e^{-\gamma_2 l}}{e^{\gamma_2 l}+r_{12}r_{23}e^{-\gamma_2 l}}.$$

$$t_{3(21)}=\frac{t_{21}t_{32}e^{-\gamma_2 l}}{1-r_{21}r_{23}e^{-2\gamma_2 l}}=\frac{t_{21}t_{32}}{e^{\gamma_2 l}+r_{12}r_{23}e^{-\gamma_2 l}}.$$

If the values of γ_1, γ_2, γ_3 and the values of r_{12}, r_{23}, t_{12}, t_{23} previously obtained are inserted in the above expressions, then multiplication of each amplitude ratio by its complex conjugate yields the corresponding intensity ratio. Considerable labour is saved if at this point we put $\mu_1=\mu_3=1$, a condition that is always satisfied in practice. The permeability μ_2 of the metal will not necessarily be unity, however, since ferromagnetic metals have residual permeabilities at centimetre wave-lengths which can be considerably greater than unity. Nevertheless μ_2 will be taken equal to unity at this point, and this assumption will be justified later. The reflection and transmission coefficients of the film can now be obtained and can be written

$$R_{1(23)}=N_1/D\,;\quad R_{3(21)}=N_2/D\,;\quad T_{1(23)}=N_3/D\,;\quad T_{3(21)}=N_4/D,$$

where

$$\begin{aligned}
D&=F_1e^{2\beta_2 l}+F_2e^{-2\beta_2 l}+F_3\cos 2\beta_2 l+F_4\sin 2\beta_2 l,\\
N_1&=F_5e^{2\beta_2 l}+F_6e^{-2\beta_2 l}+F_7\cos 2\beta_2 l+F_8\sin 2\beta_2 l,\\
N_2&=F_6e^{2\beta_2 l}+F_5e^{-2\beta_2 l}+F_7\cos 2\beta_2 l-F_8\sin 2\beta_2 l,\\
N_3&=32\beta_1^2\beta_2^2,\\
N_4&=32\beta_3^2\beta_2^2,\\
F_1&=[2\beta_2^2+2\beta_1\beta_2+\beta_1^2][2\beta_2^2+2\beta_2\beta_3+\beta_3^2],\\
F_2&=[2\beta_2^2-2\beta_1\beta_2+\beta_1^2][2\beta_2^2-2\beta_2\beta_3+\beta_3^2],\\
F_3&=2[(2\beta_2^2-\beta_1^2)(\beta_3^2-2\beta_2^2)+4\beta_1\beta_2^2\beta_3],\\
F_4&=4\beta_2[\beta_3(2\beta_2^2-\beta_1^2)+\beta_1(2\beta_2^2-\beta_3^2)],\\
F_5&=[2\beta_2^2-2\beta_1\beta_2+\beta_1^2][2\beta_2^2+2\beta_2\beta_3+\beta_3^2],\\
F_6&=[2\beta_2^2+2\beta_1\beta_2+\beta_1^2][2\beta_2^2-2\beta_2\beta_3+\beta_3^2],\\
F_7&=2[(2\beta_2^2-\beta_1^2)(\beta_3^2-2\beta_2^2)-4\beta_1\beta_2^2\beta_3],\\
F_8&=4\beta_2[\beta_3(2\beta_2^2-\beta_1^2)-\beta_1(2\beta_2^2-\beta_3^2)],
\end{aligned}$$

With the aid of these relations the absorption coefficients for the two directions of incidence can also be found. The energy flow associated with a wave is the real part of the complex Poynting vector $\frac{1}{2}(E\times H)$, so that if the electric vector of an incident wave in medium 1 is $ae^{j\omega t-\gamma_1 z}$ the energy carried by the wave is $\frac{1}{2}a^2\omega\mu_1\beta_1/c$. Of this a fraction A is absorbed by the film, while as a result of the reflection, the energy entering the film is $\frac{1}{2}(1-R_{1(23)})a^2\omega\mu_1\beta_1/c$. The energy associated with the transmitted wave in medium 3 is likewise equal to $\frac{1}{2}T_{1(23)}\omega\mu_3\beta_3/c$. On putting $\mu_1=\mu_3$ as before, the energy balance yields the absorption coefficient

$$A_1=1-R_{1(23)}-\frac{\beta_3}{\beta_1}T_{1(23)}=(\beta_1D-\beta_1N_1-\beta_3N_3)/\beta_1D.$$

Similarly for an incident wave in medium 3 the absorption coefficient is seen to be

$$A_3 = 1 - R_{3(21)} - \frac{\beta_1}{\beta_3} T_{3(21)} = (\beta_3 D - \beta_3 N_2 - \beta_1 N_4)/\beta_3 D.$$

§ 3. *Approximations.*

The above expressions, which are strictly correct for the case $\mu_2 = 1$, are capable of considerable simplification if the film thickness is much less than a quarter wave-length, measured in the film, *i. e.*, if $l \ll \lambda_2/4$ so that $2\beta_2 l \ll \pi/2$; then, on expanding the expressions for D, N_1, N_2 in ascending powers of $2\beta_2 l$ and neglecting terms involving powers of l higher than the second, we find

$$D = 8\beta_2^2[(\beta_1+\beta_3)^2 + 4\beta_2^2 l(\beta_1+\beta_3) + (4\beta_2^4 + \beta_1^2\beta_3^2)l^2],$$
$$N_1 = 8\beta_2^2[(\beta_3-\beta_1)^2 + 4\beta_2^2 l(\beta_3-\beta_1) + (4\beta_2^4 + \beta_1^2\beta_3^2)l^2],$$
$$N_2 = 8\beta_2^2[(\beta_1-\beta_3)^2 + 4\beta_2^2 l(\beta_1-\beta_3) + (4\beta_2^4 + \beta_1^2\beta_3^2)l^2].$$

The term $\beta_1^2\beta_3^2 l^2$ in these expressions can be ignored in comparison with the term $4\beta_2^4 l^2$ since the value of β_2, calculated from metallic conductivities at centimetre wave-lengths, is enormously greater than the product $\beta_1\beta_3$ for all ordinary dielectric media. The intensity ratios can therefore be written in the form :

$$R_{1(23)} = r_1^2; \quad R_{3(21)} = r_3^2; \quad T_{1(23)} = t_1^2; \quad T_{3(21)} = t_3^2,$$

where

$$r_1 = \frac{\beta_3 - \beta_1 + 2\beta_2^2 l}{\beta_1 + \beta_3 + 2\beta_2^2 l}; \quad r_3 = \frac{\beta_1 - \beta_3 + 2\beta_2^2 l}{\beta_1 + \beta_3 + 2\beta_2^2 l},$$

$$t_1 = \frac{2\beta_1}{\beta_1 + \beta_3 + 2\beta_2^2 l}; \quad t_3 = \frac{2\beta_3}{\beta_1 + \beta_3 + 2\beta_2^2 l},$$

and the absorption coefficients are given by

$$A_1 = 1 - r_1^2 - \frac{\beta_3}{\beta_1} t_1^2 = \frac{8\beta_1\beta_2^2 l}{[\beta_1 + \beta_3 + 2\beta_2^2 l]^2},$$

$$A_3 = 1 - r_3^2 - \frac{\beta_1}{\beta_3} t_3^2 = \frac{8\beta_3\beta_2^2 l}{[\beta_1 + \beta_3 + 2\beta_2^2 l]^2}.$$

Now consider the case when the metallic permeability μ_2 is different from unity. The effect of this on the results derived above is seen from the amplitude ratios of the interfaces to be equivalent to altering β_1 and β_3 to $\mu_2\beta_1$ and $\mu_2\beta_3$, respectively ; the results are then applicable to ferromagnetic metals. From Maxwell's equations, however, the real and imaginary parts α_2, β_2 of the metallic propagation constant are given by

$$\alpha_2^2 - \beta_2^2 = \epsilon_2\mu_2 4\pi^2/\lambda^2,$$
$$\alpha_2\beta_2 = 2\pi\sigma\omega\mu_2/c^2 = 4\pi^2\sigma\mu_2/c\lambda.$$

On inserting values for σ, λ it is found that $\alpha_2\beta_2 \gg \alpha_2^2-\beta_2^2$ so that $\alpha_2 \doteqdot \beta_2$ and hence $\beta_2^2=4\pi^2\sigma\mu_2/c\lambda$. Since β_2^2 contains μ_2 and the expressions derived are homogeneous in β_1, β_2^2, β_3 the permeability μ_2 does not appear in the final expressions for the reflection, transmission and absorption coefficients. Thus the expressions derived above for R, T, A are valid even for ferromagnetic metals provided that $\beta_2^2=4\pi^2\sigma/c\lambda$.

The conductivity σ is measured in E.S.U. since Gaussian units are employed. If conductivity g is measured in mho-cm. $\sigma=9\times10^{11}g$; then, defining the surface resistance S of the film as the resistance between opposite edges of 1 cm.^{2} of film $S=1/gl$, where S is measured in ohms-cm.$^{-2}$.

Hence
$$2\beta_2^2 l=\frac{2\times2\pi\times9\times10^{11}gl}{3\times10^{10}}\times\frac{2\pi}{\lambda}=\frac{\beta Z_0}{S},$$

where β is the phase constant in free space and $Z_0=120\pi$ is the surface impedance of plane waves in free space. Finally, since $\beta=2\pi/\lambda$; $\beta_1=2\pi/\lambda_1$; $\beta_3=2\pi/\lambda_3$, the expressions obtained for R_1, R_3, T_1, T_3 can all be re-written in terms of λ, λ_1, λ_3, S and Z_0.

The quantities r_1, t_1, r_3, t_3 are the real parts of $r_{1(23)}$, $t_{1(23)}$, $r_{3(21)}$, $t_{3(21)}$ and satisfy the relations

$$1-r_1-t_1=0\,; \qquad 1-r_3-t_3=0.$$

The imaginary parts of the complex amplitude ratios $r_{1(23)}$ etc. express the phase changes due to the film which accompany reflection and transmission. These are found to be negligibly small if the film thickness $l \ll \lambda_2/4$. The effect of the film separating the media 1 and 3 is thus equivalent to an interface having amplitude ratios r_1, t_1 when viewed from medium 1, and r_3, t_3 when viewed from medium 3.

§ 4. *Film on Dielectric Slab.*

The preceding results only apply when the dielectric media 1 and 3 are of infinite extent and hence in practice only to the case of a self-supporting film in air ($\beta_1=\beta_3$). Films thinner than about 10^{-5} cm. are not self-supporting and are usually prepared on a slab of solid dielectric material. If the faces of the slab are plane-parallel, the effect of multiple reflections occurring within the slab must be taken into account in computing the coefficients R, T, A. We therefore consider the case in which a metallic film of thickness l is carried on one face of a plane-parallel slab of dielectric material 3 of thickness L, and the whole is immersed in medium 1 so that the arrangement of the media is 1, 2, 3, 1. Two cases arise according to the direction of incidence.

Considering first the case when the wave is incident on the film before passing through the slab, two further amplitude ratios for the interface 3, 1 are required. These are

$$r_{31}=\frac{\beta_1-\beta_3}{\beta_1+\beta_3} \quad\text{and}\quad t_{31}=\frac{2\beta_3}{\beta_1+\beta_3}.$$

A wave $ae^{jwt-\gamma_1z}$ incident on the film then gives rise to a reflected wave $-a'e^{jwt+\gamma_1z}$ and a transmitted wave $a''e^{jwt-\gamma_1z}$ which are given by

$$ae^{jwt\gamma+{}_1z}[r_1e^{j\pi}+t_1t_3r_{31}e^{j(\pi-2\beta_3L)}\{1+r_3r_{31}e^{2j(\pi-\beta_3L)}+(r_3r_{31})^2e^{4j(\pi+\ldots-\beta_3L)}\}]$$

and $$ae^{jwt-\gamma_1z}t_1t_{31}e^{-j\beta_3L}[1+r_3r_{31}e^{2j(\pi-\beta_3L)}+(r_3r_{31})^2e^{4j_1(\pi-\beta_3L)}+\ldots].$$

The amplitude ratios $r=a'/a$, $t=a''/a$ are thus

$$r=\frac{r_1e^{j\beta_3L}+r_{31}(t_1t_3-r_1r_3)e^{-j\beta_3L}}{e^{j\beta_3L}-r_3r_{31}e^{-j\beta_3L}},$$

$$t=\frac{t_1t_{31}}{e^{j\beta_3L}-r_3r_{31}e^{-j\beta_3L}},$$

and from these the reflection, transmission and absorption coefficients are easily found to be

$$R=\frac{4\beta_3^2\beta_2^4l^2+(\beta_1^2-\beta_3^2)(\beta_1^2-\beta_3^2-4\beta_1\beta_2^2l+4\beta_2^4l^2)\sin^2\beta_3L}{4\beta_3^2(\beta_1+\beta_2^2l)^2+(\beta_1^2-\beta_3^2)(\beta_1^2-\beta_3^2+4\beta_1\beta_2^2l+4\beta_2^4l^2)\sin^2\beta_3L},$$

$$T=\frac{4\beta_1^2\beta_3^2}{4\beta_3^2(\beta_1+\beta_2^2l)^2+(\beta_1^2-\beta_3^2)(\beta_1^2-\beta_3^2+4\beta_1\beta_2^2l+4\beta_2^4l^2)\sin^2\beta_3L},$$

$$A=1-(T+R).$$

If the direction of incidence is altered so that the incident wave traverses the slab before being reflected by the film, two further amplitude ratios are required. These are

$$r_{13}=\frac{\beta_3-\beta_1}{\beta_1+\beta_3}; \qquad t_{13}=\frac{2\beta_1}{\beta_1+\beta_3}.$$

The incident wave $ae^{wt+\gamma_1z}$ now gives rise to a reflected wave $-a'e^{jwt-\gamma_1z}$ given by

$$ae^{jwt-\gamma_1z}[r_{13}e^{j\pi}+t_{13}t_{31}r_3e^{j(\pi-2\beta_3L)}\{1+r_3r_{31}e^{2j(\pi-\beta_3L)}+(r_3r_{31})^2e^{4j(\pi-\beta_3L)}l\ldots\}]$$

and a transmitted wave $a''e^{jwt+\gamma_1z}$ given by

$$ae^{jwt+\gamma_1z}t_{13}t_3e^{-j\beta_3L}\{1+r_3r_{31}e^{2j(\pi-\beta_3L)}+(r_3r_{31})^2e^{4j(\pi-\beta_3L)}+\ldots\},$$

so that the reflection and transmission amplitude ratios r', t' are

$$r'=\frac{a'}{a}=\frac{r_{13}+r_3(t_{13}t_{31}-r_{13}r_{31})e^{-2j\beta_3L}}{1-r_3r_{31}e^{-2j\beta_3L}}=\frac{r_{13}e^{j\beta_3L}+r_3e^{-j\beta_3L}}{e^{j\beta_3L}-r_3r_{31}e^{-j\beta_3L}},$$

$$t'=\frac{a''}{a}=\frac{t_{13}t_3}{e^{j\beta_3L}-r_3r_{31}e^{-j\beta_3L}}\equiv\frac{t_1t_{31}}{e^{j\beta_3L}-r_3r_{31}e^{-j\beta_3L}}=t,$$

and these yield the intensity ratios

$$T'=T,$$

$$R'=\frac{4\beta_3^2\beta_2^4l^2+(\beta_1^2-\beta_3^2)(\beta_1^2-\beta_3^2+4\beta_1\beta_2^2l+4\beta_2^4l^2)\sin^2\beta_3L}{4\beta_3^2(\beta_1+\beta_2^2l)^2+(\beta_1^2-\beta_3^2)(\beta_1^2-\beta_3^2+4\beta_1\beta_2^2l+4\beta_2^4l^2)\sin^2\beta_3L},$$

$$A'=1-(R'+T').$$

Thus although the transmission coefficient is the same for both directions of the incident light, the reflection and absorption coefficients in general depend on the direction of incidence. This results from the fact that the reflection ratios r_1, r_3 for the two sides of the film, regarded as an interface between the dielectric media 1 and 3, are not equal and opposite as they would be if the film were absent; this, in turn, is due to the loss which occurs in the film. It should be noticed that the reflection and transmission coefficients for the slab alone can be obtained from these expressions by putting $l=0$.

For the special case in which the slab thickness L is an integral number of half-wave-lengths ($\beta_3 L=n\pi$), the expressions for R, T, A simplify, since the effect of the slab is entirely eliminated, and we have

$$R=R'=\left[\frac{\beta_2{}^2 l}{\beta_1+\beta_2{}^2 l}\right]^2=\left[\frac{\beta Z_0}{\beta Z_0+2\beta_1 S}\right]^2,$$

$$T=T'=\left[\frac{\beta_1}{\beta_1+\beta_2{}^2 l}\right]^2=\left[\frac{2\beta_1 S}{\beta Z_0+2\beta_1 S}\right]^2,$$

and

$$A=A'=\frac{4\beta\beta_1 Z_0 S}{[\beta Z_0+2\beta_1 S]^2}.$$

In this case the absorption is a maximum (50 per cent.) if $S=\beta Z_0/2\beta_1$ and then $R=T=25$ per cent.

§ 5. *Film on Dielectric Slab backed by Perfect Reflector.*

If the film is carried on one face of a plane-parallel dielectric slab, the other face of which rests against a perfectly reflecting surface, the reflection coefficient of the combination is again a function of both the film thickness l and the slab thickness L. The general case in which the slab material (3) is different from the medium (1) in front of the film is easily derived from the preceding case by replacing r_{31} and t_{31} by unity and zero respectively, since the interface 3, 1 is here replaced by a new interface formed by the medium 3 and the perfect reflector. We thus have a new reflection ratio given by

$$r=\frac{r_1-(r_1 r_3-t_1 t_3)e^{-2j\beta_3 L}}{1-r_3 e^{-2j\beta_3 L}}$$

$$=\frac{(\beta_3-\beta_1+2\beta_2{}^2 l)e^{j\beta_3 L}+(\beta_1+\beta_3-2\beta_2{}^2 l)e^{-j\beta_3 L}}{(\beta_1+\beta_3+2\beta_2{}^2 l)e^{j\beta_3 L}-(\beta_1-\beta_3+2\beta_2{}^2 l)e^{-j\beta_3 L}},$$

whence the reflection coefficient is found to be

$$R=|r|^2=\frac{\beta_3{}^2\cos^2\beta_3 L+(\beta_1-2\beta_2{}^2 l)^2\sin^2\beta_3 L}{\beta_3{}^2\cos^2\beta_3 L+(\beta_1+2\beta_2{}^2 l)^2\sin^2\beta_3 L}.$$

Further, since the transmission coefficient of the composite reflector is zero, the absorption coefficient is given by

$$A=1-R=\frac{8\beta_1\beta_2{}^2 l\sin^2\beta_3 L}{\beta_3{}^2\cos^2\beta_3 L+(\beta_1+2\beta_2{}^2 l)^2\sin^2\beta_3 L}.$$

The dependence of R on L is obtained by considering the two extreme cases. On the one hand, if $L=(2n+1)\lambda_3/4$ so that $\beta_3 L=(2n+1)\pi/2$, we have

$$R=\left[\frac{\beta_1-2\beta_2{}^2 l}{\beta_1+2\beta_2{}^2 l}\right]^2=\left[\frac{\beta_1 S-\beta Z_0}{\beta_1 S+\beta Z_0}\right]^2.$$

If, in addition, the film has a surface resistance $S=\beta Z_0/\beta_1$, the reflection coefficient is zero. If these two conditions are satisfied, the composite reflector may be used to provide a matched termination for a concentric line or wave-guide.

On the other hand, if $L=n\lambda_3/2$ so that $\beta_3 L=n\pi$, the reflection coefficient is unity, and the film absorbs no energy. For intermediate values of L the reflection coefficient does not vanish, but the absorption coefficient has a maximum value when

$$S=\frac{\beta Z_0}{[\beta_1{}^2+\beta_3{}^2\cot^2\beta_3 L]^{\frac{1}{2}}}.$$

The phase shift ϕ, which accompanies the reflection from the composite reflector, is obtained as the imaginary part of the reflection ratio r, and is given by

$$\tan\phi=\frac{-2\beta_1\beta_3\sin 2\beta_3 L}{\beta_3{}^2-\beta_1{}^2+4\beta_2{}^4 l^2+(\beta_1{}^2+\beta_3{}^2-4\beta_2{}^4 l^2)\cos 2\beta_3 L}.$$

The amplitude reflection ratio for the composite reflector is thus $|r|\,e^{j(\pi+\phi)}$.

§ 6. *Resonance Condition for a Resonator loaded with a Film on a Dielectric Slab.*

If another perfectly reflecting plane be placed at a distance l_1 in front of the composite reflector of the last section, the arrangement is capable of resonance at certain frequencies. If l_1 is varied at a given frequency, resonance occurs when the energy loss in the film is a maximum. Since the energy lost in the film is proportional to $|E|^2/S$, where E is the resultant electric vector at the front surface of the film, the resonance condition is obtained by differentiating $|E|^2$ with respect to l_1.

Now if an incident wave has an electric vector of magnitude A at the film then, as a result of multiple reflections occurring in the space in front of the film, the resultant electric vector E at the film due to waves travelling in both directions is given by

$$E=A[1+re^{j\pi}+re^{j(2\pi-2\beta_1 l_1)}+r^2e^{j(3\pi-2\beta_1 l_1)}+r^2e^{j(4\pi-4\beta_1 l_1)}+\ldots]$$

$$=A\,\frac{1-r}{1-re^{-2j\beta_1 l_1}}=A\,\frac{1-|r|\,e^{j\phi}}{1-|r|\,e^{j(\phi-2\beta_1 l_1)}},$$

where r, ϕ have the values obtained in the preceding section.

Hence $$|E|^2=\frac{A^2[1+|r|^2-2\,|r|\cos\phi]}{[1+|r|^2-2\,|r|\cos(\phi-2\beta_1 l_1)}.$$

For this to be a maximum with respect to l_1,

$$\sin(\phi-2\beta_1 l_1)=0,$$

and hence
$$\tan\phi=\tan 2\beta_1 l_1,$$

is the required resonance condition.

This condition may also be obtained by forming stationary waves in the space in front of the film. Since, in general, energy is lost in reflection at the film, the reflected wave has a smaller magnitude than the incident wave and can only form a stationary wave with a part of the incident wave of equal magnitude. The condition for the stationary system to be formed is that the electric intensity at the second perfectly reflecting plane, *i. e.*, at $z=-l_1$, should be zero. This requires that

$$|A|\,r\,|e^{j\beta_1 l_1}-Are^{-j\beta_1 l_1}|=0,$$

i. e.,
$$\cos(2\beta_1 l_1-\phi)=1 \quad \text{or} \quad \sin(2\beta_1 l_1-\phi)=0 \text{ as before.}$$

Inserting the value of ϕ from the preceding section, the resonance condition is thus seen to be

$$\tan 2\beta_1 l_1=\frac{-2\beta_1\beta_3\tan\beta_3 L}{\beta_3^2-(\beta_1^2-4\beta_2^4 l^2)\tan^2\beta_3 L}.$$

This condition can be verified by putting $l=0$ when the condition reduces to

$$\frac{\tan\beta_1 l_1}{\beta_1}=-\frac{\tan\beta_3 L}{\beta_3},$$

which is the known resonance condition for a resonator loaded with a dielectric slab.

§ 7. *Application to Guided Waves.*

The foregoing results have been obtained for plane waves and can be shown to apply to certain guided waves provided that

$$\beta_1^2=\epsilon_1\beta^2-\beta_c^2,$$
$$\beta_3^2=\epsilon_3\beta^2-\beta_c^2,$$

where $\beta=2\pi/\lambda$; $\beta_c=2\pi/\lambda_c$ and λ_c is the critical wave-length appropriate to the type of wave and the wave-guide dimensions, ϵ_1, ϵ_3 being the dielectric permittivities of the media 1, 3.

The waves in question are transverse electric or H waves, and it will suffice to consider a single example, the H_0 wave peculiar to a cylindrical wave-guide. This is a wave having three components E_ϕ, H_z, H_ρ, of which the first two stand in a special relationship, defined by the component curl equation

$$-\frac{\mu}{c}\frac{\partial H_z}{\partial t}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\phi).$$

Since z does not occur in this equation the relation between E_ϕ and μH_z remains unchanged as the wave progresses. If boundary matching is carried out at an interface which is normal to the direction of z, the continuity conditions for normal B (*i. e.*, μH_z) and tangential E (*i. e.*, E_ϕ) are thus identical. The relationship

$$H_\varrho = \frac{c}{j\omega\mu} \cdot \frac{\partial E_\phi}{\partial z},$$

between H_ϱ and E_ϕ is identical with that which exists between H and E in a plane wave. Thus, for propagation in the z direction and boundary matching in planes perpendicular to this direction, the H_0 wave has effectively only two component vectors, viz., E_ϕ and H_ϱ, and the H_z component may be ignored. In applying the expressions for plane waves, however, the correct phase constants must be employed for the guided wave as indicated earlier in this section.

It may be noticed that if an annular interface were considered, the effective vectors in the H_0 wave would be E_ϕ and H_z; H_ϱ could be ignored. Similar considerations apply to H_1 waves in rectangular wave-guides.

§ 8. *Discussion.*

The approximate values of R, T, A derived in section 3 for plane waves incident on a film between two dielectric media simplify if air is the medium on both sides of the film. In this case, since $\beta=\beta_1=\beta_3$,

$$R=\left[\frac{Z_0}{Z_0+2S}\right]^2; \qquad T=\left[\frac{2S}{Z_0+2S}\right]^2; \qquad A=\frac{4Z_0S}{[Z_0+2S]^2}.$$

These expressions were first derived in this form by Czerny (1) and his pupils, who used them to evaluate the surface resistance of metallic films at wave-lengths in the infra-red. Their starting point was an expression for the amplitude reflection ratio of a film in air which had been obtained by Drude, and they did not extend the analysis to cover the more general cases considered above. An alternative derivation of the expressions for R, T has recently been given by D. Smith (2), who uses Maxwell's equations in integral form and integrates around appropriate contours in the film. The method is, however, of doubtful value, since approximations are made for which justification appears to be lacking. Thus, by ignoring the reflection which occurs at the second face of the film he is able to represent the electro-magnetic energy in the film by a single travelling wave, and in order to neglect certain integrals he gives this wave a velocity in the film which is the velocity c appropriate to free space. Since the velocity $v \ll c$ the integrals, which contain the quantity $1/v$, have a value considerably in excess of that computed by him. As he is concerned with a long wave-length (3 m.) and a poor conductor, this error does not affect his results, but for shorter wave-lengths and better conductors the error may be considerable, since $v \propto (\lambda/\sigma)^{\frac{1}{2}}$.

On the experimental side, Murmann$^{(3)}$ and Woltersdorff$^{(4)}$ have measured R and T at wave-lengths of 6–110 μ and have compared the values of S obtained from the above expressions with the measured values. Since the films were prepared on thin ($\sim 1\mu$) sheets of celluloid, these expressions are not strictly applicable to this case, which may account for some of the discrepancies noticed by these investigators. For films prepared on thicker glass bases, Woltersdorff found that R varied greatly with the wave-length. This suggests that the effect of multiple reflections occurring in the glass was responsible, in which case the expressions derived in section 4 would apply here. One other result obtained by Woltersdorff, viz. "the anomaly of the bad conductor," which was found with Bi and Te, can be explained on the basis of the present work. The film thicknesses which he used for these metals were quite comparable with the quarter-wave-length in the metal; his application of the theoretical expressions to this case was thus invalid, since it has been shown that these are obtained on the assumption that $l \ll \lambda_2/4$.

The results derived in this paper should be applicable at any wave-length for which Maxwell's theory is valid and for which the film thickness is small compared to the quarter-wave-length in the films. Two further assumptions which are implicit in the reasoning are (1) that the film is a homogeneous isotropic medium, and (2) that it has an intrinsic conductivity high enough to ensure that the phase and attenuation constants are sensibly equal. Experiments which have been made to test these conclusions at a wave-length of 1·25 cm. will be reported in a further paper.

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XVII. *On the Theory of Point Electrons.*

By T. Lewis *.

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I SHOULD like to express my thanks to Dirac for replying to certain criticisms which I made of his treatment of the theory of point electrons. His clarification of his theory disposes of most of my objections from a mathematical point of view; but, from the physical point of view, it appears to me that he is surrendering the whole position regarding the interaction of an electron with its own field. I do not object to the position taken up by Dirac, for it is in line with the position I adopted originially. But I cannot square it with the original investigations of Dirac, Pryce and others. In this short paper I shall deal with the case of a single electron in an external field, and also the case of a system of interacting electrons.

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* Communicated by the Author.

The notation adopted is the same as Dirac's.

The case of one electron.—According to Dirac (1938) the equation of motion of an electron in an *incident* field, $F_{\mu\nu\,\mathrm{in}}$, is

$$m\dot{v}_\mu = \tfrac{2}{3}\, e^2(\ddot{v}_\mu - (\mathbf{v}\ddot{\mathbf{v}})v_\mu) + ev^\nu F_{\mu\nu\,\mathrm{in}}, \quad \ldots \ldots \quad (1)$$

where the first term on the right-hand side represents the reaction of the electron with its own field. Dirac has shown that on the world line of the electron

$$F_{\mu\nu\,\mathrm{ret}} - F_{\mu\nu\,\mathrm{adv}} = \frac{4e}{3}(\ddot{v}_\mu v_\nu - \ddot{v}_\nu v_\mu),$$

so that (1) may be written

$$m\dot{v}_\mu = ev^\nu\{\tfrac{1}{2}(F_{\mu\nu\,\mathrm{ret}} - F_{\mu\nu\,\mathrm{adv}}) + F_{\mu\nu\,\mathrm{in}}\} \quad \ldots \ldots \quad (2)$$

I objected to the assumption that this equation could be derived from a variational principle, $\delta I = 0$, where

$$I = \int m\,ds + e\int A_{\mu\,\mathrm{in}} v^\mu\, ds + \tfrac{1}{2}e\int (A_{\mu\,\mathrm{ret}} - A_{\mu\,\mathrm{adv}})v^\mu\, ds. \quad \ldots \quad (3)$$

I naturally assumed that the incident field, represented by the potentials $A_{\mu\,\mathrm{in}}$, was independent of the motion of the electron and would remain unchanged for a variation of the world line of the electron. In the examples worked out by Dirac (1938), the incident field is certainly independent of the motion of the electron. When this is the case the second term on the right-hand side of (3) gives rise to the term involving $F_{\mu\nu\,\mathrm{in}}$ in (1) and (2). But it can be shown that the last term in (3),

$$\tfrac{1}{2}\int (A_{\mu\,\mathrm{ret}} - A_{\mu\,\mathrm{adv}})v^\mu\, ds = -\int e\dot{v}_\mu v^\mu\, ds \equiv 0, \quad \ldots \quad (4)$$

both for the actual path and for the displaced path, provided the electron carries its own field with it, and is not thought of as being displaced relative to its own field when applying the variational principle. It follows that variation of this term contributes nothing to the equations of motion, and the variation principle leads to equations of motion in which damping terms do not appear.

However, Dirac does not regard $F_{\mu\,\mathrm{in}}$ and $A_{\mu\,\mathrm{in}}$ as independent of the electron. In fact, the field independent of the electron is represented by the potentials A_μ, where

$$A_\mu = \tfrac{1}{2}(A_{\mu\,\mathrm{in}} + A_{\mu\,\mathrm{out}}), \quad \ldots \ldots \quad (5)$$

and $A_{\mu\,\mathrm{out}}$ is defined by the equation

$$A_{\mu\,\mathrm{out}} + A_{\mu\,\mathrm{adv}} = A_{\mu\,\mathrm{in}} + A_{\mu\,\mathrm{ret}}. \quad \ldots \ldots \quad (6)$$

We thus see that the field independent of the electron is given by

$$A_\mu = A_{\mu\,\mathrm{in}} + \tfrac{1}{2}(A_{\mu\,\mathrm{ret}} - A_{\mu\,\mathrm{in}}). \quad \ldots \ldots \quad (7)$$

Bearing these definitions in mind, the I of equation (3) becomes

$$I=\int m\,ds+\int e\,A_\mu v^\mu ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

and the variation principle gives rise to the equations

$$m\dot{v}_\mu=ev^\nu F_{\mu\nu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

We can now resolve $F_{\mu\nu}$ into parts after the manner of (7) thus,

$$F_{\mu\nu}=F_{\mu\nu\,\mathrm{in}}+\tfrac{1}{2}(F_{\mu\nu\,\mathrm{ret}}-F_{\mu\nu\,\mathrm{adv}}). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

Substituting (10) in (9) we get equation (2) and hence (1).

But since $F_{\mu\nu\,\mathrm{in}}$ is not independent of the electron's motion, the form of equation (1) is not very convenient in practice. The form of equation (9) is far simpler, especially when we consider that the one field, $F_{\mu\nu}$, which it involves is independent of the electron's motion, whereas all the terms on the right-hand side of (1) depend upon the electron's motion.

Now if $A_\mu=\frac{1}{2}(A_{\mu\,\mathrm{in}}+A_{\mu\,\mathrm{out}})$ is the field which is independent of the electron we can assume it to be known. It can be any prescribed solution of Maxwell's equations. It is what the classical physicists called the external field. If we ask the question: how will an electron move in a field represented by prescribed $F_{\mu\nu}$? the answer is clearly: according to equation (9). But where are the damping terms? It is true we can now resolve $F_{\mu\nu}$ as in equation (10), and hence obtain damping terms at the expense of introducing a field $F_{\mu\nu\,\mathrm{in}}$ which depends upon the electron's motion. In fact, $F_{\mu\nu\,\mathrm{in}}$ contains terms which, on the world line of the electron, are equal and opposite to the damping terms. This is not what the classical physicists meant by radiation damping. For them $F_{\mu\nu\,\mathrm{in}}$ was the prescribed field and the radiation reaction was something real added to the effect of the prescribed field. Also, as pointed out earlier, $F_{\mu\nu\,\mathrm{in}}$ was the prescribed field in Dirac's original investigations. However, the problem of a single electron moving in a prescribed field, whether the prescribed field be $F_{\mu\nu}$ or $F_{\mu\nu\,\mathrm{in}}$, is itself rather artificial. To see the problem in its proper perspective one should consider a system of electrons moving in one anothers' presence; for according to classical physics the prescribed field of our previous discussion is due to the motion of some other electrons or electric charges. (The position in Quantum mechanics is not quite the same, but here we are concerned with the Classical point of view only.) In this more general case the variational principle gives rise, in a very natural way, to terms which may be interpreted as the reaction of an electron with its own field, though it must not be confused with the old Classical concept of the reaction of an electron with its own field. It is true that under special conditions the equations of motion appear in the classical form *with* radiation damping. But under other circumstances, such as the case of an electron in an electrostatic field (which can be deduced from the general case by making the masses of all electrons except one very large) there appear no damping terms at all.

The case of several electrons.—The equations of motion are obtained by applying the variational principle to I, which is conveniently divided into two parts ; the first being

$$I_1=\Sigma_i\int m_i ds_i+\tfrac{1}{2}\Sigma_i \Sigma_{j\neq i}\int e_i(\mathbf{A}^{(j)}_{\text{ret}}+\mathbf{A}^{(j)}_{\text{adv}})\cdot d\mathbf{z}_i, \quad . \quad . \quad . \quad (11)$$

where (i), (j) are simply labels to distinguish the electrons. $\mathbf{A}^{(j)}_{\text{ret}}$ is the retarded 4-vector potential of the electron (j) at the position $\mathbf{z}_i$ of the electron (i), and $\mathbf{A}^{(j)}_{\text{adv}}$ is the corresponding advanced potential.

The second part of I is

$$I_2=\tfrac{1}{2}\Sigma_i e_i(\mathbf{A}_{\text{in}}+\mathbf{A}_{\text{out}})\cdot d\mathbf{z}_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where, according to Dirac $(\mathbf{A}_{\text{in}}+\mathrm{A}_{\text{out}})$ is independent of all the electrons, and may therefore be prescribed. In particular, it may have the value zero everywhere, in which case the variational principle is identical with Fokker's variational principle.

I agree with Dirac that variation of I_1 (when $I_2=0$) gives rise to equations of motion which include terms which may be interpreted as the reaction of the electron with its own field. But I maintain that I_1 accounts for the whole of this reaction—there is no other reaction even when $(\mathbf{A}_{\text{in}}+\mathbf{A}_{\text{out}})$ is not zero.

To show how the electron's own field enters into I_1 and hence into the equations of motion derived from it, it will be necessary to refer to another paper (Lewis, 1940), where I have shown that the principle of motion embodied in $\delta I_1=0$ may be derived from a more fundamental variational principle.

Writing
$$J=\frac{1}{8\pi}\iiiint(H^2-E^2)\,dx\,dy\,dz\,dt, \quad . \quad . \quad . \quad . \quad (13)$$

where **E** and **H** are the electric and magnetic vectors, respectively, Fokker's variational principle follows from

$$\delta J=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

subject to the conditions

$$\delta(ds_i)=0,\ i=1,2\ldots n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where we are concerned with the variation of J due to the variation of the world lines of electrons only. The infinite part of J is eliminated on account of the supplementary conditions (15), and the mass of each electron enters in the form of a Lagrangian multiplier.

In the first place one introduces *retarded* fields only, and the principle embodied in (14) and (15) leads to

$$\Sigma_i\int m_i\delta(ds_i)+\tfrac{1}{2}\Sigma_i \Sigma_{j\neq i} e_i\int \delta(\mathbf{A}^{(j)}_{\text{ret}}\cdot d\mathbf{z}_i)=0, \quad . \quad . \quad . \quad . \quad (16)$$

where, as before, $\mathbf{A}^{(j)}_{\text{ret}}$ is the retarded potential of the electron (j) at the position $\mathbf{z}_i$ of the electron (i).

To avoid confusion, I shall write (16) in full for the case of two electrons, in which the world-line of electron (1) only is displaced. For this special case (16) becomes

$$\int m_1\delta(ds_1)+\tfrac{1}{2}e_1\int\delta(\mathbf{A}^{(2)}_{\text{ret}}\,.\,d\mathbf{z}_1)+\tfrac{1}{2}e_2\int\delta(\mathbf{A}^{(1)}_{\text{ret}}\,.\,d\mathbf{z}_2)=0. \quad . \quad . \quad (17)$$

We note that the second term involves the retarded field of electron (2) at the position of electron (1), but that the last term involves the retarded field produced by electron (1) itself, but evaluated at the position of electron (2). It is this last term which represents the reaction of the electron with its own field. But if we write down the explicit expression for $\mathbf{A}^{(1)}_{\text{ret}}(\mathbf{z}_2)$ and take into account certain relations, this term can be transformed so that it appears to involve the advanced potential of electron (2) at the position of electron (1).

We have that

$$\mathbf{A}^{(1)}_{\text{ret}}(z_2)=\frac{e_1\dot{\mathbf{z}}_1}{(\dot{\mathbf{z}}_1\,.\,\mathbf{z}_2-\mathbf{z}_1)}=\frac{e_1 d\mathbf{z}_1}{(d\mathbf{z}_1\,.\,\mathbf{z}_2-\mathbf{z}_1)}, \quad . \quad . \quad . \quad (18)$$

where $$(\mathbf{z}_1-\mathbf{z}_2\,.\,\mathbf{z}_1-\mathbf{z}_2)=0 \text{ and } (\mathbf{z}_2)_0>(\mathbf{z}_1)_0. \quad . \quad . \quad . \quad . \quad (19)$$

From (19) it follows that

$$(d\mathbf{z}_1\,.\,\mathbf{z}_1-\mathbf{z}_2)=(d\mathbf{z}_2\,.\,\mathbf{z}_1-\mathbf{z}_2). \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Hence we get

$$\begin{aligned} e_2(\mathbf{A}^{(1)}_{\text{ret}}\,.\,d\mathbf{z}_2)&=e_2e_1\frac{(d\mathbf{z}_1\,.\,d\mathbf{z}_2)}{(d\mathbf{z}_1\,.\,\mathbf{z}_2-\mathbf{z}_1)}=e_1e_2\frac{d\mathbf{z}_2\,.\,d\mathbf{z}_1}{(d\mathbf{z}_2\,.\,\mathbf{z}_2-\mathbf{z}_1)} \\ &= e_1(\mathbf{A}^{(2)}_{\text{adv}}\,.\,d\mathbf{z}_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21) \end{aligned}$$

Substituting the last expression in (21) for the last term in (17), the latter is identical with $\delta I_1=0$, *i. e.*, it is Fokker's variational principle. Although I_1 as it appears in equation (11) does not contain the self-field of an electron explicitly, it is represented by $\mathbf{A}^{(j)}_{\text{adv}}$. Had we admitted advanced fields only in the first place we should have arrived at exactly the same result. Consistent application of the variational principle forces us to admit retarded and advanced field on the same footing.

The equations of motion can now be written in the form

$$m_i\dot{v}^{(i)}_\mu=\tfrac{1}{2}\sum_{j\neq i}e_i(\mathrm{F}^{(j)}_{\mu\nu\,\text{ret}}+\mathrm{F}^{(j)}_{\mu\nu\,\text{adv}})\overset{(i)}{v^\nu}\,. \quad . \quad . \quad . \quad . \quad (22)$$

If we now introduce, after Dirac,

$$\mathrm{F}_{\mu\nu\,\text{rad}}=\sum_i(\mathrm{F}_{\mu\nu\,\text{ret}}-\mathrm{F}_{\mu\nu\,\text{adv}}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

equation (22) becomes

$$m_i\dot{v}^{(i)}_\mu=e_i\{-\tfrac{1}{2}\mathrm{F}_{\mu\nu\,\text{rad}}+\tfrac{1}{2}(\mathrm{F}^{(i)}_{\mu\nu\,\text{ret}}-\mathrm{F}^{(i)}_{\mu\nu\,\text{adv}})+\sum_{j\neq i}\mathrm{F}^{(j)}_{\mu\nu\,\text{ret}}\}\overset{(i)}{v^\nu}\,. \quad . \quad (23)$$

We thus see that $-\frac{1}{2}\mathrm{F}_{\mu\nu\,\mathrm{rad}}$ plays the part of $\mathrm{F}_{\mu\nu\,\mathrm{in}}$ in Dirac's treatment, and, as Dirac states, it is not independent of the electron (i). $\frac{1}{2}\mathrm{F}_{\mu\nu\,\mathrm{rad}}$, without the minus sign, plays the part of Dirac's $\mathrm{F}_{\mu\nu\,\mathrm{out}}$. To get the most general case in which $(\mathrm{F}_{\mu\nu\,\mathrm{in}}+\mathrm{F}_{\mu\nu\,\mathrm{out}})\neq 0$, we need only write

$$\mathrm{F}_{\mu\nu\,\mathrm{in}}=\mathrm{F}_{\mu\nu}-\tfrac{1}{2}\mathrm{F}_{\mu\nu\,\mathrm{rad}},\ \mathrm{F}_{\mu\nu\,\mathrm{out}}=\mathrm{F}_{\mu\nu}+\tfrac{1}{2}\mathrm{F}_{\mu\nu\,\mathrm{rad}}, \quad . \quad . \quad . \quad (24)$$

where $\mathrm{F}_{\mu\nu}$ is a prescribed solution of Maxwell's equations, independent of the motion of all the electrons, so that we are justified in calling it the external field.

A very interesting special case has been discussed by Wheeler and Feynman (1945), namely, the case for which

$$\mathrm{F}_{\mu\nu\,\mathrm{rad}}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

everywhere. The authors call this the case of complete absorption. This condition gives rise to equations of motion in which the classical damping terms occur explicitly, together with the retarded fields of the other electrons. Defining $\mathrm{F}_{\mu\nu\,\mathrm{in}}$ as in (24), the equations of motion now become

$$m_i\dot{v}_\mu^{(i)}=e_i\{\mathrm{F}_{\mu\nu\,\mathrm{in}}+\tfrac{1}{2}(\mathrm{F}^{(i)}_{\mu\nu\,\mathrm{ret}}-\mathrm{F}^{(i)}_{\mu\nu\,\mathrm{adv}})+\sum_{j\neq i}\mathrm{F}^{(j)}_{\mu\nu\,\mathrm{ret}}\}\overset{(i)}{v^\nu}, \quad . \quad . \quad . \quad (26)$$

where $\mathrm{F}_{\mu\nu\,\mathrm{in}}$ is now independent of the electrons. We thus get the equations of motion in the form given by Dirac and with $\mathrm{F}_{\mu\nu\,\mathrm{in}}$ equal to a prescribed field; but it is equal to the prescribed field, $\mathrm{F}_{\mu\nu}$, only for the actual motion. For the displaced path (25) is not satisfied and $\mathrm{F}_{\mu\nu\,\mathrm{in}}$, in general, has to be replaced by the value given in (24). $\mathrm{F}_{\mu\nu\,\mathrm{rad}}=0$, is a condition which must be imposed *after* variation and not before.

We have now arrived at results which are formally in complete agreement with Dirac's results. But I believe the agreement to be formal only, for Dirac states (1938) that the field $\frac{1}{2}(\mathrm{F}_{\mu\nu\,\mathrm{in}}+\mathrm{F}_{\mu\nu\,\mathrm{out}})$ is unknown, while I maintain that it can be a prescribed field. Furthermore, in certain special cases, there is a complete absence of radiation damping as understood by the old classical physicist, as will be shown in the next paragraph.

The case of a single electron in an electrostatic field.—If we make the masses of all the electrons except one very large, so that the velocities of these electrons (or protons) may be neglected, their advanced and retarded fields are equal and both reduce to electrostatic fields, and equation (22) may be written

$$m_i\dot{v}_\mu^{(i)}=\sum_{j\neq i} f^{(j)}_{\mu\nu}\overset{(i)}{v^\nu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

where $f_{\mu\nu}$ is a static field ($f_{23}=f_{31}=f_{12}=0$). We could still write the equation in the form (23) so as to give an appearance of radiation damping, which is represented by the term $\frac{1}{2}e_i(\mathrm{F}_{\mu\nu\,\mathrm{ret}}-\mathrm{F}_{\mu\nu\,\mathrm{adv}})v_\nu^{(i)}$. But now $\mathrm{F}_{\mu\nu\,\mathrm{rad}}$ contains only one significant term, which just cancels this radiation damping term. However, it would not be quite correct to say that the electron's own field does not enter into its equation of motion. From our treatment it is clear that half the terms on the right-hand side of the equations of motion, (22) for instance, are due to the *retarded* fields of other electrons, and the other half are due to the electron's own *retarded* field acting on the other electrons. By a mathematical dodge these latter terms can be written as the *advanced* field of the other electrons acting on the particular electron under discussion. This is still the case when the fields of the other electrons reduce to electrostatic fields. But it remains true that the motion of an electron in the field of a proton, as in the case of the hydrogen atom, is governed in *effect* by the electrostatic field of the proton, without further damping terms, if the above theory is accepted. There is no tendency to spiral towards the nucleus as was thought by the classical physicists. This fact makes me doubt whether the agreement with Dirac's theory, which I mentioned earlier, is more than formal. On the other hand, if Dirac holds that $(A_{in}+A_{out})$ is the independent field, and not A_{in}, I fail to see that any results other than those arrived at in this paper are possible on the basis of the variational principle.

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XVIII. *A New Criterion of Yielding in Metals due to Complex Stresses.*

By K. H. Swainger, Ph.D., B.Sc.(Eng.) *.

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Summary.

The older criteria of yielding except those of maximum shear stress and energy of change of shape were invalidated by the experimental evidence that metals could withstand great hydrostatic pressures without yielding. The ratio of simple shear yield stress to simple tensile yield stress was found to fit the theory using energy of change of shape better than the maximum shear stress theory. The results from the mild steel tests did not fit very well. When metal yields under an applied simple shear stress and an applied simple tension then the curve found experimentally by plotting these two stresses as variables is an ellipse as predicted by the energy of deformation theory.

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* Communicated by the Author.

In the present theory the plane stress experimental ellipse is accepted and the equation to it is written in linear form by means of vectors. It is shown that any plane stress system except "two-dimensional hydrostatic pressure or tension" will reduce to a simple shear with a simple tension or compression at any point by choosing suitable directions to define the orthogonal element on which the stresses act. The experimental results on yield stress values then apply to the element with this orientation.

Any three-dimensional stress system except hydrostatic pressure or tension can be reduced to one of simple tension or compression together with two shears at any given point by a suitable choice of directions to define the element. These three stresses will define two ellipsoidal surfaces for the complete specification of yielding. If the metal is isotropic then each surface is described by revolution about the normal stress axis and then the tests with simple shear and simple tension will apply as the yield criterion of the three-dimensional system.

1. *History and Introduction.*

THE mathematical theory of elasticity evolved gradually from the efforts of workers in several fields of activity. Engineers wished to predict the behaviour of structures under load, physiologists wanted to know the strength of animal fibres and physicists wanted to understand the mechanics of the phenomena observed in the transmission of light (1). At first the strains were neglected, but gradually it was realized that an analysis of the forces operating within the deformed body due to the external loads could only be performed when the strains are considered. It was seen at an early date that the strains could be divided into two broad divisions, one approximately independent of time and the other definitely depending on time. The strains independent of time could again be divided into two components ; the elastic part which disappeared immediately with release of load and a " permanent set " which remained after the removal of load. The statement that such phenomena are observed is really an idealization of the truth for analytical convenience. In fact the elastic strain component is not absolutely independent of time since slight " creep " can occur under prolonged loading, nor is the " permanent " set so permanent because some decrease of the strain occurs when the strained body is rested. However, accepting the idea of an elastic strain range in a material it soon became evident to the early research workers that a theoretical criterion was required that would indicate when permanent set had commenced to appear in a loaded body. Coulomb suggested in 1776 that when the shearing stress at a point in the loaded body reached a critical value independent of the other stresses operating then permanent set appeared. Lamé proposed in 1833 that the greatest tensile stress at a point of the loaded body decided the onset of permanent strains. Mariotte at some earlier date, and then Poncelet in 1839 followed by Saint Venant thought the greatest tensile strain at a point was the criterion of the beginning of yield. Then in 1869 Tresca took the greatest difference of the principal stresses as the yield criterion. Beltrami suggested in 1885 that the total strain energy of the element about a point gave the criterion of yielding.

Föppl in 1899 and Guest independently in 1900 experimented with great hydrostatic pressures on solids. They found that they could not produce permanent strains in the metal specimens with the pressures at their disposal. This immediately excluded as unacceptable all the yield criteria except those of Coulomb and Tresca. Mohr suggested in 1914 that an improvement in the maximum shear stress theory could be effected by supposing that the stress normal to the plane on which the maximum shear stress operated also influenced the yielding. The experiments with great hydrostatic pressure led Huber in 1904 (7) to suggest dividing the total strain energy of an element into two parts, one producing changes in volume of the metal and the remaining component to be regarded as energy producing change of shape and governing the yield of the element. This critical energy was to be independent of the type of stress system inducing the energy. Von Mises in 1913 and Hencky in 1925 each rediscovered independently during slightly different approaches what amounts to Huber's criterion. The criterion can be stated

$$\sum_{\substack{i=1,\,2,\,3\\ j=2,\,3,\,1}} \{(S_{0,\,ii}-S_{0,\,jj})^2+6S^2_{0,\,ij}\}=2(X^{0,11})^2, \qquad . \; . \; . \; . \; . \quad (1.1)$$

where $X^{0,11}$ is the simple tensile yield stress while $S_{0,\,ii}$, $S_{0,\,ij}$ are respectively the normal and shear stresses operating on the element when it first yields.

Lode (8) in 1926 and Taylor and Quinney (2) in 1932 undertook experimental examination of the maximum shear stress and energy of change of shape theories. Each found that the theory of energy of change of shape gave the best results for a two-dimensional stress system. In particular for yielding under simple shear stress Taylor and Quinney found the results shown in Table 1.1 with my notation *.

Both the maximum shear stress and energy of deformation theories suggest that when a simple shear stress X^{12} and a simple tensile stress X^{11} operate simultaneously on the specimen and yield occurs, then on plotting these values as ordinate and abscissa respectively defines an ellipse. Taylor and Quinney devised their tests to give this stress system with torsion and longitudinal tension on thin-walled tubes. Their test results define the ellipse to a good approximation for copper and aluminium.

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* It is recognized that the following notation seems unnecessarily complicated for the immediate purpose, but something like it must be introduced later as the general argument develops so it is hoped that the reader will be patient.

The test points for mild steel show some scatter, but they still define an ellipse. Such scatter may be expected since mild steel has such an anomalous, unstable primary yielding.

The writer extended the Huber criterion to all points along the post-primary yield curve ⁽³⁾. This followed from the experimental evidence that if after yielding a simple tensile specimen is unloaded and then reloaded the secondary yield occurs at approximately the same load as that from which the specimen was unloaded. Thus we can state,

$$\sum_{\substack{i=1,\,2,\,3\\ j=2,\,3,\,1}} \{(S_{ii}-S_{jj})^2+6S_{ij}^2\}=2(X^{11})^2, \quad \ldots \ldots \quad (1.2)$$

where S_{ii}, S_{ij} are the post-yield complex stresses operating while X^{11} is the " equivalent " stress on the simple tensile curve corresponding to the complex system.

TABLE 1.1.

Simple shear yield stress $X^{0,12}=f^{0,12}\,X^{0,11}$.

Material	Parameter $f^{0,12}$		
	Experimental	Theory	
		Maximum shear stress	Energy of change of shape
Copper	0·560	0·500	$(\sqrt{3})^{-1}=0{\cdot}577$
Aluminium ..	0·564	0·500	0·577
Mild steel	0·612	0·500	0·577

It is relevant to the present paper that slight progress was made in the stress-strain problem while the purely theoretical approach considering the metal as a homogeneous lattice of atoms was pursued. Vicat in 1830 denounced the mathematicians of his day as quite ignoring experimental evidence on the behaviour of the materials discussed ⁽¹⁾. Vicat's scathing attacks led the great Saint Venant from about 1850 up to his death in 1886 to give the form to the theory of elasticity used up to the present day. He kept in mind that the experimental evidence was the thing of primary importance and the mathematical expressions must fit. The present writer endeavours to adopt this attitude of mind.

2. *A New Criterion of Yielding under a Plane Stress System.*

It is useful at this point to introduce the terminology " primary " yielding to signify the first yielding of a specimen. Similarly the yielding that occurs on reloading after unloading will be called " secondary."

The simple stress systems will be called "one-stress," a plane stress system will be a "two-stress," while the general complex stress will be a "three-stress."

2.1. *Primary Yielding under a Two-stress System.*

The tests by Taylor and Quinney were on thin-walled tubes with a longitudinal tension and a torque applied. Loads were applied in increments and removed until a permanent set was observed. A typical point on the primary yield ellipse is as in fig. 2.1, where $\bar{X}^{11}$ is the vector normal stress due to the tension and $\bar{X}^{12}$ is the vector shear stress due to the torque. A typical element of the thin wall of the tube is shown in fig. 2.2. The wall thickness is in the direction $\bar{x}^3$ of the set of unit orthogonal reference vectors $\bar{x}^i$ $(i=1, 2, 3)$. From the two-stress definition there is no stress on the plane normal to $\bar{x}^3$, and Taylor and Quinney felt that they achieved this closely. The well-known condition of rotational static equilibrium gives the complementary shear stress X^{21} arithmetically equal to the applied shear stress X^{12}. The experiments showed that each of the materials had a definite primary yield simple shear stress $X^{0,12}$. We can relate this to the simple tensile primary yield stress $X^{0,11}$ by putting

Fig. 2.1.

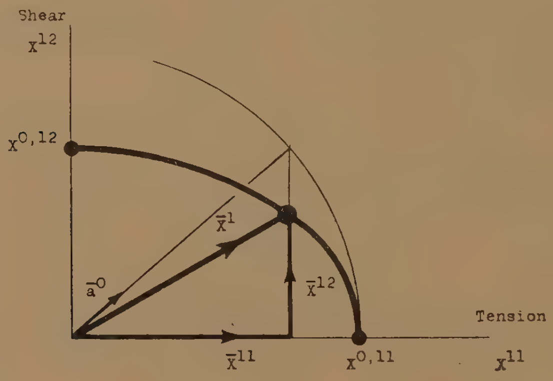
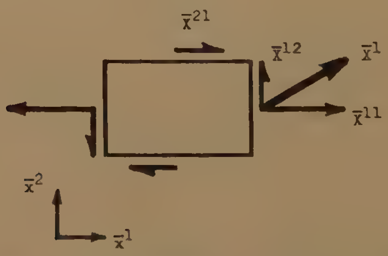


Fig. 2.2.



$$X^{0,12}=f^{0,12}\,X^{0,11}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.1)$$

where $f^{0,12}$ is a parameter.

The equation to the ellipse can be written in a linear form using vectors. Thus from fig. 2.1, with $\bar{a}^0$ in the direction of the radius of the point on the generating circle corresponding to the point on the ellipse gives

$$\frac{\bar{X}^{11}}{X^{0,11}}+\frac{\bar{X}^{12}}{X^{0,12}}=\bar{a}^0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.2)$$

The more familiar equation to the ellipse is found by forming the scalar product of each side of (2.2) with itself to give

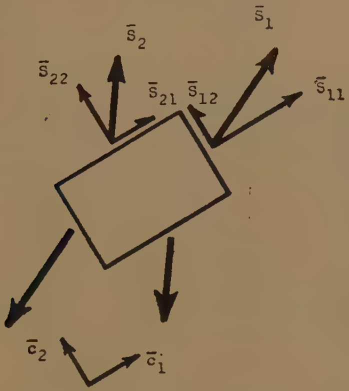
$$\left(\frac{X^{11}}{X^{0,11}}\right)^2+\left(\frac{X^{12}}{X^{0,12}}\right)^2=1.$$

Again the equation to the ellipse can be written

$$\bar{X}^1=\bar{a}^0\cdot[\bar{x}^1\bar{x}^1+f^{0,12}\,\bar{x}^2\bar{x}^2]X^{0,11}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.3)$$

where the dot indicates the scalar product of the unit vector $\bar{a}^0$ and the antecedent of each of the dyads (9). Thus *the vector stress* $\bar{X}^1$, *which is the radius vector of the ellipse and was found experimentally, can be taken as the criterion of yielding for the element* (5).

Fig. 2.3.



Now consider the two-stress system shown in fig. 2.3 with vector stresses $\bar{S}_1$, $\bar{S}_2$ operating. The vector stress $\bar{X}^i$ operating on a surface normal to the unit reference vector $\bar{x}^i$ is given by the well-known expression in terms of the stress dyadic (9). Thus,

$$\bar{X}^i=\bar{x}^i\cdot[\bar{c}_1\bar{S}_1+\bar{c}_2\bar{S}_2]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.4)$$

Now choose $\bar{x}^i$ equal to $\bar{x}^2$ in fig. 2.2 and write down the condition that the normal stress component of the stress on the face normal to this unit vector is zero. This gives

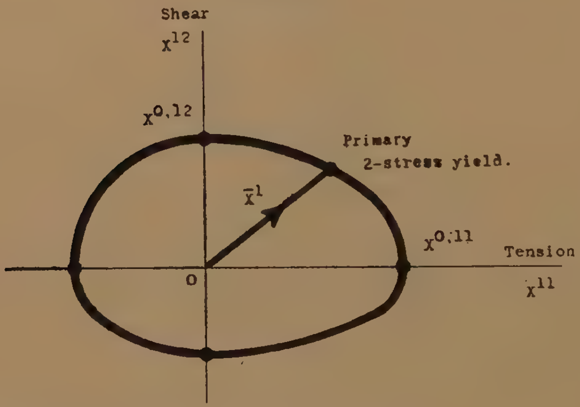
$$X^{22}=\bar{X}^2\cdot\bar{x}^2=\bar{x}^2\cdot[\bar{c}_1\bar{S}_1+\bar{c}_2\bar{S}_2]\cdot\bar{x}^2=0. \quad . \quad . \quad . \quad (2.5)$$

If the $\bar{\iota}^2$ is chosen to satisfy (2.5) then the orthogonal element defined by directions $\bar{\iota}^1$, $\bar{\iota}^2$ will be under precisely the same stress conditions as prevailed in the Taylor and Quinney tests when yield occurred. Thus as the stress system $\bar{S}_1$, $\bar{S}_2$ increases it will reach a value where the vector stress $\bar{X}^1$ reaches the value at which yield occurred in the experiments. Thus with $i=1$ in (2.4) equated to (2.3) gives at primary yield for the two-stress system,

$$\bar{x}^1 \cdot [\bar{c}_1 \bar{S}_1 + \bar{c}_2 \bar{S}_2] = \bar{\iota}^0 \cdot [\bar{x}^1 \bar{\iota}^1 + f^{0,12} \bar{\iota}^2 \bar{x}^2] X^{0,11} \quad . \quad . \quad . \quad . \quad (2.6)$$

If, however, the stress system is such that no direction $\bar{x}^2$ can be found to make $\bar{X}^{22}$ vanish, then we have the two-dimensional analogue of hydrostatic or isotropic stress under negative values, of which we have seen in paragraph 1 the metals will not yield. Thus the yield criterion embodied in equations (2.5) and (2·6) satisfy this requirement of no yielding under isotropic stress because it is excluded mathematically. However, it appears on intuitive physical grounds that yield would occur under such an isotropic two-stress since the component normal to the thin sheet is zero. If this should be so, then the value of isotropic two-stress yield would be used without invalidating the foregoing criterion.

Fig. 2.4.



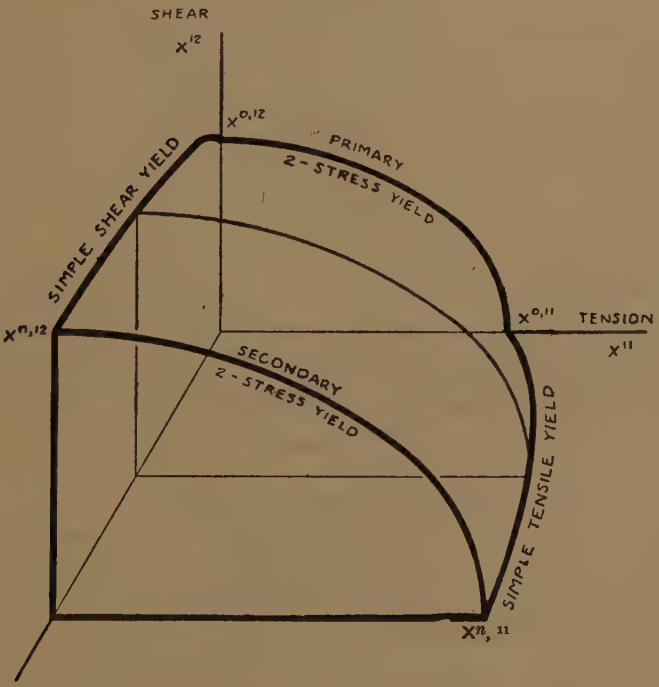
Several metals have been noted to have different yield stress values under tension or compression. Hence the positive and negative semi-axes of the ellipse are not equal in general. Again, for the same reason, one might expect that in an anisotropic metal the yield load value will not be the same when the shear stress is reversed although the normal stress X^{11} remains constant. This follows from consideration of the normal stresses " auxiliary " to the shear stresses (4), (6). Thus the complete figure for a given material will consist of four quadrants, one from each of four ellipses as in fig. 2.4. Thus the condition of isotropy is more restricted than the writer has seen stated previously. *A body is truly isotropic at a point only when the stress-strain parameters do not change their values with change of sign of the stresses.*

The suggested criterion requires no stress-strain relationships to be known as are necessary in the Huber–von Mises–Hencky theory of energy of change of shape, and the values of the parameters $f^{0,12}$, $X^{0,11}$ are experimental without appeal to hypotheses (5) and the expressions are linear in stresses.

2.2. *Secondary Yielding under a Two-stress System.*

It is convenient to have a secondary yielding criterion for use in the study of inelastic stress-strain relationships (3), (4). Thus there appears the need for another set of experiments which would define a surface of two-stress yielding shown diagrammatically in fig. 2.5 for one quadrant.

Fig. 2.5.



Thus the general relationship between the normal and shear stresses X^{11}, X^{12} inducing yielding would be shown concisely. In this more general case the equation (2.6) now becomes,

$$\bar{x}^1\cdot[\bar{c}_1\bar{S}_1+\bar{c}_2\bar{S}_2]=\bar{a}^n\cdot[\bar{x}^1\bar{x}^1+f^{n,12}\,\bar{x}^2\bar{x}^2]X^{n,11} \quad . \quad . \quad . \quad (2.7)$$

at any secondary yield point provided $\bar{x}^2$ satisfies equation (2.5).

If, as work-hardening proceeds, the parameter $f^{n,12}$ remains constant (*i.e.*, is f^{12} say), then all curves described on the surface parallel to the plane of X^{11}, X^{12} will be geometrically similar. This is the implication that the stress from the one-stress test of the material is "equivalent" to a complex stress distribution (3), as mentioned in paragraph 1, with regard to the extended Huber criterion. This restriction was not foreseen when the "equivalence" was suggested by the extended Huber criterion.

3. *A New Criterion of Yielding under a Three-dimensional Stress System.*

The yield criterion for the three-stress system $\bar{S}_1$, $\bar{S}_2$, $\bar{S}_3$ is clear from the argument for the two-stress case. Suppose, in fig. 3.1, the element is rotated about the axis defined by $\bar{c}_1$. Then some orientation can be found where S_{33} is zero. Now, about the axis defined by the new $\bar{c}_3$ direction, rotate the element until S_{22} vanishes and one is then left with the stress condition shown in fig. 3.2. This is now the simplest stress arrangement that—theoretically at least—is susceptible to experiment to *find* the values X^{11}, X^{12}, X^{13} at which the element will yield if the shear stress X^{23}, and hence also its complement X^{32}, is zero. These stress values are shown in fig. 3.3 for the first quadrant of the complete surface. The "simple" yield stresses are $X^{n,11}$, $X^{n,12}$, $X^{n,13}$, where n would be O at primary yield and it denotes the general work-hardened condition.

Fig. 3.1.

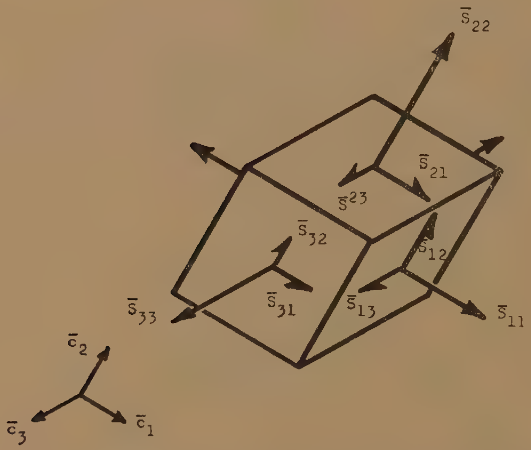
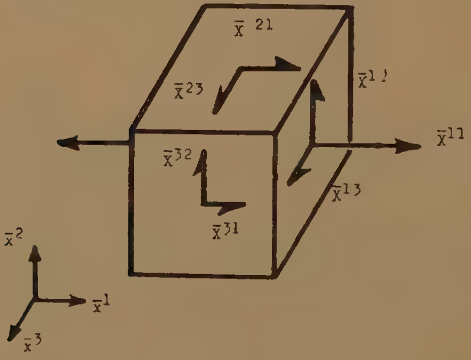

$\bar{S}_{22}$
$\bar{S}_{21}$
$\bar{S}^{23}$
$\bar{S}_{32}$
$\bar{S}_{12}$
$\bar{S}_{33}$
$\bar{S}_{31}$
$\bar{S}_{13}$
$\bar{S}_{11}$
$\bar{c}_2$
$\bar{c}_3$
$\bar{c}_1$

Fig. 3.2.


$\bar{x}^{21}$
$\bar{x}^{23}$
$\bar{x}^{12}$
$\bar{x}^{11}$
$\bar{x}^{32}$
$\bar{x}^{13}$
$\bar{x}^{31}$
$\bar{x}^2$
$\bar{x}^1$
$\bar{x}^3$

The ellipsoidal surface defined by such a system leads to an equation similar to that in (2.7),

$$\bar{x}^1\cdot[\bar{c}_1\bar{S}_1+\bar{c}_2\bar{S}_2+\bar{c}_3\bar{S}_3]=\bar{a}^n\cdot[\bar{x}^1\bar{x}^1+f^{n,12}\,\bar{x}^2\bar{x}^2+f^{n,13}\,\bar{x}^3\bar{x}^3]X^{n,11}, \quad (3.1)$$

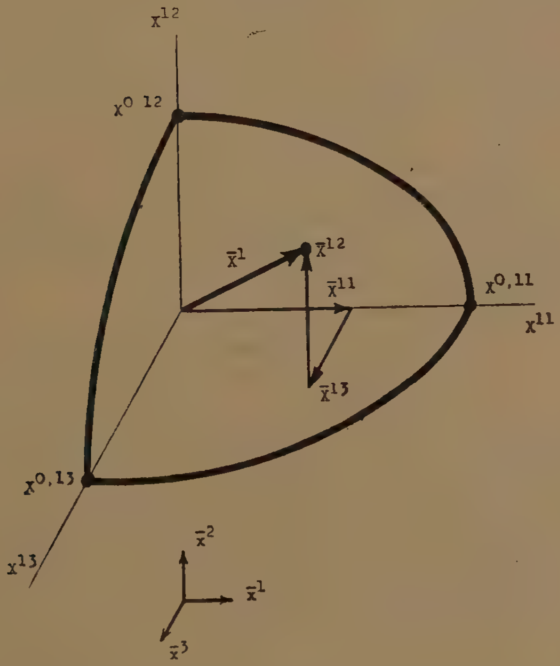
provided that, similar to equation (2.5),

$$\left.\begin{aligned} X^{22}&=\bar{x}^2\cdot[\bar{c}_1\bar{S}_1+\bar{c}_2\bar{S}_2+\bar{c}_3\bar{S}_3]\cdot\bar{x}^2=0\\ X^{33}&=\bar{x}^3\cdot[\bar{c}_1\bar{S}_1+\bar{c}_2\bar{S}_2+\bar{c}_3\bar{S}_3]\cdot\bar{x}^3=0\end{aligned}\right\}. \quad . \quad . \quad . \quad (3.2)$$

Direction $\bar{a}^n$ is now that of the radius vector to the generating spherical surface. Again the ellipsoid can be stated as

$$\frac{\bar{X}^{11}}{X^{n,11}}+\frac{\bar{X}^{12}}{X^{n,12}}+\frac{\bar{X}^{13}}{X^{n,13}}=\bar{a}^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.3)$$

Fig. 3.3.



Again (3.2) cannot be satisfied when a hydrostatic or isotropic stress operates, so that the experimental evidence about yield under such a system is recognized in the theoretical statement of equations (3.1) and (3.2).

If, however, X^{23} is not zero then a further surface of yielding must be defined. Plot the values at yield of X^{11}, modulus $(\bar{X}^{12}+\bar{X}^{13})$ and X^{23} as orthogonal co-ordinates in the directions $\bar{x}^1$, $\bar{x}^2$, $\bar{x}^3$ repectively. If X^{11} and mod. $(\bar{X}^{12}+\bar{X}^{13})$ should each be put zero in turn then the curves connecting the other two stresses at yield should, on test, prove to be ellipses or approximately this shape. Hence, if all three stresses operate simultaneously an ellipsoid should be defined. Hence we can write from the same reasoning as that which gave (3.1),

$$\bar{x}^1\cdot\left(\sum_{i=1}^{3}\bar{c}_i\bar{S}_i\right)\cdot\bar{x}^1\bar{x}^1+\bar{x}^2\cdot\left(\sum_{i=1}^{3}\bar{c}_i\bar{S}_i\right)\cdot\bar{x}^3\bar{x}^3$$

$$+\left\{\text{mod}\left[\bar{x}^1\cdot\left(\sum_{i=1}^{3}c_i\bar{S}_i\right)\cdot(\bar{x}^2\bar{x}^2+\bar{x}^3\bar{x}^3)\right]\right\}\bar{x}^2$$

$$=\bar{a}^n\cdot(\bar{x}^1\bar{x}^1+f^{n,\,123}\bar{x}^2\bar{x}^2+f^{n,\,23}\bar{x}^3\bar{x}^3)X^{n,\,11}, \quad . \quad (3.3)$$

where now $\bar{a}^n$ is the unit vector for the ancillary ellipsoid generating sphere, $f^{n,\,123}$ is the ratio between the normal one-stress yield $X^{n,\,11}$ and the value of mod. $(\bar{X}^{12}+\bar{X}^{13})$ for yield under these shears alone, while $f^{n,\,23}$ is the ratio between $X^{n,\,11}$ and the simple shear yield $X^{n,\,23}$. Thus the general test is to satisfy (3.2) and then, if X^{23} is not zero one must satisfy (3.3). In anisotropic metal the ratio $f^{n,\,123}$ will take various values depending on the ratio between the components of $(\bar{X}^{12}+\bar{X}^{13})$ at yield under this system only.

In an isotropic metal $f^{n,\,12}$ equals $f^{n,\,13}$ (*i. e.*, equals f^n say) and the quadrant of fig. 3.3 is then from a surface of revolution about the $\bar{x}^1$ axis. Hence mod $(\bar{X}^{12}+\bar{X}^{13})$ is constant and so in (3.3) $f^{n,\,123}$ is constant and becomes independent of directions as does $f^{n,\,23}$ also in an isotropic metal. Thus the ellipsoidal surface for (3.3) also becomes one of revolution about $\bar{x}^1$. Thus for an isotropic metal the Taylor & Quinney two-stress tests will apply to the three-stress system whether one uses equation (3.1) or the more general (3.3).

4. *Conclusions and Future Developments.*

The criterion of yielding that has been suggested appears to have the minimum of appeal to hypothesis and at each step to satisfy the available experimental evidence. Thus the evidence that a metal will not yield under great hydrostatic pressure is incorporated. This evidence on isotropic pressure invalidated so many of the older theories. Again, some of the theories incorporated the non-yielding under isotropic stress but fail to give the correct ratio between the simple shear yield stress and the simple tension yield stress, whereas it is of the correct value in the present theory because it is accepted from the experimental evidence. Thus the maximum shear theory fails, and while the theory on energy of change of shape is close for some metals it appears difficult to adjust it for the anomalous case of mild steel for example. Again, the Huber criterion cannot take account of the effect seen as a change in the length of the semi-axes of the ellipsoid with change of sign of stresses.

The two-stress system resulting from the tension and simple shear can be used as the yield criterion for any plane stress system so long as tests including the effects of reversal of stresses are included. The tension-shear two-stress test can give the yield criterion for a three-stress system when the metals are isotropic.

Further tests are required to give the ellipses for secondary yielding to supplement the evidence available on primary yielding of a two-stress system. A still larger programme is required to investigate the effects of a three-stress system on the yielding of anisotropic metals.

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Notation.

$\bar{c}$, $\bar{x}$, unit vectors of an orthogonal reference system.

1, 2, 3, as subscripts and indices denote orthogonal directions.

$\bar{\mathrm{S}}$, $\bar{\mathrm{X}}$, stress vectors with suitable subscripts and indices to denote direction(s). *E. g.*, $\bar{\mathrm{S}}_1$ a vector stress on the face of an element orthogonal to direction $\bar{c}_1$. $\bar{\mathrm{S}}_{12}$ the component of $\bar{\mathrm{S}}_1$ in the direction $\bar{c}_2$.

S, X, scalar moduli of $\bar{\mathrm{S}}$, $\bar{\mathrm{X}}$.

O, n, as subscripts and indices denote primary and generalized secondary yielding conditions respectively.

f, non-dimensional parameter defined as the ratio of the yield simple shear stress to the yield simple tensile stress.

$\bar{a}$, unit vector in the direction of the radius vector to the point on the generating circle or sphere corresponding to the point on the ellipse or ellipsoid.

XIX. *The Logarithmic Solutions of Linear Differential Equations.*

By N. G. DE BRUIJN, D.Sc., Natuurkundig Laboratorium, N. V. Philips' Gloeilampenfabrieken, Eindhoven (Nederland) *.

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SOME years ago Balth. van der Pol † found the following expression for $x^{\frac{1}{2}n}$ K_n $(2\sqrt{x})$, where n is an integer ($n \geqslant 0$), and K_n is the Bessel function of the second kind ‡ with imaginary argument

$$x^{\frac{1}{2}n}K_n(2\sqrt{x}) = -\tfrac{1}{2}\sum_{m=0}^{\infty}\frac{d}{dm}\left(\frac{x^m}{\Pi(m)\Pi(m-n)}\right) \quad (x>0), \quad . \quad . \quad (1)$$

which is in striking resemblance to the following formula for the function of the first kind :

$$x^{\frac{1}{2}n}I_n(2\sqrt{x}) = \sum_{m=0}^{\infty}\frac{x^m}{\Pi(m)\Pi(m-n)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Formula (1) is a very short notation for the expansion usually given :

$$K_n(2\sqrt{x}) = \tfrac{1}{2}\sum_{\nu=0}^{n-1}\frac{(-1)^{\nu+n}}{\nu\,!}(n-\nu+1)\,!\;x^{\nu}$$

$$-\tfrac{1}{2}\sum_{\nu=0}^{\infty}\frac{x^{\nu}}{\nu\,!\;(n+\nu)\,!}\,[\log x-\psi(\nu+1)-\psi(\nu+n+1)],$$

where $$\psi(u)=\Gamma'(u)/\Gamma(u).$$

Prof. van der Pol also gave a formula for $e^x\,\mathrm{li}\,(e^{-x})$ (see formula (16) below), in similar analogy with the power series for e^x. He obtained these results with the aid of operational calculus, but since K_m and I_m satisfy a single second-order linear differential equation, and the same can be said concerning $e^x\,\mathrm{li}\,(e^{-x})$ and e^x, we tried to give proofs for his formulæ, starting from the corresponding differential equations. This turned out to be possible, leading to a more general process for finding logarithmic solutions of linear differential equations, which shows some resemblance to a well-known method of Frobenius §. The latter, however, does not lead directly to simple expressions like (1).

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* Communicated by Prof. B. van der Pol.

† *Wiskundige Opgaven met de oplossingen v. h. Wiskundig Genootschap*, Amsterdam, xviii. p. 96 (1943), problem 34.

‡ See for its definition : Whittaker and Watson, 'Modern Analysis,' 4th ed. p. 374. We notice that the definition of $K_n(x)$ in Watson, 'Bessel Functions,' differs from this one by a factor $(-1)^n$.

§ *Journ. reine angew. Math.* lxxvi. pp. 214–235 (1873). Frobenius' method is discussed in most textbooks on differential equations.

Frobenius' method furnishes all solutions of a linear differential equation

$$A_k(x)y^{(k)}+A_{k-1}(x)y^{(k-1)}+\ldots+A_1(x)y'+A_0(x)y=0, \quad . \quad . \quad (3)$$

provided that $x=0$ be a regular point for this equation, *i.e.*, $x^{k-\nu}A_\nu(x)A_k^{-1}(x)$ regular at $x=0(\nu=0, 1, \ldots\ldots, k)$. The solutions are obtained as series of the type

$$\sum_{m=0}^{\infty} P_m(\log x)x^{m+\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where ρ (the *exponent*), is a real or complex number, and the P_m are polynomials (of degree $<k$). The solutions, for which all the P_m are of zero degree, are always easy to construct, but the difficulties lie in the logarithmic solutions.

In connection with van der Pol's formulæ we introduce another process for finding logarithmic solutions. This process, however, does not necessarily give *all* solutions of a given equation, but it has the advantage of being easy to explain and to handle, and to furnish simple expressions for the logarithmic solutions in many applications. But it is of no use for the construction of a complete system of linearly independent solutions in the general case, which is the primary aim of Frobenius' method.

We have to restrict ourselves by the assumption that all coefficients $A_\nu(x)(\nu=0, 1, \ldots., k)$ be polynomials, but on the other hand we may omit the conditions about the regularity of

$$x^{k-\nu}A_\nu(x)A_k^{-1}(x) \text{ for } x=0, \nu=0, 1, \ldots., k.$$

We call a series $$F \sim \sum_{m=-\infty}^{\infty} a(m)x^{m+\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

a *formal solution* of (3), when, on substituting F into (3), taking for F′, F″, ... the formal successive derivatives

$$F^{(\nu)} \sim \sum_{m=-\infty}^{\infty} a(m)(m+\rho)(m+\rho-1)\ldots(m+\rho-\nu+1)x^{m+\rho-\nu},$$

of F, the left-hand side of (3) becomes identically zero. That is to say, on taking together terms with equal powers of x, after formal multiplication by the polynomial coefficients $A_\nu(x)$, every power of x obtains the coefficient zero. For any integer $m=\mu$, the coefficient of $x^{\mu+\rho}$ becomes a linear combination of a finite number of $a(m)$'s. The condition that this expression vanishes forms the so-called *recurrence relation.*

Now suppose that a function $a(t)$ be defined and differentiable for all real values of t, and that

$$F \sim \sum_{m=-\infty}^{\infty} a(m+t)x^{m+t+\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

be a formal solution of (3) for any value of t. Then it will be clear that the derivative with respect to t for $t=t_0$,

$$\left(\frac{\partial F}{\partial t}\right)_{t=t_0} \sim \sum_{m=-\infty}^{\infty} \frac{\partial}{\partial t}\{a(m+t)x^{m+t+\rho}\}_{t=t_0}, \quad . \quad . \quad . \quad . \quad (7)$$

or in abbreviated notation

$$\sum_{-\infty}^{\infty} \frac{\partial}{\partial m}\{a(m+t_0)x^{m+t_0+\rho}\},$$

is also a formal solution of (3). For, if we substitute (7) into (3), and take together all terms showing the same power $x^{\mu+t_0+\rho}$, we get an expression, identical with the partial derivative of the expression, obtained in the same way for the series (6). The latter [*], being equal to zero, identically in t and containing only a finite number of terms, the former vanishes as well.

If a formal solution (5) is convergent for all x of any open region of the complex x-plane, it yields a solution in the ordinary sense. The same holds true for formal solutions of the type $\sum_{-\infty}^{\infty} P_m(\log x)x^{m+\rho}$, where the P_m are polynomials of degree $<k$ [†]. In both cases, namely, convergence in any open region includes absolute convergence of all formal derivatives in the same region. Henceforth we will use the term *actual solution* for any solution in the ordinary sense.

Now the importance of the expression (7) arises from the possibility that, although (6) cannot furnish actual solutions for all values of t (the equation (3) admitting only a finite number of exponents), nevertheless (7) may form an actual solution for some value of t_0.

The series (7) can be split into two parts (without loss of generality we take $t_0=0$) :

$$\left(\frac{\partial F}{\partial t}\right)_{t=0} \sim \sum_{-\infty}^{\infty} a'(m)x^{m+\rho}+\log x \sum_{-\infty}^{\infty} a(m)x^{m+\rho}, \quad . \quad . \quad . \quad (8)$$

and, if we suppose the series (7) to be convergent for all x in a certain region, it is easy to prove that both series in (8) are convergent in that region. The two series in (8) being Laurent series, the region of convergence of the series (7), if existing at all, will be a range [‡] $\alpha<|x|<\beta$, where $0\leqslant\alpha<\beta\leqslant\infty$.

We notice that the second series in (8) is the same as the series (6) for $t=0$, whose range of convergence thus includes that of (7) for $t_0=0$.

The proposed process can thus be described as follows :—construct a formal solution with a parameter t, of the type (6), and examine whether its formal derivative with respect to t, for $t=0$, has a region of convergence. If that is the case, we have obtained an actual solution. This is a logarithmic one, unless $a(m)=0$ for all integers m (which by no means implies the vanishing of the $a'(m)$).

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[*] This is the left-hand side of the recurrence relation, multiplied by the corresponding power of x.

[†] For $\log x$ we may take any branch of the multi-valued logarithmic function.

[‡] Not mentioning possible convergence in boundary points.

Sometimes it may be useful to repeat this process, and thus take the second formal derivative of (6), and so on. Another generalization arises from the remark, that it is only necessary for (6) to be a formal solution for $-\epsilon<t<\epsilon(\epsilon>0)$. In that case, we need not define $a(t)$ outside the set of intervals $|t-m|<\epsilon(m=0, \pm 1, \pm 2, \ldots)$. Moreover, $a(t)$ only needs to be differentiable at integral values of t.

In applications, the following trivial remark may be useful. If $f(t)$ is periodic with period 1, and (6) forms a formal solution, then the series

$$\sum_{-\infty}^{\infty} a(m+t)f(m+t)x^{m+t+\rho}$$

will be a formal solution as well.

EXAMPLES.

1. *Bessel's equation.*

$$x^2y''+xy'+(x^2-n^2)y=0, \quad \ldots \ldots \quad (9)$$

where n is an integer $(n \geqslant 0)$. Substituting (6) with $\rho=0$ into (9), we obtain the recurrence relation

$$(t^2-n^2)a(t)+a(t-2)=0,$$

a solution of which is $$a(t)=\frac{2^{-t}\cos\frac{\pi}{2}(t-n)}{\Pi\left(\frac{t+n}{2}\right)\Pi\left(\frac{t-n}{2}\right)}.$$

For $t=0$ the series (6) becomes

$$\sum_{-\infty}^{\infty} a(m)x^m=\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu\,!\,(\nu+n)\,!}\left(\frac{x}{2}\right)^{n+2\nu},$$

the well-known expansion of $J_n(x)$.

Partial differentiation of $\sum\limits_{-\infty}^{\infty} a(m+t)x^{m+t}$ with respect to t does not lead, for $t=0$, to a convergent expression, owing to the appearance of terms of the type

$$-\frac{\pi}{2}\frac{2^{-m}\sin\frac{\pi}{2}(m-n)\,x^m}{\Pi\left(\frac{m+n}{2}\right)\Pi\left(\frac{m-n}{2}\right)},$$

the absolute value of which tends very rapidly to infinity when m runs to $-\infty$ through negative integers $\equiv n+1 \pmod{2}$. Therefore we first multiply $a(t)$ by $\cos^2\frac{\pi}{2}(t-n)$. Multiplication by a function of period 2 is allowed here since the difference of the arguments occurring in the recurrence relation is even. The derivative of $\cos^3 \frac{\pi}{2}(t-n)$ vanishing for all integral values of t, the series (7) now becomes ($t=0$) :

$$\sum_{m=-\infty}^{\infty} \cos^3 \frac{\pi}{2}(m-n) \frac{d}{dm} \left\{ \frac{(\frac{1}{2}x)^m}{\Pi\left(\frac{m+n}{2}\right)\Pi\left(\frac{m-n}{2}\right)} \right\}$$
$$= \tfrac{1}{2} \sum_{\nu=-\infty}^{\infty} (-1)^\nu \frac{d}{d\nu} \left\{ \frac{(\frac{1}{2}x)^{n+2\nu}}{\Pi(\nu)\Pi(\nu+n)} \right\}.$$

Because this series contains only powers of x with exponents $\geqslant -n$, its convergence for all $x \neq 0$ is easy to establish. Thus we have found an actual solution of (9). Evaluation of a few coefficients shows that it is $\frac{\pi}{2} Y_n(x) = \frac{1}{2}\mathbf{Y}(x)$, where Y_n and $\mathbf{Y}_n$ denote the Bessel functions of the second kind with the notation of Hankel and Weber, respectively *.

Our formulæ for $J_n(x)$ and $\mathbf{Y}_n(x)$ show a similar resemblance as van der Pol's formulæ (1) and (2) :

$$J_n(x) = \sum_{\nu=-\infty}^{\infty} (-1)^\nu \frac{(\frac{1}{2}x)^{n+2\nu}}{\Pi(\nu)\Pi(\nu+n)}$$
$$\mathbf{Y}_n(x) = \sum_{\nu=-\infty}^{\infty} (-1)^\nu \frac{d}{d\nu} \left\{ \frac{(\frac{1}{2}x)^{n+2\nu}}{\Pi(\nu)\Pi(\nu+n)} \right\}. \quad . \quad . \quad . \quad . \quad (10)$$

The expression for $\mathbf{Y}_n(x)$, usually given †, is

$$\mathbf{Y}_n(x) = -\sum_{\nu=0}^{n-1} \frac{(n-\nu-1)\,!}{\nu\,!} \left(\frac{x}{2}\right)^{-n+2\nu} +$$
$$+ \sum_{\nu=0}^{\infty} \frac{(-1)^\nu (\frac{1}{2}x)^{n+2\nu}}{\nu\,!\ (\nu+n)\,!} \{2 \log (\tfrac{1}{2}x) - \psi(\nu+1) - \psi(\nu+n+1)\},$$

where $\psi(u) = \Gamma'(u)/\Gamma(u)$. The agreement with (10) is easily verified. Van der Pol's result (1) can be derived from (10), but of course it can also be proved independently in a similar manner.

2. *Legendre's equation.*

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{du}{d\mu}\right\} + n(n+1)u = 0, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where n is a non-negative integer. We transform it by

$$\mu = \frac{1+x}{1-x}, \quad u = (1-x)^{-n} y$$

into

$$1-x)\left(x\frac{d}{dx}\right)^2 y + 2nx\left(x\frac{d}{dx}\right) y - xn^2 y = 0,$$

whose recurrence relation reads ($\rho=0$) :

$$t^2 a(t) - (n-t+1)^2 a(t-1) = 0.$$

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* See Whittaker and Watson, 'Modern Analysis,' 4th ed. p. 372 ; Watson, 'Bessel Functions,' 2nd ed. pp. 58, 64 (1944).

† Watson, 'Bessel Functions,' p. 62.

A solution is $$a(t)=\left\{\frac{\Pi(n)}{\Pi(n-t)\Pi(t)}\right\}^2=\binom{n}{t}^2.$$

We have

$a(t)=a'(t)=0$ for $t=n+1,\ n+2,\ n+3,\ldots$ and for $t=-1,\ -2,\ldots.$ Hence

$$\left(\frac{1+\mu}{2}\right)^n\sum_{m=0}^{n}\binom{n}{m}^2\left(\frac{\mu-1}{\mu+1}\right)^m$$

and

$$\left(\frac{1+\mu}{2}\right)^n\sum_{m=0}^{n}\frac{d}{dm}\left\{\binom{n}{m}^2\left(\frac{\mu-1}{\mu+1}\right)^m\right\}$$

are solutions of (11), and so they are linear combinations of $P_n(\mu)$ and $Q_n(\mu)$ *. Simple considerations concerning the behaviour at $\mu=1,\ -1$ and ∞ show them to be equal to $P_n(\mu)$ and $-2Q_n(\mu)$, respectively. We have thus obtained

$$Q_n(\mu)=-\tfrac{1}{2}\left(\frac{1+\mu}{2}\right)^n\sum_{m=0}^{n}\frac{d}{dm}\left\{\binom{n}{m}^2\left(\frac{\mu-1}{\mu+1}\right)^m\right\}.$$

3. The equation $$xy''+(1-x)y'-y=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

has the solutions

$$y=e^x \quad\text{and}\quad y=e^{-x}\,\mathrm{li}(e^{-x}) \quad \left(-\mathrm{li}(e^{-x})=\int_x^\infty e^{-s}\frac{ds}{s}\quad\text{for}\quad x>0\right).$$

The recurrence relation reads ($\rho=0$)

$$t^2a(t)-ta(t-1)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

a solution of which is $$a(t)=\frac{1}{\Pi(t)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Thus for $t=0$ the series (6) becomes the power series for e^x. Our differentiation process now yields the formal solution

$$\sum_{-\infty}^{\infty}\frac{d}{dm}\left(\frac{x^m}{\Pi(m)}\right)=\log x\ .\sum_{0}^{\infty}\frac{x^m}{m!}+\sum_{-\infty}^{\infty}x^m\frac{d}{dm}\left(\frac{1}{\Pi(m)}\right). \quad . \quad (15)$$

This result is nowhere convergent, and we cannot make it convergent by multiplying $a(t)$ by suitable functions. Thus our method fails. But in this case we are able to show that (15) remains a formal solution of (12) after omitting all terms with $m<0$. For, (13) gives

$$2ta(t)-a(t-1)+t^2a'(t)-ta'(t-1)=0,$$

hence it follows from (14) that

$$t^2a'(t)-ta'(t-1)=0 \quad\text{for}\quad t=-1,-2,\ldots$$

Taking $b(m)=a'(m)$ for $m=-1,\ -2,\ -3,\ \ldots,$ and $b(m)=0$ for $m=0,\ 1,\ 2,\ \ldots,$ our $b(m)$ now satisfies $t^2b(t)-tb(t-1)=0$ for all integers t.

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* See for its definition: Hobson, 'Spherical Harmonics,' p. 51 (1931).

Hence $\sum_{-\infty}^{\infty} b(m)x^m$ is a formal solution of (12). Subtracting the latter from (15), we obtain the formal solution

$$\sum_{0}^{\infty} \frac{d}{dm}\left(\frac{x^m}{\Pi(m)}\right).$$

This series being convergent for all $x \neq 0$, an actual solution of (12) is found. Evaluation of a few coefficients shows that

$$\sum_{0}^{\infty} \frac{d}{dm}\left(\frac{x^m}{\Pi(m)}\right) = e^x \operatorname{li}(e^{-x}), \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

which is van der Pol's result *.

Eindhoven, August 1946.

XX. *A Diffusion Problem in which the Amount of Diffusing Substance is Finite.—II. Diffusion with Non-Linear Adsorption.*

By J. Crank, M.Sc.,
Courtaulds Ltd., Maidenhead, Berks †.

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§ 1. *Introduction.*

This is the second of a series of papers concerning the mathematical treatment of some diffusion problems associated with the commercial dye-bath. One essential feature of the commercial dye-bath is its finiteness, which means that the concentration of dye in the bath is not constant but decreases as dyeing proceeds. Furthermore, it is generally considered that dyeing is best represented by a process of diffusion with adsorption, in which the dye molecules diffuse along capillaries within the fibre and are deposited on the walls of the capillaries. In Paper I (1) of the series, formal mathematical solutions of the appropriate diffusion equations are obtained subject to the condition of a finite amount of diffusing substance, for cases in which linear relations define the rate of deposition on the capillary walls. Two methods of solution are used depending on whether the rate of deposition is comparable with, or is very much greater than, the rate of diffusion. The present paper describes a method of evaluating approximate numerical solutions when deposition is rapid compared with diffusion but the relation between the concentrations of deposited and diffusing dye is non-linear. Thus the method is applicable to adsorption isotherms of the Freundlich or Langmuir types or to even more complicated isotherms if necessary.

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* *Wiskundige Opgaven met de Oplossingen v. h. Wiskundig Genootschap*, Amsterdam, xviii. (1) p. 90 (1943), problem 33.

† Communicated by A. H. Wilson, F.R.S.

A previous paper (2) described a convenient method of evaluating numerical solutions of an equation of the type

$$\frac{\partial C}{\partial t}=\frac{\partial^2 C}{\partial x^2}-\frac{\partial S}{\partial t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which S is a non-linear function of C. With a suitable choice of variables, (1) is the equation representing the diffusion of solute into an infinite sheet of uniform material, the solute being deposited on the capillary walls at a rate much greater than that of diffusion. S is the amount of solute deposited per unit overall volume and C is the concentration of solute free to diffuse. To solve the present problem for a plane sheet, it is only necessary to modify the numerical process discussed in reference (2) by inserting the boundary condition expressing the finiteness of the amount of solute. In the present paper the same method is applied to evaluate solutions of the corresponding equation for radial diffusion into a cylinder of radius a. It will be clear that the way of introducing the condition of a finite amount of solute in the radial case applies equally well to the plane sheet case.

§ 2. *Reduced Variables.*

Suppose a circular cylinder of infinite length is placed in a solution and the solute is allowed to diffuse radially into the cylinder, and is deposited in the capillaries of the cylinder. The concentration of solute in the solution is always uniform and is initially C_0, while the cylinder is initially free from solute. Consider the cylinder to occupy the space $r \leqslant a$ (using cylindrical coordinates), while the cross-section of the bath in which it is immersed is A, excluding the space occupied by the cylinder. Considering unit length of the cylinder (and of the bath) the equation which determines the diffusion is

$$\frac{\partial C}{\partial t}=\frac{\kappa}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right)-\frac{\partial S}{\partial t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where S is a known function of C. The coefficient of diffusion κ is assumed constant, that is, independent of C, r, t. The boundary condition on $r=a$ is

$$AC+\int_0^a 2\pi r(C+S)dr=AC_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the initial condition at $t=0$,

$$C=0 \quad (0\leqslant r\leqslant a). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

On introducing new variables $c=C/C_0$, $s=S/C_0$, $\rho=r/a$, $\tau=\kappa t/a^2$, (2), (3) and (4) reduce to

$$\frac{\partial c}{\partial \tau}=\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial c}{\partial \rho}\right)-\frac{\partial s}{\partial \tau}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\frac{Ac}{\pi a^2}+\int_0^1 2\rho(c+s)d\rho=\frac{A}{\pi a^2}, \ (\rho=1), \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$c=0 \text{ at } \tau=0 \qquad (0\leqslant\rho\leqslant 1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

§ 3. *Finite Difference Ratios.*

The derivatives with respect to ρ and τ in equation (5) are to be replaced in a particular way by finite difference ratios. The range in ρ is divided into p equal intervals $\delta\rho$, that is, the cylinder is thought of as divided into concentric shells each of thickness $\delta\rho$. If c_{m-1}, c_m and c_{m+1} are concentrations at time τ at the points $(m-1)\delta\rho$, $m\delta\rho$ and $(m+1)\delta\rho$ respectively, then at the point $m\delta\rho$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial c}{\partial\rho}\right)=\frac{\partial^2 c}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial c}{\partial\rho}$$

$$=\frac{1}{2m(\delta\rho)^2}\left\{(2m+1)c_{m+1}-4mc_m+(2m-1)c_{m-1}\right\} \qquad (8)$$

approximately, for $m\neq 0$, and

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial c}{\partial\rho}\right)=\frac{4(c_1-c_0)}{(\delta\rho)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

approximately for $m=0$. Here c_1 is the concentration at the point $\delta\rho$. Considering a time interval $\delta\tau$ and writing c'_{m-1}, c'_m, c'_{m+1} for concentrations at $\tau+\delta\tau$, the relation corresponding to (8) at time $\tau+\frac{1}{2}\delta\tau$ becomes

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial c}{\partial\rho}\right)=\frac{1}{4m(\delta\rho)^2}\{(2m+1)(c_{m+1}+c'_{m+1})$$

$$-4m(c_m+c'_m)+(2m-1)(c_{m-1}+c'_{m-1})\}. \quad . \quad . \quad . \qquad (10)$$

But at $\tau+\frac{1}{2}\delta\tau$ at the point $m\delta\rho$

$$\frac{\partial c}{\partial\tau}=\frac{c'_m-c_m}{\delta\tau}, \qquad \frac{\partial s}{\partial\tau}=\frac{s'_m-s_m}{\delta\tau}$$

approximately. Equation (5) may therefore be replaced by the approximate finite difference form

$$c'_m=c_m+\frac{\delta\tau}{4m(\delta\rho)^2}\{(2m+1)(c_{m+1}+c'_{m+1})-$$

$$4m(c_m+c'_m)+(2m-1)(c_{m-1}+c'_{m-1})\}-(s'_m-s_m), \quad . \qquad (11)$$

and, for $m=0$,

$$c_0'=c_0+\frac{2\delta\tau}{(\delta\rho)^2}\{(c_1'+c_1)-(c_0'+c_0)\}-(s_0'-s_0). \quad . \quad . \quad . \qquad (12)$$

Equations (11) for different m in the range $0<m<p$, form a set of simultaneous algebraic equations for c'_m as a function of m, given c_m and s_m for each value of m. c_p is given by the boundary condition (6). The equations may be solved by an iterative process applied at successive times so that the solution proceeds in steps $\delta\tau$. In the scheme of computation adopted, a table is drawn up having a row for each step in τ and a column for each step in ρ. Values of c and s are recorded in each column.

The method is independent of the particular form of the relation between s and c, provided this is known. Assume that the solution has proceeded as far as the n^{th} step in τ, that is $\tau=n\delta\tau$, and it is desired to evaluate the $(n+1)^{\text{th}}$ step. The following operations are performed successively :—

(*a*) Estimate values of $c'_p, c'_{p-1}, \ldots, c'_m, \ldots, c_0'$, by inspection of previous differences in each column.

(*b*) Obtain $s'_p, s'_{p-1}, \ldots, s'_m, \ldots, s_0'$, using the known relation between s and c.

(*c*) Evaluate s'_{p-1} by substituting known and estimated values in the equation of the set (11) for which $m=p-1$.

(*d*) If this calculated c'_{p-1} differs from the value estimated in (*a*), adjust s'_{p-1} and recalculate c'_{p-1}. Repeat until calculated and estimated values agree.

(*e*) Solve the equation for c'_{p-2} and iterate if the new value differs from the estimated value, adjusting s'_{p-2} accordingly.

(*f*) Evaluate, by solving the appropriate equations iteratively, $c'_{p-3}, \ldots, c'_m, \ldots, c_0'$, remembering that an alteration in c'_m may necessitate resolving the equations for c'_{m-1} and c'_{m+1}.

(*g*) When the whole line has been evaluated in this way, $\int_0^1 (c+s)\rho d\rho$ is readily calculated by using one of the well-known formulæ for numerical integration. For example, Simpson's one-third rule gives

$$\int_0^1 (c+s)\rho d\rho=\tfrac{1}{3}(\delta\rho)\{p(c'_p+s'_p)+4(p-1)(c'_{p-1}+s'_{p-1})$$
$$+2(p-2)(c'_{p-2}+s'_{p-2})+\ldots+4(c_1'+s_1')\}. \quad . \quad . \quad (13)$$

(*h*) Insert $\int_0^1 (c+s)\rho d\rho$ in the boundary condition (6) to calculate c_p, that is c at $\rho=1$.

(*i*) If the value of c_p calculated in (*h*) differs from the value estimated in (*a*), steps (*b*), (*c*), etc., will have to be repeated till all values are consistent.

The iteration of a whole line may appear prohibitive but in practice the method has proved satisfactory and easy to carry out. Clearly the time required to evaluate a particular solution depends on the number of steps in ρ and τ taken to cover a given range, which in turn depends on the accuracy required.

For cases in which s is greater than c it is desirable to re-write (11) in the form

$$s'_m=s_m+\frac{\delta\tau}{4m(\delta\rho)^2}\{(2m+1)(c_{m+1}+c'_{m+1})$$
$$-4m(c_m+c'_m)+(2m-1)(c_{m-1}+c'_{m-1})\}-(c'_m-c_m). \quad . \quad . \quad (14)$$

Step (*c*) of the process of solution will now read :—

Evaluate s'_{p-1} by substituting known and estimated values in the equation of set (14) for which $m=p-1$.

Corresponding adjustments will arise at each stage, except that the evaluation of c_p in steps (*g*) and (*h*) is unchanged. This modification will apply whenever s is greater than c, and in some cases it may be sufficiently accurate to neglect the term $\partial c/\partial \tau$ completely and to write (5) in the form

$$\frac{\partial s}{\partial \tau}=\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial c}{\partial \rho}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

with a corresponding simplification of the numerical processes of solution.

§ 4. *Numerical Example.*

As an example of the use of the method of this paper a number of solutions, which are useful in attempting to interpret the results of dyeing experiments, has been obtained. Subject to certain simplifying assumptions, the equations set out above may be taken to represent the absorption of dye by a circular cross-section fibre. Experimental data on direct dyes suggest that the relation between s and c, over a limited range of values, is frequently of the form

$$s=\mathrm{R}c^m \quad (\mathrm{R}\geqslant 0, \quad 0\leqslant m\leqslant 1), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)^*$$

where R and m are constants for given conditions of dyeing. Numerical solutions have been obtained for the particular cases in which $A=25\pi a^2$, $m=1{\cdot}0$, $0{\cdot}5$, $0{\cdot}25$, $0{\cdot}0$, and the value of R in each case is chosen so that in the final equilibrium state 90 per cent. of the dye initially in the bath has entered the fibre. The values of R are readily calculated as follows.

If c_∞, s_∞ represent the values of c and s in the equilibrium state, attained theoretically after infinite time, then

$$\pi a^2(c_\infty+s_\infty)+Ac_\infty=\mathrm{A}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Substitution for s from (16) leads immediately to an expression for R, namely

$$\mathrm{R}=\{\mathrm{A}(1-c_\infty)-\pi a^2c_\infty\}/\pi a^2c_\infty^m. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

If M_∞ is the total amount of dye in the fibre at equilibrium, then

$$\mathrm{M}_\infty+\mathrm{AC}_\infty=\mathrm{AC}_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

so that

$$c_\infty=\frac{\mathrm{C}_\infty}{\mathrm{C}_0}=1-\frac{\mathrm{M}_\infty}{\mathrm{AC}_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

In the example considered here, $\mathrm{M}_\infty/\mathrm{AC}_0=0{\cdot}9$, so that $c_\infty=0{\cdot}1$. Substitution in (18) for the several values of m gives the required R values.

A form of solution convenient for comparison with dyeing measurements is the relation between $\mathrm{M}_t/\mathrm{M}_\infty$ and time, where M_t is the amount of dye in the fibre at time t. Since

$$\frac{\mathrm{M}_t}{\mathrm{AC}_0}=1-\frac{(\mathrm{C}_t)_{\rho=1}}{\mathrm{C}_0}=1-(c_t)_{\rho=1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

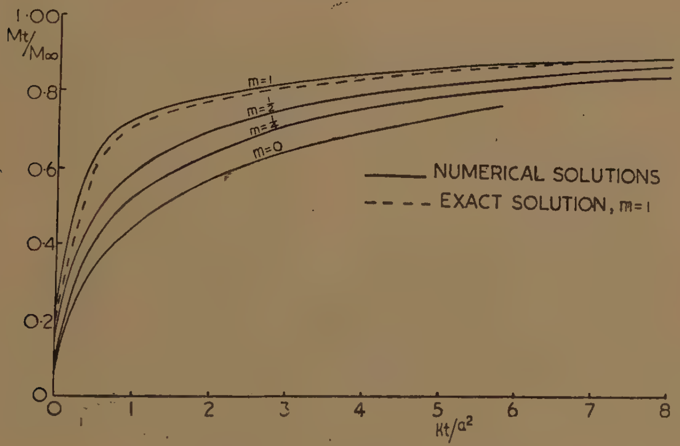
---

* No confusion is likely to arise due to the use of m as the exponent in (16) and as a suffix in § 3.

it follows that
$$\frac{M_t}{M_\infty}=\frac{1-(c)_{t\rho=1}}{1-c_\infty}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$
which is readily obtained from the computation sheet at each step in time.

Fig. 1 shows M_t/M_∞ as a function of $\tau=\kappa t/a^2$ for $m=1$, $\frac{1}{2}$, $\frac{1}{4}$, 0. The curves for the values of m other than zero were obtained by taking eight steps in $\rho(p=8)$ and a time interval defined by $\delta\tau/(\delta\rho)^2=R/4$ for small τ, changing to $\delta\tau/(\delta\rho)^2=R$ later. The broken curve obtained by evaluation of the appropriate exact solution for $m=1$ (see equation 42 of reference (1)) indicates the order of accuracy of the corresponding approximate solution. It is considered that the numerical solutions for which $p=8$ are sufficiently accurate for present purposes, but it should be realised that this accuracy can be improved by taking a larger value of p, that is by taking more steps in the range $0 \leqslant \rho \leqslant 1$. This question of accuracy is discussed more fully in reference (2), and it is sufficient to say that the order of accuracy of the numerical solutions presented here is 2 per cent. over most of the range in time. For $m=\frac{1}{2}$, a solution evaluated for $p=12$ differed by less than 1 per cent. from the solution for $p=8$, except in the neighbourhood of $\tau=0$. On evaluating the exact solution for $m=1$ it is found that a larger number of terms of a series need to be included the smaller τ is. In a similar way, if the region of small τ is to be investigated by finite difference methods, more steps in ρ and τ are necessary. Because of the singularity in the solution at $\tau=0$, it is desirable, if possible, to avoid the use of finite difference methods in the neighbourhood of this point. In many cases (*e. g.* reference (2)) conditions are such that an exact solution, or a solution in series, can be found for small τ. It has not yet been possible to do this for the present problem, and so finite difference approximations have been applied right from $\tau=0$.

Fig. 1.



Rate of absorption curves.

The special case of $m=0$ needs further consideration. Although it is difficult to conceive of a mechanism by which this value of m could arise physically, it is nevertheless interesting mathematically as a limiting case. When $m=0$, either $s=0$ (when $c=0$) or $s=\mathrm{R}$ (when $c>0$), and s changes discontinuously from 0 to R on a surface which moves inwards as diffusion proceeds. Thus, considering an element of volume $2\pi r dr$, all the solute which enters this volume will at first be deposited on the walls of the capillaries, and c, the concentration of solute free to diffuse, will remain zero. Only when an amount of solute $2\pi \mathrm{R} r dr$ has been deposited (per unit length of cylinder) will c increase. This deposition of solute without increase of c is closely analogous in heat flow work to the absorption of latent heat without increase of temperature till the change of state is complete, a problem which has been handled approximately, using finite difference methods, by Hartree (3). His treatment, which has been found to give a good representation of the behaviour except at temperatures very near the transformation point, is readily applied to the present diffusion problem for $m=0$.

The cylinder, into which diffusion proceeds, has been considered as built up of a finite number of concentric shells. If the concentration in each shell is assumed uniform, the left-hand side of the equation

$$\frac{\partial c}{\partial \tau}=\frac{1}{2m(\delta\rho)^2}\{(2m+1)c_{m+1}-4mc_m+(2m-1)c_{m-1}\} \quad . \quad . \quad (23)$$

represents the rate of change of concentration of one of the shells (actually the one between $\rho=(m\pm\frac{1}{2})\delta\rho$), and the right-hand side the difference between the amount of solute diffusing into and out of it. Thus if deposition is assumed complete throughout a shell before the concentration c in the shell increases, and if f is the fraction at time τ of the final amount R deposited, (23) may be replaced for the shell in which deposition is occurring by

$$\mathrm{R}\frac{\partial f}{\partial \tau}=\frac{1}{2m(\delta\rho)^2}\{(2m+1)c_{m+1}-4mc_m+(2m-1)c_{m-1}\}. \quad . \quad . \quad (24)$$

(24) is used to obtain f while f changes from 0 to 1, after which the concentration in this shell will increase according to (23). The curve for $m=0$ in fig. 1 was obtained in this way, taking $p=12$, *i. e.* $\delta\rho=1/12$. Fig. 1 shows that the general effect of decreasing the exponent m in the adsorption isotherm is to cause the dyeing process to proceed more slowly.

Fig. 2 shows the overall concentration of dye in the fibre as a function of the radial coordinate at a given time ($\kappa t/a^2=\tau=5$ approximately) for $m=1{\cdot}0$, $0{\cdot}5$, $0{\cdot}25$, the dyeing proceeding from a finite bath. A corresponding curve for an infinite bath and $m=1$ is also shown, the constant concentration of dye in the infinite bath being one-tenth of the initial concentration in the finite bath. The necessary exact solution for $m=1$

and the finite bath is not actually given in Paper I, but it follows readily from the equations of that paper that

$$c=c_\infty\left\{1+\sum_n \frac{4(1+\alpha)e^{-\beta q_n^2 t}}{(4+4\alpha+\alpha^2 q_n^2)}\;\frac{J_0(q_n r/a)}{J_0(q_n)}\right\}, \quad . \quad . \quad . \quad . \qquad (25)$$

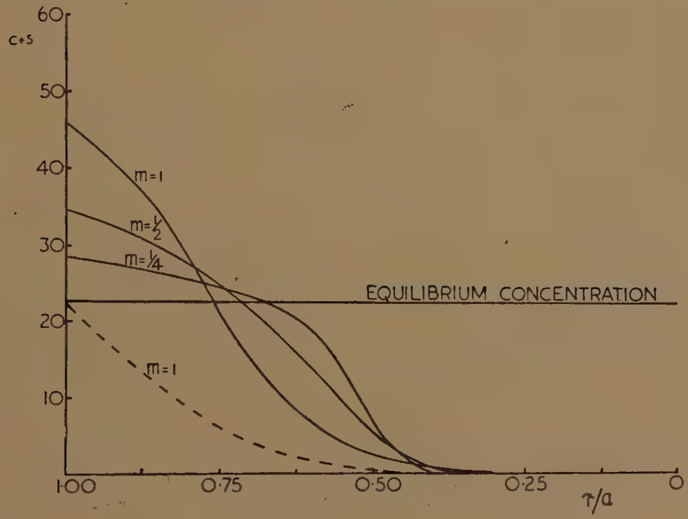
where the q_n's are positive roots other than zero of

$$\alpha q_n J_0(q_n)+2J_1(q_n)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

and

$$M_\infty=\frac{AC_0}{1+\alpha}, \quad \alpha=\frac{A}{\pi a^2(R+1)}, \quad \beta=\frac{\kappa}{(R+1)a^2}. \quad . \quad . \quad . \quad . \qquad (27)$$

Fig. 2.



Overall concentration—radial co-ordinate for different values of the exponent m.

$\kappa t/a^2=5$ in each case.

———— Finite bath, 90 per cent. exhaustion.

– – – – – – Infinite bath.

In the present example $M_\infty/AC_0=0{\cdot}9$, so that $c_\infty=0{\cdot}1$ and $\alpha=1/9$; and since $A=25\pi a^2$, we have $R+1=225$. The solution for the case of the infinite bath follows from (25) and (26) on making α infinite. These exact solutions were evaluated to obtain the curves of fig. 2 for $m=1$, and for $m=0{\cdot}5$, $0{\cdot}25$ the numerical solutions were used.

It is interesting to see that the curve for the infinite bath, which is of a form familiar in diffusion problems, differs essentially from all the others, each of which possesses a point of inflexion. Similar curves showing points of inflexion have previously been drawn by King (4), using a solution by Hopkins (5) for diffusion into a semi-infinite sheet when the diffusion coefficient is not constant but increases with increasing concentration of diffusing substance. The comparison is relevant since it follows from equations (15) and (16) that

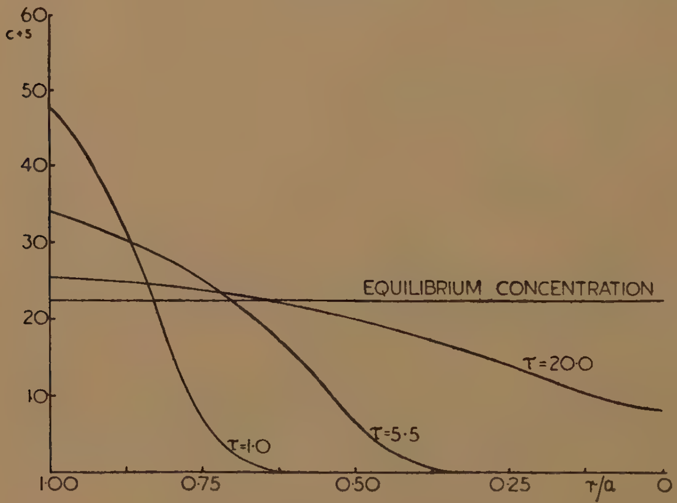
$$\frac{\partial s}{\partial \rho}=\frac{1}{\rho}\,\frac{\partial}{\partial \rho}\left(\frac{s^{(1-m)/m}}{m\mathrm{R}^{1/m}}\,\frac{\partial s}{\partial \rho}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (28)$$

which is the equation of diffusion when the diffusion coefficient is given by

$$\frac{s^{(1-m)/m}}{m\mathrm{R}^{1/m}}.$$

Thus diffusion with adsorption may be represented as simple diffusion with an effective diffusion coefficient which is a function of the concentration of diffusing substance. For $m=\frac{1}{2}$ this coefficient is proportional to s and for $m=\frac{1}{4}$ proportional to s^3, and so for these values of m the curves of fig. 2 are expected to resemble King's curves. Fig. 2, however, shows that when $m=1$, for which value the effective diffusion coefficient in (28) is constant, a point of inflexion still occurs if diffusion is proceeding from a finite bath, but not otherwise. The effect becomes increasingly noticeable as m decreases.

Fig. 3.



Overall concentration—radial co-ordinate for $s=\mathrm{R}c^{1/2}$; finite bath, 90 per cent. exhaustion.

Fig. 3 shows overall concentration curves as a function of r/a for $m=0{\cdot}5$ at various times. The decreasing concentration of the bath is clearly shown by the ordinates at $r/a=1$ at successive times, and the approach to the equilibrium value is clear.

No detailed comparison between theoretical solutions and experimental data is yet possible, as the necessary dyeing measurements have not been made.

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XXI. *New Methods in Diffraction Theory.*

By V. A. Fock,
Academy of Sciences of the U.S.S.R., Moscow *.

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The general problem of the theory of diffraction of electro-magnetic waves consists in finding a solution of Maxwell's equations, having prescribed singularities (field sources) and satisfying prescribed boundary conditions and conditions at infinity.

The solution of this problem presents serious mathematical difficulties, which arise chiefly from the necessity of taking into account the geometrical shape of the obstacles on which the wave is falling. The problem is somewhat simplified if only monochromatic waves of given frequency are considered, but the difficulties are still so great, that the problem has not yet been solved, except in cases when the obstacle is of a particularly simple form. The best known of these are the cases of a perfectly reflecting half-plane or a wedge, the cases of a sphere and a circular cylinder.

The cases of an elliptic and a parabolic cylinder have also been considered, and the field of a plane wave incident on a perfectly reflecting paraboloid of revolution (oblique incidence) has recently been obtained by the author. In the few cases enumerated a rigorous solution of the problem in the form of an infinite series of integrals has been obtained.

The aim of a theory is to give a picture reproducing all the qualitative and quantitative features of the phenomenon considered. This aim is not attained until the solution obtained is of a sufficiently simple form. If the rigorous solution has a complicated analytical form, it constitutes only the first step; a second step must be made—the derivation of formulæ suitable for numerical calculations.

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* Communicated by Soviet News Press Service.

This second step may be as difficult as the first one. To give an example, we may mention that the problem of diffraction of electromagnetic waves around a sphere was solved rigorously some 40 years ago (Mie). This problem includes that of the propagation of radio-waves along the surface of the earth. Owing to the slow convergence of the series involved, the general solution could, however, not be applied to the latter problem until 1918, when a transformation of the original series into another rapidly converging series was found (Watson). But the improved form of the solution was still unsatisfactory in some respects, being very complicated and applicable only in the region of the geometrical shadow (far beyond the line of horizon). A far more satisfactory form of the solution, applicable in all cases of practical importance, has been recently found by the author (1). Thus, the way from the rigorous theoretical solution to the approximate practical one took about 40 years of research.

To find first a rigorous solution of a diffraction problem and then to transform it into another form suitable for numerical calculations—this straightforward method is, however, of a very limited application. It can only be applied to the few problems admitting a rigorous solution in form of series of integrals.

In other cases (especially when the diffracting obstacle is of arbitrary shape) attempts have been made to reduce the problem to integral equations. These attempts have proved successful from the theoretical point of view; but with the exception of a paper by the author (2), no use has been made of the integral equations for the practical solution of the problem, the general theory of integral equations being quite useless for purposes of numerical calculation.

An approximate method, suffiiently general and leading to sufficiently simple formulæ is thus urgently needed. In the following we shall outline the principal ideas of such a method, proposed and developed by the author.

Every approximate method is based on the smallness of some parameters involved in the problem. We have to consider which of the parameters of our problem may be regarded as small.

We are usually concerned with the propagation of waves in air, *i. e.*, in a medium with properties widely different from those of the scattering bodies (obstacles). The electrical properties of these bodies are characterized by means of the complex dielectric permeability

$$\eta=\epsilon+i\frac{4\pi\tau}{\omega}.$$

(ϵ denotes as usual the dielectric constant, τ—the conductivity of the medium, ω—the frequency). Now it is essential that in most cases $|\eta|\gg 1$. Thus we may choose as one of the small parameters of the problem the inverse value of $|\eta|$ or the quantity $1 : \sqrt{|\eta|}$.

Next, the wave-length λ *in vacuo* is usually very much smaller than the radii of curvature of the scattering bodies. We thus have another small parameter—the quotient $\lambda : \mathrm{R}$, where R is the radius of curvature of the obstacle. It is convenient to take instead the quantity

$$\frac{1}{m}=\sqrt[3]{\frac{\lambda}{\pi \mathrm{R}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In addition to the two small parameters defined above, there may be others, depending on the position of the point of observation. For instance, in the problem of the propagation of radio waves along the earth surface the angle of inclination of the ray to the horizon may be regarded as small.

Let us consider the consequences of the fact that the parameters $1 : \sqrt{|\eta|}$ and $1 : m$ are small. In the limiting case $|\eta| = \infty$ (perfect conductor) a great simplification arises from the fact that the field is known beforehand inside the conductor (this field being equal to zero). We can confine ourselves to the space outside the conductor by prescribing proper boundary conditions to the field in air (the tangential components of the electrical vector should vanish at the surface). A similar situation arises if $\sqrt{|\eta|}$ is very large. The field inside the body is in this case very small except in a thin surface layer (skin-effect), and the influence of this layer may be accounted for by stating boundary conditions for the external field. These are of the form

$$\frac{4\pi}{c} jx = \sqrt{\eta}(\mathrm{E}_x - n_x \mathrm{E}_n) = n_y \mathrm{H}_z - n_z \mathrm{H}_y\,, \text{ etc.,} \quad . \quad . \quad . \quad (3)$$

where (j_x, j_y, j_z) is the surface current density vector, (n_x, n_y, n_z) the unit vector of the normal to the surface, E_n the normal component of the electric field, the meaning of the other symbols being evident. These conditions, first stated by Leontovich $^{(3)}$ in a somewhat different form, apply if $|\eta| \gg 1$ and if $\mathrm{KR}\sqrt{|\eta|} \gg 1$ ($\mathrm{K} = 2\pi : \lambda$). The latter inequality signifies that the thickness of the skin layer should be small as compared with the radius of curvature of the obstacle. Conditions (3) may be easily generalized for arbitrary values of the magnetic permeability m.

Consequently the smallness of $1 : \sqrt{|\eta|}$ permits us to confine our attention to the field outside and on the body, which constitutes an important simplification of the problem.

We now proceed to examine the influence of the smallness of the wave-length.

As well known, in the limiting case of small wave-lengths the laws of geometrical optics become valid. Particularly, the boundary of the shadow on the surface of the body becomes sharp and well defined. On the one side of the boundary—in the illuminated region—the field obeys very nearly Frensnel's laws of reflection, and on the dark side the field rapidly decreases to zero.

The approximation given by the geometrical optics is, however, not sufficient for our purposes. The point of interest for us is the diffraction phenomenon in its strict sense, *i. e.*, the bending of the ray around the obstacle. This phenomenon cannot be treated by the means of geometrical optics, and to give a theory of this phenomenon a more accurate solution of the field equations is required.

The author succeeded in finding this solution by means of a new principle which may be called The "Principle of the Local Field in the Penumbra Region."

This principle consists in the following:—The transition from light to shadow on the surface of the body takes place in a narrow strip along the boundary of the geometrical shadow. The width of this strip is of the order

$$d=\sqrt[3]{\frac{\lambda}{\pi}\mathrm{R}_0^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where R_0 is the radius of curvature of the normal section of the body by the plane of incidence. It may be proved that, with neglect of small quantities of the order $\sqrt[3]{\frac{\lambda}{\pi \mathrm{R}_0}}$, the field in this strip has a local character: it depends only on the value of the field of the incident wave in the neighbourhood of the point considered, on the geometrical shape of the body near the point and on the electrical properties of the material of the body. The field near a given point on the strip does not depend on its values at distant points and can be calculated separately.

To establish the principle of the local field and to derive explicit formulæ for this field we have used two different methods.

One of these (2) applies to the case of an absolute conductor and gives the values of the field on its surface. We start with the integral equation for the surface current density j. This is of the form

$$j=2j^{ex}+\frac{1}{2\pi}\int\left\{\frac{nx[j'x(z-z')]}{\mathrm{R}^3}f\right\}\frac{d\mathrm{S}'}{s\omega if}, \quad . \quad . \quad . \quad . \quad (5)$$

where

$$f=(1-i\mathrm{KR})e^{iKR}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The vector j^{ex} (external current density) is defined by the expression (3), where H is replaced by H^{ex}, the magnetic vector of the external field; z is the radius vector of the point of observation, z' that of the point of integration; $\mathrm{R}=|z-z'|$ is the length of the chord between z and z'; n is the value of the unit vector of the normal at z. A qualitative study of the integral equation permits to establish the principle of the local field. This principle once established, we have to find a solution of the diffraction problem for a convex body of a particular shape and to derive approximate formulæ for the field on its surface. In virtue of the principle of the local field, these formulæ hold for any other convex body having at the point considered the same values of the principal radii of curvature. (The particular body must of course be sufficiently general to possess points with any prescribed values of principal radii of curvature; actually a paraboloid of revolution has been used). Proceeding in this way we arrive at a general formula for the surface values of the tangential components of the magnetic field or, which amounts to the same, for the surface current density vector. This formula is of the form

$$j=j^{ex}\,G(\epsilon, 0)\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

where the argument ϵ in G denotes the quantity

$$\epsilon=l:\sqrt[3]{\frac{\lambda}{\pi}R_0^2}=l:d\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

l being the distance from the boundary of the geometrical shadow, measured along the ray (*i. e.*, along the line of intersection of the plane of incidence with the surface of the body) and taken positive in the direction of the shadow and negative in the opposite direction. The function $G(\epsilon, 0)$ is defined by the integral

$$G(\epsilon, 0)=e^{\frac{i\epsilon^3}{3}}\frac{1}{\sqrt{\pi}}\int_C \frac{e^{i\epsilon t}\,dt}{\omega'(t)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

where C is a contour in the complex t-plane running from infinity to zero along the line arc $t=\frac{2\pi}{3}$ and from zero to infinity along the positive real axis.

The function $\omega(t)$ may be called the complex Airy's function; it is defined by the differential equation

$$\omega''(t)=t\omega(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

and by the asymptotic behaviour for large negative values of t

$$\omega(t)=e^{\frac{i\pi}{4}}(-t)^{-\frac{1}{4}}\,.\,\exp\left[i\,\frac{2}{3}(-t)^{\frac{3}{2}}\right]. \quad . \quad . \quad . \qquad (11)$$

The function $G(\epsilon, 0)$ tends to the limit $G=2$ for large negative values of ϵ, while its modulus decreases exponentially for large positive values of ϵ. Formula (7) reproduces thus the gradual decrease of the field amplitude when passing from light to shadow.

The same results may be obtained by another method (4) which allows to generalize them in two respects. Firstly, the body need not be a perfect conductor, but may have a finite conductivity, if only the boundary conditions (3) are applicable. Secondly, the field is obtained not only on the surface of the body, but also near the surface (at distances that are small as compared with the radii of curvature). The method consists in simplifying Maxwell's equations and boundary conditions by neglecting quantities of the order of the square of the small parameters $1=\sqrt{|\eta|}$ and $1:m$. The wave equation for the amplitude is thereby replaced by a parabolic equation of Schrödinger's type. The simplified equations are valid in a limited region near a point on the penumbra strip.

The solution of these equations may be performed by means of the separation of variables and yields the field in the region considered and especially in the penumbra strip on the body. Introducing the complex quantity

$$q=\frac{im}{\sqrt{\eta}}=\frac{i}{\sqrt{\eta}}\cdot\sqrt[3]{\frac{\pi R_0}{\lambda}} \quad . \; . \; . \; . \; . \; . \quad (12)$$

(the modulus $|q|$ is thus the quotient of the two small parameters), we may write instead of (7)

$$j=j^{ex}G(\epsilon, q), \quad . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

where

$$G(\epsilon, q)=e\frac{i\epsilon^3}{3}\cdot\frac{1}{\sqrt{\pi}}\int_C\frac{e^{i\epsilon t}dt}{\omega'(t)-q\omega(t)}, \quad . \; . \; . \; . \; . \quad (14)$$

the contour C being the same as in (9). These formulæ give thus the distribution of currents on the penumbra strip on the body and generalize our previous formulæ (7) and (9). The formulæ for the field near the surface are more complicated and will not be written here.

It is to be noted that in the outward portion of the strip, where the illuminated region begins, approximate expressions can be derived from our formulæ that coincide with expressions for the field obtained by superposing the incident and the reflected wave and using Fresnel's coefficients of reflection. On the other hand, in the opposite portion of the strip the field is practically zero. Thus our formulæ constitute the missing link joining the two regions where the laws of geometrical optics may be applied. Together with Fresnel's formulæ they allow to calculate the field near and on the whole surface of the diffracting body.

In some problems this is all that is required. In the problem of propagation of waves around the earth's surface, for instance, we are only concerned with the field on heights not exceeding ten kilometres—a quantity that is small as compared with the earth's radius (6380 km.). In this instance our formulæ, if modified so as to include the case when the source is near or on the surface, give the required solution.

In other problems, however, the field at large distances from the scattering body is needed. In spite of the fact that our formulæ are valid only in the region near the surface, they provide a means to calculate the field at large distances also. Indeed, the field of the scattered wave is generated by the currents induced on the surface (in the skin-layer) by the incident wave. These currents are given by our formulæ. Thus, by applying well-known theorems on the vector potential due to a given current distribution, we may, in principle, calculate the field for arbitrary distances from the reflecting body.

The principle of the local field in the penumbra region provides thus a basis for the approximate solution of the problem of diffraction in the general case of a convex body of arbitrary shape.

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XXII. *Latent Heat of Vaporization and Composition.*

By S. T. Bowden and W. J. Jones *.

[Received February 24, 1947.]

A liquid substance conforms closely over a wide range of temperature to relations of the form

$$\frac{Ml^a}{D-d}=[L], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{M\lambda^b}{D-d}=[\Lambda], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where M represents the molecular weight of the substance in the vapour state, l the total latent heat of vaporization, λ the internal latent heat, D the corresponding density of the liquid, d the orthobaric density of the vapour, and a, b, [L] and [Λ] are specific constants for the substance. The constant [L] has been termed the normal lyoparachor and [Λ] the true lyoparachor. Except for the molecularly associated substances, water, ammonia, the alcohols and the carboxylic acids, the values of the indices, a, b, vary little from liquid to liquid, and for all normal liquids of boiling points between 200° and −100° C., the following equations hold approximately :—

$$\frac{Ml^{4/5}}{D-d}=[L], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{M\lambda^{3/4}}{D-d}=[\Lambda]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Since d is practically negligible at temperatures below the boiling point, it follows that at such temperatures the ratio of the normal lyoparachors of two non-associated liquids is the ratio of their molecular volumes under conditions in which their total latent heats are the same. Similarly, the true lyoparachor furnishes a relative measure of the molecular volumes of such liquids when their internal latent heats of vaporization are equal ⁽¹⁾.

An investigation is here described of the dependence of the values of the lyoparachors of a substance upon its composition and molecular constitution.

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* Communicated by the Authors.

The lyoparachors of all non-associated liquids of boiling points between 200° and −100° C., for which latent heats determined by the electrical (2)(3), the adiabatic (4), or the vapour-pressure method (5), and the corresponding densities (6) are given in the literature, have been calculated by means of equations (3) and (4), and are entered under the headings [L] and [Λ] in Tables I. and III. The above methods of latent-heat measurement give fairly concordant results, and where determinations by more than one of these methods are recorded, the mean value has been taken, but the older values obtained by the condensation method have not been used, since they do not agree closely with those found by the three methods here accepted (2), though this method has now been greatly improved (7).

Where the value of the internal latent heat, λ_b, at the boiling point, t_b° C., is not to be found in the literature, it has been calculated from that of the total latent heat, l_b, at the boiling point by means of the equation

$$\lambda_b = l_b - 0{\cdot}024209\left(\frac{1}{d_b} - \frac{1}{D_b}\right), \quad \ldots\ldots\ldots \quad (5)$$

where D_b and d_b represent the orthobaric boiling-point densities, and where the latter has not been experimentally determined, its value has been computed from the equation

$$\frac{1}{d_b} = \frac{82{\cdot}058(273{\cdot}1 + t_b)}{M}. \quad \ldots\ldots\ldots \quad (6)$$

Substances belonging to classes for which the lyoparachors of several members can be given are listed in Table I.: the lyoparachors of substances that are the sole representatives of their classes are included in Table III. Water, ammonia, the alcohols and the carboxylic acids have been omitted inasmuch as they do not obey equations (3) and (4).

Inspection of the values given in Table I. shows that in a straight-chain series of carbon compounds there is steady gradation in the lyoparachor from each member to its successor, that a substance with a branched carbon chain has a lower value than its straight-chain isomer, that substitution of hydrogen by another atom causes a definite change, that this change for an oxygen atom depends upon its function, being different in a ketone or carboxylate from in an ether, and that a benzene or cyclohexane ring makes a large positive contribution. From such comparisons it follows that the atomic and bond constants of the normal lyoparachor are those for an atom of carbon, c, of hydrogen, h, of nitrogen, n, of a oxygen in an ether, o, of oxygen in a ketone, Ω, of oxygen in a carboxylate, ω, of chlorine, χ, of bromine, β, of iodine, i, for a branch in a carbon chain, x, for a benzene ring, ϕ, and for a cyclohexane ring, κ. For the true lyoparachor there are corresponding constants.

The evaluation of the constants of the normal lyoparachor has been carried out by formulating for each substance $C_{n'}\,H_{n''}\ldots$ from its lyoparachor value as given in the [L] column of Table I. an equation

$$[L] = n'c + n''h \ldots + n'''x, \ldots + n''''\phi, \quad \ldots\ldots \quad (7)$$

Table I.—Densities, Latent Heats and Lyoparachors of Members of Series.

Substance	D_b	l_b	λ_b	[L]	$[L]_{\text{calc.}}$	$[\Lambda]$	$[\Lambda]_{\text{calc.}}$
Ethane	0·5410	129·00	116·94	2720	2680	1980	1950
Propane	0·5755	104·48	94·22	3170	3180	2330	2330
Butane	0·6010	93·91	84·60	3680	3670	2710	2720
Pentane	0·6093	84·92	76·83	4160	4170	3090	3100
Hexane	0·6132	80·55	72·93	4730	4670	3530	3480
Heptane	0·6139	76·23	69·05	5260	5160	3930	3860
Octane	0·6112	70·80	64·15	5680	5660	4260	4250
Isobutane	0·6010	87·46	78·51	3470	3540	2560	2620
Isopentane	0·6113	81·88	74·02	4020	4040	2990	3000
Diisopropyl	0·6262	76·42	68·89	4440	4400	3310	3290
Diisobutyl	0·6152	66·40	60·23	5360	5390	4040	4060
Benzene	0·8143	94·87	86·02	3670	3700	2720	2740
Toluene	0·7780	86·54	78·32	4270	4190	3200	3120
Ethylbenzene	0·7610	81·16	73·55	4670	4690	3520	3500
o–Xylene	0·7559	82·88	75·16	4830	4690	3600	3500
m–Xylene	0·7572	81·87	74·20	4780	4690	3560	3500
p–Xylene	0·7558	81·02	73·37	4740	4690	3540	3500
Mesitylene	0·7372	74·32	67·11	5140	5180	3840	3880
p–Cymene	0·7260	67·63	61·01	5410	5550	4060	4170
Cyclohexane	0·7198	85·63	77·29	4130	4140	3060	3090
Diethylamine	0·6685	—	—	4280	4410	3190	3260
Aniline	0·8736	103·70	93·99	4380	4430	3230	3280
Methylaniline	0·8370	101·20	92·56	5160	4920	3830	3660
Dimethylaniline	0·8122	87·48	79·88	5360	5420	4000	4050
Methyl chloride	0·9915	102·30	92·51	2070	2210	1520	1620
Ethyl chloride	0·9076	92·59	83·84	2670	2710	1980	2000
Propyl chloride	0·8549	84·32	76·29	3200	3210	2380	2380
Isopropyl chloride	0·8332	80·68	72·89	3170	3210	2360	2380
Butyl chloride	0·8127	79·77	72·27	3800	3700	2830	2770
Amyl chloride	0·7885	74·41	67·38	4250	4200	3180	3150
tert.-Amyl chloride	0·7903	68·89	62·25	4010	4070	3000	3050
Chlorobenzene	0·9813	75·84	68·86	3680	3720	2750	2790
o–Chlorotoluene	0·9395	72·64	65·90	4170	4220	3130	3170
p–Chlorotoluene	0·9351	73·13	66·35	4210	4220	3160	3170
Methylene chloride	1·3009	78·61	71·30	2150	2240	1610	1670
Chloroform	1·4085	59·02	53·47	2230	2270	1690	1720
Carbon tetrachloride	1·4814	46·43	41·92	2250	2300	1730	1770
Ethylene chloride	1·1594	77·34	70·23	2770	2740	2080	2050
sym.-Tetrachloroethane	1·4080	55·07	50·14	2960	2800	2260	2150
Cyclohexyl chloride	0·8821	74·78	67·85	4260	4170	3190	3140
Ethyl bromide	1·4134	59·92	54·26	2040	2050	1550	1550
Propyl bromide	1·2639	58·02	52·49	2510	2550	1900	1930

TABLE I (*continued*).

Substance	D_b	l_b	λ_b	[L]	[L]calc.	[Λ]	[Λ]calc.
Isopropyl bromide	1·2368	55·19	49·84	2470	2550	1870	1930
Butyl bromide	1·1505	56·76	51·35	3030	3050	2290	2310
Isoamyl bromide	1·0881	57·84	52·68	3580	3410	2730	2600
Bromobenzene	1·3055	56·58	51·29	3050	3070	2310	2330
Methyl iodide	2·2146	45·87	41·46	1390	1320	1070	1020
Ethyl iodide	1·8110	45·61	41·24	1840	1820	1420	1400
Butyl iodide	1·4333	45·90	41·57	2750	2810	2110	2160
Iodobenzene	1·5653	46·31	42·08	2810	2830	2160	2180
Ethyl ether	0·6962	83·99	76·00	3700	3850	2750	2860
Ethyl propyl ether	0·6871	82·68	75·21	4410	4350	3290	3240
Butyl ether	0·6575	67·80	61·50	5810	5840	4370	4390
Ethyl isobutyl ether	0·6711	74·78	67·97	4830	4710	3620	3530
Acetone	0·7511	123·30	112·07	3680	3580	2700	2650
Methyl ethyl ketone	0·7442	105·95	96·25	4170	4080	3090	3030
Methyl propyl ketone	0·7257	—	—	4440	4570	3340	3420
Methyl amyl ketone	0·7028	82·70	75·39	5580	5560	4180	4180
Cyclohexanone	0·8297	—	—	4470	4540	3380	3410
Methyl formate	0·9570	113·05	102·98	2760	2800	2030	2060
Ethyl formate	0·8768	97·12	88·37	3300	3300	2440	2450
Propyl formate	0·8330	88·14	80·18	3820	3790	2840	2830
Butyl formate	0·7972	86·76	79·43	4570	4290	3420	3210
Methyl acetate	0·8840	98·55	89·71	3310	3300	2450	2450
Ethyl acetate	0·8285	86·25	78·38	3780	3790	2810	2830
Propyl acetate	0·7917	79·92	72·66	4310	4290	3230	3210
Butyl acetate	0·7683	73·84	67·09	4750	4780	3560	3590
Methyl propionate	0·8412	87·36	79·60	3760	3790	2800	2830
Ethyl propionate	0·7964	79·73	72·67	4280	4290	3210	3210
Propyl propionate	0·7682	73·16	66·46	4710	4780	3540	3590
Butyl propionate	0·7489	71·74	65·39	5330	5280	4020	3970
Methyl butyrate	0·8034	78·57	71·58	4190	4290	3140	3210
Ethyl butyrate	0·7704	74·69	68·02	4780	4780	3590	3590
Propyl butyrate	0·7461	68·34	62·01	5150	5280	3880	3970
Isobutyl formate	0·7784	78·52	71·35	4320	4150	3230	3120
Isobutyl acetate	0·7589	73·77	67·15	4800	4650	3610	3500
Isobutyl propionate	0·7474	65·95	59·72	5000	5150	3760	3880
Isobutyl butyrate	0·7269	64·52	58·63	5590	5640	4230	4260
Isoamyl formate	0·7554	73·60	66·86	4820	4650	3620	3500
Isoamyl acetate	0·7430	69·05	62·79	5210	5150	3930	3880
Isoamyl propionate	0·7295	65·24	59·29	5620	5640	4250	4260
Isoamyl butyrate	0·7092	61·89	56·26	6090	6140	4610	4640
Methyl isobutyrate	0·8036	77·01	70·26	4120	4150	3100	3120
Ethyl isobutyrate	0·7681	72·07	65·56	4660	4650	3500	3500
Methyl isovalerate	0·7803	72·40	65·78	4600	4650	3460	3500
Ethyl isovalerate	0·7498	67·85	61·52	5090	5150	3830	3880
Propyl isovalerate	0·7300	64·52	58·64	5570	5640	4210	4260
Isobutyl isovalerate	0·7095	60·44	54·92	5970	6010	4530	4550

involving all its atoms, branches and rings, and then calculating from the 88 simultaneous numerical equations so obtained the values of the 12 unknowns, c, h, n, o, Ω, ω, χ, β, i, x, ϕ and κ, by the method of least squares. The values of the constants for the true lyoparachor have been calculated similarly, and both sets of constants are given in Table II.

TABLE II.—Atom and Bond Constants.

	Normal Lyoparachor $a=4/5$	True Lyoparachor $b=3/4$
Carbon	−1193·6	−804·5
Hydrogen	844·8	593·4
Nitrogen	−112·7	−48·3
Oxygen in an ether	178·3	146·5
Oxygen in a ketone	2092·1	1505·4
Oxygen in a carboxylate	903·6	649·1
Chlorine	873·9	643·1
Bromine	217·2	189·5
Iodine	−17·2	40·5
Branch in a carbon chain	−132·7	−94·7
Benzene-ring bonds	5788·6	4002.4
Cyclohexane-ring bonds	1166·1	793·7

The lyoparachor values obtained by the summation of these constants are given in the columns headed $[\mathrm{L}]_{\text{calc.}}$ and $[\Lambda]_{\text{calc.}}$ in Table I.; the average difference between the values so calculated and those obtained directly by means of equations (3) and (4) is 1·6 per cent., which shows that both the normal and the true lyoparachors are fairly closely additive properties determined by the atoms and bonds of the molecule.

TABLE III.—Densities, Latent Heats and Lyoparachors of Miscellaneous Compounds.

Substance	D_b	l_b	λ_b	[L]	$[\Lambda]$
Ethyl carbonate	0·8462	—	—	4130	3160
Fluorobenzene	0·9431	79·30	71·92	3380	2520
Tin tetrachloride	1·9803	30·90	27·96	2060	1610
Nitromethane	1·0236	135·00	122·86	3020	2210
Trichloroethylene	1·3518	57·25	51·82	2490	1880
Tetrachloroethylene	1·4487	50·06	45·36	2630	2010
Acetylene	0·6232	198·00	183·62	2880	2090
Pyridine	0·8842	—	—	4040	3030
Furane	0·8600	95·35	86·50	3040	2250
Ethylene bromide	1·9246	46·24	41·98	2100	1610
Carbon dioxide	—	—	—	1330	970
Carbon disulphide	1·2250	—	—	2200	1640
Nitrous oxide	1·2260	—	—	1350	950

From the lyoparachors given in Table III., provisional, but rather uncertain, values for the constants of other atoms, groups, bonds and rings can, using Table II., be obtained by subtraction, and these are contained in Table IV. In computing the furane-ring constants, the oxygen atom has been assumed to be of the ether type. Ethylene bromide has been relegated to Table III. since it is the only polybromo-compound of known lyoparachor and would seem to require constants for bromine different from those found for the monobromo-compounds.

The marked effect of chain branching, the several constants required for oxygen according to its mode of combination, and the abnormality of the alcohols and carboxylic acids show that the lyoparachor is more sensitive to constitutive influences than is the ordinary parachor [P] based on surface tension (8).

Table IV.—Provisional Values of Atom and Bond Constants.

	Normal Lyoparachor $a=4/5$	True Lyoparachor $b=3/4$
Oxygen in a carbonate	550	420
Fluorine	530	380
Tin	—1440	—960
Nitro group	1680	1230
Double bond	1470	1010
Triple bond	3580	2510
Pyridine-ring bonds	5900	4130
Furane-ring bonds	4260	2950

It may be noted that the value of the true lyoparachor is in general about three-quarters of that of the normal lyoparachor, so that very roughly

$$[\mathit{\Lambda}]=0{\cdot}75\,[\mathrm{L}]. \qquad (8)$$

Moreover, it is found that the parachor is given by

$$[\mathrm{P}]=\frac{2\lambda^{3/4}\mathrm{V}_c\mathrm{T}_c}{3l^{4/5}\mathrm{T}_b}, \qquad (9)$$

where V_c is the molar critical volume, T_c the absolute critical temperature and T_b the normal boiling point of the liquid. On multiplying numerator and denominator by $\mathrm{M}/(\mathrm{D}-d)$, equation (9) becomes

$$[\mathrm{P}]=\frac{2[\mathit{\Lambda}]\mathrm{V}_c\mathrm{T}_c}{3[\mathrm{L}]\mathrm{T}_b}. \qquad (10)$$

The applicability of this equation to some representative substances is shown in Table V., where $[\mathrm{P}]_{\mathrm{calc.}}$ is the value of the parachor calculated from equation (10), and [P] is the experimental value (9).

TABLE V.—Lyoparachor Ratios and the Parachor.

Substance	$[\Lambda]/[L]$	V_c	T_c/T_b	$[P]_{calc.}$	$[P]$
Ethane	0·728	142·8	1·65	114	111
Pentane	0·743	310·2	1·52	234	231
Hexane	0·745	367·5	1·49	272	273
Heptane	0·747	428·1	1·45	309	311
Octane	0·750	490·2	1·43	351	350
Diisobutyl	0·754	482·7	1·44	349	345
Benzene	0·741	256·4	1·59	201	206
Cyclohexane	0·741	307·5	1·56	237	239
Methyl chloride	0·734	136·4	1·66	111	109
Ethyl chloride	0·742	195·4	1·59	154	152
Chlorobenzene	0·747	308·1	1·56	239	244
Chloroform	0·758	231·3	1·60	187	183
Carbon tetrachloride......	0·769	275·7	1·59	225	220
Ethylene chloride	0·751	236·2	1·57	186	188
Bromobenzene	0·757	323·4	1·56	255	258
Iodobenzene	0·769	350·7	1·56	281	281
Ethyl ether	0·743	282·0	1·52	212	211
Acetone	0·734	216·6	1·54	163	162
Methyl formate	0·736	172·2	1·60	135	138
Ethyl acetate	0·743	286·3	1·55	220	216
Propyl acetate	0·749	345·3	1·49	257	255
Methyl butyrate	0·749	340·2	1·47	250	254

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University College,
Cardiff.

XXIII. *A Paradox in Electrodynamics.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

IN a letter, W. A. Tripp (1) referred to a paper by the present writer entitled " An Historical Note on a Paradox in Electrodynamics" (2), and suggested an alternative solution : the formula given by Tripp for the magnetic force between two moving electrons is :—$F=e_1e_2(V_x^2-\frac{1}{2}V_y^2)/r^2$, where V_x and V_y are the components of the relative velocity between the charges taken respectively perpendicular to, and in, the line joining them. This formula, as stated, follows the analysis of Weber (3), and if adopted, no paradox arises for the forces between the electrons in the case discussed. However, Helmholtz (4) has shown that the Weber formula can lead to a dynamically impossible situation, so if Weber's formula is adopted another paradox arises.

However, Weber's theory has been revised and altered by Ritz (5), who assumed a ballistic theory associated with a periodic vibration of the electron. This assumption of a vibrating electron was entirely arbitrary when proposed by Ritz in 1908, but is now well-established and forms the basis of electron diffraction, the frequency of an electron at rest is now known to be :—$1{\cdot}24\times10^{20}$ vibrations per second. Ritz derived a formula for the total force between two electrons, which gave results in keeping with all the standard electrodynamical experiments, and which also agreed with Kaufmann's experiments (6) on high speed electrons or beta-rays, without assuming that the mass of the electron increased with the velocity.

The formula is

$$F_x=ee'\left\{\frac{\cos(r,x)}{r^2}+\frac{\cos(r,x)}{4r^2c^2}\,[(3-K)U^2-3(1-K)Ur^2]-\right.$$
$$\left.\frac{(K+1)U_xU_r}{2c^2r^2}-\frac{W_x'+W_r'\cos(r,x)}{2c^2r^2}\right\},$$

with corresponding expressions for F_y and F_z, where r is the actual distance between the electrons at a given instant, U is velocity of one electron relative to the other, W′ is the acceleration of the electron e' with reference to a Newtonian set of axes, c is the velocity of light and K is an arbitrary constant. O'Rahilly (7) suggested that $K=3$. It can be seen from the formula that if the electrons are at relative rest, the total force between them reduces to Coulomb's law .and if the electrons are in relative motion to each other without acceleration, the action on one electron is always equal and opposite to the reaction on the other ; but if either or both electrons are accelerating with reference to a Newtonian set of axes, the action-reaction law fails. On the other hand, it is known that when an electron accelerates there is an energy emission of $2\frac{e^2}{c^2}W^2$ and a radiation reaction of $\frac{2}{3}\frac{e^2}{c^2}\dot{W}$, so that Ritz's law is in keeping with experimental facts.

The theories of Ritz and Einstein are able to account for many electrodynamic phenomena by entirely different methods, and for the paradoxical case previously considered it would appear to the writer that the theory of Ritz is to be preferred to that of Einstein.

Yours faithfully,
IAN A. ROBERTSON.

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2, Sutherland Street,
Glasgow, W.2.

XXIV. *Notices respecting New Books.*

Osborne Reynolds and his work in Hydraulics and Hydrodynamics. By A. H. GIBSON. [Pp. x+33.] Science in Britain Series. (Published for the British Council. London: Longmans, Green and Co., 1946.) Price 1*s.* 6*d.*

AMONG the scientists who flourished in the latter half of the last century, Osborne Reynolds always seems a lonely figure. Much of his work in physics and engineering fell rather outside the main lines of advance at the time, and full recognition of its value, and in engineering of its relevance, came only at a later date. This is well illustrated by his investigations on the condensation of mixtures of air and steam, and on the action of the heating surface in steam boilers; the principles arrived at are now appreciated as being of basic importance in practical design problems. Reynolds is perhaps best known for his work on steam line and turbulent flow, but this is only part of extensive inter-related investigations on such matters as vortex motion, lubrication, and the use of models in the study of tidal and river action.

In his more purely physical work penetrating insight is shown, no less than mathematical power, together with an alertness in getting at what Reynolds calls "the secrets nature has concealed by her method of deadening curiosity by leaving them exposed." The slipperiness of ice is thus exposed, but it was left to Reynolds to see in it something unusual and calling for explanation; simple, when once it has been given. Treading on firm moist sand lays bare another curious property, "dilatancy," as Reynolds called it, and which, too, he brilliantly explained. He pondered much on this property, and his thoughts and studies took shape in that amazing *tour de force*, his memoir on the submechanics of the universe, in which is presented a mechanical theory of matter and aether, of gravitation and light. There is something of tragedy here, for this culminating work, to which Reynolds devoted so much time and effort, fell on a cold and unresponsive world. It was not simply that the memoir was difficult to follow; it did not seem worth trying to follow. For a mechanical aether had long been dying; it was soon to be dead and buried and little mourned. In the later years of his working life Reynolds might well have made contributions to science of greater and more lasting value had he been less independent and self-sufficient; but then, too, much of the essential originality of his earlier work might have been lost.

The life and work of Reynolds provide a fascinating theme. In this pleasantly produced illustrated booklet, Professor Gibson has touched but little on the life. The survey is mainly of the scientific work, and for the reader with even a little knowledge of engineering or physics, it gives a clear impression of its quality and extent. This alone, however, is most welcome, for of those short of the greatest who contributed greatly to science in the last century, there are few more liable to be overlooked, and more worthy to be remembered, than Osborne Reynolds. E. C. S.

An Introduction to Analytical Geometry. By A. Robson. [Pp. viii+215.] (Cambridge University Press, 1947). Price 10*s*. 6*d*.

The outstanding feature of this book is the broad point of view taken by the author. An attempt has been made to fuse pure and analytical methods and the subject matter has been developed in a manner which frees the student from the unnecessary restrictions imposed by a more traditional treatment.

The first two chapters deal with Homography and Involution. Then there is a discussion on the use of symbols in projective geometry, and here the influence of Professor H. F. Baker's book, *Introduction to Plane Geometry*, is in evidence. Various properties of conics are deduced and systems of conics and ranges on a conic are discussed. The focus of an algebraic curve is defined as a point of intersection of two tangents drawn to the curve from the circular points at infinity. Various examples are given before the restriction to conics is made. The normals to a conic and properties of these are studied and finally some special homogeneous co-ordinates are introduced.

This text is important in that it points the way to a new treatment of analytical geometry both in schools and the first year at the Universities.
L. S. G.

Functions of a Complex Variable. By T. M. MacRobert. [Pp. xv+390.] Third Edition. (Macmillan & Co., Ltd.). Price 18*s*.

This third edition of Professor MacRobert's well-known book differs from the previous edition, of 1933, in that an appendix (20 pages) on Generalized Hypergeometric Functions has been added together with 125 further miscellaneous examples. The appendix contains important material taken from various original papers of recent years and the examples should appeal to any student who requires further exercise in the theory of functions of a complex variable.

The subject matter of the book may be conveniently divided into three parts: general theory, special functions and linear differential equations. The special functions whose properties are developed include Gamma and Beta functions, Weierstrassian Elliptic functions, Jacobian Elliptic functions, Legendre and Bessel functions. Regular integrals of linear differential equations are treated and the method of solving by means of definite integrals is explained.

The issue of this new edition will fill the need of students who are experiencing difficulty in obtaining a suitable text-book on the subject. L. S. G.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

XXV. *The Two-Dimensional Potential Problem of Seepage into a Cofferdam*.*

By E. N. Fox, Ph.D. and J. McNamee, B.Sc.†.

[Received July 27, 1947.]

1. Introduction.

Problems of seepage flow through porous media and around impermeable boundaries are frequently encountered in the design of engineering structures and in hydrology. Towards the end of the last century a number of classical solutions had been obtained under somewhat sweeping assumptions which are not realized in field applications, and the work of the early investigators was neglected until about the nineteen-twenties when analytical work in this class of problem was stimulated by its association with experimental techniques.

A fundamental law to govern the flow of fluids in porous media was formulated by Darcy (1) in 1856. It was subsequently shown by Forchheimer (2) (and independently by Slichter (3)) that steady-state flow problems may be reduced to the determination of a function which is harmonic within a region and is defined on the boundaries of the region.

The analysis of seepage problems is thus included in the general theory of harmonic functions. In particular, steady state two-dimensional problems require the determination of an analytic function of a complex variable and are susceptible of attack by the various techniques which have been elaborated for the determination of plane harmonic functions. A comprehensive review of analytical and experimental methods is given by Muscat (4). For the applications of the theory in engineering, reference may be made to Khosla, Bose and McKenzie-Taylor (5) or Casagrande (6).

The particular problem considered by the authors was suggested by the engineering problem of a cofferdam embedded in a porous medium and is illustrated in fig. 1. It is shown in § 2.2 that it can be reduced to the basic potential problem of fig. 2, where ϕ and ψ are conjugate potential and flow functions. The formal solution of a slightly more general problem including our problem as a special case has been given by Malhotra(7), but so far as can be ascertained no calculations pertinent to our problem were carried out, except for very simple cases which are in turn special cases of our problem (8), (9), (10). The main purpose of the present note is to give the results of basic calculations from which flow lines and equipotentials can easily be plotted over the whole range of physical

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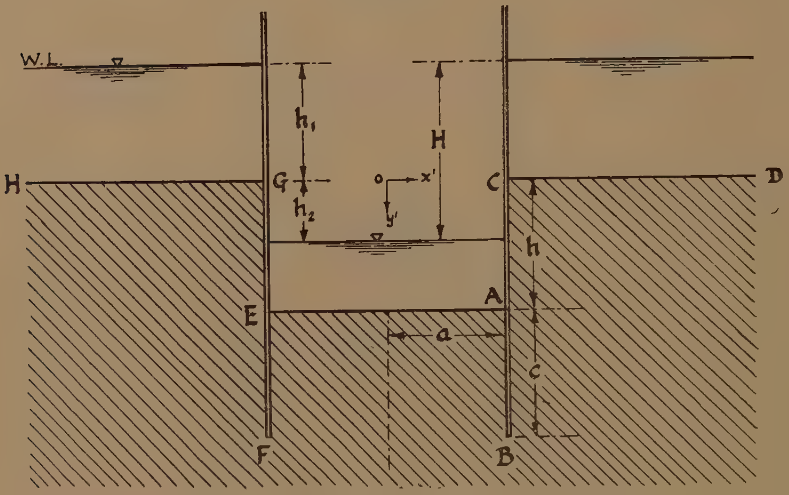
† Communicated by Sir Edward Appleton.

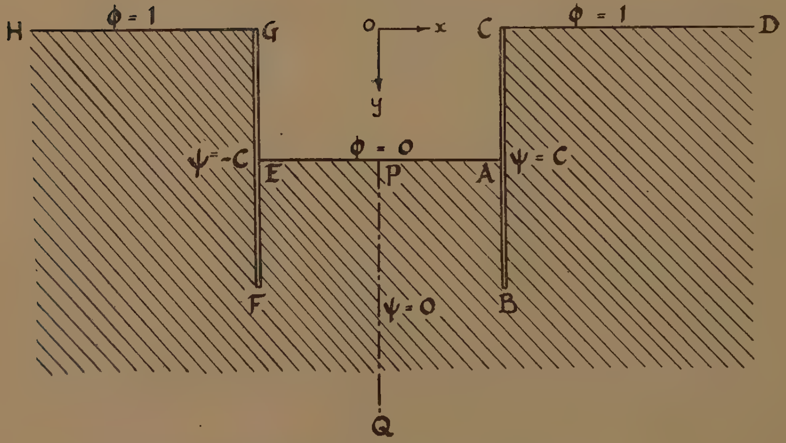
parameters. These calculations have necessitated some recasting of the form of Malhotra's formal solution and for sake of completeness we include the derivation of this formal solution *ab initio* in the present paper. Much of the further analysis is concerned with the Jacobian Zeta function of complex argument and the resulting tables given in the appendix may be of interest over a wider field than the particular problem for which the work has been carried out.

Fig. 1.



General cofferdam problem.

Fig. 2.



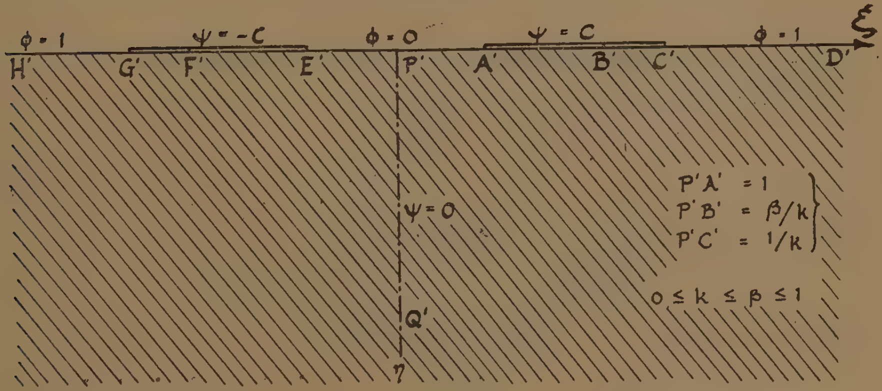
Potential problem in z-plane.

2. DERIVATION OF SOLUTION.

2.1. *Notation.*

x', y'	rectangular cartesian coordinates (fig. 1).
x, y	non-dimensional cartesian coordinates (fig. 2).
p	hydrostatic pressure.
v_x, v_y	components of velocity of flow in fig. 1.
κ	permeability of medium.
ρg	density of water.
$h_1, h_2, H,$ $h, a, c,$	lengths defined as in fig. 1.
$q=a/h$	
$s=c/h$	
Φ	potential function.
ϕ	non-dimensional potential function.

Fig. 3.



Potential problem in ζ-plane.

ψ	non-dimensional flow function.
ξ, η	rectangular cartesian coordinates in transformed problem (fig. 3).
z	$=x+iy$
ζ	$=\xi+i\eta$
χ	$=\psi+i\phi$.
$k=\sin\alpha$	modulus of Jacobian Zeta function.
$k'=\cos\alpha$	
K, K′, E, E′	complete elliptic integrals of first and second kinds.
A, β, ϵ	parameters arising in solution.
ψ_1	$=\frac{K'}{K}\psi$.
f_1	real part of Jacobian Zeta function of complex argument (equation (32)).
f_2	imaginary part of Jacobian Zeta function plus linear term (see equation (32)).

G'	exit gradient at point A (fig. 1).
$G=\frac{h}{H}G'$	exit gradient at point A (fig. 2).
ϕ_B	value of ϕ at pile tip B (fig. 2).

Certain subsidiary notation is defined where introduced.

2.2. *The basic equations.*

The general problem considered is illustrated in fig. 1, and if p is the hydrostatic pressure in the water at any point and v_x, v_y the components of velocity of flow, then by Darcy's law we have as basic equations

$$\left.\begin{aligned} v_x&=-\frac{\kappa}{\rho g}\frac{\partial p}{\partial x'} \\ v_y&=-\frac{\kappa}{\rho g}\left(\frac{\partial p}{\partial y'}-\rho g\right) \end{aligned}\right\}, \quad \text{. . . . . . .} \quad (1)$$

where κ is permeability and y' is measured downwards.

Defining a function Φ by

$$\left.\begin{aligned} v_x&=-\kappa\frac{\partial \Phi}{\partial x'} \\ v_y&=-\kappa\frac{\partial \Phi}{\partial y'} \end{aligned}\right\}, \quad \text{. . . . .} \quad (2)$$

then the equation of continuity for incompressible fluids,

$$\frac{\partial v_x}{\partial x'}+\frac{\partial v_y}{\partial y'}=0, \quad \text{. . . . .} \quad (3)$$

leads to

$$\frac{\partial^2 \Phi}{\partial x'^2}+\frac{\partial^2 \Phi}{\partial y'^2}=0. \quad \text{. . . . .} \quad (4)$$

Comparing (1) and (2) we can write

$$\Phi=\frac{p}{\rho g}-y'. \quad \text{. . . . .} \quad (5)$$

Taking atmospheric pressure as our zero of pressure, the boundary conditions of our problem (fig. 1) may be written in terms of Φ as

$$\left.\begin{aligned} &\Phi=h_1 \text{ along HG and CD,} \\ &\Phi=-h_2 \text{ along EA,} \\ &\frac{\partial \Phi}{\partial n}=0 \text{ round GFE and ABC,} \end{aligned}\right\}, \quad \text{. . . . . .} \quad (6)$$

where $\frac{\partial}{\partial n}$ indicates differentiation along the normal to the boundary.

To reduce to non-dimensional form we write

$$\left.\begin{aligned} \Phi+h_2&=(h_1+h_2)\phi=\mathrm{H}\phi,\\ x'&=hx,\\ y'&=hy,\\ a&=hq,\\ c&=hs, \end{aligned}\right\}, \quad \ldots\ldots \quad (7)$$

where ϕ will also satisfy the Laplace equation and we introduce the conjugate (flow) function ψ defined by

$$\left.\begin{aligned} \frac{\partial\psi}{\partial x}&=\frac{\partial\phi}{\partial y},\\ \frac{\partial\psi}{\partial y}&=-\frac{\partial\phi}{\partial x}, \end{aligned}\right\}, \quad \ldots\ldots\ldots \quad (8)$$

whence the Laplace equation will be satisfied as usual by

$$\chi=\psi+i\phi=f(x+iy), \quad \ldots\ldots\ldots \quad (9)$$

where f is at present an undetermined analytic function of the complex variable $z=x+iy$.

The problem of fig. 1 thus reduces to a consideration of the non-dimensional potential problem of fig. 2 in which the function f of equation (9) has to be found to satisfy the boundary conditions.

$$\left.\begin{array}{l} \phi=1 \text{ when } y=0 \quad |x|\geqslant q\\ \phi=0 \text{ when } y=1 \quad |x|\leqslant q\\ \psi=\pm\mathrm{C} \text{ round ABC and GFE} \end{array}\right\}, \quad \ldots\ldots \quad (10)$$

where C is a constant.

As defined by equation (8) the function ψ is indeterminate to the extent of an additive arbitrary constant which does not affect the flow. Without loss of generality we have made the problem determinate by choosing the flow line $\psi=0$ to correspond to the line PQ in fig. 2, which we know to be a flow line by symmetry. The function ψ is then odd in x while ϕ is even in x.

We have therefore to solve the potential problem of fig. 2 and thence from equations (2), (5) and (7) the velocity and pressure for the general problem of fig. 1 will be given by

$$\left.\begin{aligned} v_x&=-\kappa\frac{\mathrm{H}}{h}\frac{\partial\phi}{\partial x},\\ v_y&=-\kappa\frac{\mathrm{H}}{h}\frac{\partial\phi}{\partial y}, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (11)$$

$$\frac{p}{\rho g}=y'-h_2+\mathrm{H}\phi. \quad \ldots\ldots\ldots\ldots \quad (12)$$

2.3. *The formal solution.*

To solve the basic problem of fig. 2 the Schwarz-Christoffel transformation is applied as usual to transform the contour of fig. 2 in the z-plane into the axis $\eta=0$ of the ζ-plane in fig. 3. The transformation is

$$z-i=\mathrm{A}\int_0^{\zeta}\frac{(\beta^2-k^2\,\zeta^2)\,d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

where the constant of integration has been chosen so that P of fig. 2 transforms to P′ of fig. 3 and the symmetry of the problem is preserved.

The unit of length in the ζ-plane is arbitrary and we have chosen it as P′A′; the constants A, B, k of equation (13) are then determined by the conditions that A, B, C of fig. 2 transform into A′, B′, C′ of fig. 3. Some care is necessary in defining the radicals in (13) by continuation, and the resulting relations giving A, B, k in terms of q and s are then easily obtained in the form

$$\left.\begin{aligned} q&=\mathrm{A}\int_0^1\frac{(\beta^2-k^2\,\xi^2)\,d\xi}{\sqrt{(1-\xi^2)(1-k^2\,\xi^2)}},\\ s&=\mathrm{A}\int_1^{\beta/k}\frac{(\beta^2-k^2\,\xi^2)\,d\xi}{\sqrt{(\xi^2-1)(1-k^2\,\xi^2)}},\\ -1&=\mathrm{A}\int_1^{1/k}\frac{(\beta^2-k^2\,\xi^2)\,d\xi}{\sqrt{(\xi^2-1)(1-k^2\,\xi^2)}} \end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

These can be expressed in terms of elliptic integrals to give, after some manipulation, the relations

$$\left.\begin{aligned} \mathrm{A}&=\frac{2}{\pi}(\mathrm{K}+q\mathrm{K}'), && \text{(i)}\\ \beta^2&=\frac{\mathrm{E}'}{\mathrm{K}'}-\frac{1}{\mathrm{AK}'}, && \text{(ii)}\\ s&=-1+\frac{\mathrm{F}(k',\epsilon)}{\mathrm{K}'}+\mathrm{E}'\mathrm{A}\left[\frac{\mathrm{E}(k',\epsilon)}{\mathrm{E}'}-\frac{\mathrm{F}(k',\epsilon)}{\mathrm{K}'},\right] && \text{(iii)} \end{aligned}\right\}, \quad . \qquad (15)$$

where

$$\left.\begin{aligned} \sin^2\epsilon&=\frac{1-\beta^2}{k'^2},\\ k'^2&=1-k^2, \end{aligned}\right\} 1\geqslant\beta\geqslant k\geqslant 0, \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

while E, F, E′, K, K′ have their usual significance as tabulated for example in Jahnke and Emde (11).

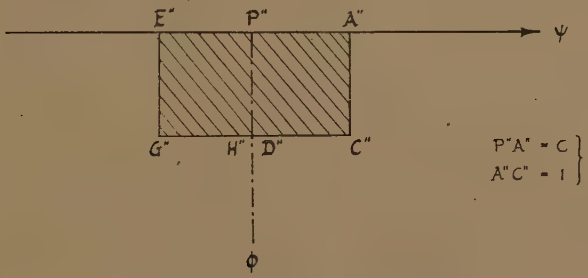
Equations (13), (15) and (16) give the relation between any point (x, y) in fig. 2 and a corresponding point (ξ, η) in fig. 3, and we need by symmetry consider only the right-hand half $\xi \geqslant 0$ and find $\psi + i\phi$ as a function of $\xi + i\eta$ to satisfy the boundary conditions

$$\left.\begin{array}{lll} \psi=0 & \xi=0, & \eta\geqslant 0, \\ \phi=0 & \eta=0, & 0\leqslant\xi\leqslant 1, \\ \psi=\mathrm{C} & \eta=0, & 1\leqslant\xi\leqslant 1/k, \\ \phi=1 & \eta=0, & \xi\geqslant 1/k. \end{array}\right\} \quad \ldots \ldots \quad (17)$$

In the reference (8) available to the authors, Malhotra's solution is produced in summarized form and the solution of this ζ-plane problem is given without indicating its derivation. Such derivation could proceed logically in the following manner.

To solve the problem of fig. 3 we see that we must obtain the transformation linking the ξ axis H′G′E′P′A′C′D′ in the ζ-plane of fig. 3 with the rectangle A″C″G″E″ in the χ-plane of fig. 4. Conversely this is

Fig. 4.



Potential problem in χ-plane.

simply the Schwarz-Christoffel transformation which turns the rectangle in the χ-plane into the ξ-axis in the ζ-plane, and it will be of the form

$$\chi=\mathrm{B}\int_0^{\zeta} \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}, \quad \ldots \ldots \quad (18)$$

where the lower limit of zero ensures the symmetry by transforming P′ to P″, and the transformation of A″ to A′ and C″ to C′ gives

$$\mathrm{C}=\mathrm{BK},$$

$$\mathrm{C}+i=\mathrm{B}(\mathrm{K}+i\mathrm{K}'),$$

whence

$$\left.\begin{array}{l} \mathrm{B}=1/\mathrm{K}', \\ \mathrm{C}=\mathrm{K}/\mathrm{K}'. \end{array}\right\} \quad \ldots \ldots \quad (19)$$

The transformation (18) can them be written conversely as

$$\zeta=\mathrm{S}n(\mathrm{K}'\chi, k), \quad \ldots \ldots \quad (20)$$

where Sn is the usual Jacobian elliptic function, while (19) also gives us now an expression for the constant C which has previously remained undetermined.

From (13) and (18) we can now eliminate the intermediate variable ζ to obtain a direct relation between z and χ, which by use of (15) may be written in the form

$$z=i+q\frac{\mathrm{K}'}{\mathrm{K}}\chi+\mathrm{A}Zn(\mathrm{K}'\chi, k). \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

This completes the formal solution of the problem of fig. 2, since the parameters A, k, K, K′ can be evaluated in terms of the basic parameters q and s by means of equations (15) and (16). Save for a different notation, equations (21), (15) and (16) correspond to Malhotra's formal solution of the problem.

2.4. *The Jacobian Zeta function of complex argument.*

In order to plot the equipotentials and flow lines in the basic problem of fig. 2, it is necessary to separate out the real and imaginary parts of equation (21). Considering, by virtue of symmetry, only the right-hand half $x \geqslant 0$ in fig. 2, we note that ϕ varies between 0 and 1 whereas ψ varies between 0 and K/K′. It is therefore convenient to introduce a function ψ_1 in place of ψ, where

$$\psi_1=\frac{\mathrm{K}'}{\mathrm{K}}\psi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The argument of the Zeta function in (21) then becomes $\mathrm{K}\psi_1+i\mathrm{K}'\phi$, where both ϕ and ψ_1 range between 0 and unity in the right-hand half of fig. 2.

When values of ϕ and ψ were first examined on the profile it was found that equation (21) led at first sight to relations differing from those given by using (13) and (20) to obtain elliptic integrals which were subsequently converted to a Zeta function relation. This difference could only be reconciled by the existence of the particular relation (28) below between $Zn(u, k)$ and $Zn(iu+\mathrm{K}'-i\mathrm{K}, k)$, and a direct proof of this relation was then sought and obtained as follows.

Let u and v be complex variables related by

$$u=v+\mathrm{K}+i\mathrm{K}', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23\,a)$$

then

$$iv=iu+\mathrm{K}'-i\mathrm{K}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23\,b)$$

Whence, using Jacobi's imaginary transformation (Whittaker and Watson ⁽¹²⁾, page 519), we have

$$Zn(iu+\mathrm{K}'-i\mathrm{K}, k')=Zn(iv, k')$$

$$=i\frac{\mathrm{sn}(v, k)\,\mathrm{dn}(v, k)}{\mathrm{cn}(v, k)}-iZn(v, k)-i\frac{\pi v}{2\mathrm{KK}'}. \quad . \quad . \quad . \quad (24)$$

But from Whittaker and Watson⁽¹²⁾, page 520, Example 4, we have

$$\mathrm{E}(v+\mathrm{K}+i\mathrm{K}', k)=\mathrm{E}(v, k)-\frac{\mathrm{sn}(v, k)\,\mathrm{dn}(v, k)}{\mathrm{cn}(v, k)}+\mathrm{E}+i(\mathrm{K}'-\mathrm{E}'), \quad . \quad . \quad (25)$$

whence, using the well known relations

$$\left.\begin{aligned} Zn(v, k) &= \mathrm{E}(v, k) - \frac{v\mathrm{E}}{\mathrm{K}}, \\ \mathrm{EK}' + \mathrm{E}'\mathrm{K} - \mathrm{KK}' &= \pi/2 \end{aligned}\right\}, \quad . \ . \ . \ (26)$$

we obtain the relation

$$Zn(v+\mathrm{K}+i\mathrm{K}', k) = Zn(v, k) - \frac{\mathrm{sn}(v, k)\,\mathrm{dn}(v, k)}{\mathrm{cn}(v, k)} - \frac{i\pi}{2\mathrm{K}}. \quad . \ (27)$$

Hence, multiplying (24) by i and subtracting from (27) and using (23 a), we obtain after re-arragement and cancelling,

$$Zn(u, k) = \frac{\pi}{2\mathrm{K}'}\left(1 - \frac{u}{\mathrm{K}}\right) + iZn(iu+\mathrm{K}'-i\mathrm{K}, k'). \quad . \ . \ (28)$$

If we now put

$$u = \mathrm{K}\psi_1 + i\mathrm{K}'\phi \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (29)$$

in equation (28) and re-arrange we obtain

$$Zn(\mathrm{K}\psi_1 + i\mathrm{K}'\phi, k) + \frac{i\pi}{2\mathrm{K}}\phi$$
$$= i\left[Zn\{\mathrm{K}'(1-\phi) - i\mathrm{K}(1-\psi_1), k'\} - \frac{i\pi}{2\mathrm{K}'}(1-\psi_1)\right]. \quad (30)$$

On account of this relation it is simpler to consider the real and imaginary parts of the left-hand side of (30) as the basic functions rather than the real and imaginary parts of the Zeta function itself. Further, it is convenient in view of existing tables, to write as usual

$$k = \sin\alpha, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ (31)$$

and regard α as the basic parameter for purposes of calculation. We therefore define functions f_1 and f_2 by the relation

$$f_1(\psi_1, \phi, \alpha) + if_2(\psi_1, \phi, \alpha) = Zn(\mathrm{K}\psi_1 + i\mathrm{K}'\phi, k) + \frac{i\pi}{2\mathrm{K}}\phi. \quad . \ . \ (32)$$

A similar relation holds, of course, with $-i$ for i, and if we then substitute in (30) for the Zeta functions from (32) we find the very simple reciprocal relations

$$\left.\begin{aligned} f_1(\psi_1, \phi, \alpha) &= f_2\left(1-\phi, 1-\psi_1, \frac{\pi}{2}-\alpha\right), \\ f_2(\psi_1, \phi, \alpha) &= f_1\left(1-\phi, 1-\psi_1, \frac{\pi}{2}-\alpha\right) \end{aligned}\right\}, \quad . \ . \ . \ (33)$$

which are of course essentially one and the same relation *. The numerical evaluation of f_1 and f_2 is considered later.

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* The simplicity of the reciprocal relation and its proof suggests that it is probably not new, although no mention could be found in standard references on elliptic functions which have been consulted.

2.5. *Final form of solution.*

The formal solution of our problem was given by equation (21) and using equations (22) and (32) this becomes

$$\begin{aligned} x+iy&=i+q\psi_1+iq\frac{K'}{K}\phi+AZn(K\psi_1+iK'\phi, k) \\ &=i+q\psi_1+iq\frac{K'}{K}\phi+Af_1(\psi_1, \phi, \alpha) \\ &\quad +iAf_2(\psi_1, \phi, \alpha)-\frac{iA\pi}{2K}\phi. \qquad (34) \end{aligned}$$

Hence, using the first of equations (15), the solution is

$$\left.\begin{aligned} x&=q\psi_1+Af_1(\psi_1, \phi, \alpha), \\ y&=1-\phi+Af_2(\psi_1, \phi, \alpha) \end{aligned}\right\}, \qquad (35)$$

or, using (33), we may express it solely in terms of the function f_1 as

$$\left.\begin{aligned} x&=q\psi_1+Af_1(\psi_1, \phi, \alpha), \\ y&=1-\phi+Af_1\left(1-\phi, 1-\psi_1, \frac{\pi}{2}-\alpha\right) \end{aligned}\right\}. \qquad (36)$$

2.6. *Limiting cases.*

(*a*) $\alpha\to 0$.

It can be shown from the equations (15) that $\alpha\to 0$ corresponds in fig. 1 to the physical condition $EA/BC\to 0$. The case $\alpha=0$ itself has no practical significance, but we have carried the calculations at this end of the range down to $\alpha=1°$ to cover the most extreme cases likely to occur in practice.

(*b*) $\alpha\to\pi/2$.

When $\alpha\to\pi/2$ it can be shown from equations (15) that in fig. 1 the ratio $EA/BC\to\infty$. As $\alpha\to\pi/2$ we thus tend to the limiting case, where the flow round either pile becomes independent of the presence of the other pile. In the limit the solution for such flow will then become that for the simple case of a single sheet pile, which is a special case of the weir problems considered by Khosla, Bose and McKenzie-Taylor (5). This limiting solution for the right-hand pile in fig. 2 may be written

$$\left.\begin{aligned} x_1&\to\psi'-\frac{1}{\pi\cos\theta}\cos\pi\phi\sinh\pi\psi', \\ y&\to 1-\phi+\frac{1}{\pi\cos\theta}\sin\pi\phi\cosh\pi\psi' \end{aligned}\right\} q\to\infty, \qquad (37)$$

where

$$\left.\begin{aligned} x_1&=x-q, \\ \psi'&=\frac{K}{K'}-\psi, \\ \tan\theta-\theta&=\pi s \end{aligned}\right\}. \qquad (38)$$

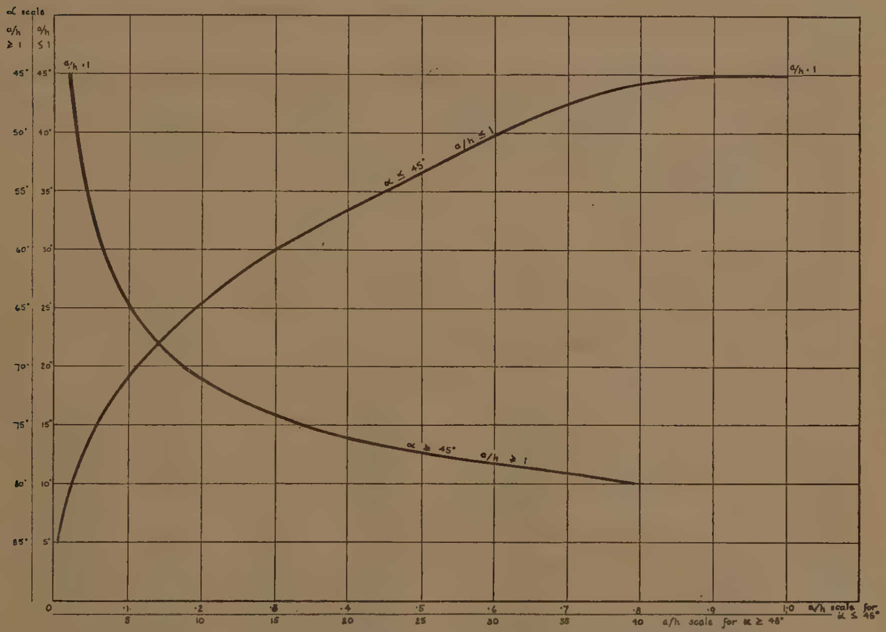
The use of x_1 merely means of course that we shift the origin for convenience to the point C, while similarly the introduction of ψ' means a revised choice of the arbitrary constant in the flow function to avoid a meaningless infinity, the pile CB being now chosen as the flow line $\psi'=0$.

Since equations (37) and (38) involve only relatively elementary functions the flow lines and equipotentials for the limiting case can be easily plotted.

3.1. *Numerical evaluation of parameters.*

It will be seen that our final form of solution as given by equation (36) involves the parameters A and α, which are to be evaluated from equations (15) and (16) in terms of the basic physical ratios $q=a/n$ and $s=c/h$.

Fig. 5.



Limit curves of the parameter α for given values of a/h.

For facility of computation it was better to invert this problem by commencing with given values of q and α. Thence, A was evaluated from the first of equations (15), β from the second of (15), ϵ from (16), and finally s from the third of (16).

It was revealed in the course of computation that for a given value of q the modular angle α has an upper limit determined by the physical condition that C and ($s=c/h$) cannot be negative. The calculations show that for given q the value of s decreases steadily as α increases, and at the upper limit for α we have $s=0$ corresponding to coincidence of the points A and B in figs. 1 and 2. This also implies coincidence of the points A′ and

B′ in fig. 3, so that at this upper limit for α, putting $\beta=k$ and eliminating A between the first and second of equations (15), the limiting value of α for given q is determined by the relation

$$q=\frac{E-k'^2K}{E'-k^2K'}. \qquad (39)$$

This limiting relation is plotted in fig. 5.

Initially it was decided to compute the parameters for the ranges $q=0{\cdot}5$ (0·5), 5·0 (1·0), 10·0; $\alpha=5^\circ\,(5^\circ)\alpha_1$, where α_1 is the limiting α given in fig. 5. Later it was decided to cover a wider range of physical conditions by calculating also for the ranges $q=0{\cdot}5$, 1·0, 1·5 ; $\alpha=1^\circ\,(1^\circ)\,4^\circ$ and $q=2{\cdot}0$; $\alpha^\circ=2^\circ$, 3°, 4°.

The computation of the parameters was carried out at the Building Research Station, using Jahnke and Emde's tables (11) of the elliptic integrals of the first and second kinds, which are accurate to four decimal places.

The results are given in Table I., where four significant figures have been retained in the tabulation of A and s, and these are accurate to within three units in the last digit. Values of ϵ are also given which are accurate to a maximum error of 0·05°.

TABLE I.

Values of the Parameters.

q	α (degrees)	ϵ (degrees)	A	s	ϕ_B	G
0·50	1	70·05	2·730	1·013	0·68	0·579
	2	69·20	2·510	0·806	0·64	0·666
	3	68·65	2·382	0·686	0·62	0·732
	4	68·30	2·291	0·601	0·59	0·790
	5	68·00	2·222	0·535	0·57	0·844
	10	67·50	2·011	0·331	0·50	1·11
	15	67·95	1·899	0·213	0·43	1·45
	20	69·20	1·829	0·132	0·37	1·96
	25	71·50	1·785	0·072	0·31	2·93
	30	75·35	1·760	0·031	0·23	5·48
1·00	1	67·80	4·460	2·092	0·70	0·290
	2	66·60	4·020	1·686	0·67	0·333
	3	65·80	3·763	1·449	0·65	0·366
	4	65·20	3·581	1·283	0·63	0·393
	5	64·75	3·441	1·154	0·61	0·419
	10	63·40	3·015	0·758	0·55	0·541
	15	63·05	2·780	0·530	0·50	0·677
	20	63·35	2·626	0·371	0·45	0·857
	25	64·40	2·520	0·251	0·41	1·12
	30	66·20	2·446	0·156	0·36	1·55
	35	69·20	2·397	0·083	0·30	2·42
	40	74·30	2·368	0·029	0·22	5·08
	45	90·00	2·361	0	0	∞

TABLE I. (*cont.*).
Values of the Parameters.

η	α (degrees)	ϵ (degrees)	A	s	ϕ_B	G
1·50	1	66·90	6·190	3·177	0·70	0·193
	2	65·45	5·529	2·571	0·68	0·221
	3	64·55	5·144	2·221	0·66	0·243
	4	63·85	4·871	1·973	0·64	0·262
	5	63·25	4·661	1·782	0·63	0·279
	10	61·50	4·019	1·198	0·57	0·358
	15	60·70	3·661	0·861	0·53	0·442
	20	60·60	3·423	0·627	0·49	0·548
	25	61·05	3·255	0·449	0·45	0·692
	30	62·10	3·133	0·308	0·41	0·900
	35	64·05	3·045	0·195	0·37	1·26
	40	67·15	2·986	0·105	0·31	1·96
	45	72·40	2·951	0·039	0·23	4·01
	50	85·65	2·939	0·000	0·06	7·87
2·00	2	64·85	7·039	3·459	0·68	0·166
	3	63·85	6·525	2·994	0·66	0·182
	4	63·10	6·161	2·666	0·65	0·196
	5	62·45	5·881	2·414	0·63	0·209
	10	60·40	5·023	1·641	0·58	0·267
	15	59·35	4·542	1·197	0·54	0·328
	20	58·95	4·220	0·889	0·51	0·403
	25	59·05	3·989	0·655	0·48	0·501
	30	59·75	3·819	0·469	0·44	0·638
	35	61·10	3·693	0·319	0·40	0·850
	40	63·35	3·602	0·196	0·36	1·22
	45	67·05	3·541	0·100	0·30	2·00
	50	73·50	3·507	0·029	0·21	4·79
2·50	5	61·90	7·100	3·047	0·64	0·167
	10	59·70	6·026	2·086	0·59	0·213
	15	58·45	5·423	1·536	0·55	0·261
	20	57·85	5·017	1·154	0·52	0·318
	25	57·75	4·724	0·865	0·49	0·392
	30	58·15	4·505	0·634	0·46	0·493
	35	59·15	4·341	0·447	0·43	0·642
	40	60·90	4·218	0·293	0·39	0·882
	45	63·75	4·131	0·169	0·34	1·33
	50	68·50	4·076	0·072	0·27	2·47
	55	78·25	4·051	0·009	0·14	10·4
3·00	5	61·55	8·320	3·680	0·64	0·139
	10	59·15	7·030	2·524	0·60	0·177
	15	57·80	6·304	1·876	0·56	0·217
	20	57·10	5·815	1·421	0·53	0·263
	25	56·80	5·459	1·076	0·50	0·322
	30	57·00	5·192	0·802	0·47	0·402
	35	57·75	4·995	0·580	0·44	0·514
	40	59·15	4·834	0·395	0·41	0·692
	45	61·45	4·721	0·244	0·37	1·00
	50	65·25	4·645	0·123	0·31	1·66
	55	72·20	4·602	0·035	0·22	4·09

TABLE I. (*cont.*).
Values of the Parameters.

q	α (degrees)	ϵ (degrees)	A	s	ϕ_B	G
3·50	5	61·25	9·540	4·314	0·65	0·119
	10	58·80	8·034	2·979	0·60	0·152
	15	57·35	7·185	2·216	0·57	0·185
	20	56·50	6·612	1·690	0·54	0·224
	25	56·10	6·194	1·289	0·51	0·274
	30	56·15	5·878	0·971	0·48	0·339
	35	56·70	5·636	0·712	0·46	0·431
	40	57·80	5·450	0·498	0·42	0·569
	45	59·75	5·312	0·322	0·39	0·800
	50	62·95	5·213	0·178	0·34	1·26
	55	68·50	5·153	0·069	0·26	2·54
	60	78·65	5·129	0·002	0·14	24·1
4·00	5	61·05	10·76	4·948	0·65	0·104
	10	58·50	9·038	3·426	0·60	0·133
	15	57·00	8·067	2·558	0·57	0·162
	20	56·05	7·409	1·958	0·54	0·196
	25	55·55	6·929	1·502	0·52	0·238
	30	55·45	6·565	1·142	0·49	0·293
	35	55·85	6·283	0·847	0·47	0·370
	40	56·80	6·067	0·603	0·44	0·484
	45	58·45	5·902	0·401	0·40	0·667
	50	61·15	5·782	0·235	0·36	1·01
	55	65·90	5·704	0·106	0·29	1·84
	60	75·45	5·666	0·018	0·17	6·63
4·50	5	60·90	11·98	5·582	0·65	0·0927
	10	58·30	10·04	3·873	0·61	0·118
	15	56·70	8·947	2·933	0·58	0·143
	20	55·65	8·207	2·227	0·55	0·173
	25	55·10	7·664	1·717	0·52	0·210
	30	54·90	7·251	1·313	0·50	0·258
	35	55·20	6·931	0·982	0·47	0·324
	40	55·95	6·683	0·709	0·45	0·420
	45	57·40	6·492	0·482	0·41	0·573
	50	59·75	6·351	0·294	0·37	0·841
	55	63·85	6·255	0·145	0·32	1·44
	60	71·60	6·202	0·038	0·22	3·84
5·00	5	60·75	13·20	6·217	0·65	0·0835
	10	58·10	11·05	4·322	0·61	0·106
	15	56·45	9·829	3·243	0·58	0·129
	20	55·35	9·004	2·496	0·55	0·155
	25	54·65	8·399	1·932	0·53	0·188
	30	54·50	7·938	1·485	0·50	0·231
	35	54·65	7·579	1·119	0·48	0·288
	40	55·25	7·299	0·816	0·45	0·372
	45	56·50	7·082	0·565	0·42	0·500
	50	58·65	6·920	0·355	0·39	0·722
	55	62·25	6·806	0·186	0·34	1·19
	60	68·85	6·739	0·061	0·25	2·71

TABLE I. (*cont.*).
Values of the Parameters.

q	α (degrees)	ϵ (degrees)	A	s	ϕ_B	G
6·00	5	60·55	15·64	7·486	0·65	0·0695
	10	57·80	13·05	5·218	0·61	0·0883
	15	56·05	11·59	3·928	0·58	0·107
	20	54·90	10·60	3·036	0·55	0·129
	25	54·15	9·869	2·363	0·53	0·156
	30	53·80	9·310	1·829	0·51	0·190
	35	53·80	8·874	1·394	0·49	0·236
	40	54·20	8·531	1·031	0·47	0·302
	45	55·20	8·263	0·730	0·44	0·400
	50	56·90	8·057	0·478	0·41	0·563
	55	59·80	7·908	0·274	0·36	0·877
	60	65·00	7·812	0·114	0·30	1·70
	65	77·05	7·769	0·010	0·15	8·67
7·00	5	60·40	18·08	8·755	0·65	0·0596
	10	57·60	15·06	6·114	0·61	0·0756
	15	55·80	13·35	4·613	0·58	0·0918
	20	54·55	12·19	3·577	0·56	0·110
	25	53·75	11·34	2·795	0·54	0·133
	30	53·25	10·68	2·173	0·52	0·162
	35	53·15	10·17	1·669	0·50	0·200
	40	53·40	9·763	1·248	0·48	0·254
	45	54·15	9·443	0·898	0·45	0·333
	50	55·60	9·195	0·605	0·42	0·461
	55	58·05	9·011	0·364	0·38	0·695
	60	62·40	8·885	0·174	0·33	1·24
	65	71·35	8·818	0·038	0·22	3·76
8·00	5	60·25	20·52	10·03	0·65	0·0521
	10	57·45	17·07	7·011	0·61	0·0661
	15	55·60	15·12	5·300	0·58	0·0802
	20	54·30	13·79	4·119	0·56	0·0964
	25	53·40	12·81	3·227	0·54	0·116
	30	52·90	12·06	2·520	0·52	0·141
	35	52·65	11·46	1·944	0·50	0·174
	40	52·80	11·00	1·467	0·48	0·219
	45	53·40	10·62	1·068	0·46	0·286
	50	54·60	10·33	0·733	0·43	0·390
	55	56·70	10·11	0·457	0·40	0·576
	60	60·40	9·959	0·236	0·35	0·978
	65	67·75	9·868	0·070	0·26	2·40
9·00	5	60·20	22·96	11·30	0·66	0·0463
	10	57·30	19·08	7·908	0·61	0·0587
	15	55·40	16·88	5·984	0·59	0·0712
	20	54·10	15·38	4·659	0·56	0·0855
	25	53·15	14·28	3·659	0·54	0·103
	30	52·55	13·43	2·866	0·52	0·125
	35	52·30	12·76	2·222	0·51	0·153
	40	52·35	12·23	1·686	0·49	0·193

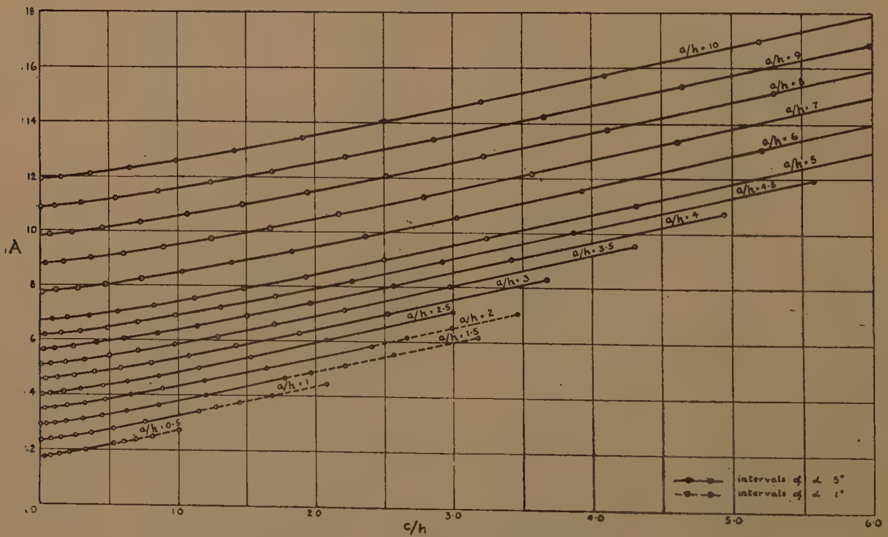
TABLE I. (*cont.*).

Values of the Parameters.

q	α (degrees)	ϵ (degrees)	A	s	ϕ_B	G
9·00	45	52·80	11·80	1·238	0·47	0·250
	50	53·80	11·47	0·862	0·44	0·339
	55	55·65	11·21	0·551	0·41	0·491
	60	58·90	11·03	0·298	0·37	0·806
	65	65·15	10·92	0·108	0·29	1·76
	70	85·25	10·88	0·000	0·05	70·9
10·00	5	60·10	25·40	12·57	0·66	0·0417
	10	57·20	21·08	8·804	0·62	0·0529
	15	55·30	18·64	6·671	0·59	0·0641
	20	53·95	16·98	5·202	0·57	0·0769
	25	52·95	15·75	4·093	0·55	0·0923
	30	52·30	14·80	3·213	0·53	0·112
	35	51·95	14·06	2·498	0·51	0·137
	40	51·95	13·46	1·903	0·49	0·172
	45	52·30	12·98	1·408	0·47	0·222
	50	53·15	12·61	0·993	0·45	0·299
	55	54·80	12·32	0·646	0·42	0·429
	60	57·65	12·11	0·364	0·38	0·685
	65	63·15	11·97	0·149	0·31	1·39
	70	77·15	11·91	0·010	0·15	8·94

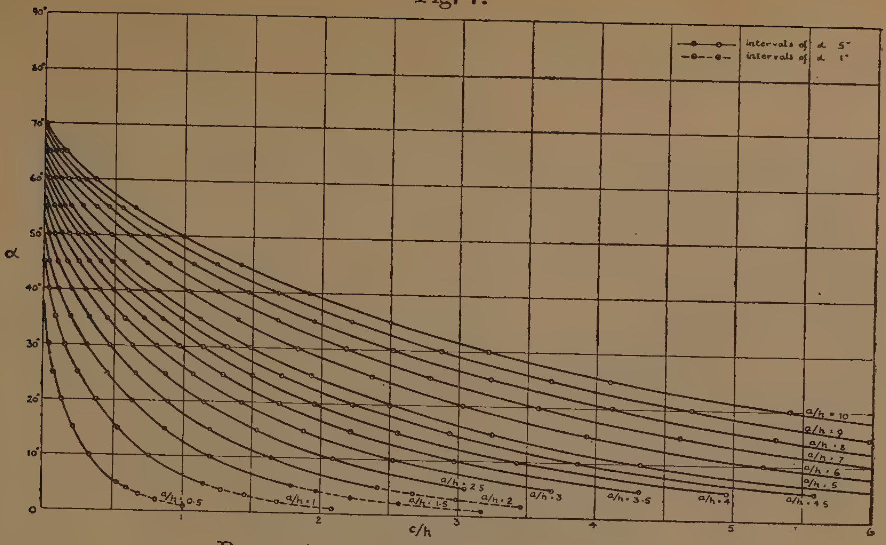
The parameters A, α, ϵ, β^2 are plotted in figs. 6–9 as functions of the physical ratios $q=a/h$, $s=c/h$.

Fig. 6.



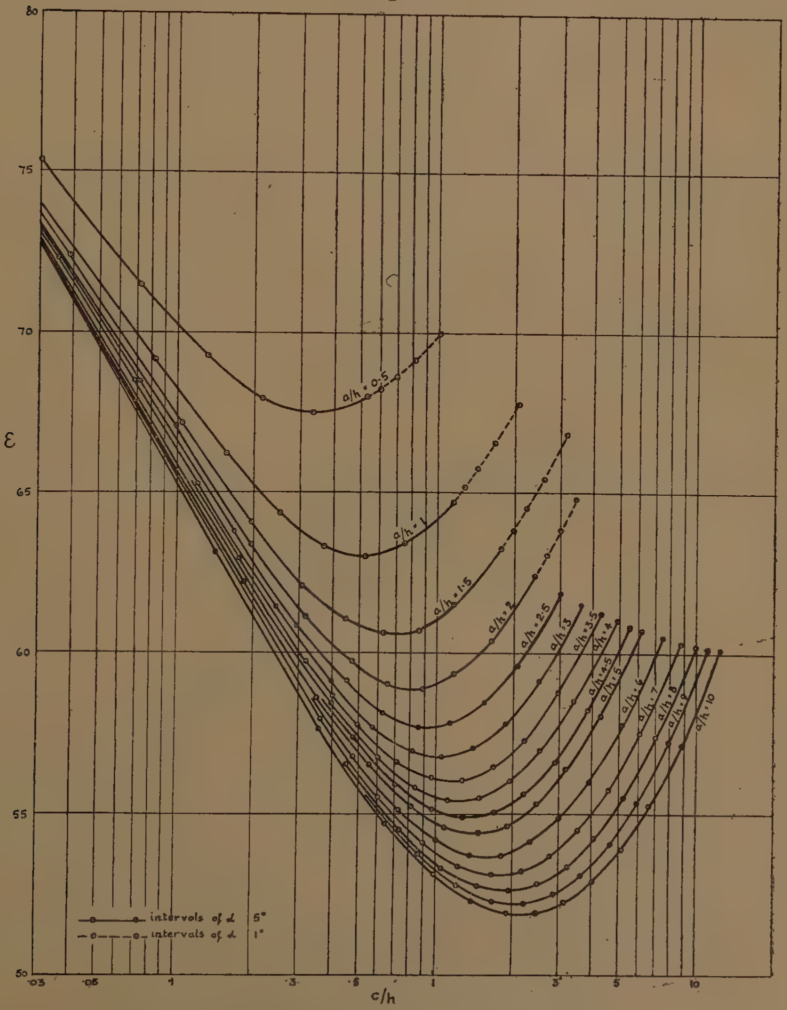
Parameter A versus c/h for constant a/h.

Fig. 7.



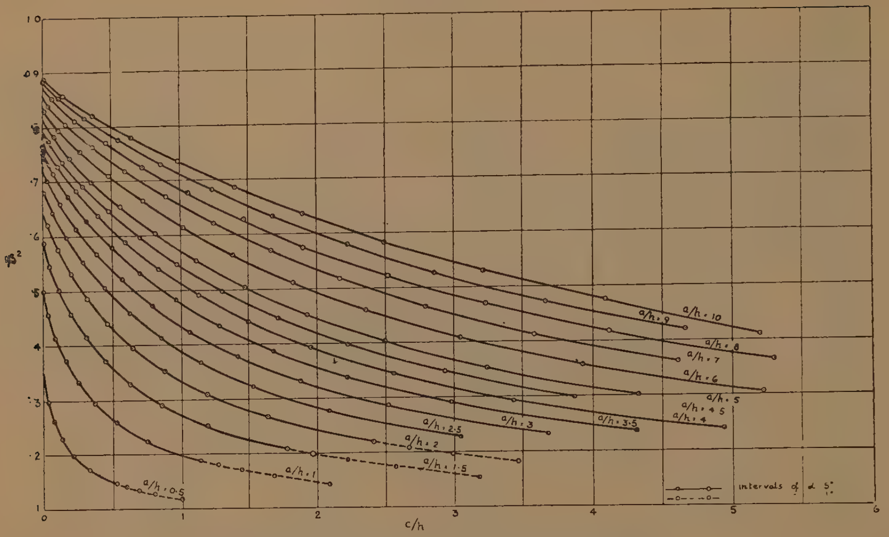
Parameter a versus c/h for constant a/h.

Fig. 8.



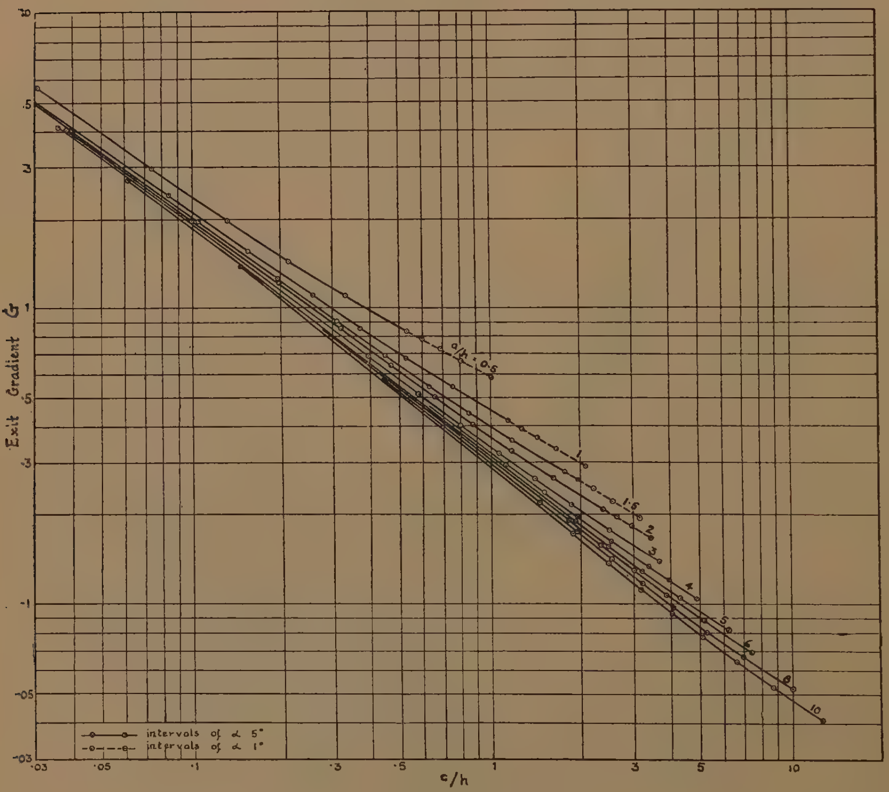
Parameter ϵ versus c/h for constant a/h.

Fig. 9.



Parameter β^2 versus c/h for constant a/h.

Fig. 10.



Exit gradient at point A.

3.2. *Hydraulic exit gradient.*

A quantity of particular physical interest is the hydraulic exit gradient at the point A (fig. 1) given by

$$G' = \frac{H}{h}\left(\frac{\partial\phi}{\partial y}\right)_A . \qquad (40)$$

This is best evaluated by considering the two equations (13) and (20) rather than equation (21).

Thus on ABC in fig. 2, corresponding to A′B′C′ in fig. 3, we have from (13)

$$\left.\begin{aligned} x &= q, \\ y &= 1 + A\int_1^{\xi}\frac{(\beta^2-k^2\xi^2)\,d\xi}{\sqrt{(\xi^2-1)(1-k^2\xi^2)}} \end{aligned}\right\} . \qquad (41)$$

While from (20)

$$\left.\begin{aligned} \psi &= K/K', \\ \phi &= \frac{1}{K'}\int_1^{\xi}\frac{d\xi}{\sqrt{(\xi^2-1(1-k^2\xi^2)}} \end{aligned}\right\} . \qquad (42)$$

Thus

$$\left.\begin{aligned} \frac{\partial\phi}{\partial\xi} &= \frac{1}{K'}\,\frac{1}{\sqrt{(\xi^2-1)(1-k^2\xi^2)}}, \\ \frac{\partial y}{\partial\xi} &= \frac{A(\beta^2-k^2\xi^2)}{\sqrt{(\xi^2-1)(1-k^2\xi^2)}} \end{aligned}\right\} \text{ along ABC.}$$

Whence

$$\frac{\partial\phi}{\partial y} = \frac{1}{AK'(\beta^2-k^2\xi^2)} \text{ along ABC.} \qquad (43)$$

But point A corresponds to A′ where $\xi=1$, whence from (40) and (43)

$$G = \frac{h}{H}G' = \frac{1}{AK'(\beta^2-k^2)} . \qquad (44)$$

Values of G are included in Table I. and are plotted in fig. 10. Three significant figures have been retained in the tabular values subject to a maximum error of two units in the last digit.

3.3. *Value of ϕ at pile tip.*

When calculating values of x and y along equipotentials there is an ambiguity for points along AB which is most easily resolved if the value of

ϕ at the pile tip B is known. Equipotentials with lower values of ϕ will then start from the left side of AB (fig. 2), while those with higher values of ϕ will start from the right-hand side.

Now the pile tip B corresponds to the point B′ in fig. 3, where $\xi=\beta/k$. Hence from (42) we have

$$\phi_B=\frac{1}{K'}\int_1^{\beta/k}\frac{d\xi}{\sqrt{(\xi^2-1)(1-k^2\xi^2)}}$$

$$=1-\frac{F(k',\epsilon)}{K'}, \quad \ldots \ldots \quad (45)$$

in terms of the parameter ϵ defined in (16).

Values of ϕ_B are included in Table I. and are given to two significant figures, subject to a maximum error of one unit in the second digit.

3.4. *Numerical evaluation of f_1 and f_2.*

Excluding the special cases $\alpha=0$, $\alpha=\pi/2$, which have been discussed already (§ 2.6), we require calculations for $0<\alpha<\pi/2$, $0\leqslant\phi\leqslant1$, $0\leqslant\psi_1\leqslant1$.

To obtain formulæ for f_1 and f_2 suitable for calculation purposes we start from the known Fourier series for the Jacobian Zeta function, namely

$$Zn(u, k)=\frac{\pi}{K}\sum_{n=1}^{\infty}\frac{\sin(n\pi u/K)}{\sinh(n\pi K'/K)}. \quad \ldots \ldots \quad (46)$$

If we now write

$$\left.\begin{aligned}\lambda&=\frac{\pi}{2}\frac{K'}{K},\\ \lambda'&=\frac{\pi}{2}\frac{K}{K'}\end{aligned}\right\}, \quad \ldots \ldots \quad (47)$$

and substitute (46) in equation (32), which defines f_1 and f_2, we obtain

$$\left.\begin{aligned}f_1(\psi_1, \phi, \alpha)&=\frac{\pi}{K}\sum_{n=1}^{\infty}\frac{\sin n\pi\psi_1\cosh 2n\lambda\phi}{\sinh 2n\lambda},\\ f_2(\psi_1, \phi, \alpha)&=\frac{\pi}{K}\left\{\frac{\phi}{2}+\sum_{n=1}^{\infty}\frac{\cos n\pi\psi_1\sinh 2n\lambda\phi}{\sinh 2n\lambda}\right\}\end{aligned}\right\} \quad \ldots \quad (48)$$

The series in (48) are absolutely convergent for $0\leqslant\phi<1$.

Similarly, by using the reciprocal relations (33) we can obtain alternative series by replacing K, λ, ψ_1, ϕ, in (48) by K′, λ', $1-\phi$ and $1-\psi_1$ respectively; these alternative series are absolutely convergent provided $\psi_1>0$. Expressed otherwise, we can regard (48) as giving the basic series and decide in any particular case whether it is quicker to calculate f_1 (or f_2) for given ψ_1, ϕ, α or to calculate the equal f_2 (or f_1) for arguments $(1-\phi)$, $(1-\psi_1)$ and complementary modular angle. This alternative holds whatever expressions we obtain for f_1 and f_2.

Even with this alternative, however, the series (48) are not very convenient when both $1-\phi$ and ψ_1 are small and more rapidly convergent series can be obtained by making, in effect, a subsidiary summation of the predominant contributions from each term of (48).

Thus, using a temporary restriction that $\phi<1$, i. e. excluding $\phi=1$, the series in (48) are absolutely convergent and expanding cosech $2n\lambda$ in each term as an absolutely convergent series in $e^{-2\eta\lambda}$, we can then re-group terms to obtain

$$f_1(\psi_1, \phi, \alpha)=\frac{\pi}{\mathrm{K}}\sum_{r=1}^{\infty}[f_3(2r-1-\phi, \psi_1)+f_3(2r-1+\phi, \psi_1)], \quad . \quad . \quad (49)$$

$$f_2(\psi_1, \phi, \alpha)=\frac{\pi\phi}{2\mathrm{K}}+\frac{\pi}{\mathrm{K}}\sum_{r=1}^{\infty}[f_4(2r-1-\phi, \psi_1)+f_4(2r-1+\phi, \psi_1)], \quad . \quad (50)$$

where

$$\left.\begin{aligned} f_3(\mu, \psi_1)&=\sum_{n=1}^{\infty} e^{-2n\lambda\mu}\sin n\pi\psi_1, \\ f_4(\mu, \psi_1)&=\sum_{n=1}^{\infty} e^{-2n\lambda\mu}\cos n\pi\psi_1 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

These series for f_3 and f_4 are absolutely convergent and can in fact be summed easily to give

$$f_3(\mu, \psi_1)=\frac{\sin \pi\psi_1}{2\{\cosh 2\lambda\mu-\cos \pi\psi_1\}}, \quad . \quad . \quad . \quad . \quad (52)$$

$$f_4(\mu, \psi_1)=\frac{\cos \pi\psi_1-e^{-2\lambda\mu}}{2\{\cosh 2\lambda\mu-\cos \pi\psi_1\}}. \quad . \quad . \quad . \quad . \quad (53)$$

These equations have been derived on the assumption that $\phi<1$, but they can easily be seen to be valid also for $\phi=1$ provided $\psi_1>0$. Thus the Jacobian Zeta function is analytic over the whole range of our calculations except for a simple pole at $i\mathrm{K}'$ corresponding to $\phi=1$, $\psi_1=0$. The functions f_1 and f_2 are thus finite and continuous except at this point where they become infinite.

Similarly, except at this point $\phi=1$, $\psi_1=0$, the functions f_3 and f_4 given by (52) and (53) are finite and continuous and so are the absolutely convergent series on the right-hand side of equations (49) and (50). Since both sides of these equations are thus finite and continuous except at the point $\phi=1$, $\psi_1=0$ and are equal for $0\leqslant\psi_1\leqslant1$, $0\leqslant\phi<1$, it follows by continuation that they will also be equal when $\phi=1$ provided $\psi_1>0$.

The singularity at $\phi=1$, $\psi_1=0$ gives infinite f_1 and f_2 corresponding physically to the fact that $\phi\rightarrow1$, $\psi_1\rightarrow0$ at all points distant from the cofferdam.

From equation (49) it is easily shown that the ratio of two successive terms is always $\leqslant e^{-4\lambda}$, and even when α is as large as 85° this ratio is $<0{\cdot}08$. As all terms are positive the series (49), with f_3 given by (52), is thus eminently suitable for calculation of f_1 over the whole ranges for ϕ and ψ_1 except for a small part of the upper end of the range for α. While individual terms in the series (50) for f_2 also converge ultimately with the same rapidity as those in the series (49) for f_1, the occurrence of both positive and negative terms slows down the overall convergence for calculation to a given percentage accuracy. For small values of α, however, the convergence of individual terms is so rapid, e. g. $e^{-4\lambda}<10^{-6}$ for $\alpha\leqslant 5°$, that partial cancellation of the leading terms has little influence on overall convergence. The series (50) is thus very suitable for calculation of f_2 when α is small, and by virtue of the reciprocal relations (33) it will thus give f_1 for $\alpha\sim 90°$ i. e. for that part of the range where the series (49) is tending to become slowly convergent. Together, therefore, the two series (49) and (50), combined with the reciprocal relations (33), give reasonable convergence for calculation of f_1 and f_2 over the whole ranges of α, ϕ and ψ_1.

Initially it was decided to cover the ranges $\alpha=5°$ (5°) 85°; $\phi=0$ (0·1) 1·0; $\psi_1=0$ (0·1) 1·0, and these calculations were carried out by the Mathematics Division of the National Physical Laboratory, using the series (49). Later it was decided to cover a wider range of physical parameters by extending the calculations to smaller values of α; $\alpha=1°$ (1°) 4°. These further calculations were carried out at the Building Research Station by using the series (49) for f_1 and (50) for f_2, the latter being then expressed as values of f_1 for $\alpha=86°$ (1°) 89° by use of the reciprocal relation (33).

The results of calculations are shown in Table II., which gives the function f_1 to three significant figures subject to a maximum error of unity in the third figure. By virtue of the reciprocal relations (33) this table could also be used to read off f_2 for the complementary α by insertion of scales for ϕ upwards and ψ_1 to left, commencing at the bottom right-hand corner of each sub-table for given α. It has been thought better, however, to give our solution in the final form (36), which involves f_1 only, and then to refer to f_1 only in the table to avoid the slight possibility of confusion and error which is always present in the use of a table with dual scales. We could equally have given these results in terms of f_2 only, but f_1 is slightly the more basic, since it is the real part of the Jacobian Zeta function, whereas we must subtract a linear ϕ term from f_2 to obtain the imaginary part of the Zeta function.

TABLE II.

Values of $f_1(\psi_1, \phi, a)$.

$a=1°$

ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	(—4) ·235	(—3) ·0389	(—3) ·105	(—3) ·307	(—2) ·0912	(—2) ·272	(—1) ·0820	(—1) ·255	·0882	·441	6·31
0·2	·448	·0739	·199	·585	·173	·517	·155	·480	·161	·698	3·08
0·3	·616	·102	·274	·805	·239	·710	·213	·649	·209	·761	1·96
0·4	·724	·120	·323	·946	·280	·832	·248	·746	·230	·709	1·38
0·5	·762	·126	·339	·994	·294	·873	·259	·766	·225	·606	1·00
0·6	·724	·120	·322	·945	·280	·828	·244	·712	·200	·485	·726
0·7	·616	·102	·274	·804	·238	·702	·206	·593	·161	·361	·509
0·8	·448	·0739	·199	·584	·173	·509	·149	·424	·112	·239	·325
0·9	·235	·0389	·105	·307	·0907	·267	·0780	·221	·0572	·119	·158
1·0	0	0	0	0	0	0	0	0	0	0	0

Note 1.—This is essentially a table of the complex Jacobian Zeta function (see equations (32) and (33) in text).

Note 2.—To avoid repetition of noughts after the decimal point throughout many of the columns, we use the notation that figures in parenthesis at the beginning of a column indicate the power of ten by which all values in this column are to be multiplied to give f_1.

TABLE II. (*cont.*).

$a=2°$

ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	(−3) ·0941	(−3) ·140	(−2) ·0321	(−2) ·0814	(−2) ·210	(−1) ·0548	(−1) ·145	·0402	·126	·579	6·31
0·2	·179	·266	·0610	·155	·400	·104	·275	·0752	·226	·870	3·08
0·3	·246	·366	·0840	·213	·549	·143	·374	·101	·287	·902	1·96
0·4	·290	·430	·0987	·250	·645	·167	·434	·114	·307	·809	1·38
0·5	·305	·452	·104	·263	·676	·174	·450	·116	·293	·674	1·00
0·6	·290	·430	·0986	·250	·642	·165	·422	·106	·256	·530	·726
0·7	·246	·366	·0839	·212	·545	·140	·355	·0878	·202	·390	·509
0·8	·179	·266	·0609	·154	·395	·101	·255	·0623	·139	·256	·325
0·9	·0941	·140	·0320	·0810	·208	·0530	·133	·0322	·0709	·127	·158
1·0	0	0	0	0	0	0	0	0	0	0	0

$a=3°$

ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	(−3) ·212	(−3) ·297	(−2) ·0620	(−2) ·144	(−1) ·0344	(−1) ·0829	(−1) ·204	·0530	·157	·688	6·31
0·2	·403	·564	·118	·274	·0653	·157	·385	·0984	·278	·994	3·08
0·3	·554	·777	·162	·377	·0896	·215	·523	·131	·347	·995	1·96
0·4	·652	·913	·190	·442	·105	·251	·603	·147	·364	·871	1·38
0·5	·685	·959	·200	·464	·110	·262	·622	·147	·342	·714	·999
0·6	·652	·912	·190	·441	·104	·247	·581	·134	·294	·556	·726
0·7	·554	·776	·162	·375	·0885	·208	·486	·110	·230	·407	·509
0·8	·403	·564	·117	·272	·0641	·151	·348	·0774	·157	·266	·325
0·9	·212	·296	·0617	·143	·0337	·0789	·182	·0400	·0798	·131	·158
1·0	0	0	0	0	0	0	0	0	0	0	0

TABLE II. (*cont.*).

$\alpha=4^\circ$

ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	(—2) ·0377	(—2) ·0507	(—2) ·0998	(—2) ·217	(—1) ·0488	(—1) ·111	(—1) ·261	·0649	·185	·781	6·31
0·2	·0716	·0965	·188	·411	·0925	·211	·491	·120	·324	1·09	3·07
0·3	·0986	·133	·259	·566	·127	·288	·664	·158	·397	1·07	1·96
0·4	·116	·156	·304	·664	·149	·335	·763	·176	·410	·916	1·37
0·5	·122	·164	·320	·696	·155	·349	·783	·175	·381	·742	·999
0·6	·116	·156	·304	·661	·147	·328	·727	·158	·324	·574	·726
0·7	·0985	·133	·258	·561	·125	·276	·605	·128	·252	·418	·509
0·8	·0716	·0963	·187	·407	·0902	·199	·433	·0900	·171	·272	·325
0·9	·0376	·0506	·0985	·214	·0473	·104	·225	·0463	·0863	·134	·158
1·0	0	0	0	0	0	0	0	0	0	0	0

$\alpha=5^\circ$

ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	(—2) ·0588	(—2) ·0770	(—2) ·143	(—2) ·297	(—1) ·0638	(—1) ·140	(—1) ·316	·0759	·210	·867	6·30
0·2	·112	·146	·271	·565	·121	·265	·593	·140	·365	1·18	3·07
0·3	·154	·201	·373	·776	·166	·362	·799	·183	·441	1·12	1·96
0·4	·181	·237	·438	·910	·194	·420	·913	·202	·450	·951	1·37
0·5	·190	·249	·460	·954	·203	·436	·933	·199	·413	·764	·998
0·6	·181	·236	·437	·905	·192	·409	·862	·178	·348	·587	·725
0·7	·154	·201	·371	·767	·162	·344	·715	·144	·269	·426	·509
0·8	·112	·146	·269	·556	·117	·247	·510	·101	·182	·277	·324
0·9	·0587	·0767	·141	·292	·0614	·129	·265	·0517	·0916	·137	·158
1·0	0	0	0	0	0	0	0	0	0	0	0

TABLE II. (*cont.*).

$\alpha = 10°$

ϕ / ψ_1	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	(—2) ·236	(—2) ·284	(—1) ·0449	(—1) ·0804	(—1) ·151	(—1) ·292	·0589	·128	·326	1·22	6·27
0·2	·448	·539	·0853	·152	·285	·549	·109	·230	·539	1·48	3·05
0·3	·616	·742	·117	·209	·389	·740	·144	·291	·616	1·31	1·95
0·4	·723	·870	·137	·244	·451	·848	·162	·311	·596	1·06	1·37
0·5	·760	·913	·144	·254	·467	·868	·161	·297	·525	·826	·992
0·6	·722	·867	·136	·240	·438	·804	·146	·258	·429	·625	·721
0·7	·613	·736	·115	·203	·368	·668	·119	·204	·324	·449	·506
0·8	·445	·534	·0836	·146	·265	·476	·0838	·140	·216	·290	·322
0·9	·234	·281	·0439	·0768	·138	·248	·0432	·0714	·108	·142	·157
1·0	0	0	0	0	0	0	0	0	0	0	0

$\alpha = 15°$

ϕ / ψ_1	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	(—1) ·0531	(—1) ·0613	(—1) ·0890	(—1) ·146	(—1) ·253	·0457	·0866	·178	·433	1·51	6·21
0·2	·101	·116	·169	·275	·475	·0850	·158	·313	·685	1·68	3·03
0·3	·139	·160	·231	·375	·643	·113	·206	·385	·747	1·41	1·93
0·4	·163	·187	·270	·436	·740	·128	·225	·399	·696	1·11	1·35
0·5	·170	·196	·282	·452	·759	·129	·221	·370	·595	·853	·983
0·6	·162	·186	·266	·425	·706	·118	·196	·316	·477	·640	·714
0·7	·137	·158	·225	·357	·588	·0972	·158	·245	·355	·457	·501
0·8	·0994	·114	·162	·257	·421	·0688	·110	·167	·235	·295	·319
0·9	·0522	·0599	·0851	·134	·219	·0356	·0562	·0843	·117	·145	·156
1·0	0	0	0	0	0	0	0	0	0	0	0

TABLE II. (*cont.*).

$\alpha=20°$

ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	(−1) ·0946	(−1) ·107	(−1) ·146	(−1) ·224	·0368	·0633	·115	·229	·536	1·77	6·12
0·2	·179	·202	·276	·422	·0688	·117	·208	·392	·813	1·83	2·98
0·3	·246	·277	·376	·573	·0924	·154	·265	·469	·851	1·47	1·90
0·4	·288	·323	·438	·661	·105	·172	·284	·472	·768	1·14	1·33
0·5	·301	·338	·455	·682	·107	·171	·273	·429	·642	·866	·970
0·6	·285	·319	·428	·636	·0986	·154	·239	·359	·507	·646	·705
0·7	·242	·270	·360	·532	·0815	·125	·190	·276	·374	·459	·494
0·8	·175	·195	·260	·381	·0580	·0881	·131	·186	·246	·296	·315
0·9	·0918	·102	·136	·199	·0301	·0453	·0667	·0934	·122	·145	·154
1·0	0	0	0	0	0	0	0	0	0	0	0

$\alpha=25°$

ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	(−1) ·148	(−1) ·164	(−1) ·215	(−1) ·316	·0496	·0822	·145	·280	·638	2·00	6·01
0·2	·281	·310	·406	·592	·0923	·151	·257	·469	·927	1·94	2·93
0·3	·384	·424	·552	·799	·123	·196	·322	·545	·935	1·51	1·87
0·4	·449	·494	·639	·916	·138	·215	·339	·535	·821	1·15	1·31
0·5	·468	·514	·661	·937	·139	·211	·320	·476	·675	·868	·954
0·6	·442	·484	·619	·868	·127	·188	·276	·393	·527	·644	·693
0·7	·374	·408	·519	·722	·104	·152	·217	·298	·386	·458	·486
0·8	·270	·295	·373	·515	·0738	·106	·148	·200	·253	·294	·310
0·9	·141	·154	·195	·268	·0381	·0541	·0752	·0999	·125	·144	·151
1·0	0	0	0	0	0	0	0	0	0	0	0

TABLE II. (*cont.*).

$a=30°$

ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	(—1) ·214	(—1) ·234	(—1) ·297	·0420	·0638	·103	·176	·332	·740	2·20	5·88
0·2	·406	·442	·559	·0785	·118	·186	·308	·543	1·03	2·00	2·87
0·3	·554	·602	·757	·105	·155	·238	·378	·614	1·00	1·53	1·83
0·4	·644	·699	·872	·120	·173	·258	·391	·589	·860	1·15	1·28
0·5	·670	·724	·897	·122	·173	·250	·362	·514	·696	·863	·933
0·6	·630	·679	·836	·112	·156	·220	·308	·418	·539	·638	·678
0·7	·531	·571	·698	·0924	·127	·176	·239	·315	·392	·453	·476
0·8	·383	·411	·500	·0656	·0892	·122	·163	·210	·256	·290	·303
0·9	·200	·215	·260	·0340	·0459	·0620	·0821	·105	·126	·142	·148
1·0	0	0	0	0	0	0	0	0	0	0	0

$a=35°$

ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	(—1) ·293	(—1) ·317	·0392	·0538	·0794	·125	·209	·386	·840	2·38	5·73
0·2	·554	·597	·0735	·100	·146	·223	·360	·615	1·12	2·05	2·79
0·3	·754	·810	·0991	·133	·190	·282	·432	·676	1·06	1·53	1·78
0·4	·873	·936	·114	·150	·209	·301	·438	·634	·887	1·14	1·25
0·5	·904	·966	·116	·151	·206	·287	·399	·544	·709	·851	·910
0·6	·847	·903	·107	·138	·184	·250	·335	·438	·544	·628	·661
0·7	·711	·755	·0893	·113	·149	·197	·258	·327	·394	·445	·464
0·8	·511	·542	·0637	·0801	·104	·136	·175	·217	·256	·285	·296
0·9	·267	·283	·0331	·0414	·0534	·0691	·0877	·108	·126	·139	·144
1·0	0	0	0	0	0	0	0	0	0	0	0

TABLE II. (*cont.*).

$a=40°$

ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	·0386	·0413	·0501	·0670	·0964	·148	·243	·442	·937	2·53	5·55
0·2	·0727	·0777	·0936	·124	·175	·262	·412	·684	1·20	2·07	2·71
0·3	·0985	·105	·125	·163	·226	·326	·484	·730	1·09	1·51	1·73
0·4	·114	·121	·143	·183	·246	·342	·480	·670	·901	1·12	1·21
0·5	·117	·124	·145	·182	·240	·322	·431	·567	·713	·834	·883
0·6	·109	·115	·133	·165	·212	·277	·358	·451	·544	·615	·642
0·7	·0911	·0957	·110	·135	·170	·217	·273	·335	·393	·435	·451
0·8	·0653	·0685	·0782	·0947	·118	·148	·184	·221	·255	·279	·288
0·9	·0340	·0356	·0405	·0488	·0604	·0752	·0921	·110	·125	·136	·140
1·0	0	0	0	0	0	0	0	0	0	0	0

$a=45°$

ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	·0493	·0524	·0625	·0817	·115	·173	·280	·500	1·04	2·66	5·35
0·2	·0925	·0982	·116	·150	·207	·302	·464	·750	1·27	2·07	2·61
0·3	·125	·132	·155	·196	·264	·370	·533	·777	1·12	1·49	1·67
0·4	·143	·151	·174	·217	·284	·381	·518	·698	·907	1·09	1·17
0·5	·146	·154	·176	·215	·273	·354	·458	·583	·711	·814	·854
0·6	·136	·142	·160	·193	·239	·301	·376	·460	·540	·599	·622
0·7	·113	·117	·132	·156	·190	·234	·285	·339	·389	·424	·437
0·8	·0805	·0837	·0932	·109	·132	·159	·191	·223	·252	·272	·279
0·9	·0418	·0434	·0482	·0561	·0670	·0805	·0955	·111	·124	·133	·136
1·0	0	0	0	0	0	0	0	0	0	0	0

TABLE II. (*cont.*).

$\alpha=50^\circ$

ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	·0616	·0651	·0765	·0981	·136	·200	·319	·559	1·13	2·75	5·13
0·2	·115	·121	·141	·179	·242	·344	·516	·811	1·32	2·05	2·50
0·3	·154	·162	·187	·232	·303	·413	·578	·817	1·13	1·45	1·60
0·4	·176	·184	·209	·253	·322	·419	·551	·719	·905	1·06	1·13
0·5	·179	·186	·209	·248	·305	·383	·480	·592	·703	·789	·821
0·6	·164	·170	·189	·220	·265	·323	·391	·464	·532	·581	·599
0·7	·136	·140	·154	·177	·209	·249	·294	·341	·382	·411	·421
0·8	·0965	·0995	·109	·123	·144	·169	·196	·224	·247	·264	·269
0·9	·0500	·0515	·0560	·0633	·0732	·0851	·0980	·111	·122	·129	·132
1·0	0	0	0	0	0	0	0	0	0	0	0

$\alpha=55^\circ$

ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	·0757	·0797	·0924	·117	·158	·230	·360	·620	1·23	2·82	4·88
0·2	·141	·148	·170	·211	·279	·389	·569	·868	1·36	2·01	2·38
0·3	·188	·196	·222	·269	·344	·457	·620	·848	1·13	1·40	1·53
0·4	·212	·220	·246	·291	·359	·455	·580	·732	·894	1·02	1·08
0·5	·213	·221	·243	·281	·337	·410	·498	·596	·691	·761	·787
0·6	·195	·200	·218	·248	·289	·342	·402	·464	·521	·560	·575
0·7	·160	·164	·177	·198	·227	·262	·301	·340	·374	·397	·405
0·8	·113	·116	·124	·137	·155	·177	·200	·223	·242	·255	·259
0·9	·0583	·0597	·0637	·0702	·0788	·0889	·0998	·110	·119	·125	·127
1·0	0	0	0	0	0	0	0	0	0	0	0

Table II. (*cont.*).

$\alpha=60^\circ$

ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	·0921	·0965	·111	·138	·183	·263	·404	·685	1·32	2·85	4·61
0·2	·170	·178	·202	·246	·318	·435	·620	·918	1·38	1·95	2·25
0·3	·225	·233	·261	·310	·386	·499	·657	·869	1·12	1·35	1·45
0·4	·251	·259	·285	·330	·396	·487	·602	·737	·874	·982	1·02
0·5	·250	·257	·279	·315	·367	·433	·511	·595	·673	·730	·750
0·6	·226	·231	·248	·275	·312	·358	·409	·461	·507	·538	·550
0·7	·184	·188	·199	·218	·243	·273	·305	·337	·364	·382	·388
0·8	·129	·132	·139	·150	·166	·184	·202	·221	·236	·246	·249
0·9	·0666	·0677	·0712	·0766	·0838	·0921	·101	·109	·116	·120	·122
1·0	0	0	0	0	0	0	0	0	0	0	0

$\alpha=65^\circ$

ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	·111	·116	·132	·162	·212	·299	·452	·753	1·40	2·84	4·31
0·2	·204	·212	·238	·285	·362	·482	·670	·962	1·38	1·87	2·11
0·3	·266	·275	·303	·353	·430	·540	·690	·882	1·10	1·29	1·36
0·4	·293	·302	·327	·370	·433	·517	·619	·735	·848	·933	·966
0·5	·289	·295	·315	·349	·395	·453	·520	·589	·651	·695	·711
0·6	·258	·263	·277	·301	·333	·371	·413	·455	·491	·515	·524
0·7	·208	·211	·221	·237	·257	·282	·307	·332	·353	·367	·371
0·8	·145	·147	·153	·163	·175	·189	·204	·218	·229	·236	·239
0·9	·0745	·0755	·0782	·0826	·0882	·0947	·101	·108	·113	·116	·117
1·0	0	0	0	0	0	0	0	0	0	0	0

TABLE II. (*cont.*).

$\alpha=70°$

ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	·134	·139	·157	·190	·246	·340	·506	·823	1·47	2·80	3·97
0·2	·243	·252	·279	·329	·409	·533	·719	·996	1·37	1·77	1·96
0·3	·313	·322	·350	·400	·475	·579	·716	·884	1·06	1·21	1·27
0·4	·339	·347	·371	·411	·469	·543	·631	·726	·815	·881	·905
0·5	·329	·335	·352	·382	·421	·470	·524	·578	·626	·659	·671
0·6	·290	·294	·306	·326	·352	·382	·415	·446	·473	·490	·497
0·7	·232	·234	·242	·254	·270	·289	·308	·326	·341	·351	·354
0·8	·161	·162	·167	·174	·183	·193	·204	·214	·222	·227	·229
0·9	·0820	·0827	·0847	·0879	·0920	·0967	·101	·106	·109	·112	·112
1·0	0	0	0	0	0	0	0	0	0	0	0

$\alpha=75°$

ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	·163	·169	·188	·225	·286	·389	·568	·896	1·53	2·70	3·60
0·2	·291	·300	·329	·381	·463	·586	·767	1·02	1·33	1·64	1·78
0·3	·367	·376	·403	·451	·521	·616	·736	·874	1·01	1·13	1·17
0·4	·390	·397	·418	·454	·503	·565	·636	·710	·776	·824	·841
0·5	·371	·375	·390	·414	·445	·483	·524	·564	·597	·621	·629
0·6	·322	·325	·334	·349	·368	·390	·414	·436	·454	·466	·470
0·7	·254	·256	·261	·270	·281	·294	·307	·319	·329	·336	·338
0·8	·175	·176	·179	·183	·189	·196	·203	·210	·215	·218	·219
0·9	·0887	·0891	·0904	·0925	·0952	·0982	·101	·104	·106	·108	·108
1·0	0	0	0	0	0	0	0	0	0	0	0

TABLE II. (*cont.*).

$\alpha=80°$

ϕ / ψ_1	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	·202	·208	·228	·271	·339	·451	·640	·977	1·58	2·54	3·18
0·2	·353	·362	·391	·444	·525	·644	·808	1·02	1·27	1·50	1·59
0·3	·431	·440	·465	·508	·569	·650	·746	·850	·951	1·03	1·05
0·4	·446	·451	·469	·497	·535	·582	·633	·686	·730	·761	·772
0·5	·413	·417	·427	·444	·466	·493	·520	·546	·567	·582	·587
0·6	·352	·354	·360	·370	·382	·396	·410	·424	·435	·442	·444
0·7	·274	·275	·278	·283	·290	·297	·305	·312	·317	·321	·322
0·8	·186	·187	·189	·191	·195	·198	·202	·206	·208	·210	·211
0·9	·0942	·0944	·0951	·0962	·0977	·0992	·101	·102	·103	·104	·104
1·0	0	0	0	0	0	0	0	0	0	0	0

$\alpha=85°$

ϕ / ψ_1	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	·264	·272	·297	·343	·419	·543	·741	1·07	1·57	2·26	2·64
0·2	·443	·453	·481	·533	·606	·708	·841	1·00	1·16	1·30	1·35
0·3	·516	·523	·543	·576	·622	·680	·743	·809	·866	·909	·922
0·4	·509	·513	·524	·543	·566	·595	·624	·652	·675	·692	·697
0·5	·456	·458	·464	·473	·485	·498	·512	·524	·534	·541	·544
0·6	·379	·380	·383	·387	·393	·399	·405	·411	·416	·419	·420
0·7	·290	·291	·292	·294	·296	·299	·302	·305	·307	·308	·309
0·8	·195	·196	·196	·197	·198	·200	·201	·202	·203	·204	·204
0·9	·0982	·0983	·0985	·0989	·0993	·0998	·100	·101	·101	·101	·101
1·0	0	0	0	0	0	0	0	0	0	0	0

TABLE II. (*cont.*).

$a=86°$

ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	·284	·292	·318	·365	·444	·569	·768	1·09	1·58	2·18	2·50
0·2	·469	·478	·506	·555	·626	·724	·847	·990	1·14	1·25	1·29
0·3	·537	·543	·562	·592	·633	·684	·740	·796	·846	·880	·892
0·4	·524	·527	·537	·552	·573	·596	·621	·644	·664	·676	·681
0·5	·465	·467	·471	·479	·488	·499	·510	·520	·528	·533	·535
0·6	·384	·385	·387	·390	·395	·399	·404	·409	·412	·414	·415
0·7	·293	·293	·294	·296	·297	·300	·302	·304	·305	·306	·307
0·8	·197	·197	·197	·198	·199	·200	·201	·201	·202	·203	·203
0·9	·0988	·0988	·0990	·0992	·0995	·0999	·100	·101	·101	·101	·101
1·0	0	0	0	0	0	0	0	0	0	0	0

$a=87°$

ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	·308	·317	·343	·393	·473	·600	·798	1·11	1·56	2·08	2·35
0·2	·500	·508	·535	·582	·649	·739	·850	·976	1·10	1·19	1·23
0·3	·562	·567	·583	·610	·645	·689	·736	·782	·823	·850	·860
0·4	·539	·542	·550	·562	·579	·598	·617	·636	·651	·660	·664
0·5	·474	·475	·479	·484	·491	·499	·507	·515	·521	·525	·526
0·6	·389	·389	·391	·393	·396	·400	·403	·406	·409	·410	·411
0·7	·295	·295	·296	·297	·298	·300	·301	·302	·304	·304	·304
0·8	·198	·198	·198	·199	·199	·200	·200	·201	·201	·202	·202
0·9	·0993	·0993	·0994	·0995	·0997	·0999	·100	·100	·100	·101	·101
1·0	0	0	0	0	0	0	0	0	0	0	0

TABLE II. (*cont.*).

$a=88°$

ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	·341	·350	·378	·429	·511	·639	·833	1·12	1·52	1·96	2·16
0·2	·539	·547	·572	·615	·676	·756	·851	·955	1·05	1·13	1·15
0·3	·590	·594	·608	·630	·659	·693	·728	·766	·796	·816	·823
0·4	·556	·558	·564	·573	·585	·599	·613	·626	·637	·644	·646
0·5	·483	·483	·486	·490	·494	·500	·505	·510	·514	·517	·517
0·6	·393	·393	·394	·396	·398	·400	·402	·404	·405	·406	·407
0·7	·297	·297	·298	·298	·299	·300	·301	·301	·302	·302	·303
0·8	·199	·199	·199	·199	·200	·200	·200	·201	·201	·201	·201
0·9	·0996	·0997	·0997	·0998	·0999	·100	·100	·100	·100	·100	·100
1·0	0	0	0	0	0	0	0	0	0	0	0

$a=89°$

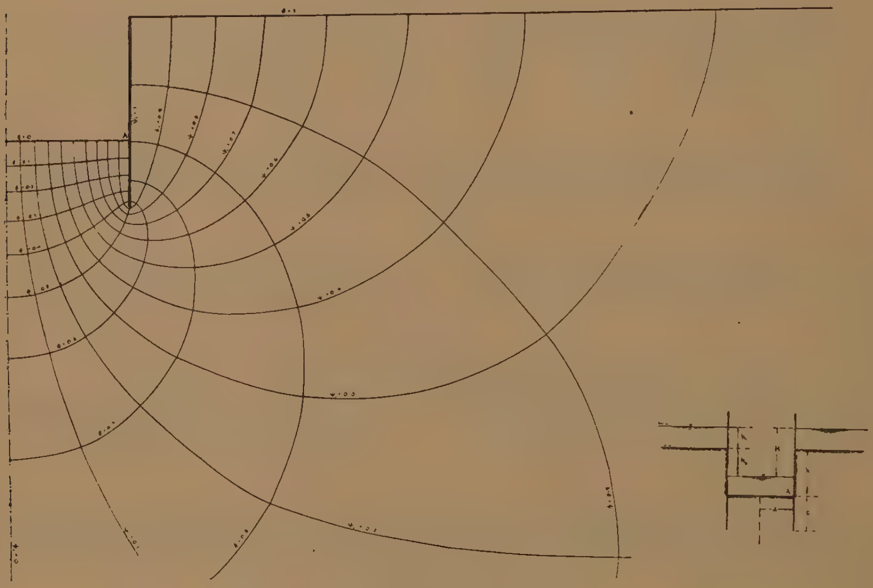
ψ_1 \ ϕ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0	0	0	0	0	0	0	0	0	0	∞
0·1	·396	·405	·434	·487	·570	·696	·879	1·14	1·46	1·78	1·92
0·2	·596	·603	·625	·661	·711	·774	·847	·923	·991	1·04	1·06
0·3	·626	·629	·639	·654	·674	·697	·721	·744	·763	·775	·780
0·4	·574	·576	·579	·585	·592	·600	·608	·615	·621	·625	·626
0·5	·491	·492	·493	·495	·497	·500	·503	·505	·507	·508	·509
0·6	·397	·397	·398	·398	·399	·400	·401	·402	·402	·403	·403
0·7	·299	·299	·299	·299	·300	·300	·300	·301	·301	·301	·301
0·8	·200	·200	·200	·200	·200	·200	·200	·200	·200	·200	·200
0·9	·0999	·0999	·0999	·0999	·100	·100	·100	·100	·100	·100	·100
1·0	0	0	0	0	0	0	0	0	0	0	0

4. Conclusion.

The results given in Tables I. and II. and figs. 6 and 7 enable the equipotential and flow patterns to be plotted for different physical conditions as illustrated in figs. 11–14. Figs. 11 and 12 typify cofferdams of normal dimensions, fig. 13 a cofferdam of deep penetration and fig. 14 a cofferdam with no penetration.

Pressures in the medium obtained from our results may be used in conjuction with the usual earth-pressure theories to investigate the stability of the cofferdam walls. A further possible application lies in the consideration of the stability of the cofferdam floor. For instance, the failure of the floor may not necessarily commence with failure of infinitesimal element at A, for which the exit gradient in fig. 10 is a relevant criterion ; instead, failure may first occur due to uplift of a finite mass of medium as a whole, and a knowledge of the potential and flow pattern is necessary to investigate this possibility.

Fig. 11.

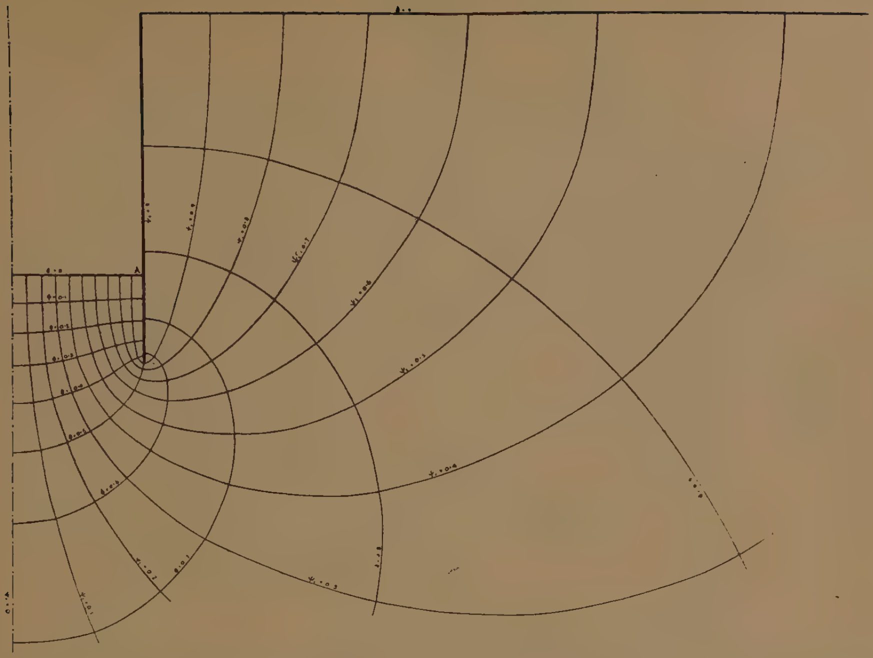


Equipotential and flow lines for
$a/h=1{\cdot}0$ $(\alpha=15^\circ)$ $c/h=0{\cdot}530$.

Exit gradient at point $A=0{\cdot}677\ H/h$.

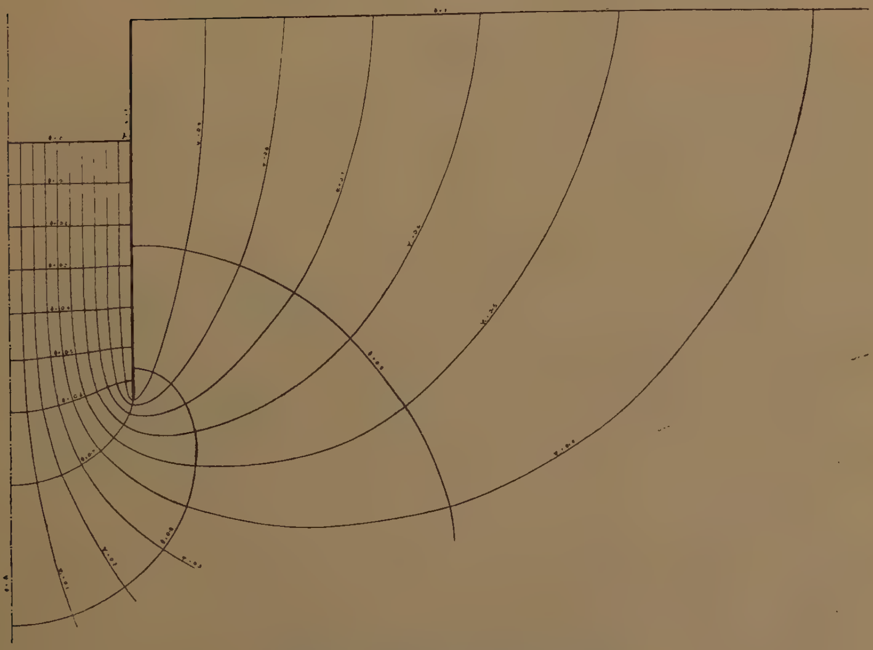
Finally, the analysis and numerical results given for the complex Jacobian Zeta function may prove of interest and application in fields other than the specific problem of seepage for which the results were derived

Fig. 12.


A
φ=0
φ=0·1
φ=0·2
φ=0·3
φ=0·4
φ=0·5
φ=0·6
φ=0·7
ψ₁=0·1
ψ₁=0·2
ψ₁=0·3
ψ₁=0·4
ψ₁=0·5
ψ₁=0·6
ψ₁=0·7
ψ₁=0·8
ψ₁=0·9
ψ₁=0·0

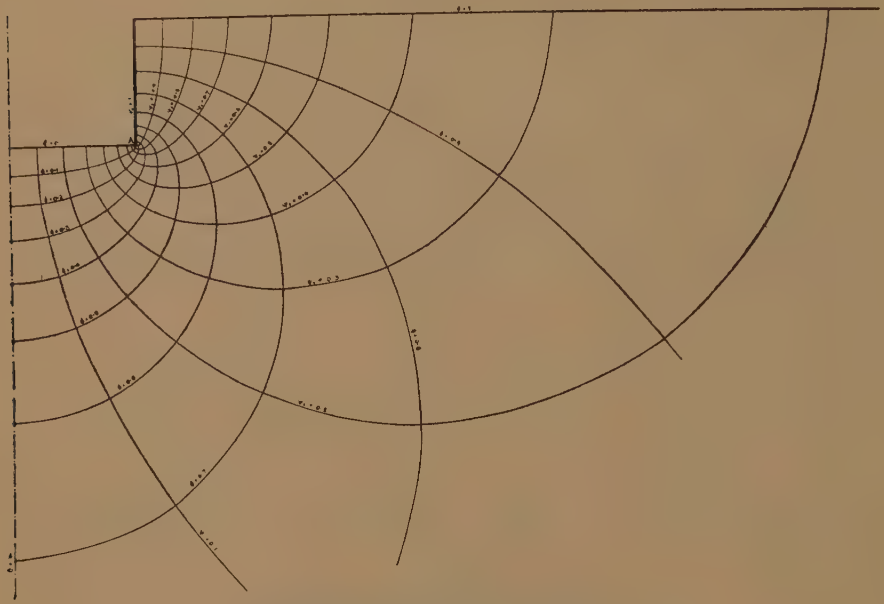
Equipotential and flow lines for
$a/h=0{\cdot}5$ ($\alpha=10°$) $c/h=0{\cdot}331$.
Exit gradient at point A=1·11 H/*h*.

Fig. 13.



Equipotential and flow lines for
$a/h=1{\cdot}0$ ($\alpha=1°$) $c/h=2{\cdot}09$
Exit gradient at point A=0·290 H/*h*.

Fig. 14.



Equipotential and flow lines for
$a/h=1{\cdot}0 \quad (\alpha=45^\circ) \quad c/h=0$
Exit gradient at point A is infinite.

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XXVI. *A Note on Duhamel's Integral, with Particular Reference to the Transient Response of Cascaded Filter Sections.*

By D. G. TUCKER, Ph.D., A.M.I.E.E. (Post Office Research Station) *.

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SUMMARY.

THE superposition theorem is generally applied by means of Duhamel's integral, and it is shown that the solution of practical transient-response problems may be aided by the use of power-series in this integral. This enables a generalized power-series expression of the output transient from cascaded circuits to be obtained, and the method is used to determine the build-up response of cascades of Zobel-type filter sections.

1. INTRODUCTION.

Transient problems in transmission circuits are rarely simple, and the mathematical treatment of the transient response of circuits such as filters is usually either very complicated, and as a result of little practical value, or alternatively very approximate. A type of problem of frequent occurrence in practice is the transient response of two circuits in cascade. It is required to find, let us say, the build-up curve (*i. e.* amplitude against time) at the output of the second circuit when a Heaviside unit step signal is applied at the input of the first. Provided the response of both circuits individually to a unit step signal is known, the desired result, $F_2(t)$, is given by Duhamel's Integral, based on the superposition theorem (1) :—

$$F_2(t)=\int_0^t E(u)\frac{d}{d(t-u)}\phi(t-u)du, \qquad \ldots \ldots (1)$$

where $E(t)$ is the response of the first circuit,
$\phi(t)$ is the response of the second circuit,
u is a variable of integration.

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* Communicated by the Author.

Unfortunately, the two individual responses may not be very simple functions, with the result that the integration above may be difficult or very laborious, or both. The individual responses may even be known only as a graph, no function having been deduced for them. In such cases mechanical (or step-by-step) integration may be resorted to, but if this is considered undesirable, the following development of Duhamel's integral can be used.

Any continuous function may be represented by a power series, so that by far the majority of likely transient responses may be represented by a power series with t, the time after switching on the unit step signal, as the independent variable. The response of two circuits in cascade is then also a power series, with coefficients related to those of the single circuits, as will be seen in the next section. The problem in practice is then to determine the coefficients of the power series for the single circuits. If the functions are not known, this can be done by substituting amplitude values at different times and solving simultaneous equations. The accuracy of match is determined by the number of terms considered in the series—n terms enable n points on the curve to be fitted. If the functions are known, they will usually consist of a combination of exponential and sine or cosine terms. These can be replaced by their series expansions, leading directly to the single series for the whole function. However, unless a large number of terms is taken, it will generally prove better in practice to determine an empirical power series of a limited number of terms, even when the exact function is known. This will be demonstrated in Section 3. The reason is, of course, that the functions are being studied over only a limited range.

2. APPLICATION OF POWER SERIES TO THE DUHAMEL INTEGRAL.

Let the amplitude-time response of one circuit be

$$\mathrm{E}(t) = \sum_{n=1}^{\infty} a_n t^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and that of the other circuit be

$$\phi(t) = \sum_{m=1}^{\infty} b_m t^m . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

No constant terms (n or $m=0$) are included, since, in practice, the ransient always has zero amplitude at time $t=0$.

Then

$$\mathrm{E}(u) = \sum_{n=1}^{\infty} a_n u^n ,$$

$$\frac{d}{d(t-u)} \phi(t-u) = \sum_{m=1}^{\infty} m b_m (t-u)^{m-1} .$$

Therefore the output amplitude-time response is

$$\begin{aligned} F_2(t) &= \int_0^t \left[\sum_{n=1}^{\infty} a_n u^n\right]\left[\sum_{m=1}^{\infty} m b_m (t-u)^{m-1}\right] du \\ &= \int_0^t \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m a_n b_m u^n (t-u)^{m-1}\, du \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m a_n b_m t^{m+n} \int_0^1 V^n (1-V)^{m-1} dV \quad . \quad . \quad . \quad (4) \end{aligned}$$

$$\left(\text{on putting } \frac{u}{t} = V\right).$$

The integral $\int_0^1 V^n(1-V)^{m-1}dV$ is the Beta-function $B(n+1, m)$ which is evaluated in text-books, such as Copson[(2)], on 'Functions of a Complex Variable.' For real positive integral values of n and m a simpler evaluation is possible, as shown in the appendix, giving the result

$$\int_0^1 V^n(1-V)^{m-1}dV = \frac{\underline{\angle n}\,\underline{\angle m-1}}{\underline{\angle n+m}}.$$

Equation (4) then becomes

$$F_2(t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n b_m t^{m+n} \frac{\underline{\angle n}\,\underline{\angle m}}{\underline{\angle m+n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\begin{aligned} &= \frac{1}{2}a_1b_1t^2 + \left(\frac{1}{3}a_1b_2 + \frac{1}{3}a_2b_1\right)t^3 + \left(\frac{1}{4}a_1b_3 + \frac{1}{6}a_2b_2 + \frac{1}{4}a_3b_1\right)t^4 \\ &\quad + \left(\frac{1}{5}a_1b_4 + \frac{1}{10}a_2b_3 + \frac{1}{10}a_3b_2 + \frac{1}{5}a_4b_1\right)t^5 \\ &\quad + \left(\frac{1}{6}a_1b_5 + \frac{1}{15}a_2b_4 + \frac{1}{20}a_3b_3 + \frac{1}{15}a_4b_2 + \frac{1}{6}a_5b_1\right)t^6 \\ &\quad + \left(\frac{1}{7}a_1b_6 + \frac{1}{21}a_2b_5 + \frac{1}{35}a_3b_4 + \frac{1}{35}a_4b_3 + \frac{1}{21}a_5b_2 + \frac{1}{7}a_6b_1\right)t^7 \\ &\quad + \text{, etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6) \end{aligned}$$

It is easily seen how these terms are made up. For any power n the first and last terms have denominator n, i. e.

$$\text{first term of coefficient of } t^n \text{ is } \frac{1}{n}a_1b_{n-1}$$

$$\text{and last} \quad ,, \quad ,, \quad \frac{1}{n}a_{n-1}b_1.$$

The denominator of any other term is the sum of the corresponding denominator and the one before it in the previous row.

One important conclusion can be seen directly from equations (5) or (6); the slope of the output function is zero at time $t=0$, since

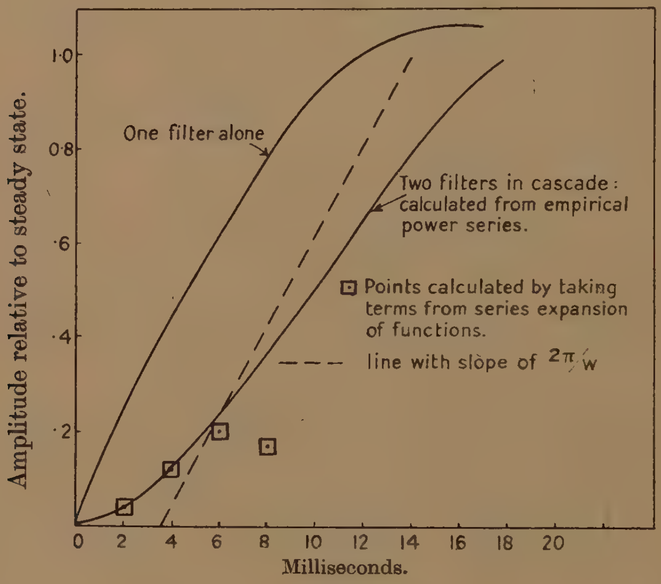
$$\frac{d}{dt}F_2(t) = a_1b_1t + (a_1b_2 + a_2b_1)t^2 +, \text{ etc.}$$

This means that, however rapidly the two circuits build up individually, the build-up of the cascaded circuits always starts with zero slope. The shape of the output transient may therefore be quite different from the responses of the individual circuits.

3. Build-up Characteristics of Filter Cascades.

The series form of the expression for the transient response of cascaded circuits proves very useful when dealing with filters of the conventional Zobel type (3), and therefore three examples of such filters will be given to illustrate the usefulness of the preceding theory. The first example will assume no knowledge of the functional form of the response of each circuit, and the second will derive the power series by expanding the known functions. It will become clear that the former process is preferable. In all cases we shall consider the build-up of the envelope of a suddenly applied signal of the mid-band frequency of a band-pass filter.

Fig. 1.



3.1. *Determination by Empirical Power Series; Two Indirectly-coupled Single-section Filters with* m=0·884 *and Band-width* W=600 *rads/sec.*

The build-up envelope of the individual filter section with derivation parameter $m=0{\cdot}884$ and W=600 rads/sec. has been given as an example in an earlier paper (4), and is reproduced in fig. 1. If we decide to take four terms in a power series to represent this curve, then four points on the curve must be taken in order to determine the coefficients, and suitable values are as follows :—

t (milliseconds)	4	8	12	16
Amplitude (relative to steady-state)	0·45	0·77	1·01	1·06

If these points are put in the equation

$$\text{amplitude}=a_1t+a_2t^2+a_3t^3+a_4t^4$$

and the four resulting simultaneous equations solved, we obtain

$$a_1=1{\cdot}42\times10^{-1},\qquad a_3=0{\cdot}75\times10^{-3},$$
$$a_2=-1{\cdot}01\times10^{-2},\qquad a_4=-0{\cdot}26\times10^{-4}.$$

Substituting these values for both $\mathrm{E}(t)$ and $\phi(t)$, since both circuits are identical, equation (6) gives us the response of the two circuits thus :—

$$\mathrm{F}_2(t)=1{\cdot}01\times10^{-2}t^2-0{\cdot}96\times10^{-3}t^3+0{\cdot}70\times10^{-4}t^4-0{\cdot}30\times10^{-5}t^5$$
$$+0{\cdot}062\times10^{-6}t^6-0{\cdot}011\times10^{-7}t^7+0{\cdot}001\times10^{-8}t^8, \quad . \quad . \quad (7)$$

where t is in milliseconds.

This curve is plotted in fig. 1, up to $t=16$ milliseconds ; we are not justified in using the series for higher values of t, since the original series was chosen to represent its curve only up to $t=16$ milliseconds, but this limit takes the curve up to 90 per cent. of its steady-state value, which is usually sufficient. It will be observed how the signal builds up very slowly at first by comparison with the response of one filter alone. The accuracy of the response given by equation (7) was checked by making a step-by-step numerical integration in the manner described by Kallman, Spencer and Singer (5), and the agreement was within about 2 per cent.

It should be emphasized that the results obtained here apply to two filter sections coupled in such a way that both are terminated by their design resistance ; this implies that the two sections cannot be joined together directly, but only via an attenuator or amplifier. The difference when the sections are directly coupled will be illustrated in Section 3.3.

3.2. *Determination by Expanding the known Response Functions : Same Filters as in* 3.1.

It is known (4) that the build-up envelope of a band-pass * filter section terminated in the design resistance is

$$\mathrm{E}(t)=1-\epsilon^{-\mathrm{A}_1}-\mathrm{G}\epsilon^{-\mathrm{A}_2t}\sin\mathrm{A}_3t, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where the steady-state amplitude is taken as unity, and A_1, A_2 and A_3 are functions of the band-width W (rads/sec.) and derivation parameter m ; G is a function of m.

If this function is used directly in Duhamel's integral, the process of integration becomes very lengthy, as there are nine separate terms to be integrated, and the algebraic manipulation required is considerable. Moreover, the resulting expression is complicated, and the numerical evaluation is quite as laborious as the whole process illustrated in Section 3.1.

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* All the work discussed here applies exactly to the build-up of a d.c. signal in a low-pass filter if W is taken as *twice* the cut-off frequency (*i. e.* the pass-band extends from $-\omega_c$ to $+\omega_c$, and the mid-band frequency is zero).

Now,

$$\epsilon^{-A_1t}=1-A_1t+\frac{A_1^2t^2}{\angle\underline{2}}-\frac{A_1^3t^3}{\angle\underline{3}}+\text{etc.},$$

$$\epsilon^{-A_2t}=1-A_2t+\frac{A_2^2t^2}{\angle\underline{2}}-\frac{A_2^3t^3}{\angle\underline{3}}+\text{etc.},$$

and

$$\sin A_3t=A_3t-\frac{A_3^3t^3}{\angle\underline{3}}+\frac{A_3^5t^5}{\angle\underline{3}}-\text{etc.},$$

and it is an obvious step to use these expansions to obtain a power series to use in the method of section 2. It is necessary to make an arbitrary limit to the number of terms taken: for our example, let us decide to take terms up to t^5 in E(t) and up to t^7 in $F_2(t)$.

Then

$$E(t)=(A_1-GA_3)t-\left(\frac{1}{2}A_1^2-GA_2A_3\right)t^2+\left(\frac{1}{6}A_1^3-\frac{1}{2}GA_2^2A_3+\frac{1}{6}GA_3^3\right)t^3$$
$$-\left(\frac{1}{24}A_1^4-\frac{1}{6}GA_2^3A_3+\frac{1}{6}GA_2A_3^3\right)t^4$$
$$+\left(\frac{1}{120}A_1^5-\frac{1}{24}GA_2^4A_3+\frac{1}{12}GA_2^2A_3^3-\frac{1}{120}GA_3^5\right)t^5,\ \text{etc.} \quad . \quad . \quad . \quad (9)$$

Take the case of two sections of $m=0{\cdot}884$ and W=600 rads/sec., as discussed previously. Substituting these values in the expressions

$$A_1=\frac{W}{2m},\quad A_2=\frac{mW}{4},\quad A_3=\frac{W}{2}\sqrt{1-\frac{m^2}{4}},$$

we find $A_1=340$, $A_2=133$ and $A_3=270$. Also for this m value, $G=0{\cdot}6$ (see the previous reference (4)). Thus, if we take t in milliseconds instead of seconds, we obtain the coefficients for E(t) as follows :—

$a_1=1{\cdot}8\times10^{-1}$, $\quad a_4=-7{\cdot}5\times10^{-4}$,
$a_2=-3{\cdot}6\times10^{-2}$, $\quad a_5=4{\cdot}5\times10^{-5}$.
$a_3=7{\cdot}0\times10^{-3}$,

On putting these figures in equation (6), we obtain as the response of two sections :—

$$F_2(t)=0{\cdot}36\times10^{-2}t^2-0{\cdot}84\times10^{-3}t^3+3{\cdot}1\times10^{-4}t^4-4{\cdot}9\times10^{-5}t^5$$
$$+4{\cdot}8\times10^{-6}t^6-3{\cdot}0\times10^{-7}t^7,\ \text{etc.} \quad . \quad . \quad (10)$$

If this equation is compared with equation (7), which is the expression obtained for the same response by the empirical method, it will be seen that equation (10) converges much less rapidly, and is therefore more likely to give errors. It actually gives good results only up to $t=6$ milliseconds ; values obtained from it are indicated in fig. 1. That it is of little use above $t=6$ is indicated by the fact that even at this point equation (9) gives a 10 per cent. error in the value of E(t) as compared with the functional expression (8).

3.3. *Two-section Filter, one Section with* m=1, *the other with* m=0·884, *both with Band-width* W=600 *rads/sec.*

In the previous examples the two filter sections have been so connected in circuit that there could be no mutual interaction, *i. e.* both sections are still individually terminated at both ends in the design resistance. In practice, a more important case is where the output terminals of the first section are directly joined to the input terminals of the second. This affects the shape of the pass-band loss-frequency response. It has been shown (6) that a non-dissipative single section of filter with $m=1$ (for example) has an insertion loss of 3 db. at its cut-off frequencies when terminated by its design resistance, although classical theory (3) shows that when the section is terminated in its image impedances the response is discontinuous at cut-off, with zero loss throughout the pass-band. Thus we can see that, as a first approximation, in a multi-section filter in which all junctions are matched on an image impedance basis, the only loss in the pass-band is still the 3 db. due to the constant-resistance termination at each end. In other words, the addition of further sections after the first makes little difference to the shape of the pass-band loss-frequency response, and we might reasonably expect the main part of the transient build-up curve to be little altered by the addition of further sections. Experiment (4) has shown that this is true ; the slope of the build-up curve around the half-steady-state amplitude point is more or less unaffected by the number of sections and is given by the expression $2\pi/\mathrm{W}$. A line of this slope has been drawn in fig. 1 and it can be seen that the main part of the single-section build-up does approximate to this slope, but that two sections joined indirectly (so that the loss-frequency responses are entirely additive) have a considerably smaller slope.

The conventional treatment (4), (7) of transient response in filters is by the assumption of " ideal " characteristics which lead to a build-up envelope,

$$\frac{1}{2}+\frac{1}{\pi}\mathrm{S}i\,.\,\frac{\mathrm{W}}{2}(t-t_0), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

where S*i* is the " sine-integral," defined by

$$\mathrm{S}i(x)=\int_0^x \frac{\sin u}{u}\,du$$

and t_0 is the time at which the half-steady-state amplitude is reached. This has the objections : (*a*) t_0 is defined by the rate of change of phase-shift with frequency, assumed uniform over the pass-band. In practice, the rate is not uniform, so that t_0 cannot be evaluated accurately by this method ; (*b*) The function S*i* is skew-symmetrical about zero, and has oscillations extending to both $+\infty$ and $-\infty$. Oscillations after build-up are found in practice if the filter is not too dissipative, but oscillations before build-up are not physically reasonable, especially before $t=0$.

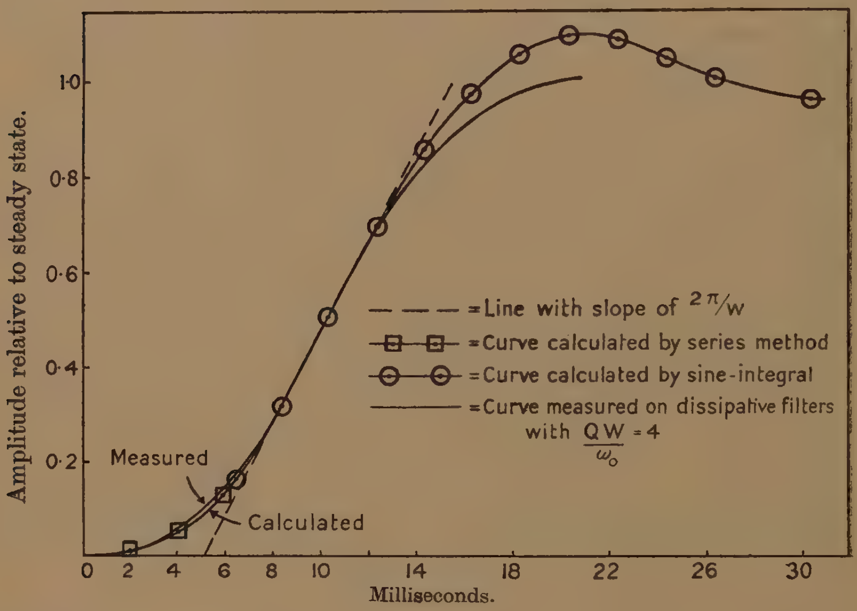
On the other hand, the expression (11) gives the slope $2\pi/\mathrm{W}$ at the half-amplitude, and over most of the build-up does represent the actual performance of multi-section filters with reasonable accuracy (4).

It is now suggested that a complete build-up curve which is physically reasonable throughout * may be obtained by using the relation (11) over all the range of times after some point where the build-up is actually becoming established, and using the theory given in previous sections of this paper to express the initial stages of the build-up. The error due to the fact that the filter sections are not cascaded without mutual interaction will be small over this limited range of small values of t.

For small values of t either of the methods described in sections 3.1 and 3.2 may be employed. Using the latter method, we have for the first filter

$$A_1=300,\ A_2=150,\ A_3=260,\ G=1{\cdot}0\ (\text{for } m=1),$$

Fig. 2.



leading to

$$a_1=0{\cdot}4\times10^{-1},\ a_2=-0{\cdot}6\times10^{-2},\ a_3=4{\cdot}5\times10^{-3},\ a_4=-6{\cdot}3\times10^{-4},$$
$$a_5=3{\cdot}8\times10^{-5}$$

for t in milliseconds.

For the second filter section ($m=0{\cdot}884$) the values have already been given in section 3.2, but the power series coefficients must be written as $b_1 \ldots b_5$. Applying equation (6), we obtain for the total response,

$$F_2(t)=0{\cdot}36\times10^{-2}t^2-0{\cdot}84\times10^{-3}t^3+3{\cdot}1\times10^{-4}t^4-4{\cdot}9\times10^{-5}t^5$$
$$+4{\cdot}8\times10^{-6}t^6-3{\cdot}0\times10^{-7}t^7+1{\cdot}5\times10^{-8}t^8-\text{ etc.},$$

which gives the curve shown in fig. 2, indicated by squares enclosing dots.

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* The need for such a curve has already been pointed out with emphasis by Bell (8). The method of Carson and Zobel (4, 9) gives a physically reasonable result, but is extremely complicated both in the analysis and in computation.

If the curve of equation (11) is plotted on the same sheet (but without regard to the exact value of t_0), it can be joined up smoothly with the initial curve, as shown in fig. 2, giving a build-up which is entirely reasonable physically. That this curve is a correct representation of the build-up of a real filter is shown by the close agreement with a measured build-up curve, which was obtained by means of a Cambridge oscillograph from a two-section filter with m values and band-width as used in the example. The only serious discrepancy is at the top of the build-up, where, owing to the high dissipation in the real filter ($Q \times W/\omega_0=4$, where ω_0 is mid-band angular frequency) there is no oscillatory effect noticeable. Apart from this, the agreement is within the limits of experimental accuracy. In a non-dissipative filter, however, it is likely that complete agreement would be obtained.

4. Conclusions.

It has been shown that the solution of transient response problems in cascaded circuits may be facilitated, in cases which are not readily treated by direct integration of functions in Duhamel's integral, by the use of power series to represent the response of the individual circuits. The overall transient response is then a power series. If the individual circuits have amplitude-time responses given by

$$\sum_{n=1}^{\infty} a_n t^n \text{ and } \sum_{m=1}^{\infty} b_m t^m,$$

then the overall response is shown to be

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n b_m t^{m+n} \frac{\angle n \, \angle m}{\angle n+m}.$$

The application of this method to cascades of filters is given by way of example, and it is shown that, owing to the restricted range considered in practice, empirical series of a limited number of terms give better results than expansions of exponential and trigonometric functions in representing the circuit responses.

The author wishes to acknowledge the helpful criticisms and suggestions (most of which have been incorporated in the paper) made by Miss M. Slack on reading his manuscript.

6. Appendix.

Evaluation of the Integral of Equation (4).

$$I_{n,(m-1)}=\int_0^1 V^n(1-V)^{m-1}\,dV$$

$$=\left[(1-V)^{m-1}\cdot\frac{V^{n+1}}{n+1}\right]_0^1+\int_0^1 (m-1)(1-V)^{m-2}\cdot\frac{V^{n+1}}{n+1}\,dV,$$

on integrating by parts

$$=\frac{m-1}{n+1}\,I_{(n+1),(m-2)}.$$

Continuing this process, we obtain

$$I_{n,\,(m-1)}=\frac{(m-1)(m-2)\,.\,.\,.\,2\,.\,1}{(n+1)(n+2)\,.\,.\,.\,(n+m-1)}\,I_{(n+m-1),\,0}.$$

Now,

$$I_{(n+m-1),\,0}=\int_0^1 V^{n+m-1}\,dV=\left[\frac{V^{n+m}}{n+m}\right]_0^1=\frac{1}{n+m}.$$

$$\therefore\ I_{n,\,(m-1)}=\frac{\angle\underline{m-1}}{(n+1)(n+2)\,.\,.\,.\,(n+m)}=\frac{\angle\underline{n}\,\angle\underline{m-1}}{\angle\underline{n+m}}.$$

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XXVII. *A Note on the Forms of the β-phase Regions in Certain Ternary Alloys of Copper.*

By G. V. Raynor

(Department of Metallurgy, University of Birmingham)*.

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I. *Introduction.*

In many binary alloys with copper, silver or gold as solvent, electron compounds with body-centred cubic, close-packed hexagonal, or β-manganese crystal structures are formed at an electron-atom ratio of 3/2. The factors affecting the formation of these phases were discussed by Hume-Rothery, Reynolds and Raynor ⁽¹⁾. Increasing solute valency favours the formation of hexagonal or β-manganese structures, whereas increasing temperature favours the body-centred cubic phases. The formation of the body-centred cubic structure is also favoured by increasing the size-factor of the solute with respect to the solvent. The size-factor is defined as $\dfrac{(d\text{ solute} - d\text{ solvent})\ 100}{d\text{ solvent}}$, where d is the closest distance of approach of atoms in the crystals of the elements.

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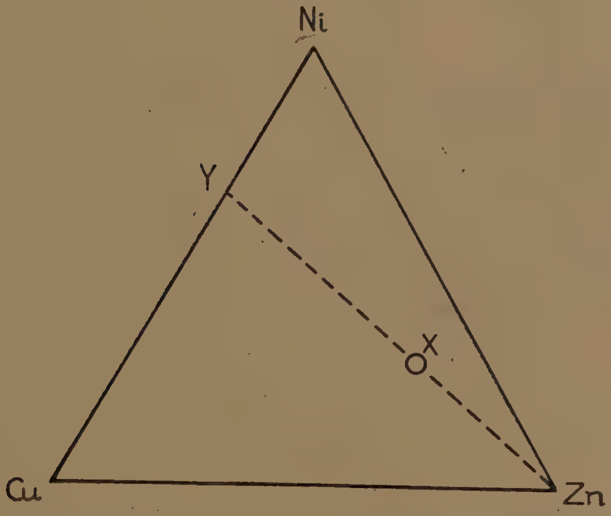
* Communicated by the Author.

For the systems copper-zinc, copper-aluminium and copper-tin, which are of interest in interpreting the structures of several important copper-rich ternary alloys, the size-factor is positive and increases in the order given. The 3/2 electron compounds (β-phases) are all body-centred cubic, and data on their homogeneity ranges are summarised in fig. 1 of reference (1). The phase based on CuZn is stable to room temperature, and possesses long range order below 453° C.–470° C. The phases based on Cu_3Al and Cu_5Sn, however, decompose eutectoidally at 570° C. and 586° C. respectively. The stabilization of the cubic structure by increasing size-factor thus appears to outweigh the effect of increasing valency, which tends to stabilize the alternative structures. The low-temperature limit of the cubic phase also appears to be influenced by the size-factor, and possibly also by the solute valency. Neglecting temporarily the latter influence, eutectoidal decomposition at some temperature between room temperature and 570° C. would be expected for solutes whose size-factor was intermediate between those of zinc and aluminium with respect to copper, while for larger size-factors the lower limit of stability of the cubic phase should be in the range 570° C.–586° C. This suggestion cannot be tested with binary alloys, but support is given by an examination of the effect of aluminium, tin and nickel on the β-phase of the copper-zinc system.

Fig. 1.



II. *Size-Factors in the Binary and Ternary Systems.*

The size-factors of zinc, aluminium and tin with respect to copper are respectively $+4{\cdot}23$, $+6{\cdot}23$ and $+9{\cdot}64$. d_{Zn} is taken as the closest distance of approach of atoms in the zinc crystal, while d_{Sn} is the inter-atomic distance in grey tin. The value of d_{Al} has been previously discussed ⁽²⁾, ⁽³⁾, ⁽⁴⁾, and in this paper a value of 2·71 A. is used ⁽²⁾.

Size-factors for ternary systems cannot be evaluated as simply. If the solvent does not form a continuous series of solid solutions with either of the solutes, an assessment of the size-factor may be made on the assumption that, to a first approximation, the effective atomic diameter of the "solute," in an alloy containing x atoms of solute A and y atoms of solute B, is $\dfrac{xd_{\mathrm{A}}+yd_{\mathrm{B}}}{x+y}$. Then, if the size-factors with respect to copper are respectively $\mathrm{F_A}$ and $\mathrm{F_B}$ for elements A and B, the effective size-factors for alloys of the ternary system may be expressed as $\dfrac{x\mathrm{F_A}+y\mathrm{F_B}}{x+y}$. Where the solvent forms continuous solutions with one solute, as in the system copper-zinc-nickel ⁽⁷⁾, the effective size-factor must be referred, not to copper, but to the primary copper-nickel solid solution of the appropriate composition, *i. e.*, that which lies at the intersection of the copper-nickel axis of the ternary diagram with a straight line drawn from the zinc-rich corner of the diagram through the composition of the alloy considered. Thus, in fig. 1, alloy X is regarded as an alloy of zinc and copper-nickel solid solution of composition Y. The effective size-factor is

$$\frac{\left[d_{\mathrm{Zn}}-\left\{\dfrac{yd_{\mathrm{Cu}}+xd_{\mathrm{Ni}}}{x+y}\right\}\right]100}{\left\{\dfrac{yd_{\mathrm{Cu}}+xd_{\mathrm{Ni}}}{x+y}\right\}},$$

where x and y are respectively the atomic percentages of nickel and copper in the alloy considered.

III. *The Ternary Systems.*

The copper-zinc β-phase is stable to room temperature. This feature is maintained for copper-zinc-tin alloys with a constant tin content of 1·05 weight per cent. (fig. 2, *a*). With a constant tin content of 2·01 weight per cent., a eutectoid-type reaction appears at 460° C. (fig. 2, *b*). Further additions of tin progressively raise the lower limit of the β-phase to 570° C. at a constant tin content of 10 weight per cent. ⁽⁵⁾. At this stage the diagram is strikingly similar to that for binary copper-aluminium alloys, as shown in fig. 3, in which the phase diagrams are plotted in terms of electron : atom ratios for ease of comparison.

In the copper-zinc-aluminium system a eutectoid-type reaction appears between aluminium contents of 4 and 6 weight per cent. of aluminium.

Fig. 2.

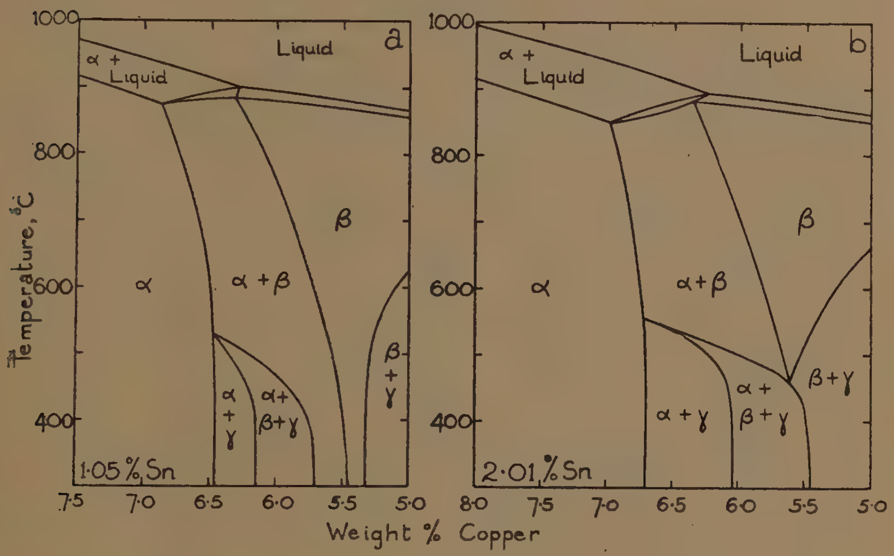
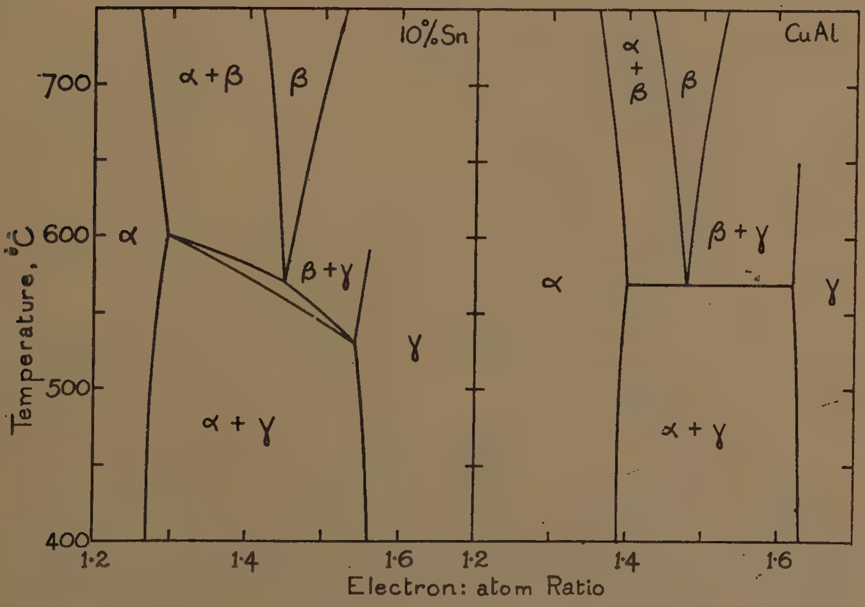


Fig. 3.



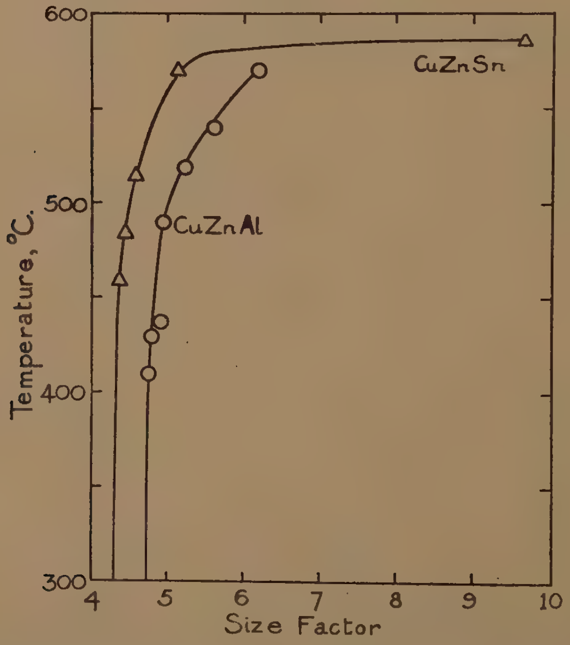
At the latter composition the diagram is similar to fig. 2, *b*. The critical temperature is progressively raised by further aluminium additions.

The compositions and temperatures of the points representing the lower limit of stability of the β-phase in the two ternary systems are summarized in Table I.

Table I.

Composition of lower limit of β-phase.								Temperature °C.
Weight per cent.				Atomic per cent.				
Cu	Zn	Sn	Al	Cu	Zn	Sn	Al	
56·00	41·99	2·01	—	57·20	41·71	1·09	—	460
56·75	40·15	3·10	—	58·24	40·06	1·70	—	484
58·25	36·75	5·00	—	60·26	36·97	2·77	—	515
63·00	27·00	10·00	—	66·59	27·75	5·66	—	570
68·00	26·00	—	6·00	63·30	23·53	—	13·17	437·5
74·70	17·60	—	7·70	67·93	15·56	—	16·50	518·8
81·30	9·30	—	9·40	72·27	8·04	—	19·69	540
65·00	30·70	—	4·30	61·92	28·43	—	9·65	410
65·00	30·00	—	5·00	61·34	27·53	—	11·12	430
65·20	28·50	—	6·30	60·50	25·72	—	13·78	490

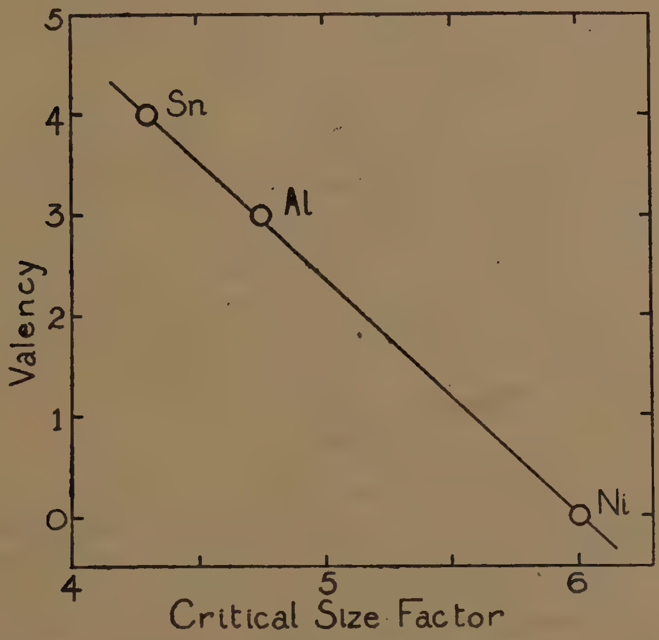
Fig. 4.



In fig. 4 the temperature limit is plotted against the effective size-factor, calculated from the compositions in Table I. The critical temperature rises rapidly as the size-factor increases to +5·5 to +6·0. The effect of small tin additions may be interpreted as due to the increase in effective size-factor caused by the large tin atom. Also, the resemblance shown in fig. 3 between aluminium-copper alloys and copper-zinc-tin alloys at 10 weight per cent. of tin may be attributed to the raising of the effective size-factor to a value comparable with that for the binary alloys.

The rate of increase of critical temperature with increasing size-factor is similar for both systems; the critical size-factor range, however, is lower for tin additions than for aluminium additions. This suggests dependence on the valency of the added element, since if only the size-factor were effective the curves should superimpose. Fig. 4 indicates that additional "strain factors" are introduced by the addition to copper-zinc alloys of elements of different valencies. There are no data for the influence of additions of suitable di- and monovalent metals. For copper-zinc-nickel alloys, however, departure from the copper-zinc type of diagram begins at 32 weight per cent. Ni, 52·5 weight per cent. Zn, and 15·5 weight per cent. Cu (7); further nickel additions raise the critical temperature rapidly, but the data are incomplete. The critical composition corresponds with an effective size-factor of +6·0; the corresponding values for the copper-zinc-tin and copper-zinc-aluminium systems are +4·3 and +4·75 (fig. 4). If the critical size-factor is plotted against the valency of the added element, taking the valency of nickel as zero in this type of alloy (8), the three points lie on a straight line (fig. 5).

Fig. 5.



The lower limit of stability of the body-centred cubic phase, in these alloys formed by the addition of a third metal to the copper-zinc system, is thus very sensitive to the magnitude of the size-factor, as suggested by the binary diagrams. The critical size-factor at which rapid raising of this limit occurs depends on the valency of the third element. The similarity of the slopes of the curves of fig. 4 suggests that, for a given system, the lower limit is determined by an overall "strain factor" composed of a variable size-factor, depending only on the atomic diameters of the components and the composition, and a superimposed "electron factor," proportional to the valency of the added element.

IV. ACKNOWLEDGMENTS.

This work forms part of a research programme under the general supervision of Professor D. Hanson, D.Sc., to whom the author's thanks are due for his interest and support.

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XXVIII. *Equilibrium Relationships in Ternary Alloys.*

By G. V. RAYNOR, Department of Metallurgy, University of Birmingham*.

[Received June 30, 1947.]

1. *Introduction.*

HUME-ROTHERY (1) has discussed some aspects of the theory of ternary primary solid solutions. The influences of the electron/atom ratio and of the distortions produced in the matrix by the solute metals were examined. In particular, it was shown that, when a compound A_xB_y is precipitated from C-rich ABC alloys, it may be expected that solid solubility isothermals will have the form

$$[A]^x\,[B]^y = \text{constant K},$$

and that $\frac{d \log K}{d\theta} = -\frac{Q}{R\theta^2}$, where θ is the absolute temperature and Q is analogous to a heat of formation. The present paper examines the effect of compound formation on the solid solubility isothermals of certain copper- and silver-rich ternary systems, where the extent of the solubility would be governed, in the absence of compound formation, by valency considerations. The conditions under which binary compounds may be expected to enter into equilibrium with the solid solution based on the third element of the ternary system are also considered.

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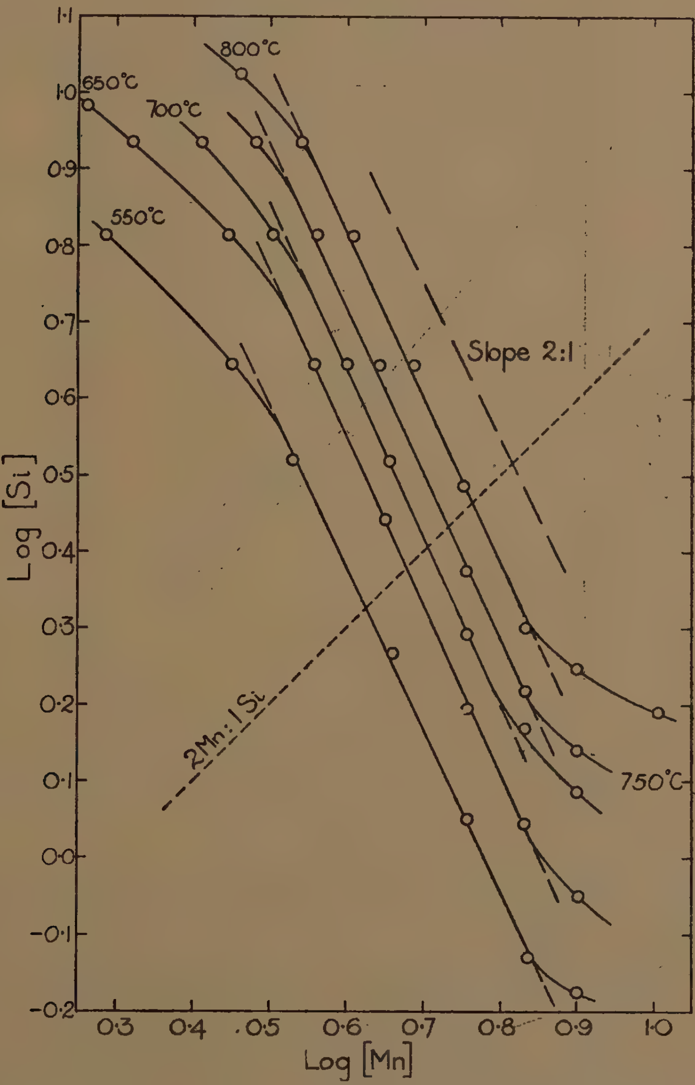
* Communicated by the Author.

2. *The Systems Copper–Manganese–Silicon and Copper–Nickel–Tin.*

The copper-rich copper–manganese–silicon alloys have been examined by Smith and Hibberd (2). The compound Mn_2Si enters into equilibrium with the copper-rich solid solution; it may therefore be expected that,

Fig. 1.

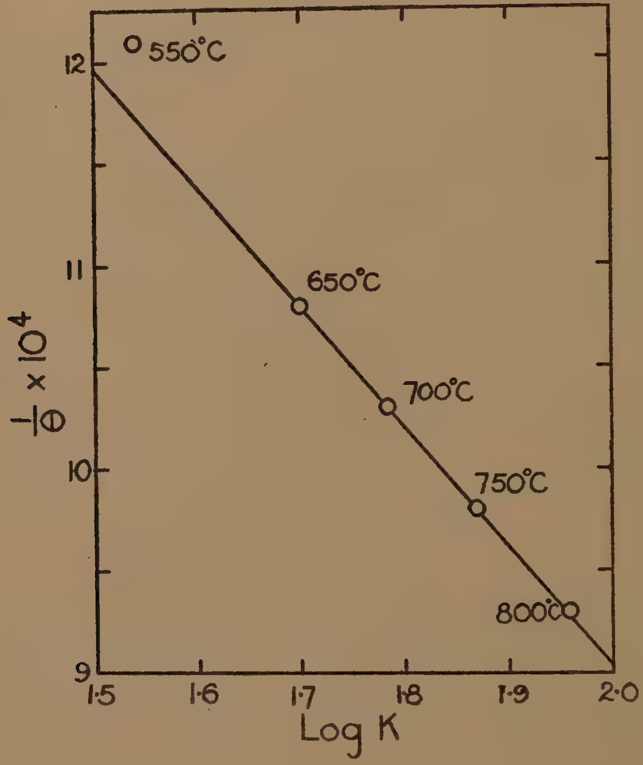


for compositions along the $\alpha/(\alpha+Mn_2Si)$ phase boundary, the relationship

$$[Mn]^2\,[Si]=K$$

will be obeyed, where [Mn] and [Si] are respectively the atomic percentages of manganese and silicon. Fig. 1 shows the results of Smith and Hibberd for five temperatures, plotted as log [Si] against log [Mn]. The central portions of the curves, in the region where the ratio of manganese to silicon atoms is close to 2 : 1, are linear and of the required slope of $-2/1$. For low manganese and high silicon contents, and *vice versa*, the curves deviate from linearity; further reference to this is made below. Fig. 2 shows log K plotted against the reciprocal of the absolute temperature. Except for the point obtained from the 550° C. isothermal, the points lie on a straight line, so that the relation $\frac{d \log K}{d\theta} = -\frac{Q}{R\theta^2}$ is obeyed. Q has the value $7 \cdot 6 \times 10^3$ calories.

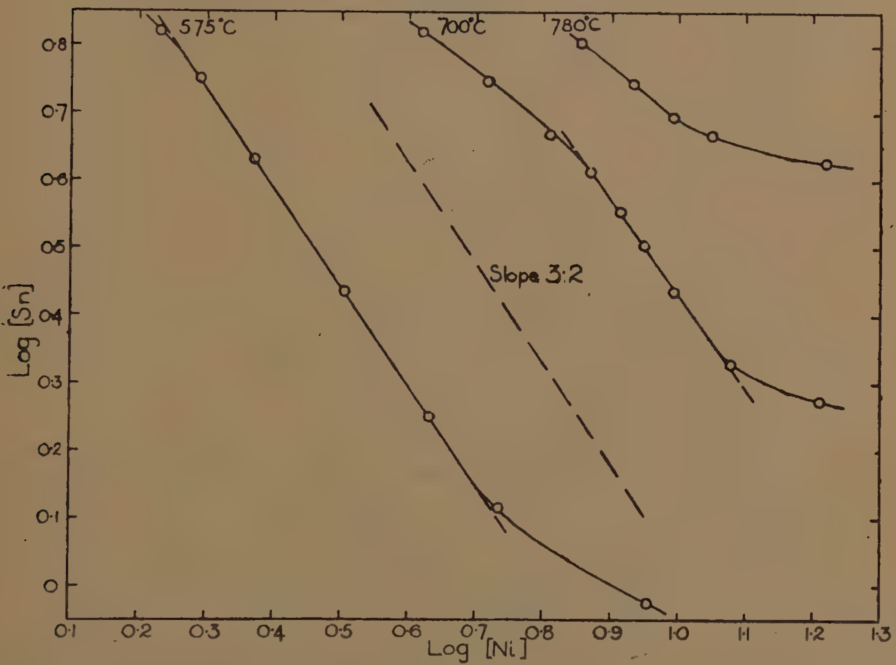
Fig. 2.



The $\alpha/(\alpha+Mn_2Si)$ isothermals may thus be interpreted as governed mainly by the nature and composition of the inter-metallic compound, in the composition range in which the solute atoms are present in roughly the same proportions as in the compound. For a large deviation from the required ratio of manganese to silicon atoms this is no longer true, and the boundary tends to approach more closely towards the position it would occupy were it conditioned only by electronic and size-factor considerations.

The work of Eash and Upthegrove (3) shows that, over the major portion of the primary solid solubility curves for the copper–nickel–tin system at all temperatures investigated, the copper-rich solution is in equilibrium with a phase θ. This phase was not identified, but, as pointed out by Hume-Rothery (1), the solubility isothermals have the characteristic form associated with stable compound formation. Figure 3 shows the results of Eash and Upthegrove, measured as accurately as possible from their published diagrams, plotted as log [Sn] against log [Ni]. At 575° C. the curve is linear over an appreciable composition range, with a slope of $-3/2$. At 700° C. linearity is only approximate, over a small composition range, and at 780° C. linearity is lost. The slope of $-3/2$ suggests that the θ phase is to be identified with Ni_3Sn_2, which has the nickel arsenide structure and crystallizes at 1264° C. at a sharp maximum on the binary liquidus (4). The non-linearity of the logarithmic curves at the higher temperatures may be due to the development of disorder in Ni_3Sn_2 as the temperature rises.

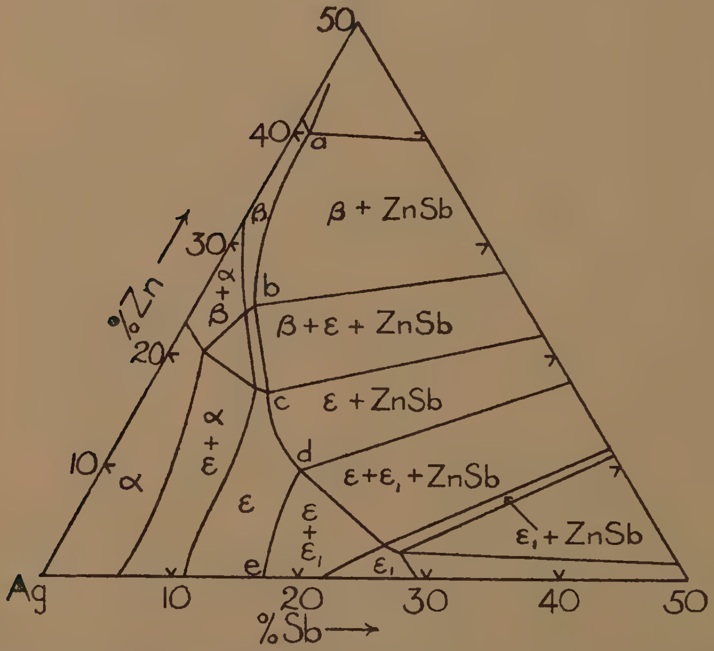
Fig. 3



3. *The System Silver–Zinc–Antimony.*

The results of Guertler and Rosenthal (5) for this system are summarized, in the form of an isothermal at 400° C., in fig. 4. In this case the stable compound ZnSb enters into equilibrium, not with the primary solution, but with electron compounds whose formation depends on a ratio of 3 electrons to 2 atoms. At 400° C., the 3/2 electron compounds of the systems silver–zinc and silver–antimony are respectively body-centred cubic and hexagonal in structure, and the zinc–antimony-rich phase boundaries are of particular interest. If the precipitation of ZnSb were the main factor governing the form of the boundaries *ab* and *cd* of fig. 4, log [Zn] plotted against log [Sb] would give a straight line of unit slope; if the electron/atom ratio were the dominant factor, no such linearity would be expected. Fig. 5 shows log [Zn] plotted against log [Sb], and includes also curves of constant electron/atom ratio. For the boundary *de* (equilibrium between ϵ phase and Ag_3Sb), deviation from the electron : atom ratio 1·62 is slight. For the boundary *cd*, however (equilibrium between ϵ phase and ZnSb), the curve is linear with the required unit slope; deviation from a curve of constant electron/atom ratio is marked, and the presence of the compound outweighs the electron factor. Along *ab* (equilibrium between β phase and ZnSb) the curve is not linear and deviates little from the constant electron/atom ratio of 1·5. Along *cb* (equilibrium between both electron compounds and ZnSb) there is a smooth transition between the curves for boundaries *cd* and *ab*. It may be inferred that, provided too great a deviation from the electron/atom ratio characteristic of the electron compound is not caused, the nature and composition of the compound controls the form of the isothermal boundary. Thus, *cd* corresponds with electron/atom ratios of 1·5 to 1·6, which are not far removed from the value for 3/2 electron compounds. If, however, the boundary resulting from $[Zn][Sb]=K$ leads to too great a deviation from the characteristic

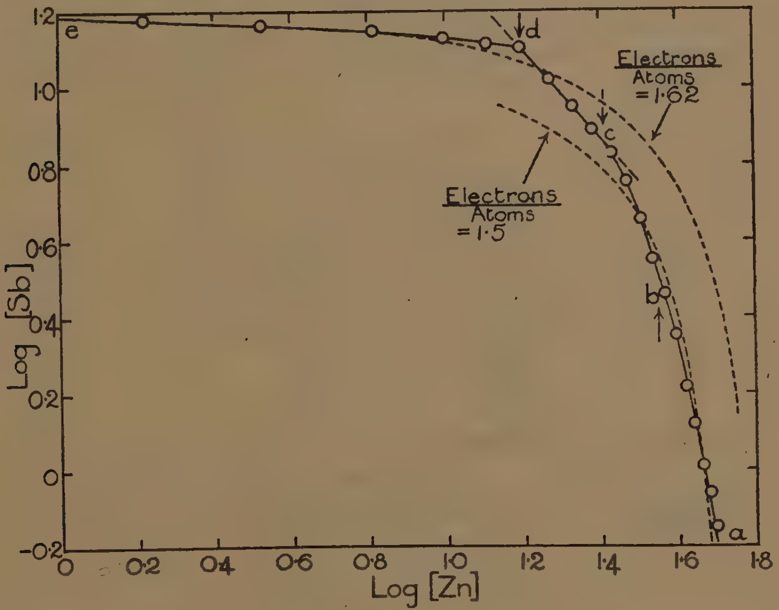
Fig. 4.



electron/atom ratio, the effect of the latter predominates. Thus, for *ab*, [Zn][Sb]=K would give rise to an isothermal along which the electron/atom ratio would deviate considerably from 1·5, as may be appreciated from fig. 5.

Where a compound A_xB_y enters into equilibrium with an electron compound or a solid solution, the extent of which is normally governed by electronic factors, the resultant phase boundary is to be regarded as a compromise between tendencies to preserve a constant electron/atom ratio, and to obey the relation $[A]^x\,[B]^y=K$. The extent to which the latter is effective will depend essentially on the stability of the compound.

Fig. 5.



The extents to which the β and ε phases (fig. 4) will respectively dissolve antimony and zinc are also of interest. The former extends from the silver–zinc axis to the approximate composition 34·7 atomic per cent. zinc and 3·54 atomic per cent. antimony, while the latter extends from the silver–antimony axis to the approximate composition 24·15 atomic per cent. zinc and 7·78 atomic per cent. antimony. In the binary silver-rich alloys, small size-factors * favour the formation of hexagonal 3/2 electron compounds, while increasing size-factor favours the body-centred cubic structure (6). Thus the silver–antimony and silver–tin 3/2 electron compounds (size-factors +0·52 and −2·98 respectively) are hexagonal, while in the silver–aluminium system (size-factor −5·65) a body-centred cubic phase occurs, at high temperatures, in addition to a hexagonal phase. For the silver–zinc system (size-factor −7·77) the body-centred cubic structure predominates, while the hexagonal structure is stable only below 300° C. The hexagonal structure becomes relatively unstable, compared with the cubic structure, between size-factors of −5·65 and −7·77. In the ternary system, the size-factor varies from +0·52 along the AgSb axis to −7·77 along the AgZn axis. Assuming the effective size-factor to be the average of the contributions of the two solutes, the effective size-factor for a composition x atomic per cent. zinc and y atomic per cent. antimony may be written $\frac{xF_{Zn}+yF_{Sb}}{x+y}$, where F_{Zn} and F_{Sb} are the size-factors in the binary systems with respect to silver. On this basis, the ternary hexagonal phase extends to a size-factor of −5·75, while the body-centred cubic structure extends to a size-factor of −7·0. Fig. 6 summarizes the data for the ternary system and the corresponding binary systems. The hexagonal phase in the ternary system becomes unstable with respect to the cubic structure at a size-factor very close to that at which the hexagonal phase disappears, at high temperatures, in the binary systems. This suggests that in certain cases we may infer the homogeneity limits of 3/2 electron compounds in ternary systems from the variation of the effective size-factor in the ternary system, and the structures of binary alloys at corresponding size-factors.

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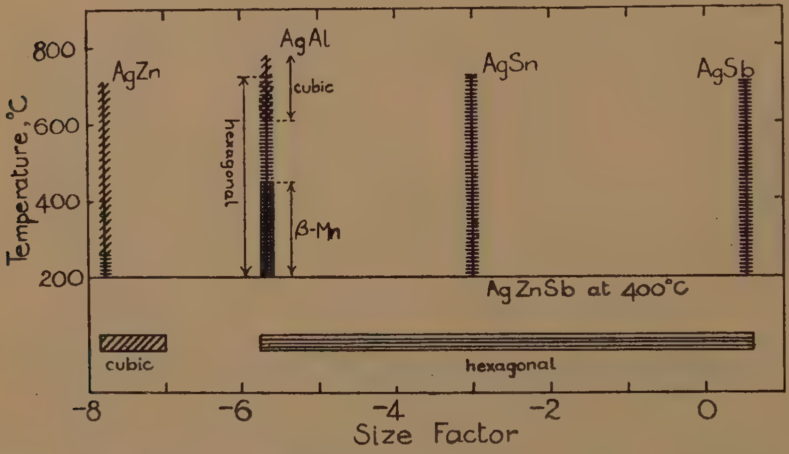
* The size-factor for a given solute with respect to silver is defined as $\frac{(d\text{ solute}-d_{\text{Ag}})\,100}{d_{\text{Ag}}}$, where *d* is the closest distance of approach in the metallic crystal.

Fig. 6.

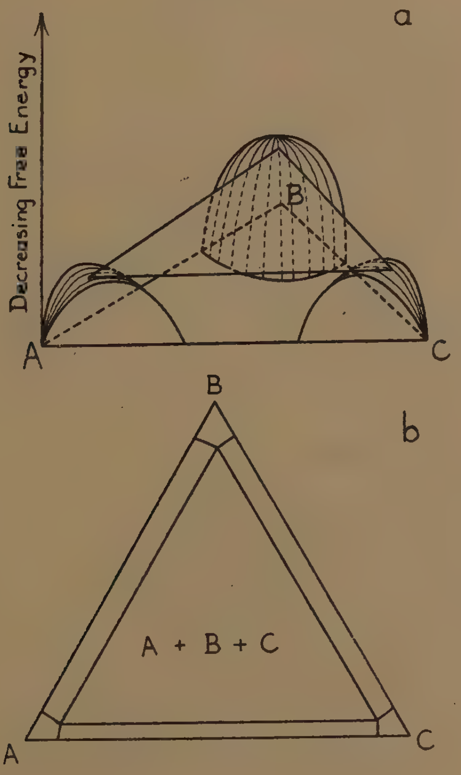


4. *Conditions for Equilibrium between an Inter-metallic Compound,* A_xB_y, *and the C-rich Solid Solution in the System* ABC.

The precipitation of an inter-metallic compound from a primary solid solution is often associated with age-hardening effects; it is thus of interest to consider the factors which may determine whether this type of equilibrium may be expected in a given case.

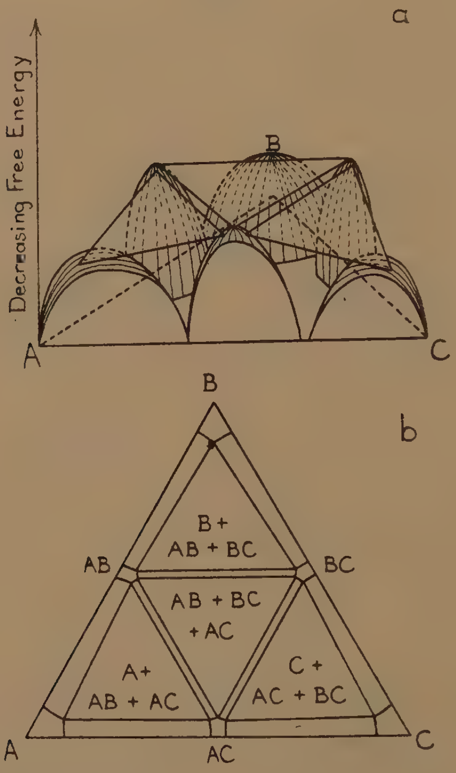
Ternary equilibrium may be interpreted in terms of free energy principles. Each homogeneous phase corresponds, in a free energy-composition model, with a surface convex in the direction of decreasing free energy. The compositions of the phases in 3-phase equilibrium at a given temperature are represented by the tangential points of contact of a plane with the free energy surfaces corresponding to the phases concerned. Thus fig. 7 *a* shows a perspective sketch of the free energy model for a system ABC in which no binary or ternary compounds are formed; fig. 7 *b* shows the corresponding isothermal diagram. In fig. 8 *a* a possible free energy model is shown for a system ABC containing the compounds AB, BC, and CA, while fig. 8 *b* shows the corresponding isothermal. The tangential planes must be disposed so that the entire surface contains no re-entrant portions, *i. e.*, so that the minimum free energy is achieved for all compositions. If the free energy of BC is now made much lower than those for AB and AC, the arrangement of planes in fig. 8 *a* no longer gives a minimum free energy for all compositions; the arrangement changes to that of fig. 9 *a*, which (fig. 9 *b*) involves equilibrium between BC and the A-rich solid solution. Thus, if BC is to come into equilibrium with the A-rich solid solution, its free energy

Fig. 7.



must be very low compared with the other phases present. In general free energy values are not available; quantitative expression is therefore difficult. Heats of formation of many inter-metallic phases are, however, known, and it is suggested that these may be used as a measure of the relative free energy values of the compounds. Only binary compounds with high heats of formation are thus expected to enter into equilibrium with the solid solution of the two components in a third metal. Typical ternary systems may now be considered; heat of formation data are taken from reference (7).

Fig. 8.



(*a*) Copper–Aluminium–Nickel.

No intermediate phases are formed in the copper–nickel system, and the heats of formation in the copper–aluminium system rise to a maximum of 5·35 k.cal./g. atom for Cu_9Al_4. In the nickel–aluminium system the heats of formation rise to the high value of 17·0 k.cal./g. atom for NiAl. This phase would therefore be expected to enter into equilibrium with the copper-rich solid solution, and the work of Alexander (8) agrees with this; over a portion of the solubility isothermals, the copper-rich solution and NiAl are in equilibrium. Ni_3Al, with the relatively high heat for formation 9·4 k.cal./g. atom, also comes into equilibrium with the copper-rich alloys.

(*b*) Copper-silicon alloys with Cobalt, Nickel and Manganese.

In each of these systems, copper-rich alloys precipitate binary compounds of silicon and the transitional metal. Copper–silicon–cobalt and copper–silicon–nickel alloys respectively precipitate Co_2Si and Ni_2Si (9), while the precipitation of Mn_2Si from sopper–silicon–manganese alloys was discussed above. This is understandable from the high heats of formation of the cobalt–silicon and nickel–silicon compounds :

Co_2Si	9·2 k.cal./g. atom.	Ni_2Si	11·1 k,cal./g. atom.
$CoSi$	12·0 ,, ,,	Ni_3Si_2	10·7 ,, ,,
		$NiSi$	10·3 ,, ,,

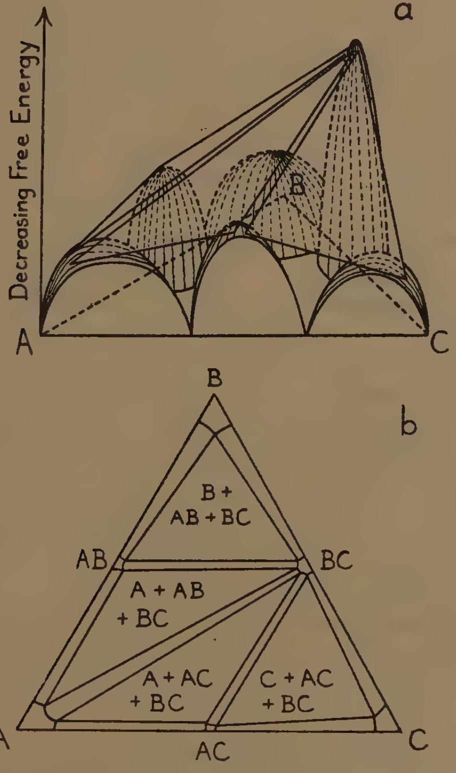


Fig. 9.

No data exist for the heats of formation of copper-silicon alloys, but these are not likely to be particularly high. The heat of formation of Mn_2Si is also unknown, but since the iron silicides have high values (9·6 k.cal./g. atom for FeSi), it is probable that the values for the manganese–silicon system are also high.

(*c*) Copper–Nickel–Tin.

In the copper–tin system, heats of formation rise to a maximum of 1·8 k.cal./g. atom for Cu_3Sn, but in the nickel–tin system, a maximum of 7·5 k.cal./g. atom occurs at Ni_3Sn_2, which, in agreement with the general principle, enters into equilibrium with the copper-rich solid solution.

(*d*) Copper–Tin–Magnesium.

A quasi-binary section is formed between the copper-rich solid solution and Mg_2Sn (10) (11). Though the heat of formation of Mg_2Sn is not very high (5·6 k.cal./g. atom), it is appreciable higher than that of any of the copper–tin intermediate phases, so that equilibrium between Mg_2Sn and the copper–rich solution is possible.

(*e*) Silver–zinc–antimony and Silver–cadmium–antimony.

The heats of formation for the silver–zinc and silver–cadmium alloys do not exceed 1·8 and 1·46 k.cal./g. atom respectively. For the silver–antimony system, heats of formation are unknown, but are unlikely to be markedly higher than for the other silver alloys considered. The values for the zinc–antimony and cadmium–antimony alloys rise to maxima, for ZnSb and CdSb, of approximately 1·8 k.cal./g. atom. In the ternary systems, there are thus no phases of outstandingly high heat of formation ; the compounds ZnSb and CdSb are thus not likely to enter into equilibrium with the primary silver-rich solid solution. Fig. 4 confirms this for silver–zinc–antimony alloys. For silver–cadmium–antimony alloys, CdSb comes into equilibrium with the ternary 3/2 electron compound, stretching from the silver–antimony to the silver–cadmium axis, but not with the primary solid solution (12).

It appears, therefore, that in ternary alloys of copper and silver, binary compounds between the two solute metals may come into equilibrium with the primary solid solution if their heats of formation are appreciably higher than the heats of formation of any phases occurring in the binary systems with copper or silver. If the heats of formation of all the intermediate phases which occur in the three binary systems are comparable, then it is unlikely that such an equilibrium will occur. It may be regarded as empirically established that, provided no phases of very high heat of formation occur in the binary systems with copper or silver, compounds between the solute elements may come into equilibrium with the primary solid solution if their heats of formation exceed 5 to 6k.cal./g. atom.

5. *Acknowledgments.*

This work forms part of a programme carried out under the general supervision of Professor D. Hanson, D.Sc., to whom the author's thanks are due for his interest and support.

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XXIX. *Daniel Bernoulli's Mechanical Problem.*

By Ian N. Sneddon

(The Department of Natural Philosophy, the University of Glasgow) *.

[Received October 14, 1946.]

1. *Introduction.*

The problem of determining the motion of a heavy chain suspended from one end, under given initial or boundary conditions—first treated by Daniel Bernoulli †—is of historical importance, as in Bernoulli's analysis the Bessel function of zero order appeared for the first time. In a recent textbook ‡ various aspects of the problem are treated by the method of the Laplace transform which is used to eliminate the time variable from the partial differential equation governing the motion, thus reducing it to an ordinary differential equation which can be handled more easily. The final result is obtained as a contour integral, and is sometimes difficult to evaluate.

An examination of the equation of motion suggests, however, that boundary and initial value problems are most easily solved by the use of the method of the Hankel transform and of the "finite" Hankel transform introduced in recent papers §. In both instances the transform used removes the spatial coordinate from the partial differential equation of motion and reduces the problem to an initial value problem for an ordinary differential equation of the second order and very simple type. It should be emphasised that, like the method of "finite" Fourier transforms developed recently in America || in the solution of physical

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* Communicated by the Author.

† For references see :—G. N. Watson, 'The theory of Bessel functions' 2nd ed. p. 3. Cambridge, 1944.

‡ H. S. Carslaw and J. C. Jaeger, 'Operational methods in applied mathematics.' Oxford, 1941.

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problems, the method of "finite" Hankel transforms does not solve problems intractable to the more usual Laplace transform method. The main reason for its use is that it greatly reduces the amount of formal calculation involved in the solution of any particular problem; it has the additional advantage that it does not involve the evalutation of a contour integral at any stage of the analysis. The object of the present paper is to illustrate, with examples drawn from Bernoulli's equation of the motion of a heavy chain, the ease with which the method of "finite" Hankel transforms can be used to solve rather complicated problems of the "Bessel-function" type, and to indicate the corresponding analysis in the case of the ordinary Hankel transform.

2. *The Equation of Motion.*

We shall consider the small transverse oscillations of a heavy chain of uniform line density ρ, suspended vertically from one end. Taking the origin of coordinates at the position of equilibrium of the *lower* end of the chain and the x-axis along the equilibrium position of the chain—pointing *upwards* so that the fixed end has coordinates $x=l$, $y=0$—we find that the tension T at the point whose abscissa is x is given by

$$\mathrm{T}=g\rho x.$$

If y denotes the horizontal deflection of the chain from its equilibrium position then the equation of motion of an element of length δx is

$$\rho\delta x\frac{\partial^2 y}{\partial t^2}-\delta x\,.\,p(x,\,t)=\frac{\partial}{\partial x}\left(\mathrm{T}\frac{\partial y}{\partial x}\right)\delta x, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $p(x,\,t)$ denotes the intensity of the external transverse force. If we divide throughout by δx and then change the independent variable x to z by means of the substitution

$$z=(4x/g)^{\frac{1}{2}},$$

we find that equation (1) reduces to the form

$$\left(\frac{\partial^2}{\partial z^2}+\frac{1}{z}\frac{\partial}{\partial z}-\frac{\partial^2}{\partial t^2}\right)y=-\frac{1}{\rho}p(\tfrac{1}{4}gz^2\,;\,t). \quad . \quad . \quad . \quad . \quad (2)$$

To solve this equation let

$$\bar{y}_\beta=\int_0^\beta zy\mathrm{J}_0(\xi_i z)\,dz, \quad \bar{p}_\beta=\int_0^\beta zp(\tfrac{1}{4}gz^2\,;\,t)\mathrm{J}_0(\xi_i z), \quad . \quad . \quad . \quad (3)$$

where the quantity ξ_i is as yet undefined, and $\beta=(4l/g)^{\frac{1}{2}}$. Then

$$\int_0^\beta z\left(\frac{\partial^2 y}{\partial z^2}+\frac{1}{z}\frac{\partial y}{\partial z}\right)\mathrm{J}_0(\xi_i z)\,dz$$

$$=\left[z\frac{\partial y}{\partial z}\mathrm{J}_0(\xi_i z)\right]_0^\beta-\xi_i\left[yz\mathrm{J}_0{}'(\xi_i z)\right]_0^\beta-\xi_i^2\bar{y}_\beta.$$

Now $y=0$ at $z=\beta$ and $z\partial y/\partial z=0$ at $z=0$, so that if we choose ξ_i to be a root of the transcendental equation

$$J_0(\xi_i\beta)=0, \qquad (4)$$

we have

$$\int_0^\beta z\left(\frac{\partial^2 y}{\partial z^2}+\frac{1}{z}\frac{\partial y}{\partial z}\right)J_0(\xi_i z)\,dz=-\xi_i^2\bar{y}_\beta.$$

Hence if we multiply equation (2) throughout by $zJ_0(\xi_i z)$, and integrating with respect to z from 0 to β and make use of this last result, we obtain

$$\left(\frac{d^2}{dt^2}+\xi_i^2\right)\bar{y}_\beta=\frac{1}{\rho}\bar{p}_\beta(\xi_i). \qquad (5)$$

The solution of this ordinary linear differential equation determines the value of $\bar{y}_\beta$; from this solution we derive the expression for y from the Fourier-Bessel expansion

$$y=\frac{2}{\beta^2}\sum_i\frac{\bar{y}_\beta J_0(\xi_i z)}{[J_1(\xi_i\beta)]^2}, \qquad (6)$$

the sum extending to all positive roots of the transcendental equation (4).

If the chain is very long so that l may be taken to be infinite, we may replace equations (3) by the pair

$$\bar{y}=\int_0^\infty zy\,J_0(\xi z)\,dz\,; \quad \bar{p}=\int_0^\infty zp(\tfrac{1}{4}gz^2,\,t)J_0(\xi z)\,dz. \qquad (7)$$

Proceeding as before, we find that y satisfies the differential equation (5) but with ξ_i replaced by ξ. y is now determined in terms of $\bar{y}$ by the Hankel inversion theorem

$$y=\int_0^\infty \xi\bar{y}\,J_0(\xi z)\,d\xi. \qquad (8)$$

3. *Free Vibrations of a Chain of Finite Length.*

If the vibrations are free then we may take

$$p(x,\,t)=\bar{p}_\beta(\xi_i,\,t)=0,$$

so that the appropriate solution of the differential equation (5) is

$$\bar{y}_\beta=A(\xi_i)\cos\xi_i t+B(\xi_i)\sin\xi_i t,$$

where the constants of integration are to be determined from the prescribed initial conditions. For example, suppose that

$$y=f(x), \quad \partial y/\partial t=h(x),$$

when $t=0$. Then the initial values of $\bar{y}_\beta$ and $d\bar{y}_\beta/dt$ at $t=0$ are :

$$\bar{y}_\beta=\int_0^\beta zf(gz^2/4)J_0(\xi_i z)\,dz, \quad \frac{d\bar{y}_\beta}{dt}=\int_0^\beta zh(gz^2/4)J_0(\xi_i z)\,dz,$$

whence we find that

$$\bar{y}_\beta = \cos \xi_i t \int_0^\beta z f(\tfrac{1}{4}gz^2) J_0(\xi_i z)\, dz + \frac{\sin \xi_i t}{\xi_i} \int_0^\beta z h(\tfrac{1}{4}gz^2) J_0(\xi_i z)\, dz.$$

Inverting this result by means of the formula (6), we find that the subsequent transverse displacement of the chain is given by

$$y = \frac{2}{\beta^2} \sum_i \cos \xi_i t \frac{J_0(\xi_i z)}{\{J_1(\xi_i \beta)\}^2} \int_0^\beta z f(\tfrac{1}{4}gz^2) J_0(\xi_i z)\, dz + \frac{2}{\beta^2} \sum_i \frac{\sin \xi_i t}{\xi_i} \cdot \frac{J_0(\xi_i z)}{\{J_1(\xi_i \beta)\}^2} \int_0^\beta z h(\tfrac{1}{4}gz^2) J_0(\xi_i z)\, dz,$$

where the sums are taken over the positive roots of the equation (4). Transforming to the original variables we have finally

$$y = \frac{1}{l} \sum_i \cos \left\{ \tfrac{1}{2}\alpha_i t \left(\frac{g}{l}\right)^{\frac{1}{2}} \right\} \frac{J_0\{\alpha_i (x/l)^{\frac{1}{2}}\}}{[J_1(\alpha_i)]^2} \int_0^l f(\eta) J_0\{\alpha_i (\eta/l)^{\frac{1}{2}}\}\, d\eta + \frac{2}{\sqrt{lg}} \sum_i \frac{\sin \{\frac{1}{2}\alpha_i t (g/l)^{\frac{1}{2}}\}}{\alpha_i} \cdot \frac{J_0\{\alpha_i (x/l)^{\frac{1}{2}}\}}{[J_1(\alpha_i)]^2} \int_0^l h(\eta) J_0\{\alpha_i (\eta/l)^{\frac{1}{2}}\}\, d\eta, \quad . \; . \; . \quad (9)$$

the summations now extending over all the positive roots of the equation

$$J_0(\alpha_i) = 0. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (10)$$

For example, if we take the initial velocity of the chain to be zero and assume that it is released from the position

$$y = f(x) = \epsilon(1 - x/l), \quad 0 \leqslant x \leqslant l,$$

then

$$\frac{1}{l} \int_0^l f(\eta) J_0\{\alpha_i (\eta/l)^{\frac{1}{2}}\}\, d\eta = \epsilon \int_0^1 \xi(1 - \xi^2) J_0(\alpha_i \xi)\, d\xi = \frac{2\epsilon}{\alpha_i^2} J_2(\alpha_i).$$

If we put $x = \alpha_i$, $n = 1$ in the recurrence relation

$$\frac{2n}{x} J_n(x) = J_{n\ 1}(x) + J_{n+1}(z)$$

we have, by virtue of equation (10), that

$$J_2(\alpha_i) = \frac{2J_1(\alpha_i)}{\alpha_i}.$$

Hence, finally, we obtain for the transverse displacement

$$y = 2\epsilon \sum_i \cos \{\tfrac{1}{2}\alpha_i t (g/l)^{\frac{1}{2}}\} \frac{J_0\{\alpha_i (x/l)^{\frac{1}{2}}\}}{\alpha_i^3 J_1(\alpha_i)},$$

where the sum is taken over all the positive roots of equation (10).

4. *Free Vibrations of a very long Chain.*

As an example of the free vibrations of a very long chain we shall consider the motion of the semi-infinite chain $0 \leqslant x \leqslant \infty$, which is initially drawn aside to the position

$$y = f(x) = \epsilon(1 + x/a)^{-\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and then released from rest at the instant $t=0$. The problem is to determine the equation governing the form of the chain at any subsequent time. As in section 3, we may take $\bar{p}(\xi, t)$ to be zero in equation (5) and obtain the solution

$$\bar{y} = \bar{f}(\xi) \cos \xi t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where

$$\bar{f}(\xi) = \int_0^\infty z f(\tfrac{1}{4}gz^2) J_0(\xi z)\, dz = \epsilon\alpha \int_0^\infty \frac{z J_0(\xi z)}{\sqrt{\alpha^2 + z^2}}\, dz = \epsilon\alpha e^{-\alpha\xi}/\xi$$

and $\alpha^2 = 4a/g$. Inserting this value of $\bar{f}(\xi)$ into equation (12) and inverting by means of the Hankel formula (8), we obtain

$$y = \epsilon\alpha \int_0^\infty e^{-\alpha\xi} \cos \xi t J_0(\xi z)\, d\xi = \epsilon \mathscr{R}l \left\{ \int_0^\infty e^{-\left(1+\frac{it}{\alpha}\right)\xi} J_0(\xi z/\alpha)\, d\xi \right\}.$$

Making use of the well-known result

$$\int_0^\infty e^{-wx} J_0(\rho x)\, dx = (\rho^2 + w^2)^{-\frac{1}{2}},$$

we have finally that

$$y = \epsilon R^{-\frac{1}{2}} \cos \tfrac{1}{2}\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where

$$R^2 = \left(\frac{x}{a} + 1 - \frac{gt^2}{4a}\right)^2 + \frac{gt^2}{a}, \quad \phi = \tan^{-1} \left\{ \frac{(gt^2/a)^{\frac{1}{2}}}{\dfrac{x}{a} + 1 - \dfrac{gt^2}{4a}} \right\}.$$

5. *Forced Vibrations of a Chain of Finite Length.*

Suppose that an external force of amount $\rho\psi(t)$ acts uniformly over the segment $(a-\epsilon, a+\epsilon)$ of the chain. Then in an obvious notation

$$\bar{p}_\beta = \frac{\rho}{2\epsilon} \psi(t) \int_{\alpha_1}^{\alpha_2} z J_0(\xi_i z)\, dz = \rho\psi(t) \frac{\alpha_2 J_1(\xi_i \alpha_2) - \alpha_1 J_1(\xi_i \alpha_1)}{2\epsilon\xi_i}.$$

If ϵ is small

$$a_1/\alpha_2 = 1 - \epsilon/a$$

and

$$\alpha_2 J_1(\alpha_2\xi_i) - \alpha_1 J_1(\alpha_1\xi_i) = \alpha_2 J_1(\xi_i\alpha_2) \left[1 - \left(1 - \frac{\epsilon}{a}\right) \frac{J_1\left\{\xi_i\alpha_2\left(1 - \frac{\epsilon}{a}\right)\right\}}{J_1(\xi_i\alpha_2)} \right]$$

$$= \frac{\epsilon}{a} \xi_i \alpha_2^2 J_0(\xi_i\alpha_2),$$

since

$$J_1'(z)=J_0(z)-\frac{1}{z}J_1(z)$$

and so

$$J_1\left\{\xi_i\alpha_2\left(1-\frac{\epsilon}{a}\right)\right\}=J_1(\xi_i\alpha_2)-\xi_i\alpha_2\frac{\epsilon}{a}\left\{J_0(\xi_i\alpha_2)-\frac{J_1(\xi_i\alpha_2)}{\xi_i\alpha_2}\right\}.$$

Thus, if ϵ tends to zero, *i. e.*, if the force is concentrated at the point $x=a$, we obtain

$$\bar{p}_\beta=\psi(t)\frac{2\rho}{g}J_0\{\xi_i(4a/g)^{\frac{1}{2}}\}, \qquad \ldots\ldots\ldots (14)$$

and the appropriate solution of equation (5) is

$$\bar{y}_\beta=\frac{2}{g}J_0\{\xi_i(4a/g)^{\frac{1}{2}}\}\frac{1}{\xi_i}\int_{-\infty}^{t}\psi(\eta)\sin(t-\eta)\xi_i\,d\eta,$$

and so, by equation (6),

$$y=\frac{4}{g\beta^2}\sum_i\frac{J_0\{\xi_i(4a/g)^{\frac{1}{2}}\}J_0(\xi_i z)}{\xi_i[J_1(\xi_i\beta)]^2}\int_{-\infty}^{t}\psi(\eta)\sin\xi_i(t-\eta)\,d\eta.$$

In terms of the original variables x and t we have finally

$$y=\frac{2}{\sqrt{lg}}\sum_i\frac{J_0\{\alpha_i(a/l)^{\frac{1}{2}}\}J_0\{\alpha_i(x/l)^{\frac{1}{2}}\}}{\alpha_i[J_1(\alpha_i)]^2}\int_{-\infty}^{t}\psi(\eta)\sin\{\tfrac{1}{2}\alpha_i(g/l)^{\frac{1}{2}}(t-\eta)\}\,d\eta,$$

the sum being taken over all the positive roots of equation (10).

On the other hand, if the transverse force acts over the whole length of the chain equation (14) is replaced by

$$\bar{p}_\beta=\rho\psi(t)\frac{1}{l}\int_0^\beta zJ_0(\xi_i z)\,dz=\frac{2\rho}{(lg)^{\frac{1}{2}}}\psi(t)\frac{J_1(\xi_i\beta)}{\xi_i}.$$

The analysis proceeds as before and we obtain

$$y=\frac{4}{(lg)^{\frac{1}{2}}}\sum_i\frac{J_0\{\alpha_i(x/l)^{\frac{1}{2}}\}}{\alpha_i^2J_1(\alpha_i)}\int_{-\infty}^{t}\psi(\eta)\sin\{\tfrac{1}{2}\alpha_i(g/l)^{\frac{1}{2}}(t-\eta)\}\,d\eta.$$

As an example, suppose that an external force of magnitude $P\rho\sin\omega t$ is applied to the lower end of the chain initially at rest in its equilibrium position. In this case we may take $a=0$, so that equation (14) reduces to

$$\bar{p}_\beta=2P\rho\sin\omega t/g$$

and the appropriate solution of equation (5) is

$$\bar{y}_\beta=\frac{2P}{g}\cdot\frac{\omega\sin\xi_i t-\xi_i\sin\omega t}{\xi_i(\omega^2-\xi_i^2)},$$

so that, using equation (6), we have

$$y=\frac{8P\omega}{g}\left(\frac{l}{g}\right)^{\frac{1}{2}}\sum_i\frac{\sin\{\tfrac{1}{2}\alpha_i t(g/l)^{\frac{1}{2}}\}J_0\{\alpha_i(x/l)^{\frac{1}{2}}\}}{\alpha_i(4l\omega^2/g-\alpha_i^2)[J_1(\alpha_i)]^2}$$

$$-\frac{4P}{g}\sin\omega t\sum_i\frac{J_0\{\alpha_i(x/l)^{\frac{1}{2}}\}}{(4l\omega^2/g-\alpha_i^2)[J_1(\alpha_i)]^2},$$

the sum extending over all the positive roots of equation (10).

6. *The Forced Vibrations of a Very Long Chain.*

If the chain is of infinite length the analysis proceeds exactly as in the last section as far as equation (14). Then

$$\bar{y}=\frac{2}{g}\cdot\frac{J_0\{\xi(4a/g)^{\frac{1}{2}}\}}{\xi}\int_{-\infty}^{t}\psi(\eta)\sin\xi(t-\eta)\,d\eta,$$

and so by equation (8)

$$y=\frac{2}{g}\int_0^\infty J_0\{\xi(4a/g)^{\frac{1}{2}}\}J_0\{\xi(4x/g)^{\frac{1}{2}}\}\,d\xi\int_{-\infty}^{t}\psi(\eta)\sin\xi(t-\eta)\,d\eta.$$

Interchanging the order of the integrations we have

$$y=\frac{2}{g}\int_{-\infty}^{t}\psi(\eta)\,d\eta\int_0^\infty J_0\{\xi(4a/g)^{\frac{1}{2}}\}J_0\{\xi(4x/g)^{\frac{1}{2}}\}\sin\xi(t-\eta)\,d\eta.$$

If the force is applied to the lower end of the chain $a=0$, and making use of the result

$$\int_0^\infty J_0(ct)\sin bt\,dt=\begin{cases}0 & b<c\\ (b^2-c^2)^{-\frac{1}{2}} & b>c\end{cases},$$

we obtain for the transverse displacement of the chain at any instant

$$y=\frac{2}{g}\int_{t-(4x/g)^{\frac{1}{2}}}^{t}\frac{\psi(\eta)}{\{(t-\eta)^2-4x/g\}}\,d\eta.$$

XXX. *Effect of Refraction upon Accuracy in X-ray Powder Diffraction Patterns.*

By J. SHEARER, B.A., M.Sc.,
Department of Physics, University of Western Australia *.

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ABSTRACT.

THE effect of refraction upon the deviation of an X-ray by a crystal particle is examined in the two cases that the ray enters and emerges (*a*) by the same face and (*b*) by different faces. The approximate error introduced into Debye-Scherrer diffraction patterns by neglecting refraction is then calculated for platy materials like clays and micas. The percentage error may be very approximately independent of wave-length and material.

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* Communicated by the Author.

Introduction.

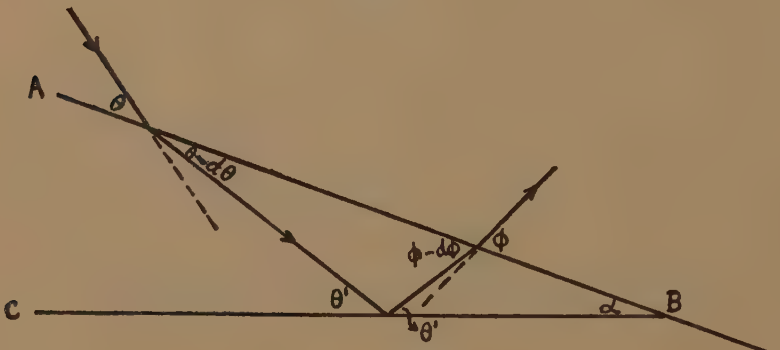
The values of $\frac{d}{n}$ for the lines in an X-ray diffraction pattern of a crystalline powder, recorded on a cylindrical film, are commonly obtained by measuring the deviation for each line and adopting as the Bragg angle one-half the deviation. Refraction is a possible source of error which, although small, may not always be negligible.

Theory.

(*a*) Consider an isolated ray diffracted by a crystal and entering and emerging by the same face.

AB in fig. 1 is the external face of a crystal; θ and ϕ are the incident and emergent grazing angles of an X-ray reflected by the Bragg planes

Fig. 1.



BC. The Bragg formula holds accurately in the form $n\lambda=2d\sin\theta'$. The error $d\theta_1'$ in the Bragg angle deduced from the total deviation is positive and of magnitude $\frac{1}{2}(d\theta+d\phi)$. If μ is the refractive index,

$$\frac{\cos\theta}{\cos(\theta-d\theta)}=\frac{\cos\phi}{\cos(\phi-d\phi)}=\mu=1-\delta. \quad \ldots\ldots \quad (1)$$

The maximum value $d\theta$ (or $d\phi$) can assume is θ_m, where θ_m is the maximum grazing angle for total reflection. This is the smallest value θ can assume. For Cu Kα radiation and a crystalline powder of refractive index similar to that of calcite (conditions that will be assumed throughout) $\delta=9\times10^{-6}$ and $\theta_m=\sqrt{2\delta}=0{\cdot}0042=15$ minutes. $d\theta$ is therefore always small, and, if θ is large compared with $d\theta$, $d\theta$ from equation (1) reduces approximately to $\delta\cot\theta$; and, if ϕ is large compared with $d\phi$, $d\phi$ from equation (1) reduces approximately to $\delta\cot\phi$. Therefore, when θ and ϕ are relatively large,

$$d\theta_1'=\frac{\delta}{2}(\cot\theta+\cot\phi). \quad \ldots\ldots \quad (2)$$

In the figure $d\theta_1'$ is positive, since both $d\theta$ and $d\phi$ are positive. In the case of α zero (that is, $\theta=\phi$ and therefore the passage of the ray is symmetrical) $d\theta$ and $d\phi$ are equal and positive. We then have

$$d\theta_1'=\delta\cot\theta$$

and therefore approximately

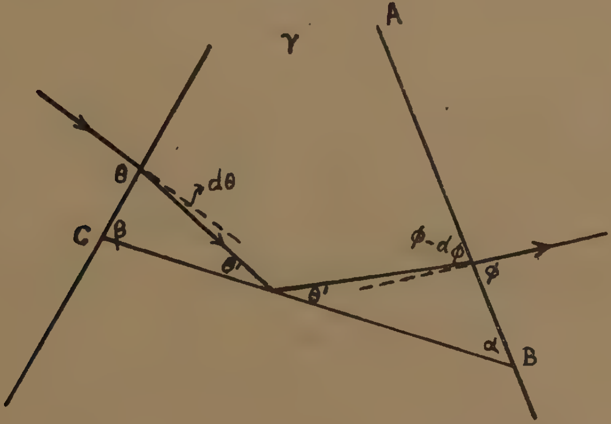
$$d\theta_1'=\delta\cot\theta'.$$

The maximum value α can assume is θ'. In that case $\theta=d\theta=\theta_m$, and equation (2) no longer holds. Assuming ϕ is relatively large, so that $d\phi$ is negligible, we have in this case

$$d\theta_1'=\tfrac{1}{2}d\theta=\tfrac{1}{2}\sqrt{2\delta}.$$

Under conditions of radiation and material assumed this is approximately 2×10^{-3} radian. If $\frac{d}{n}$ be large and therefore θ' small, this error could amount to several per cent.

Fig. 2.



If $\theta'>\frac{\pi}{4}$ then, when α lies between zero and $\left(\frac{\pi}{2}-\theta'\right)$, $d\phi$ is positive; and when α lies between $\left(\frac{\pi}{2}-\theta'\right)$ and θ', $d\phi$ is negative but numerically smaller than $d\theta$. If $\theta'<\frac{\pi}{4}$ only positive values of $d\phi$ occur in the case of a ray entering and emerging by the same face. In all cases, therefore, where the ray enters and emerges by the same face (that is, $\alpha<\theta'$) $d\theta_1'$ is positive. This generalization applies equally for rays travelling in the opposite direction to that of the ray in fig. 1.

(*b*) In the case that $\alpha>\theta'$ Bragg reflections from these planes can occur only when the incident ray enters or the emergent ray leaves by a face of the crystal other than AB. Fig. 2 is drawn to apply to the case where the incident and emergent rays enter and leave by different faces.

Equation (1) now assumes the form

$$\frac{\cos\theta}{\cos(\theta+d\theta)}=\frac{\cos\phi}{\cos(\phi-d\phi)}=\mu=1-\delta.$$

But under the conditions specified in case (*a*),

$$d\theta=-\delta\cot\theta$$

and also

$$d\phi=\delta\cot\phi.$$

Therefore, since the error $d\theta_2'$ in the Bragg angle deduced from the total deviation is of magnitude $\frac{1}{2}(-d\theta+d\phi)$,

$$d\theta_2'=\frac{\delta}{2}(\cot\theta+\cot\phi). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equations (2) and (3) are identical in form.

Under conditions of fig. 2 consider symmetrical passage only (that is $\alpha=\beta$). Then

$$d\theta_2'=\delta\cot\theta.$$

This is negative when θ is greater than $\frac{\pi}{2}$ (that is, γ is less than $2\theta'$); zero when θ is equal to $\frac{\pi}{2}$ (that is, γ equal to $2\theta'$); and positive when θ is less than $\frac{\pi}{2}$ (that is, γ greater than $2\theta'$). The greatest positive value occurs when γ is equal to $(\pi+2\theta')$. θ is now very small and equation (3) no longer holds. We have in this case

$$d\theta_2'=\tfrac{1}{2}(d\theta+d\phi)=\theta_m=\sqrt{2\delta}.$$

Under conditions of radiation and material assumed this is approximately 4×10^{-3} radian. This could amount to an error of several per cent. if θ' is small.

A common case will be that in which γ is of the order of zero. The error $d\theta_2'$ is therefore commonly negative. The maximum negative value it can assume will occur when θ is approximately $(\pi-\theta_m)$. γ is then approximately zero and θ' is approximately $\frac{\pi}{2}$. We then have

$$d\theta_2'=-\tfrac{1}{2}(d\theta+d\phi)=-\theta_m=-\sqrt{2\delta}.$$

Under conditions of radiation and material assumed this is approximately 4×10^{-3} radian numerically, which is of the order of 2 in 1000. Under γ zero conditions, therefore (that is, for symmetrical passage, α and β each equal to $\left.\frac{\pi}{2}\right)$, the percentage error in $d\theta_2'$ is always very small and negative.

Application to Debye-Scherrer Diffraction Patterns.

When an X-ray beam falls on a crystal particle, enveloping it, and undergoing diffraction, portion of the beam will in general be deviated through $(2\theta'+2d\theta_1')$ and portion through $(2\theta'+2d\theta_2')$. But many crystalline particles (for example, clays and micas) are platy in character, that is, of thickness negligible compared with the cross-sectional dimensions and with basal planes parallel to the broad external faces. The whole of the X-ray beam falling on such a particle and reflected from the basal planes (say, 001) may therefore be considered to enter and emerge by the same face, and the total error $d\theta'$ will be approximately $d\theta_1'$, that is, $\delta \cot \theta'$. Moreover, $\frac{d}{n}$ values of basal spacings being relatively large, with basal spacings will be associated small values of θ' and so large values of $d\theta_1'$.

In the case of platy particles of small but not negligible thickness, $d\theta_2'$ would be generally small compared with $d\theta_1'$, and the error involved in measuring to the centre of a diffraction line contributed to both under conditions of fig. 1 and under conditions of fig. 2 would be less than $\delta \cot \theta'$.

In the case of platy particles reflections from (010) and (100), where (001) are the Miller indices for basal planes, will occur under conditions of fig. 2, with γ of the order of zero and α and β of the order of $\frac{\pi}{2}$. The total error will be approximately $d\theta_2'$, which will be much less than $\delta \cot \theta'$.

$\delta \cot \theta'$ is therefore a measure of the maximum error that might commonly occur. It would be expected to occur for basal reflections from platy particles. Under the conditions of radiation and material assumed this would amount to a positive error of $\frac{1}{3}$ per cent. in θ' in the case of $\frac{d}{n}$ equal to 15 A. This represents a negative error of 0·05 A. in $\frac{d}{n}$.

In view of the fact that particles of more regular dimensions will in general have no very large spacings, $d\theta_1'$ in the case of such particles will be of the same order of numerical magnitude as $d\theta_2'$, but of opposite sign, and so the total error $d\theta'$ will be approximately zero.

If the Miller indices of external faces and the Miller indices of the Bragg planes giving a reflection are known for a particular crystalline powder, $d\theta'$ could be approximately calculated, allowance being made for the fact that any orientation about an axis perpendicular to the Bragg planes is permissible in the case of a particle giving a diffracted beam. Such calculation, of course, requires an approximate knowledge of the refractive index of the crystalline powder for the wave-length used.

With regard to the correction calculated for basal reflection from platy particles, it is of interest to examine briefly the dependence of this correction upon wave-length and upon material. For basal reflections

θ' is small and the correction becomes approximately $\frac{\delta}{\theta'}$. That is,

$$d\theta' = \frac{\delta}{\theta'}$$

or

$$\frac{d\theta'}{\theta'} = \frac{\delta}{\theta'^2}$$

But from Bragg's Law, when θ' is small,

$$\theta'^2 \propto \lambda^2.$$

Therefore

$$\frac{d\theta'}{\theta'} \propto \frac{\delta}{\lambda^2}$$

Now $\frac{\delta}{\lambda^2}$ is constant for a given substance (anomalous dispersion being neglected) and may not vary much from substance to substance. The calculated percentage error for basal reflections from platy substances may therefore be very approximately independent of wave-length and of material irradiated over a fairly wide range.

Two subsidiary points of interest arise out of the above discussion :—

In the first place, in the case of either diffracted beams or undiffracted beams, many crystalline particles in the path of the beam will refract (without diffraction) the portion of the beam traversing them. The effect of this multiple refraction will be to broaden symmetrically a diffracted or undiffracted beam traversing a crystalline powder *.

In the second place, multiple diffraction may occur. An intense diffracted beam may act as an incident beam giving a secondary diffracted beam of sufficient intensity to be recorded.

XXXI. *Milne's Cosmology and the Origin of Cosmic Rays.*

By E. P. George,
Birkbeck College, University of London †.

[Received May 15, 1947.]

In the course of the development of his cosmological theories, E. A. Milne has suggested a possible origin of cosmic rays (1). This may be briefly summarized by saying that if a particle at any time in its career is situated non-symmetrically in Milne's system of "equivalent" particle-observers, then it will be continually accelerated towards the apparent centre of this system. As a result of this acceleration, particles should be incident on the top of the earth's atmosphere travelling with velocities up to that of light. The more energetic of these particles are identified with the primary component of cosmic radiation.

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* R. von Nardroff : "Refraction of X-rays by small particles," Physical Review, xxviii. p. 240 (1926).

† Communicated by the Author.

The purpose of this note is to point out that not only does Milne's cosmology give a plausible qualitative picture of the origin of the corpuscular component, but also an explicit expression for the energy spectrum may be found which it should be possible to compare with experiment.

Thus, let the number of primary particles entering the atmosphere in the energy range E to E+dE be $n(E)dE$, and correspondingly let the distribution in terms of velocities be $n(v)dv$.

Then

$$n(E)dE=n(v)dv, \qquad (1)$$

and

$$n(E)=n(v)\,\frac{dv}{dE}. \qquad (2)$$

Now,

$$E=\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

and

$$\frac{dE}{dv}=\frac{m_0v}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}. \qquad (3)$$

Substitute (3) in (2) and we get

$$n(E)=\frac{n(v)\left(1-\frac{v^2}{c^2}\right)^{3/2}}{m_0v}$$

$$=\frac{n(v)m_0{}^2c^6E^{-3}}{v}. \qquad (4)$$

Now, for the purpose in hand, we are only concerned with $n(v)$ for $v\sim c$. According to Milne (2), the velocity spectrum is given by

$$n(v)=\psi\left(\frac{1}{1-\frac{v^2}{c^2}}\right)\frac{v^3}{\left(1-\frac{v^2}{c^2}\right)^{5/2}}, \qquad (5)$$

and for $v\sim c$ we may take (3)

$$\psi(x)=\frac{N}{x^{5/2}}, \qquad (6)$$

where N is a constant.

Substituting (6) in (5) gives the velocity spectrum for velocities in the neighbourhood of c:

$$n(v)=Nv^3. \qquad (7)$$

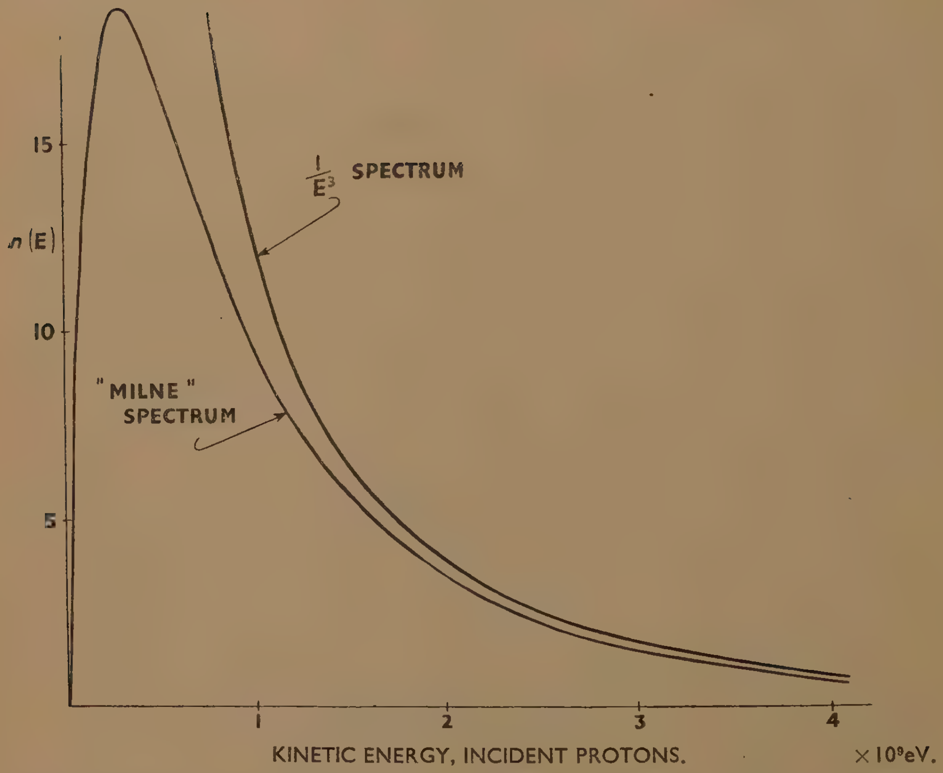
Substitution of (7) in (4) now gives the energy spectrum of the high energy primaries

$$n(\mathrm{E})=\mathrm{N}v^2m_0{}^2c^6\mathrm{E}^{-3},$$

or

$$n(\mathrm{E})=\frac{\mathrm{const.}\left(1-\dfrac{m_0{}^2c^4}{\mathrm{E}^2}\right)}{\mathrm{E}^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Fig. 1.



The present evidence appears to be in favour of the hypothesis that the primary corpuscular radiation consists of a mixture of protons on the one hand, and positive and negative electrons on the other (4). The spectrum (8) would apply equally well to either, but the low energy cut off, due to the term in the numerator, would occur at different energies.

At sea-level, and alternatively any height in small or medium magnetic latitudes where primary energies $\gg m_0c^2$ of even protons are involved, the term in brackets is of the order of unity, giving

$$n(\mathrm{E})=\frac{\mathrm{const.}}{\mathrm{E}^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This is very similar to the spectrum deduced from observations; in fact Janossy ⁽⁴⁾ finds $n(\mathrm{E}) \sim 1/\mathrm{E}^{2\cdot5}$ and Heisenberg ⁽⁵⁾ obtains $1/\mathrm{E}^{2\cdot9}$. Clearly, the exponent is not known accurately, and it may well be 3, as required by (9) above.

The spectrum (9) may be expected to extend up to $\mathrm{E} \sim 10^{17}$ eV., for charged particles, but not beyond, as it has been shown by Pomeranchuk ⁽⁶⁾ that an energy of this order of magnitude is the maximum with which a charged particle may arrive at the earth's surface, due to the large probability of energy loss by radiation in the field of the earth's magnetic dipole possessed by particles of larger energy.

It is known, further, from the measurements of Carmichael and Dymond ⁽⁷⁾, that there is a cut-off on the low energy end of the primary spectrum for energies $<2\cdot10^9$ eV. The spectrum being given by ⁽⁴⁾

$$\left.\begin{aligned} n(\mathrm{E}) &= \frac{\mathrm{A}}{\mathrm{E}^{\gamma}} \ldots \mathrm{E} > 2\cdot10^9 \text{ eV.} \\ &= 0 \ldots \mathrm{E} < 2\cdot10^9 \text{ eV.} \\ & \qquad (\gamma \sim 2\cdot5 \text{ to } 3) \end{aligned}\right\} \quad \ldots\ldots \quad (10)$$

This low energy cut-off has been attributed to the blocking effect of the sun's magnetic field ⁽⁸⁾. An interesting feature of the spectrum (8) is that for the case of protons it also shows a low-energy cut-off, due to the term in the numerator. The low energy of the spectrum (8) is plotted out in fig. 1. It is not the same as the observed spectrum (10), but is clearly very similar. This disagreement may be due to the use of the approximation (6). A more exact expression for Milne's function $\psi(x)$ is required before a critical discussion of the exact shape of the low energy end of the Milne spectrum is possible.

References.

(1) E. A. Milne, ' Relativity Gravitation and World Structure,' chap. xii.
(2) E. A. Milne, *ibid.*, p. 189.
(3) E. A. Milne, *ibid.*, p. 191.
(4) Jánossy, ' Communications Dublin Institute for Advanced Studies ' (1947).
(5) Heisenberg, *Erg. Exakt. Naturwiss.* xvii. p. 1 (1938).
(6) I. Pomernachuk, *Journ. Phys. U.S.S.R.* ii. p. 65 (1940).
(7) Carmichael and Dymond, Proc. Roy. Soc. A, clxxi. p. 321 (1939).
(8) Jánossy, *Zeitschr. Phys.* civ. p. 430 (1937).

XXXII. *Notices respecting New Books.*

Sequential Analysis. By A. Wald. [Pp. xii+212.] (John Wiley and Sons, 1947.) Price 24*s*. 0*d*.

This book, which belongs to Wiley's Series on Mathematical Statistics, gives an account of the Sequential Probability Ratio Test, which was devised by the author in 1943 for the purpose of testing statistical hypotheses. An important feature of this test is that the number of observations required is not predetermined but depends on the outcome of the observations already made. This reduces considerably the number of observations necessary compared with other procedures.

Part I. presents the general theory of the test and applications are given in Part II. These include the testing of the mean of a binomial distribution and the difference between the means of two such distributions, and also tests of various properties of normal distributions. The problem of multi-valued decisions and estimation is briefly treated in Part III., but this field is at present largely undeveloped.

The book is readable by those possessing only an ordinary knowledge of mathematics, and topics requiring advanced mathematical treatment are relegated to the Appendix.

L. S. G.

Kinetic Theory of Liquids. By J. Frenkel. [Pp. xi+488.] (Oxford University Press, 1946. Price 40*s*.)

The International Series of Monographs on Physics, published by the Oxford University Press, is well known and the present book is the latest addition to the impressive list of books already issued. In the kinetic theory of liquids a new point of view is presented in that the analogy between the liquid and the gaseous state is now largely discarded; and instead the approximation of the liquid to the solid state is emphasized. The author has played a large part in the development of this theory during the past twenty years, and this book is the first readily accessible account of the subject. It should appeal to a wide circle of physicists and chemists who are concerned with matter in the condensed state.

The first two chapters are given to that part of the theory of ideal crystal lattices which bridges the gap between the liquid and solid states. The properties of crystals at elevated temperatures are considered and the perturbation of alternation and orientation order in crystals is studied. Chapter III deals with the mechanism of fusion and the relation between the solid and liquid states. The hole theory of the liquid state is presented. Chapters IV and V deal with motion in liquids—heat motion and the orientation and rotational motion of molecules. The theories of Debye and of Onsager concerning the polarization of dipolar liquids in an electric field are discussed and also Kirkwood's improvement of these theories. Surface tension and allied phenomena, such as monomolecular adsorbed layers are dealt with in Chapter VI. The molecular kinetic theory of liquid films is given. In Chapter VII the kinetics of phase transitions is studied, and Chapter VIII deals with the properties of solutions and high polymeric substances, including rubber-like substances.

Recently Born and Green commenced publication of a series of papers on the kinetic theory of liquids. These point the way to new mathematical methods for dealing with the theory. It appears that this work has been carried through without any or much reference to the Russian work, and it will be interesting to see, as more of this theory is unfolded, in what way these two accounts are connected.

L. S. G.

[The Editors do not hold themselves responsible for the views expressed by their correspondents.]

XXXIII. *The Constitution of the Aluminium-rich Ternary Alloys of Aluminium and Manganese with Zinc and Cadmium.*

By D. W. Wakeman and G. V. Raynor*.

[Received October 11, 1947.]

[Plate I.]

Synopsis.

The constitution of the aluminium-manganese-zinc alloys was examined, in the range 0–6 per cent. manganese and 0–40 per cent. zinc, by the establishment of a 500° C. isothermal. With increasing zinc, $MnAl_6$, T_1, and $MnAl_4$ enter successively into equilibrium with the aluminium-zinc solid solution (α). T_1 is a ternary compound analogous to corresponding compounds in the systems aluminium-manganese-nickel and aluminium-manganese-copper. Solidus curves, at constant manganese contents, agree with the results in the solid state. ($\alpha+T_1+MnAl_6$) alloys melt at 542° C. to 545° C., while ($\alpha+T_1+MmAl_4$) alloys melt at 508° C. Similar work on aluminium-manganese-cadmium alloys at 600° C. reveals no ternary compound. Thus nickel, copper and zinc are within the zone of favourable atomic size for the formation of ternary compounds of the type considered, while cadmium, with a larger atom, is outside. The formation of growth twins in T_1 crystals was observed and is described.

Introduction.

In the alloys of aluminium and manganese with copper (1), nickel (2) and zinc (3) analogous ternary compounds occur in equilibrium with the aluminium-rich solid solution. In these compounds two conditions are satisfied :—

(*a*) a ratio of four atoms to one of combined solutes; and

(*b*) an electron : atom of 1·85, calculated on the basis of a theory of the rôle of transitional metals (1), (2) which considers the vacancies in the atomic orbitals postulated in Paulings' theory of these elements (4).

Condition (*a*) depends on atomic size relations. In the ternary systems mentioned above, manganese, copper, nickel and zinc have similar atomic diameters. The substitution of cadmium for zinc, while leaving electronic considerations unaltered, would alter the size relationships. Both zinc and cadmium have close packed hexagonal structures, the inter-atomic distances being 2·6595 and 2·9070 A. for zinc and 2·9731 and 3·2872 A. for cadmium. The larger size factor for cadmium might thus prevent the occurrence of a compound of the type considered in the system aluminium-manganese-cadmium. The present paper is presented in two sections. Part I. describes an extension of previous work on the aluminium-manganese-zinc alloys ⁽³⁾ to equilibrium conditions below and just above the solidus, to confirm that the ternary compound enters into equilibrium with the aluminium-rich solid solution as in the aluminium-manganese-copper and aluminium-manganese-nickel systems. Part II. describes work on the aluminium-manganese-cadmium system, to examine the effect of substituting cadmium for zinc.

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* Communication from the Department of Metallurgy, The University of Birmingham.

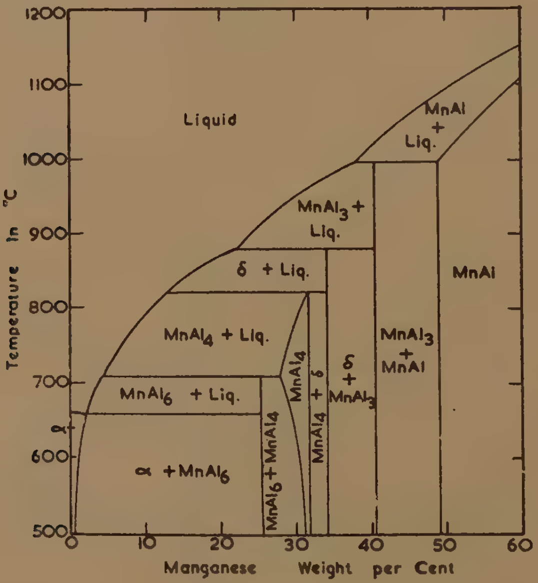
PART I.—THE SYSTEM ALUMINIUM-MANGANESE-ZINC.

(1) *Previous Work.*

(a) *The Binary Systems.*

The equilibrium diagrams of the systems aluminium-manganese and aluminium-zinc have been previously discussed ⁽³⁾, and are reproduced as figs. 1 and 2.

Fig. 1.



(b) *The Ternary System.*

Butchers, Little, Raynor and Hume-Rothery ⁽⁵⁾, ⁽⁶⁾, ⁽⁷⁾, ⁽⁸⁾ have determined the solid solubility, solidus and liquidus surfaces for a limited range of compositions (0 to 2 per cent. manganese, 0 to 8 per cent. zinc). The present authors ⁽³⁾ carried out a more extensive study, but this work was confined to the constitution of alloys in the semi-liquid state under conditions of slow cooling. Alloys in the range 0 to 3 per cent. manganese and

0 to 95 per cent. zinc were examined micrographically, and by X-ray and chemical analysis of crystals separated from slowly cooled melts. The surfaces of primary separation are shown in fig. 3. A notable feature is the long eutectic valley extending from the aluminium-manganese eutectic to a ternary eutectic at 95 per cent. zinc and 0·05 per cent. manganese (378° C.). The phases $MnAl_6$, T_1, $MnAl_4$, T_2, T_3 and $MnAl_3$ come successively into equilibrium with α on passing along the eutectic valley from the aluminium-manganese axis to the ternary eutectic. The compositions of these phases

Fig. 2.

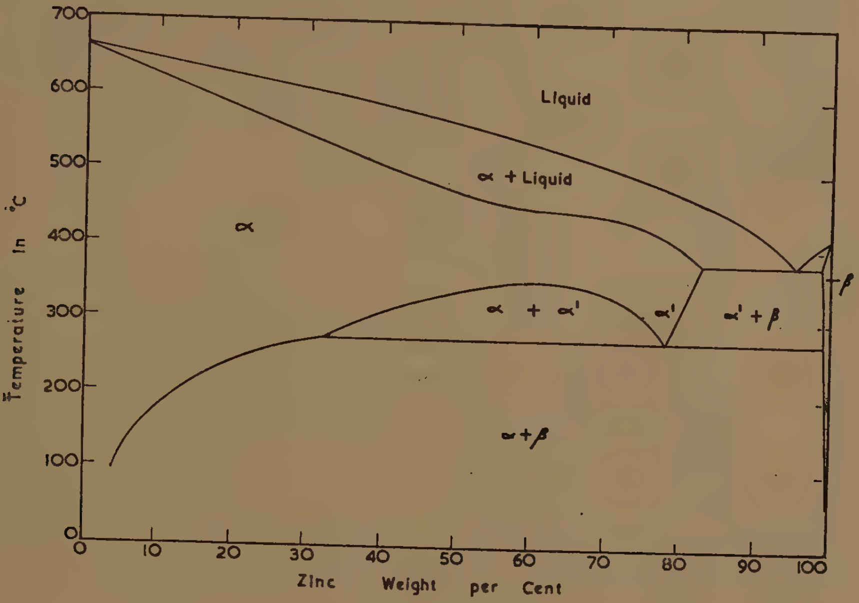
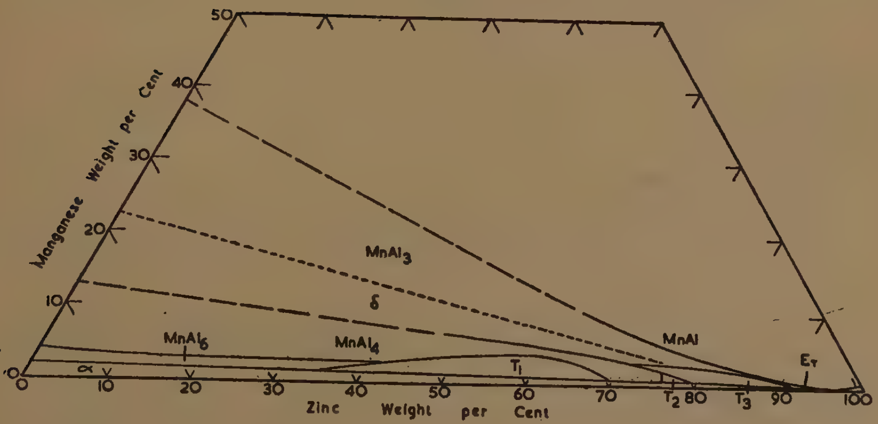


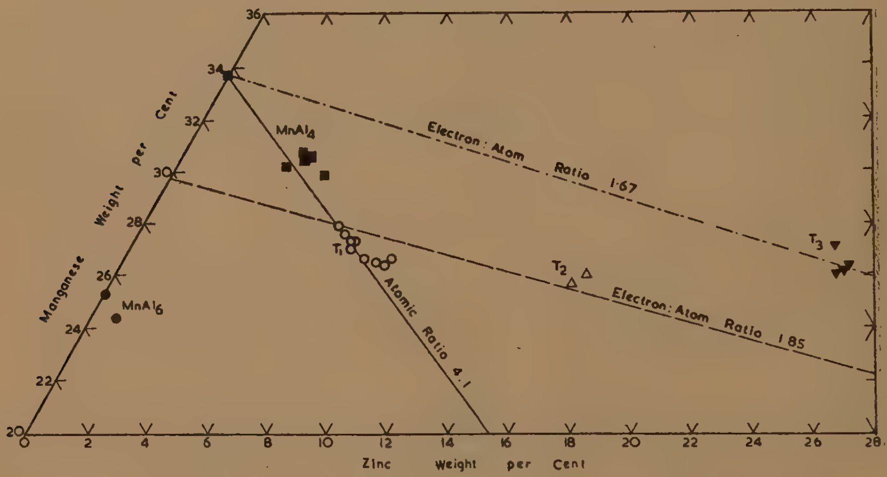
Fig. 3.



are shown in fig. 4, in which analysed compositions of extracted crystals are plotted. $MnAl_6$ and $MnAl_4$ can dissolve small quantities of zinc; T_1, T_2 and T_3 are ternary compounds, of which T_1 is a phase of the type discussed in the Introduction. T_2 corresponds to the formula Mn_2ZnAl_9 and an electron : atom ratio, according to the theory of the rôle of transitional metals, of 1·81. This formula can also be expressed as $(Mn \cdot Zn)Al_3$, suggesting analogy with $CoAl_3$, which has the almost identical electron : atom ratio of 1·82. T_3 does not correspond with a simple formula, but its electron : atom ratio is similar to that of $MnAl_4$.

The zinc-rich alloys were examined by Gebhardt (9), whose results have been utilized in fig. 3.

Fig. 4.



(2) *Experimental Methods.*

(a) *Materials Used.*

Alloys were prepared from the following materials :—

(i) Pure zinc, kindly presented by the Imperial Smelting Corporation, Ltd., in which only lead (0·0001 per cent.), cadmium (0·00005 per cent.) and calcium (0·0001 per cent.) were detected spectrographically.

(ii) Super-pure aluminium, kindly presented by the British Aluminium Co., Ltd.

(iii) An aluminium-manganese alloy (silicon 0·004 per cent., iron 0·0025 per cent., copper 0·001 per cent. and manganese 5·25 per cent.) prepared by the British Aluminium Co., Ltd.

(iv) An aluminium-manganese-zinc alloy (manganese 5·7 per cent. and zinc 67 per cent.), prepared by the authors by reduction of Analar manganous chloride with an aluminium-zinc alloy.

(b) *Preparation of Alloys.*

Weighted amounts of the master alloys and aluminium were melted in alumina-lined crucibles, and the zinc quickly stirred in with an alumina stirrer. Alloys with more than 90 per cent. aluminium, and exploratory alloys with less aluminium, were cast into a bronze chill mould, giving ingots 3″ × $\frac{5}{8}$″, weighing 20 g. Experience showed that equilibrium was very difficult to attain on annealing alloys containing less than 90 per cent. aluminium, and the apparatus shown in fig. 5 was used to obtain more rapid chilling. Microstructures of alloys of the same composition cast in both this mould, the method of use of which is evident from fig. 5, and the bronze mould, showed that specimens from the former possessed finer structures ; these specimens attained equilibrium more quickly on annealing. Fine intergranular contraction cavities, however, made detection of traces of chilled liquid somewhat uncertain after annealing above the solidus.

Fig. 5.

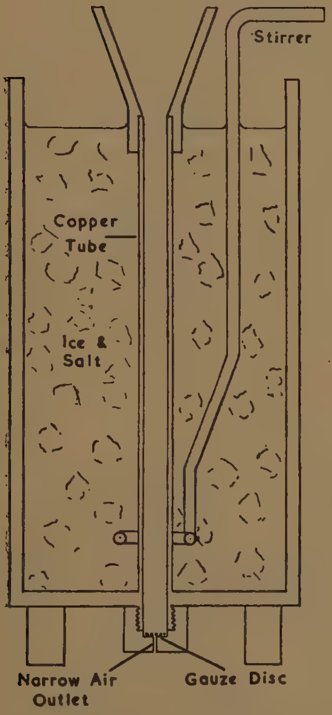
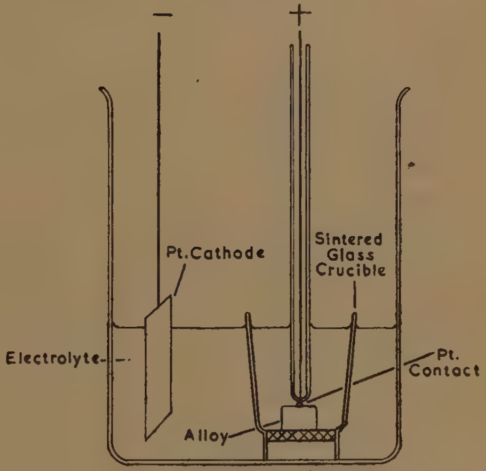


Fig. 6.



(c) *Annealing Technique.*

Specimens from the chill-cast ingots were packed, well separated, in alumina powder in an alumina-lined crucible ; the crucible top was tightly closed by an aluminium stopper, through a hole in which an alumina thermocouple sheath was inserted for accurate location of the measuring thermocouple. The whole was heated in a tubular furnace accurately controlled by an automatic regulator. For work above the solidus, specimens were placed in alumina powder in a fireclay boat ; temperatures were measured with the bare thermocouple hot-junction inserted in the powder among the specimens. All annealing treatments were terminated by quenching in cold water.

(d) *X-ray Examination.*

To assist in phase identification, crystals of intermediate phases present in actual annealed microsections were separated from the matrix by electrolytic extraction, using the apparatus of fig. 6. The electrolyte was 5 per cent. aqueous hydrochloric acid, and a current density of 0·1 amp./in.^{2} was used. Minute crystals of intermetallic compound collected on the sintered glass pad ; these were filtered, and washed with water and alcohol by gentle suction. After drying, the residue was mounted on a hair in the usual manner, and exposed to Iron Kα radiation in a 9 cm. Debye-Scherrer camera. Separation of the compound from the matrix was rarely complete, and faint face-centred cubic diffraction lines were usually visible. Results, however, were much better than could be obtained from filings of the original alloy, and films could be used for certain identification of the compound present by comparison with films obtained from pure specimens of the phases occurring in this system.

(e) *Micrography of the Alloys.*

Specimens were prepared in the usual manner ; all particles visible in unetched surfaces were pale grey, and clear phase differentiation was impracticable. The most consistent results were obtained by etching with aqueous solutions containing 2 to 5 per cent. of sodium hydroxide and an equivalent amount of sodium carbonate. Optimum compositions, temperatures and etching times varied with the zinc contents of the alloys. Chilled liquid in alloys quenched from above the solidus was satisfactorily revealed by an aqueous solution containing 0·5 per cent. of hydrofluoric acid, 2·5 per cent. of nitric acid, and 1·5 per cent. of hydrochloric acid.

(3) *Experimental Results.*

(a) *Nomenclature.*

Specimens are identified by their compositions ; " alloy 0·5/4·5 " refers to an alloy containing 0·5 per cent. by weight of manganese and 4·5 per cent. weight of zinc.

(b) *Constitution in the Solid State.*

(i) Alloys with more than 90 per cent. aluminium.

Fig. 8 shows an isothermal section at 600° C. ; all specimens received 7 days annealing treatment, followed by a further 7 days, which produced

Fig. 7.

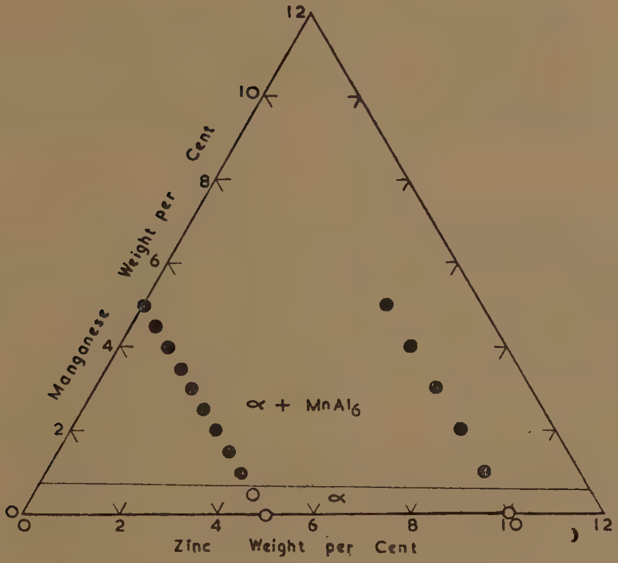
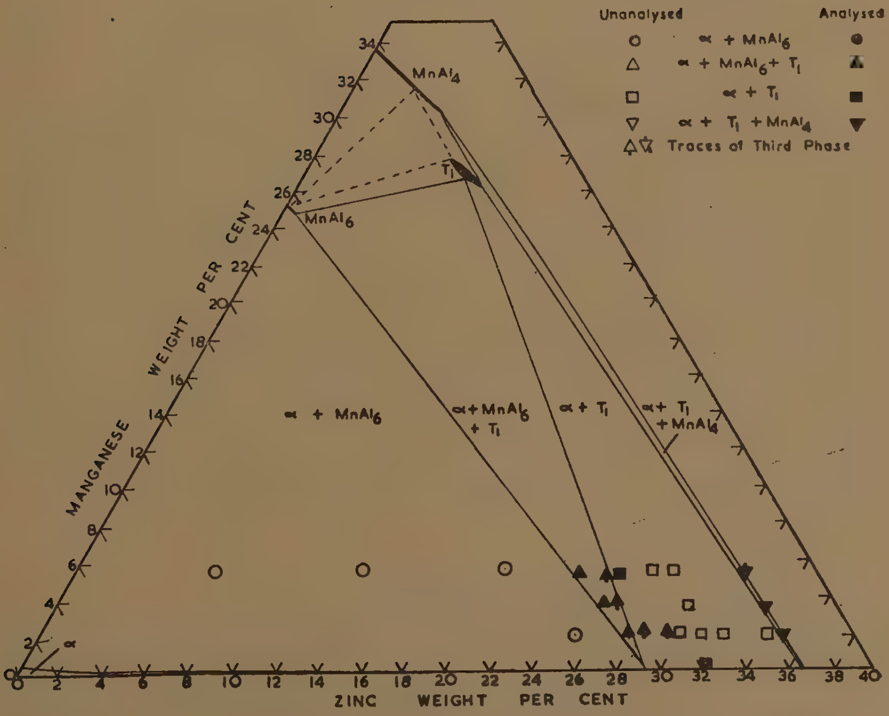

Manganese Weight per Cent
∝ + MnAl6
∝
Zinc Weight per Cent

Fig. 8.


Unanalysed
Analysed
○ ∝ + MnAl6 ●
△ ∝ + MnAl6 + T1 ▲
□ ∝ + T1 ■
▽ ∝ + T1 + MnAl4 ▼
△☆ Traces of Third Phase
MnAl4
T1
MnAl6
MANGANESE WEIGHT PER CENT
∝ + MnAl6
∝+MnAl6 + T1
∝ + T1
∝+ T1 +MnAl4
∝
ZINC WEIGHT PER CENT

no change in microstructure. The second phase in duplex alloys was identified micrographically and by X-rays as $MnAl_6$. Alloy 0·5/4·5 (homogeneous at 600° C.) was hammered and re-annealed at 530° C. for 23 days, and precipitation of $MnAl_6$ occurred in the grain boundaries.

These results are consistent with the more detailed work of Little, Raynor and Hume-Rothery (7) and no further work on these alloys was done.

(ii) Alloys with less than 90 per cent. aluminium.

Alloys containing 5·6 per cent. manganese and 6 to 60 per cent. zinc, cast in the bronze mould, were annealed at 500° or 400° C. (according to composition) for 10 days. The microstructures were unsatisfactory; the structure was coarse and many intermetallic compound crystals were rimmed by different material. Prohibitive times would have been necessary for the attainment of equilibrium, and all alloys on which the present conclusions are based were cast in the special mould described in Section 2 (b).

Alloys containing 35 to 75 per cent. zinc were annealed for 70 days at 400° C. After this treatment all contained three or more phases and were not in equilibrium. In view of the complexity of the system and the difficulty of attaining equilibrium, work on alloys containing more than 40 per cent. zinc was discontinued.

Alloys containing up to 40 per cent. zinc were annealed at 500° C. for 70 days. Fig. 8 shows the results of micrographic examination. Alloys 2·2/28·25 and 2·1/29·25 contained very small traces of $MnAl_6$, and it was concluded that in complete equilibrium these alloys would lie in the $(\alpha+T_1)$ field. Alloys 2·0/35·4, 2·0/36·5, 2·0/38·0 and 2·9/39·0 were also not in equilibrium and contained traces of chilled liquid; these points are omitted from fig. 8. Critical alloys were analyzed and are distinguished in the figure by black symbols. The ranges of homogeneity of the phases $MnAl_6$, T_1 and $MnAl_4$ in fig. 8 are taken from previous work (3). The phase boundaries are approximate at high manganese contents, but are accurately located at the lower manganese contents by the plotted points.

Diffraction patterns were obtained from small crystals of the second phase extracted from alloys 2·0/25 and 2·0/32. The patterns, in agreement with the microscopic work, corresponded exactly with those obtained respectively from pure $MnAl_6$ and T_1 crystals available from previous work (3).

Figs. 9 to 12 (Pl. I.) show typical microstructures of these alloys.

(c) *The Solidus.*

(i) Alloys with more than 90 per cent. aluminium.

Specimens already annealed for 7 days at 600° C. were held at successively higher temperatures for 1 to 1½ hours and quenched in water. Results are given in figs. 13 and 14 in the form of vertical sections through the ternary model at 95 and 90 per cent. aluminium.

Fig. 13.

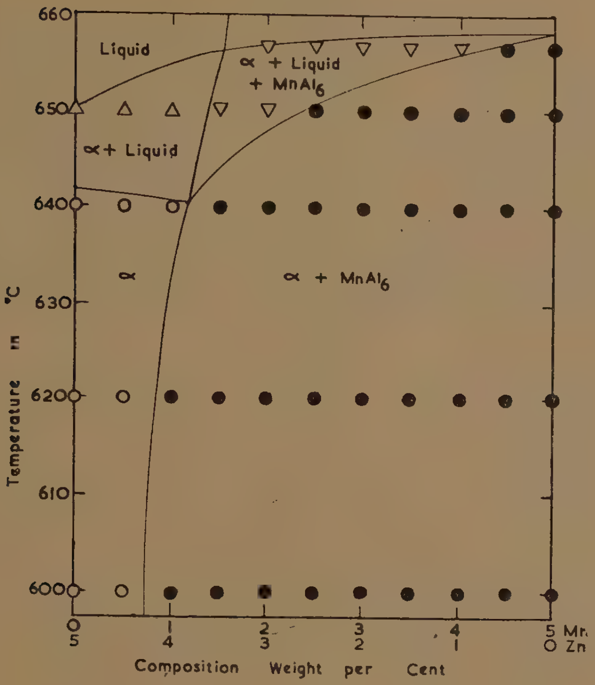
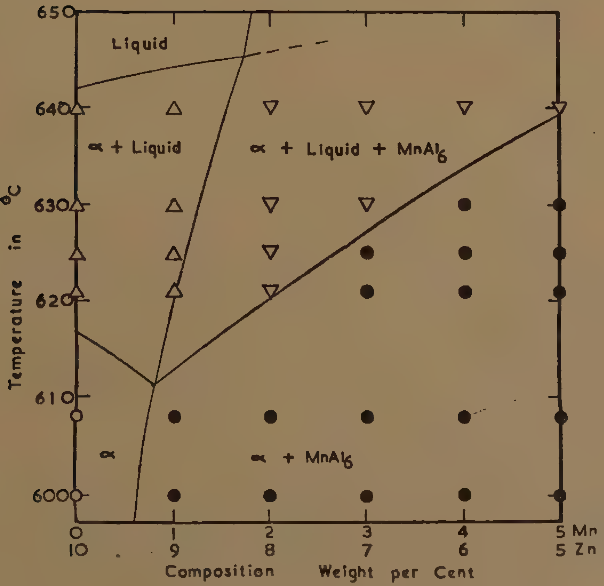

Liquid
∝ + Liquid + MnAl$_6$
∝ + Liquid
∝
∝ + MnAl$_6$
Temperature in °C
660
650
640
630
620
610
600
0 5
1 4
2 3
3 2
4 1
5 Mn 0 Zn
Composition Weight per Cent

Fig. 14.

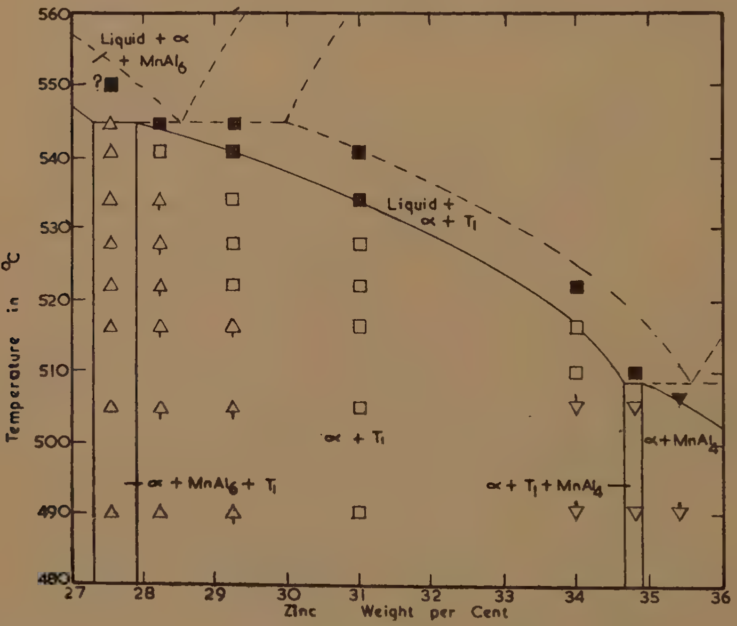

Liquid
∝ + Liquid
∝ + Liquid + MnAl$_6$
∝
∝ + MnAl$_6$
Temperature in °C
650
640
630
620
610
600
0 10
1 9
2 8
3 7
4 6
5 Mn 5 Zn
Composition Weight per Cent

(ii) Alloys with less than 90 per cent. aluminium.

Difficulty was encountered in determining the solidus owing to the presence of fine intergranular cracks ; the first traces of liquid appeared in a somewhat similar form. Solidus temperatures are thus considered to lie between that temperature at which chilled liquid was certainly observed and that at which no chilled liquid was visible. The melting temperature deduced are consequently expected to be slightly higher than the true solidus.

The specimens, previously annealed at 500° C. for 26 days, were held at successively higher temperatures for 18 to 24 hours and the results are shown, for constant manganese contents of 2, 3·5 and 5·1 per cent. manganese, in figs. 15, 16 and 17. It is to be noted that some of the alloys had not attained full equilibrium at 500° C. but relaxed towards equilibrium with rise in temperature. This is indicated by the symbols used in figs. 15, 16 and 17 and equilibrium was probably established at and just below the solidus. The results of the solidus work agree well with those of work in the solid state, and confirm the disposition of the phase-fields in fig. 8.

Fig. 15.



(iii) Twinning in the T_1 phase.

In preliminary work, alloys 2·0/32 and 2·0/35 were annealed, above the solidus, at 528° C. for 26 days and quenched. The microstructures showed large crystals of T_1 and much chilled liquid. On etching for 20 seconds in a solution containing 3 per cent. sodium hydroxide and 5 per cent. sodium carbonate (30° C.), T_1 crystals showed a structure resembling twinning (fig. 18, Pl. I.). Large T_1 crystals in slowly cooled alloys did

Fig. 16.

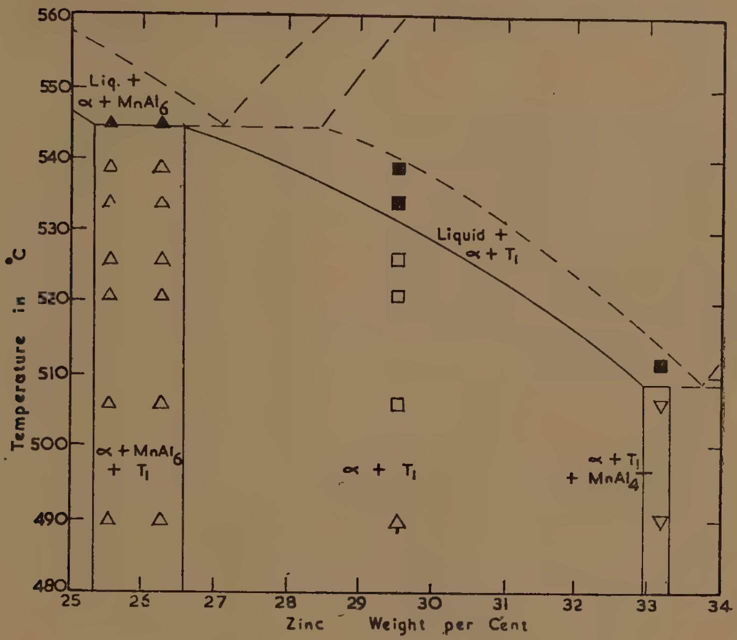
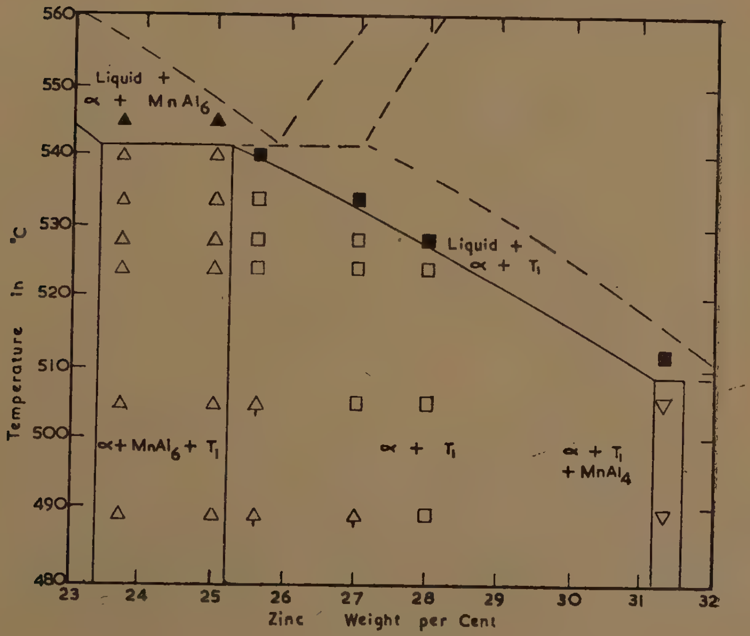

Temperature in °C
560
550
540
530
520
510
500
490
480
Liq. +
∝ + MnAl$_6$
Liquid +
∝ + T$_1$
∝ + MnAl$_6$
+ T$_1$
∝ + T$_1$
∝ + T$_1$
+ MnAl$_4$
25
26
27
28
29
30
31
32
33
34
Zinc Weight per Cent

Fig. 17.


Temperature in °C
560
550
540
530
520
510
500
490
480
Liquid +
∝ + MnAl$_6$
Liquid +
∝ + T$_1$
∝ + MnAl$_6$ + T$_1$
∝ + T$_1$
∝ + T$_1$
+ MnAl$_4$
23
24
25
26
27
28
29
30
31
32
Zinc Weight per Cent

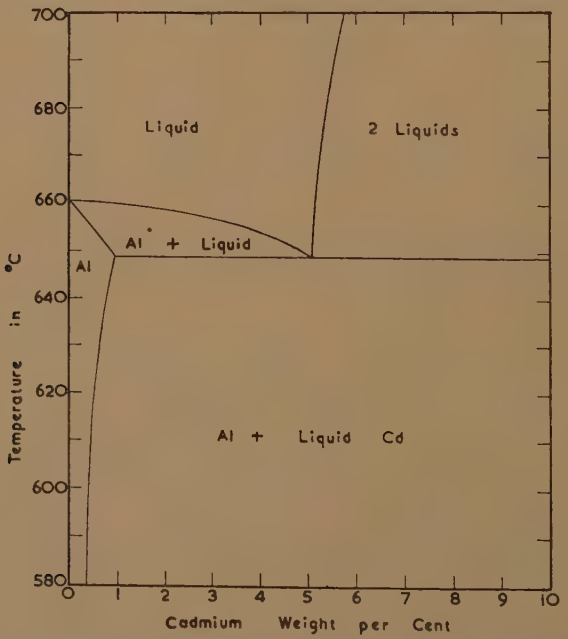
not show this structure; although small T_1 crystals in alloys annealed in the solid state showed variable colouration (fig. 11), no characteristic twin markings were shown. Attempts to increase the amount of twinning by deformation and further annealing failed, and the twins were thus assumed to be growth twins. The twinning occurred only when the growth was slow and constrained, as on heating up into the partly liquid state, and not when the crystals had grown rapidly and freely as in deposition from the melt at a high temperature. Micrographical detection of twin formation in brittle, hard intermetallic compounds is rare. To the authors' knowledge, the only other recorded case occurs with the analogous compound in the aluminium-magnesium-copper system. Petri (11) has revealed twins in the latter phase by examination with polarized light; the use of polarized light in the present case did not, however, reveal the twinning in T_1.

PART II.—THE SYSTEM ALUMINIUM-MANGANESE-CADMIUM.

(1) *Previous Work.*

The equilibrium diagram for aluminium-rich aluminium-cadmium alloys is shown in fig. 19, which is based on the diagram given by Hansen (12);

Fig. 19.



no more recent data are available. There are no intermediate phases, and liquid aluminium can dissolve just over 5 per cent. cadmium at its melting point. There is a monotectic at 5 per cent. cadmium and 649° C., at which temperature the solubility of cadmium in aluminium is 0·97 per cent. The solubility falls to 0·2 per cent. cadmium at 150° C., while liquid cadmium solidifies at 321° C.

No previous work on the ternary system has been noted.

(2) *Experimental Methods.*

The technique was in general similar to that described in Part I. Only differences will be considered here.

(a) *Materials used.*

The aluminium and aluminium-manganese alloys were from the same samples as previously described. The cadmium, kindly presented by the Imperial Smelting Corporation, Ltd., was spectroscopically pure.

(b) *Micrographic Methods.*

Annealed alloys contained hard $MnAl_6$ particles and soft cadmium particles in a matrix of nearly pure aluminium, and were difficult to prepare. Grinding was carried out on paraffined emery papers, and specimens were polished with Brasso and water on a blanket felt pad, followed by an alumina suspension on selvyt cloth.

Constituents could be differentiated in the unetched specimens. The cadmium particles were best revealed by a 1 per cent. solution of nitric acid in glycerol, while an aqueous solution of 3 per cent. sodium hydroxide and 3 per cent. sodium carbonate was used to colour the $MnAl_6$ particles.

(3) *Experimental Results.*

(a) *Slowly-cooled Alloys.*

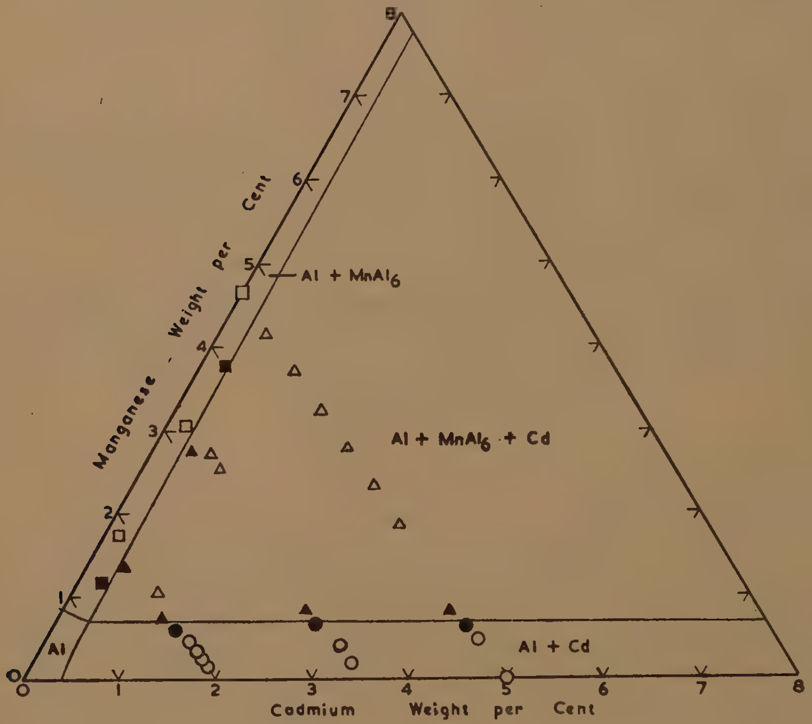
An ingot containing 3 per cent. manganese and 2 per cent. cadmium was slowly cooled from the liquid (750° C.) to room temperature at a controlled rate of 1·5° C. per minute. The microstructure showed large hollow needles of primary $MnAl_6$ and small cadmium globules in an aluminium-rich matrix The $MnAl_6$ crystals were extracted by anodic solution of the matrix in 10 per cent. aqueous hydrochloric acid. The largest and soundest crystals were hand-picked from the extract, and crushed in an agate mortar. The ductile matrix particles originally enclosed in hollow crystals were carefully picked out with forceps under a binocular microscope. The remaining powder was analyzed ; no cadmium was detected, while the manganese content was 25·33 per cent., almost exactly that required for $MnAl_6$. The identification of the crystals as $MnAl_6$ is thus confirmed, and the solubility of cadmium in them must be very small.

(b) *The* 600° C. *Isothermal.*

Fig. 20 summarizes the results obtained for alloys annealed for 10 days at 600° C. and quenched. It should be noted that at 600° C. the cadmium is liquid, but solidifies on quenching. Critical analyzed alloys are denoted by black symbols ; analysis showed that in general the intended compositions were closely attained. These critical alloys were given a further annealing treatment of 14 days, but no changes in microstructure were observed.

In the alloys studied, cadmium is present as round or lenticular areas at compositions of 0·4 per cent. or more of cadmium ; all alloys with more than 0·7 per cent. manganese contained $MnAl_6$ as a third constituent. No other phases, in addition to the matrix, were observed. The micrographic characteristics of the alloys, and the presence of cadmium in alloys of low cadmium content and high manganese content, confirm that no ternary compound exists in equilibrium with the primary solid solution at 600° C. It is improbable that such a compound exists at lower temperatures.

Fig. 20.



DISCUSSION.

In the course of earlier work (3), involving the separation of primary crystals of the intermetallic compounds of the aluminium-manganese-zinc system, it was found that a ternary compound existed, analogous to those already found in the systems aluminium-manganese-nickel and aluminium manganese-copper. From the results obtained, it was concluded that, with increasing zinc content, the phases $MnAl_6$, T_1 (the ternary compound) and $MnAl_4$ should come progressively into equilibrium with the primary solid solution. The present work, in which equilibrium relations in the solid state have been studied, confirms this, and the disposition of the phase fields and the melting points of the alloys are in good agreement with the previous work.

The ternary compound T_1, like the analogous compounds in the other systems mentioned, is based on an electron : atom ratio of 1·85, assuming for the purposes of this calculation that aluminium atoms each supply three electrons to the structure, copper atoms supply one electron, and zinc atoms supply two electrons, whereas the transitional metal atoms each absorb from the structure sufficient electrons to fill the vacancies in the atomic orbitals postulated by Pauling. If this were the only condition to be satisfied, it would be expected that the system aluminium-manganese-cadmium should also contain a ternary compound of this type. The ternary compounds under discussion, however, are also based upon an atomic ratio of four aluminium atoms to one atom of combined solutes. It might therefore be expected that the formation of this type of compound would be sensitive to the atomic size of the third element introduced.

From the present work, it is clear that, while nickel, copper and zinc as third elements lie within the zone of favourable atomic sizes for the formation of this type of compound, cadmium lies outside. The absence of the ternary compound from the aluminium-manganese-cadmium system is a reflection of the relatively large size of the cadmium atom as compared with that of zinc.

ACKNOWLEDGEMENTS.

This research was carried out in the Metallurgy Department of the University of Birmingham, under the general supervision of Professor D. Hanson, D.Sc., to whom the authors' thanks are due for this interest and encouragement. The authors must also acknowledge the valuable assistance of Mr. G. Welsh in the experimental work, and the analytical work of Mr. P. Wingrove.

The authors express their gratitude to the Department of Scientific and Industrial Research, the Royal Society, the Chemical Society, and Imperial Chemical Industries, Ltd., for general financial assistance.

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XXXIV. *The Variation with Vapour Pressure of the Properties of certain Electronic Semi-Conductors.*

By C. A. Hogarth, B.Sc., A.Inst.P., Queen Mary College, London *.

[Received July 27, 1947.]

Abstract.

The variation with vapour pressure of the Hall effect and thermo-electric power of semi-conductors having an ionic lattice structure is discussed. Formulæ representing the variations are given and, where possible, are correlated with experiment.

Introduction.

The electrical properties of electronic semi-conductors are largely dependent upon the concentration of impurity present in a given specimen. For the majority of these substances which have an ionic lattice structure, the electrical properties are due to a stoichiometric excess of one or other of the constituents, *e. g.*, excess zinc in ZnO, or iodine in CuI. From measurements of the electrical properties of this class of substances, and by finding their variation with vapour pressure, where the surrounding vapour is one of the constituents of the specimen, information concerning the electron energy levels of the substance under investigation can be deduced. The type of experiments that have so far been performed include the determination of the electrical properties of certain oxides in air or oxygen (1), of CuI in iodine vapour (2), (3), (4), of α–Ag_2S and Cu_2S in sulphur vapour (1), and of thin films of PbS in various gases (5).

The electrical properties of these substances depend to a large extent on the concentration of charge carriers, which may be either electrons excited to the conduction band from an adjacent impurity level (excess conduction), or "holes," caused by the excitation of electrons from a full conduction band to an adjacent impurity level (defect conduction). Determination of the number of charge carriers per unit volume and its dependence on vapour pressure enables us to predict the corresponding variation in electrical properties.

Schottky, Wagner, and their co-workers (4), (6), (7), (8), (9), (10) have shown, by an argument based on chemical thermodynamics and application of the law of mass action, how the concentration of charge carriers depends on the pressure of the surrounding vapour for various substances. In terms of the free electron theory of metals and semi-conductors, conductivity is defined by the following equation (11) :—

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* Communicated by the Author.

$$\sigma = \frac{4\pi n_f l_0 e^2}{3\sqrt{2\Pi m^* kT}}, \quad \ldots \ldots \ldots \ldots \quad (1)$$

where σ is the conductivity, n_f is the concentration of charge carriers, l_0 is their mean free path (assumed constant over the range of velocities for which the distribution function is integrable), e is the electronic charge, k is Boltzmann's constant, T is the absolute temperature, and m^* is the mass of the charge carrier, generally assumed to be approximately equal to the electronic mass.

By application of the law of mass action it may be shown that the concentration of charge carriers is proportional to a power of the vapour pressure of the surrounding atmosphere, and increases with decreasing vapour pressure for excess conductors, but decreases with decreasing vapour pressure for defect conductors. The effect in each case is reversible. The experimental results are usually plotted in accordance with the assumption that log σ is a linear function of log P_v, since the mass action considerations mentioned above lead to the following law :—

$$n_f = K\, P_v^{1/n}, \quad \ldots \ldots \ldots \ldots \quad (2)$$

where K is a mass action constant, raised to the power $1/|n|$ and n, which is a constant for a particular system, is also calculable from mass action considerations, together with certain assumptions concerning the degree of thermal ionization of the substance and the origin of the charge carriers. The sign of n is found to be positive for defect and negative for excess semi-conductors. Although experiments do show that log σ is a linear function of log P_v, the value of n calculated from conductivity measurements and the use of equation (2) does not as a rule agree very well with that calculated from mass action considerations. Substances for which the log P_v, log σ graph is found to be a straight line include ZnO, CdO, NiO, CuI, Cu_2O, Cu_2S and UO_2.

A few investigations have been made on the effect of changes of vapour pressure on the value of the thermo-electric power and the Hall coefficient of semi-conductors, and more extensive experiments are in progress. It is therefore desirable to explore the theory of these effects on similar lines.

HALL EFFECT.

The Hall coefficient, R, is defined in terms of the free electron theory, by the following equation (11) :—

$$R = \frac{3\pi}{8}\,\frac{1}{n_f|e|c}, \quad \ldots \ldots \ldots \ldots \quad (3)$$

where R has the algebraic sign of the charge on the carrier, *i. e.*, negative for excess and positive for defect semi-conductors.

(a) *Excess semi-conductors, showing the normal Hall effect.*

On changing the vapour pressure the value of n_f changes in accordance with equation (2), so that the numerical value of R is given by :—

$$|R|=\frac{3\pi}{8|e|c}\frac{P_v^{1/n}}{K} \qquad (4)$$

That is to say,

$$\log|R|=\frac{1}{n}\log P_v+\text{const.} \qquad (5)$$

The effect of P_v on the value of R has been studied by various experimenters but the results are not sufficiently detailed to point to a definite law. The experiments themselves present many difficulties, both in technique and interpretation, because a specimen tends to alter its electrical properties as a result of heat treatment or on treatment in a strong magnetic field. The specimen ought ,therefore, to be maintained within a narrow range of temperature and at a constant field strength whilst measurements over a range of pressure are made.

(b) *Defect semi-conductors, showing the anomalous Hall effect.*

In this case substitution of equation (2) in equation (3), taking n as positive, gives

$$|R|=\frac{3\pi}{8|e|c}\frac{P_v^{-1/n}}{K} \qquad (6)$$

Or

$$\log|R|=-\frac{1}{n}\log P_v-\text{const.} \qquad (7)$$

Bädeker (2) and Steinberg (12) have shown that the absolute value of the Hall coefficient for CuI does increase with decreasing vapour pressure.

THERMOELECTRIC POWER, $\left(\frac{dE}{dT}\right)$.

The theory of the thermo-electric power of semi-conductors has been developed in a simple form by Seitz (11), using the Wilson (13) model of a semi-conductor, in which there are n_b bound electronic states per unit volume, of energy-$\Delta\epsilon$, and these levels are completely occupied at absolute zero by n_e electrons. For intrinsic semi-conduction some electrons would be thermally excited from a full energy band to the next empty energy band, at temperatures above absolute zero. For an excess semi-conductor the mechanism would be similar except that electrons would be excited to the empty energy band from an adjacent impurity level. The number of electrons per unit volume, $n(\epsilon)$, having energy ϵ, is given by

$$n(\epsilon)=\frac{g(\epsilon)}{e^{\frac{\epsilon-\epsilon'}{kT}}+1}, \qquad (8)$$

where $g(\epsilon)$ is the density of levels and ϵ' is the normalizing parameter, corresponding to the numerical constant in the classical case, if Maxwellian instead of Fermi-Dirac statistics are used. ϵ' is given by the following expression :—

$$\epsilon'=-\frac{\Delta\epsilon}{2}+\frac{kT}{2}\log\frac{n_b 2}{C(kT)^{3/2}\sqrt{\pi}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where, for a unit volume,

$$C=\frac{4\pi(2m^*)^{3/2}}{h^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Generally, in an electronic semi-conductor, the density of charge carriers is small and classical statistics can be used with a fair degree of accuracy. By equating the classical and Fermi distributions, an equation giving the relation between n_b and n_f is obtained :—

$$n_f=n_b^{1/2}\left[\frac{\sqrt{\pi}}{2}(kT)^{3/2}C\right]^{1/2}e^{-\Delta\epsilon/2kT}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

For a thermo-junction of semi-conductors 1/2, where E is the Seebeck e.m.f., $\Pi_{1\rightarrow2}$ is the Peltier heat, and σ_{T_1} and σ_{T_2} are the Thomson coefficients, the following equations may be shown to hold :—

$$\sigma_T=-\frac{T}{e}\frac{d}{dT}\left(2k-\frac{\epsilon'}{T}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$\frac{d}{dT}\left(\frac{\Pi_{1\rightarrow2}}{T}\right)=\frac{\sigma_{T_1}-\sigma_{T_2}}{T}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and

$$E=\int_{T'}^{T''}\frac{\Pi_{1\rightarrow2}}{T}dT. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

For the system metal/semi-conductor/metal, the thermo-electric power (Seebeck e.m.f. per °C.), should be identified with $\frac{\Pi_{1\rightarrow2}}{T}$. Relating the value of the thermo-electric power of such a system to its Thomson coefficients, and assuming the value of σ_T for the metal to be negligible compared with the value for the semi-conductor, as may be confirmed experimentally, we obtain

$$\frac{\Pi_{1\rightarrow2}}{T}=\frac{dE}{dT}=\frac{\epsilon'}{Te}\pm\text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

by substituting equations (12) and (13).

Substituting for equations (9) and (11) in equation (15) the following relation is obtained :—

$$\frac{\Pi_{1\rightarrow2}}{T}=-\frac{k}{e}\left(\frac{\Delta\epsilon}{kT}+\log\frac{n_f}{n_b}\right)+B, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

B being a constant of integration.

Since we are primarily concerned with the concentration of free electrons (or holes), rather than bound electronic states, we substitute for n_b in terms of equation (11) in equation (16), obtaining

$$\frac{dE}{dT}=-\frac{k}{e}\log\frac{\left[\frac{\sqrt{\pi}}{2}(kT)^{3/2}C\right]}{n_f}+B. \quad . \quad . \quad . \quad . \quad (17)$$

This equation shows the dependence of thermo-electric power on the concentration of charge carriers. The result was obtained by considerations of free electrons but may be shown to be valid for free holes. We now substitute for n_f in equation (17) in terms of equation (2).

(a) *Excess semi-conductors,* $\frac{dE}{dT}$ *is negative.*

Writing $\frac{\sqrt{\pi}}{2}(kT)^{3/2}C=A$, we have

$$-\frac{k}{e}\log\frac{A}{n_f}=-\frac{k}{|e|}\left[\log\frac{A}{K}+\frac{1}{n}\log P_v\right]. \quad . \quad . \quad (18)$$

So that

$$\frac{dE}{dT}+\frac{k}{|e|}\log\frac{A}{K}-B=-\frac{k}{|e|n}\log P_v. \quad . \quad . \quad . \quad (19)$$

Thus we see that as P_v decreases and n_f increases, $\left|\frac{dE}{dT}\right|$ will decrease according to a logarithmic law.

The graphical form of the relations is shown in fig. 1.

(b) *Defect semi-conductors,* $\frac{dE}{dT}$ *is positive.*

In this case $\frac{dE}{dT}$ and the constant n are both positive and we have as our result

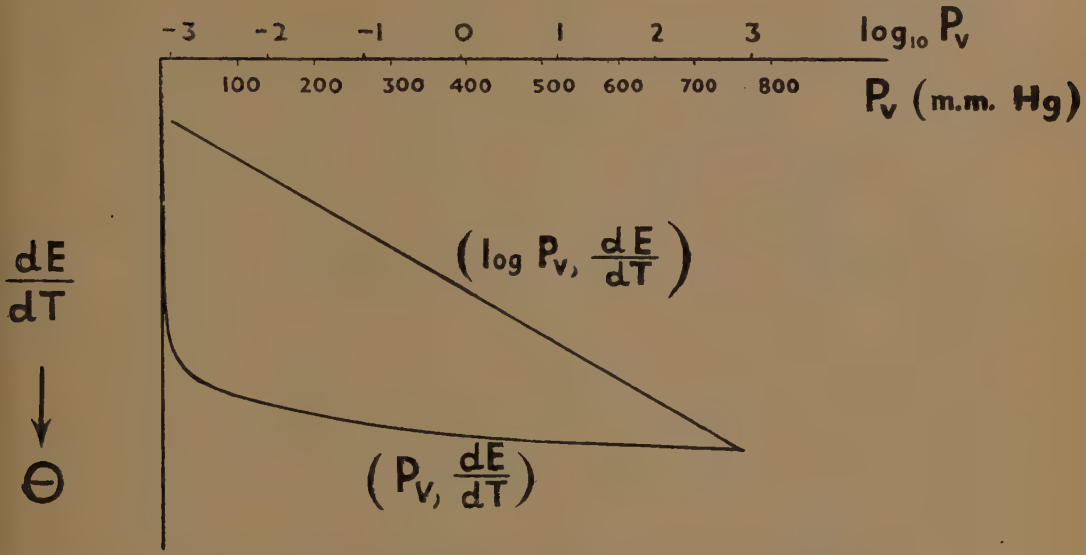
$$\frac{dE}{dT}-\frac{k}{|e|}\log\frac{A}{K}-B=-\frac{k}{|e|n}\log P_v. \quad . \quad . \quad . \quad . \quad (20)$$

Thus the result of decreasing the value of P_v, and hence the value of n_f, will be to increase $\left|\frac{dE}{dT}\right|$ in accordance with equation (20).

The graphical form of the relation is shown in fig. 2.

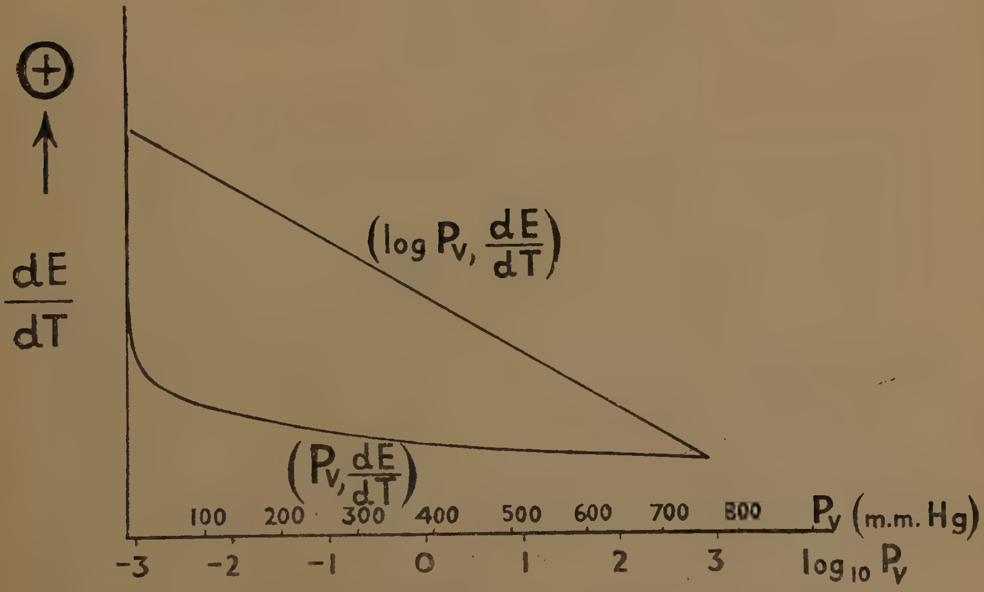
If, as is sometimes stated in the literature (14), the signs of the thermo-electric power for excess and defect semi-conductors are the opposite to those taken in this paper, *i. e.*, $\frac{dE}{dT}$ for an excess semi-conductor is

Fig. 1.



The variation of thermo-electric power with vapour pressure for an excess semi-conductor.

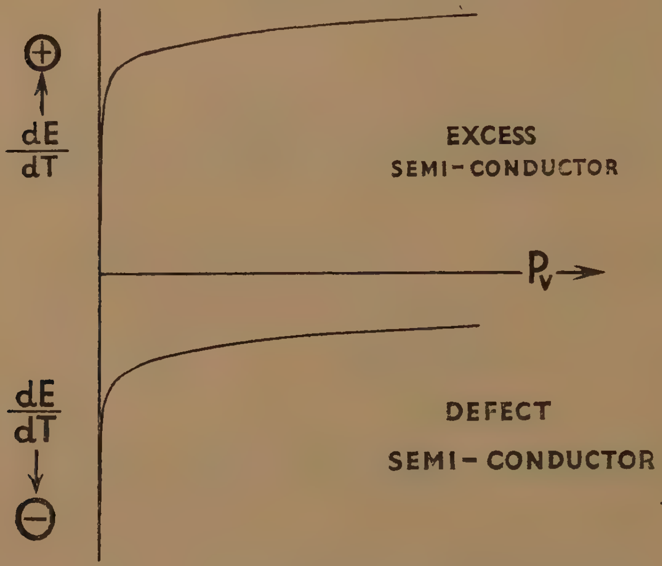
Fig. 2.



The variation of thermo-electric power with vapour pressure for a defect semi-conductor.

defined to be positive and for a defect conductor as negative, the variation will be as shown in fig. 3 and will still be in agreement with the argument outlined above and with the experimental facts.

Fig. 3.



The variation of thermo-electric power of semi-conductors with vapour pressure, using the reverse sign convention.

Correlation with Experiment.

The most complete determination of $\frac{dE}{dT}$ as a function of vapour pressure was made by Nagel and Wagner (4) for CuI. Their experiments pointed to a logarithmic variation, which was to be expected from considerations of Wagner's differential thermo-electric circuit (7). As a further test of this concept, von Baumbach and Wagner measured $\frac{dE}{dT}$ for NiO (10) and ZnO (8) at different oxygen pressures. They found that $\left|\frac{dE}{dT}\right|$ decreased with decreasing P_v for ZnO (excess semi-conductor), but increased with decreasing P_v for NiO (defect semi-conductor). The qualitative results are therefore in agreement with the conclusions of this paper.

The author (with J. P. Andrews) has made an investigation of $\frac{dE}{dT}$ for CdO as a function of oxygen vapour pressure. The experimental results are represented by the empirical formula :—

$$\frac{dE}{dT}=-\left[C_T \log P_v+\text{const.}\right], \quad \ldots\ldots \quad (21)$$

which is of the same form as equation (19) for an excess semi-conductor. On the basis of the ideas developed in this paper, C_T can be identified with $\frac{k}{|e|n}$ and a value for n thereby obtained. C_T was not constant but increased with temperature, so that the calculated value of n decreased. The values of n obtained were not more than approximately in agreement with the theoretical value of n $(=6)$. Because of the abnormally high concentration of free electrons in CdO as shown by the very high conductivity and low thermo-electric power, good agreement between the calculated and theoretical values of n can hardly be expected, since classical statistics should not be used in this case for the distribution of electron energies, and these statistics are employed in the derivation of equation (19).

The author has calculated by this method values of n from the single readings of $\left|\frac{dE}{dT}\right|$ for ZnO (8) and NiO (10) at different oxygen pressures (1 atm. and 0·21 atm.). The values obtained are 4·8 for NiO and 6·1 for ZnO, showing better agreement with the theoretical value of 6 than was obtained by conductivity measurements.

ACKNOWLEDGMENTS.

I should like to express my thanks to Professor J. P. Andrews for suggesting the investigation, and to the Department of Scientific and Industrial Research for their maintenance grant.

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XXXV. *On the Performance of the Push-Pull Relaxation Oscillator (Multivibrator).*

By R. Feinberg, University of Manchester *.

[Received January 22, 1947.]

Summary.

The operation of the push-pull relaxation oscillator (multivibrator) is interpreted as the action of an electronic switch of the two-way type which periodically charges and discharges two capacitors in alternate sequence. Based on this interpretation a simple expression is derived for the approximate computation of the oscillation frequency in the symmetrical case of oscillator arrangement. In the mathematical treatment an approximately equivalent linear relation is substituted for the non-linear characteristics of the thermionic vacuum tubes used as the electronic switch. The frequencies predicted by the approximate treatment agree within 1 to about 15 per cent. with various results obtained by experiment.

Introduction.

The theory of the push-pull relaxation oscillator circuit (multivibrator(1)) has caused interest in view of the importance of the circuit for a number of purposes and the complexity of the oscillator operation. Various lines of approach(2–12) have been taken to interpret the oscillator operation and to compute the oscillation frequency and the oscillator voltages and currents.

This communication presents a mathematical analysis of the oscillator circuit based on a simplified interpretation of the oscillator operation and, in addition, on a linear relation introduced as an approximately equivalent substitute for the non-linear characteristics of the vacuum tubes.

Oscillator Operation.

Fig. 1 shows the circuit diagram of a push-pull relaxation oscillator, and fig. 2 gives an equivalent diagram in which the combined action of the two thermionic vacuum tubes A and B is represented symbolically by the electronic two-way switch S. The switch S in position A means that the tube A is conductive and tube B is non-conductive. S in position B indicates that B is conductive and A is non-conductive.

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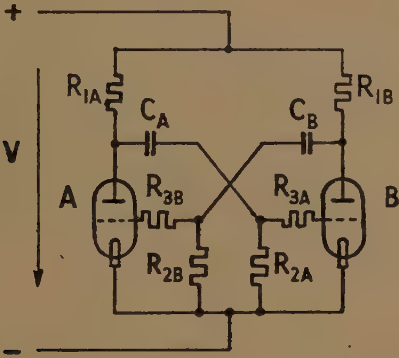
* Communicated by the Author.

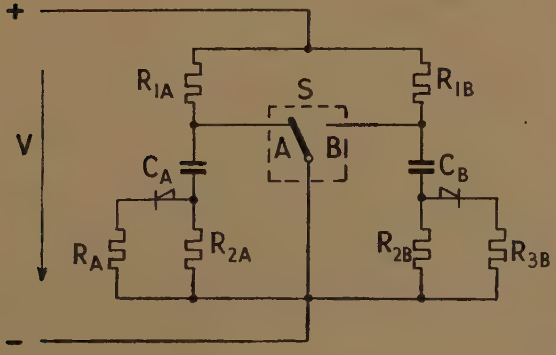
The electronic switch S is operated by the charging and discharging currents of the two capacitors C_A and C_B. During the time when S is in position A the current charging C_B produces a positive voltage at the grid of tube A, thus keeping A conductive (fig. 1), which means that in fig. 2 S is kept in position A. The current discharging C_A passes through tube A and resistor R_{2A} (see fig. 1), causing the grid of tube B to have a negative voltage with the result that the tube is non-conductive. This condition persists as long as tube B remains non-conductive.

Fig. 1.



Circuit diagram of the push-pull relaxation oscillator.

Fig. 2.



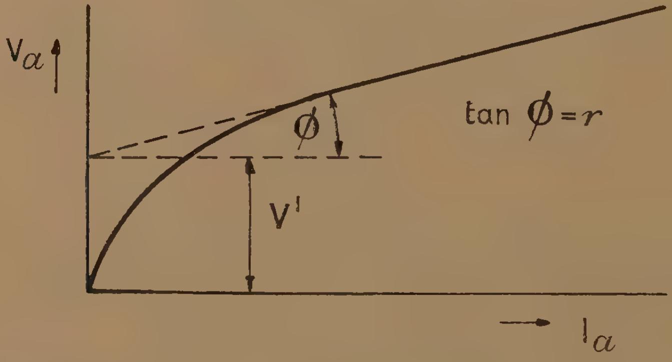
Equivalent diagram to illustrate the operation of the push-pull relaxation oscillator.

Tube B ceases to be non-conductive at the instant when the current discharging C_A has dropped to a critical value determined by the magnitude of the resistance R_{2A}, the instantaneous value of the anode voltage of B, and the tube characteristics of B. At this critical instant the capacitor C_B begins to discharge through B. The discharging current makes the sign of the grid voltage of tube A become negative. This reversal of the sign of the grid voltage initiates a chain-action of electrical events affecting tubes A and B, with the result that B becomes practically instantly fully conductive and A changes into non-conductivity. This chain-action of electrical events means, in terms of the equivalent diagram of fig. 2, that the electronic switch S suddenly changes its contact-making position from A to B.

During the time when S is in position B the current charging C_A keeps the tube B conductive, and the current discharging C_B keeps the tube A non-conductive. When C_B has sufficiently discharged a chain-action of electrical events is initiated similarly as before but in the reverse direction, with the result that, in terms of fig. 2, the electronic switch reverts practically instantly into position A.

The frequency of oscillator operation is thus identical with the frequency of electronic switch action. For computing the frequency, and consequently for finding the shapes of oscillator voltages and currents, it is therefore necessary to determine the time intervals between successive electronic switch actions. In other words it is necessary to analyse the grid voltages of the tubes A and B during their periods of negative sign.

Fig. 3.



Linear approximation for the non-linear V_a/I_a—characteristic of a tube at $V_g=0$.

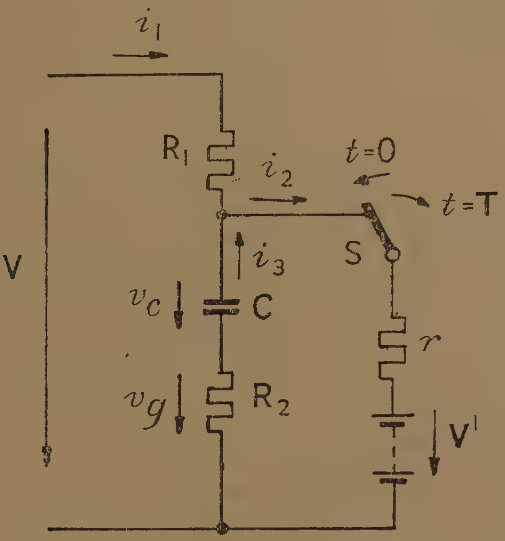
Analysis of the Grid-voltages of the Tubes.

Let the oscillator arrangement be symmetrical, *i. e.*, with the notations of fig. 1 it is $R_{1A}=R_{1B}=R_1$, $R_{2A}=R_{2B}=R_2$, $R_{3A}=R_{3B}=R_3$. $C_A=C_B=C$; in addition tubes A and B are assumed to have identical characteristics. It is further assumed that the oscillator operates in such a frequency range that the circuit performance is not affected by stray-capacitances, self-capacitances and self-inductances of the circuit components, nor by the interelectrode capacitances of the tubes.

For a simple mathematical treatment of the problem the non-linear anode voltage (V_a)/anode current (I_a) characteristic of a tube is replaced by the approximately equivalent linear relation $V_a=V'+rI_a$, as indicated in fig. 3. In making this substitution it is assumed that the grid voltage of a tube is zero throughout the conductive period of the tube action, and further that a tube operates essentially on the linear part of its characteristics. If the tube operates mainly on the curved part of the characteristic the values of V' and r of the equivalent linear relation for the characteristic have to be modified accordingly.

Based on these simplifying postulates fig. 4 represents the circuit condition during the discharging period of a capacitor C. The switch S is supposed to close at the beginning $t=0$ of the discharging period and to open at the end when $t=T$. The current i_3 discharging C produces across the resistor R_2 the grid control voltage v_g applied to the other tube, which is non-conductive.

Fig. 4.



Circuit when the capacitor C is discharging.

The shape of the grid-control voltage v_g is shown in fig. 5. The voltage changes exponentially from the initial value V_{g0} at $t=0$ to the final value v_{gT} at $t=T$, at which instant the tube controlled becomes conductive and the electron switch is "triggered." It is convenient to denote v_{gT} as the triggering grid-voltage of the electron switch.

The expression for $v_g=-i_3R_2$ is

$$v_g=V_{g0}\exp\left(-\frac{t}{RC}\right), \quad \dots\dots\dots \quad (1)$$

in which

$$R=R_2+\frac{R_1r}{R_1+r} \quad \dots\dots\dots \quad (2)$$

is the resistance in the path of the discharging current i_3, see fig. 4. The initial value V_{g0} is governed by the triggering grid-voltage v_{gT} and the discharging period T in the relation obtained from equation (1):

$$V_{g0}=v_{gT}\exp\left(\frac{T}{RC}\right), \quad \dots\dots\dots \quad (3)$$

so that equation (1) can be written in the form

$$v_g = v_{gT} \exp\left(\frac{T}{RC} - \frac{t}{RC}\right). \qquad (4)$$

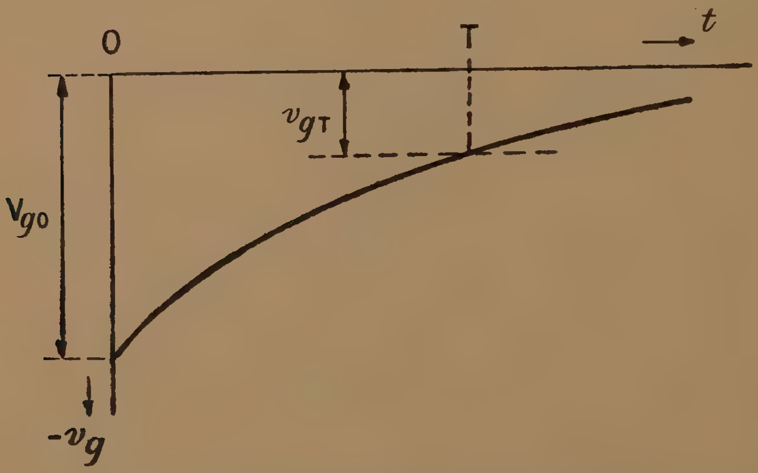
The discharging period T determines the oscillation frequency of the oscillator circuit.

Calculation of the Oscillation Frequency.

Since the oscillator arrangement under investigation is symmetrical the discharging period T of a capacitor is equal to its charging period in the steady state of oscillator operation. The oscillator frequency f is, therefore, related to T by

$$f = \frac{1}{2T}. \qquad (5)$$

Fig. 5.



The negative period of the grid-control voltage v_g.

According to equation (3) we have

$$T = RC \ln \frac{V_{g0}}{v_{gT}}. \qquad (6)$$

The problem of calculating T means to express V_{g0} and v_{gT} in terms of the values of the circuit components.

The value of V_{g0} is derived from the circuit equations obtained from fig. 4 :—

$$v_g = -i_3R_2, \qquad (7)$$

$$i_3 = i_2 - i_1, \qquad (8)$$

$$V = i_1R_1 + i_2r + V', \qquad (9)$$

$$V = i_1R_1 - \frac{1}{C}\int i_3dt - i_3R_2, \qquad (10)$$

and becomes

$$V_{g0}=-\frac{V-V'}{1+\frac{r}{R_1}+\frac{r}{R_2}}\left[1-\frac{V-X}{V-V'}\left(1+\frac{r}{R_1}\right)\right]. \quad . \quad . \quad . \quad (11)$$

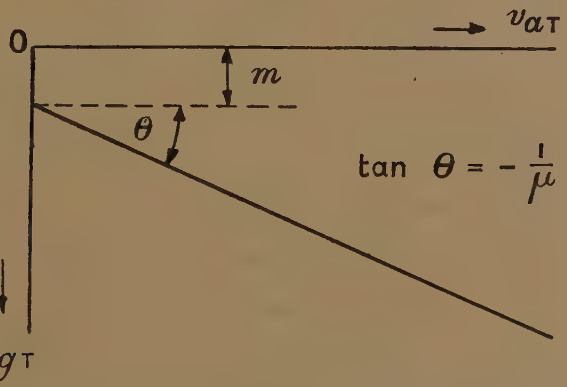
The unknown quantity X stands for the unknown potential difference across the capacitor C at the beginning $t=0$ of the discharging period.

The value of v_{gT} is obtained from the graph shown in fig. 6, which represents the triggering grid-voltage in relation to the voltage v_{aT} denoting the anode voltage of a tube at the instant of triggering. Expressed in terms of tube characteristics, fig. 6 shows the grid-voltage/anode-voltage characteristic of the tube at zero anode current. With the notation of fig. 6, we have

$$v_{gT}=m-\frac{v_{aT}}{\mu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where μ is the amplification factor of the tube at zero anode current.

Fig. 6.



Triggering-voltage characteristic of the electronic switch (*i. e.*, grid-voltage/anode voltage characteristic of a tube at zero anode current).

To eliminate the unknown quantity v_{aT} in equation (12) requires a consideration of the circuit condition during the charging period of the capacitor C. This circuit condition is indicated in fig. 7. Switch S is supposed to open at the beginning $t=0$ of the charging period and to close at the end $t=T$. (Owing to the symmetry of the oscillator arrangement the same notations of time can be used for the discharging and the charging periods.) With the notations of fig. 7 and the abbreviation

$$R^*=\frac{R_2R_3}{R_2+R_3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

the anode voltage v_a of a tube during the non-conductive period is given by

$$v_a=v_c+i_1R^*, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

with

$$i_1 = \frac{V-Y}{R_1+R^*} \exp\left[-\frac{t}{C(R_1+R^*)}\right] \quad . \quad . \quad . \quad . \quad . \quad (15)$$

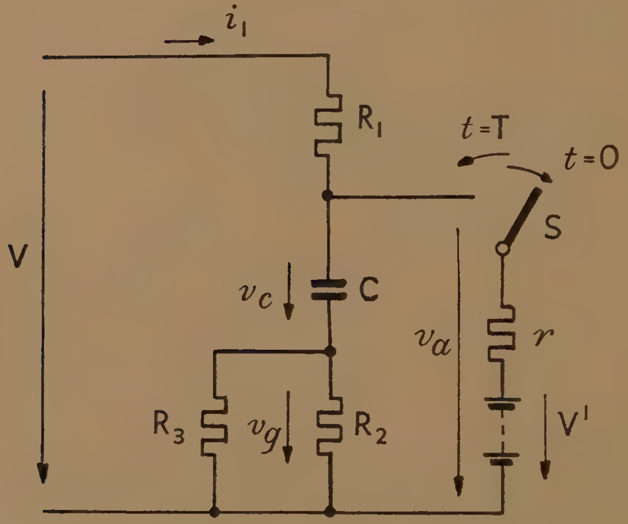
being the current charging C, and

$$v_c = V-(V-Y)\exp\left[-\frac{t}{C(R_1+R^*)}\right] \quad . \quad . \quad . \quad . \quad . \quad (16)$$

being the voltage across C. The unknown quantity Y stands for the unknown value of v_c at the beginning $t=0$ of the charging period. Since the electronic switch is triggered at the instant $t=T$ when $v_a=v_{aT}$, we obtain with equations (15) and (16) from equation (14)

$$v_{aT} = V - \frac{V-Y}{1+\frac{R^*}{R_1}} \exp\left[-\frac{T}{C(R_1+R^*)}\right], \quad . \quad . \quad . \quad (17)$$

Fig. 7.



Circuit when the capacitor C is charging up.

and thus, from equation (12),

$$v_{gT} = m - \frac{1}{\mu}\left\{V - \frac{V-Y}{1+\frac{R^*}{R_1}} \exp\left[-\frac{T}{C(R_1+R^*)}\right]\right\} \quad . \quad . \quad . \quad (18)$$

For evaluating equation (6) there remain to be determined the unknown value X of the voltage across the capacitor C at the beginning of the discharging period and the unknown value Y of the voltage across C at the beginning of the charging period. Since the electronic switch changes over to the other contact-making position in a vanishingly short time, the voltage across C at the end of a discharging period is equal to the voltage Y at the beginning of the subsequent charging period, and the voltage at the end of a charging period is equal to the voltage X at the beginning of the preceding discharging period provided that the oscillator is in steady operation. These boundary conditions establish two equations for the unknown quantities X and Y.

One equation for X and Y is obtained from equation (16), which gives the voltage across C during the charging period. For $t=\mathrm{T}$ when $v_c=\mathrm{X}$ we have

$$\mathrm{X}=\mathrm{V}-(\mathrm{V}-\mathrm{Y})\exp\left[-\frac{\mathrm{T}}{\mathrm{C}(\mathrm{R}_1+\mathrm{R}^*)}\right]. \qquad (19)$$

The second equation for X and Y is based on the relation

$$v_c=\mathrm{V}-\frac{\mathrm{V}-\mathrm{V}'}{1+\frac{r}{\mathrm{R}_1}}+\frac{\mathrm{V}-\mathrm{V}'}{1+\frac{r}{\mathrm{R}_1}}\left[1-\frac{\mathrm{V}-\mathrm{X}}{\mathrm{V}-\mathrm{V}'}\left(1+\frac{r}{\mathrm{R}_1}\right)\right]\exp\left(-\frac{t}{\mathrm{RC}}\right), \qquad (20)$$

which gives the voltage across C during the discharging period and is derived from equation (7) to (10). For $t=\mathrm{T}$ when $v_c=\mathrm{Y}$ we have

$$\mathrm{Y}=\mathrm{V}-\frac{\mathrm{V}-\mathrm{V}'}{1+\frac{r}{\mathrm{R}_1}}+\frac{\mathrm{V}-\mathrm{V}'}{1+\frac{r}{\mathrm{R}_1}}\left[1-\frac{\mathrm{V}-\mathrm{X}}{\mathrm{V}-\mathrm{V}'}\left(1+\frac{r}{\mathrm{R}_1}\right)\right]\exp\left(-\frac{\mathrm{T}}{\mathrm{RC}}\right). \qquad (21)$$

To eliminate X and Y from equations (11) and (18) introduce the abbreviations

$$a=1-\frac{\mathrm{V}-\mathrm{X}}{\mathrm{V}-\mathrm{V}'}\left(1+\frac{r}{\mathrm{R}_1}\right), \qquad (22)$$

$$k=\frac{\mathrm{V}_{g0}}{v_{g\mathrm{T}}}, \qquad (23)$$

$$n=\frac{r}{\mathrm{R}_1}, \qquad (24)$$

$$p=\frac{r}{\mathrm{R}_2}, \qquad (25)$$

$$q=\frac{\mathrm{R}^*}{\mathrm{R}_1}, \qquad (26)$$

$$s=\frac{\mathrm{R}}{\mathrm{R}_1+\mathrm{R}^*}. \qquad (27)$$

Equation (11) then becomes, with equations (22), (24) and (25),

$$\mathrm{V}_{g0}=-\frac{\mathrm{V}-\mathrm{V}'}{1+n+p}\,a, \qquad (28)$$

or, with equation (23) to replace V_{g0} by k,

$$v_{g\mathrm{T}}=-\frac{\mathrm{V}-\mathrm{V}'}{1+n+p}\,\frac{a}{k}. \qquad (29)$$

To produce another expression for $v_{g\mathrm{T}}$ rewrite equation (19),

$$\exp\left[-\frac{\mathrm{T}}{\mathrm{C}(\mathrm{R}_1+\mathrm{R}^*)}\right]=\frac{\mathrm{V}-\mathrm{X}}{\mathrm{V}-\mathrm{Y}}, \qquad (30)$$

and substitute in equation (18), which becomes with equation (26)

$$v_{gT}=m-\frac{1}{\mu}\left(V-\frac{V-X}{1+q}\right). \qquad (31)$$

From equations (22) with (24)

$$V-X=(V-V')\frac{1-a}{1+n}, \qquad (32)$$

therefore

$$v_{gT}=m-\frac{1}{\mu}\left\{V-\frac{V-V'}{1+q}\frac{1-a}{1+n}\right\}. \qquad (33$$

Now eliminate v_{gT} by equating (29) and (33) and obtain as relation between a and k

$$\frac{V-V'}{1+n+p}\frac{a}{k}=\frac{1}{\mu}\left\{V-\frac{V-V'}{1+q}\frac{1-a}{1+n}\right\}-m. \qquad (34)$$

An expression for a as an explicit function of k is derived from equation (30). Write for the left-hand side of equation (30), with equations (3), (23) and (27),

$$\exp\left[\frac{T}{C(R_1+R^*)}\right]=k^s, \qquad (35)$$

and for the right-hand side of equation (30), with equations (32), (21) to (24), and (3),

$$\frac{V-Y}{V-X}=\frac{1-\frac{a}{k}}{1-a}. \qquad (36)$$

Equating (35) and (36) and solving for a gives

$$a=\frac{k^{s+1}-k}{k^{s+1}-1}. \qquad (37)$$

The solution for k from equations (34) and (37) is to be inserted in the expression for the oscillation frequency

$$f=\frac{1}{2RC\ln k}, \qquad (38)$$

which is obtained from equation (5) with (6) and (23). The value for k has to be found by graphical solution of equations (34) and (37).

For finding the solution for k it is convenient to plot a graph giving the left-hand side of equation (34) as a function of ln k, and another graph representing the right-hand side of equation (34) as a function of ln k. The value of a to be used in equation (34) is computed from equation (37). The intersection of the two graphs gives the value of k which subsequently is inserted in equation (38). For practical evaluation it is helpful if a of equation (37) is plotted as a function of s with ln k as parameter, as shown in fig. 8.

The factor s represents according to equation (27) the ratio of the time constant of the discharging circuit to the time constant of the charging circuit for a capacitor. If this ratio is large in the sense indicated in fig. 8, according to which the value of the parameter $\ln k$ has to be taken into account, then

$$a=1, \quad \ldots\ldots \quad (39)$$

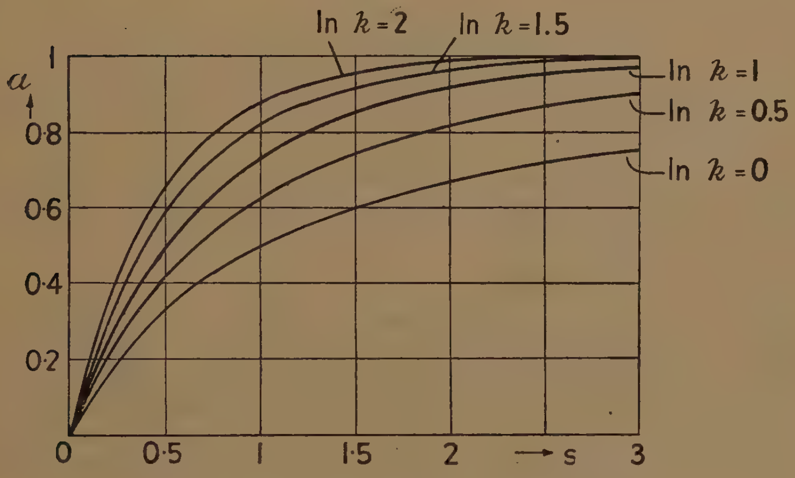
which means according to equation (32) that $X=V$, *i. e.*, the capacitor C is fully charged at the end of the charging period. In this case equation (34) is simplified to

$$\frac{V-V'}{1+n+p}\frac{1}{k}=\frac{V}{\mu}-m, \quad \ldots\ldots \quad (40)$$

which gives

$$k=\frac{V-V'}{1+n+p}\frac{1}{\frac{V}{\mu}-m}. \quad \ldots\ldots \quad (41)$$

Fig. 8.



Graphical representation of equation (37).

Oscillator Currents and Voltages.

During the discharging period of a capacitor the currents i_1, i_2, and i_3 of fig. 4 are, according to equations (7) to (10) and with equation (32),

$$i_1=\frac{V-V'}{r+R_1}\left[1-\frac{a}{1+\frac{R_2}{r}+\frac{R_2}{R_1}}\exp\left(-\frac{t}{CR}\right)\right], \quad \ldots \quad (42)$$

$$i_2=\frac{V-V'}{r+R_1}\left[1+\frac{R_1}{r}\frac{a}{1+\frac{R_2}{r}+\frac{R_2}{R_1}}\exp\left(-\frac{t}{CR}\right)\right], \quad \ldots \quad (43)$$

$$i_3=\frac{V-V'}{r}\frac{a}{1+\frac{R_2}{r}+\frac{R_2}{R_1}}\exp\left(-\frac{t}{CR}\right). \quad \ldots \quad (44)$$

The current during the charging period of C is, in accordance with fig. 7,

$$i_1=\frac{V-V'}{R_1+R^*}\,\frac{1-\frac{a}{k}}{1+\frac{r}{R_1}}\,\exp\left[-\frac{t}{C(R_1+R^*)}\right]. \quad . \quad . \quad . \quad (45)$$

The anode voltage v_a of a tube is obtained from the relation

$$v_a=V-i_1R_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

During the conductive period of the tube v_a is described with equation (42) by the expression

$$v_a=V-\frac{V-V'}{1+\frac{r}{R_1}}\left[1-\frac{a}{1+\frac{R_2}{r}+\frac{R_2}{R_1}}\exp\left(-\frac{t}{RC}\right)\right], \quad . \quad . \quad (47)$$

and during the non-conductive period with equation (45) the anode voltage becomes

$$v_a=V-\frac{V-V'}{1+\frac{R^*}{R_1}}\,\frac{1-\frac{a}{k}}{1+\frac{r}{R_1}}\,\exp\left[-\frac{t}{C(R_1+R^*)}\right]. \quad . \quad . \quad (48)$$

The values of a and k to be inserted in equations (42) to (48) are those obtained as the solutions of equations (34) and (37), or in the case of $s \gg 1$, equations (39) and (41).

Experiments.

Table I. gives data of the tubes used in the experiments, and Tables II. and III. show the results of experiment, compared to the results predicted by the approximate treatment of this paper. The approximately equivalent circuit values V′ and r of a tube in a particular quantitative circuit constellation have been derived by applying a method demonstrated in fig. 9.

TABLE I.

Data of Vacuum Triodes Used in the Experiments.

Type	μ	m volts
VR 37	25·0	−1·0
VR 38	11·6	−1·2

Fig. 9 shows the V_a/I_a—characteristic of a tube. The heavily-drawn part of the characteristic indicates the maximum extent of the operating part of the tube in the quantitative circuit constellation under consideration. The maximum anode current flows when the capacitor C in fig. 4

is charged to the supply voltage V at the beginning of the discharging period, and the minimum anode current when the discharge is completed. The tangent to the V_a/I_a—characteristic at the maximum anode current has been taken to determine V′ and *r*.

TABLE II.

Experiments with $V=240$ volts and $R_1=R_3=10{,}000$ ohms.

Tube	R_2 ohms	*r* ohms	V′ volts	C μF	*f* calcul.	*f* observ.
VR 38	2050 000	5300	24	0·46	0·28	0·30
	500 000	5300	24	0·46	1·16	1·08
				0·00055	970	1100
	50 000	5300	24	6·0	0·87	0·81
				0·00055	9420	8660
VR 37	500 000	11000	24	0·46	0·95	0·98
				0·00055	795	910

TABLE III.

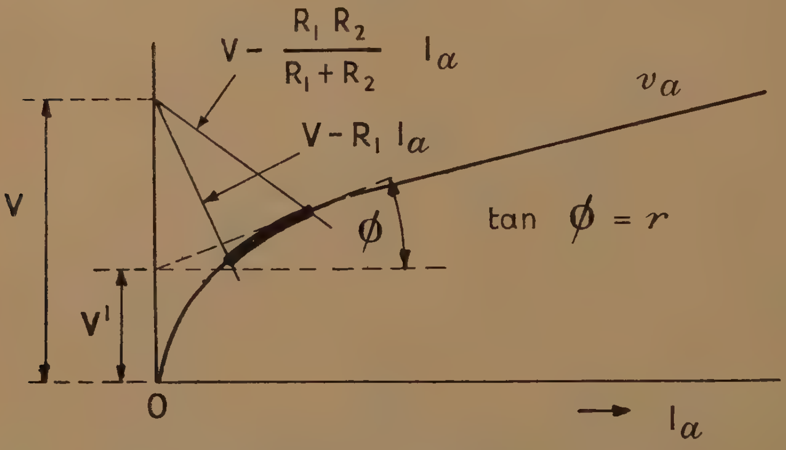
Experiments with $V=118$ volts and $R_3=10{,}000$ ohms.

			VR 37				VR 38			
R_1 ohms	R_2 ohms	C μF	*r* ohms	V′ volts	*f* calc. c/s.	*f* obs. c/s.	*r* ohms	V′ volts	*f* calc. c/s.	*f* obs. c/s.
10 000	2050 000	0·46	14 400	12	0·26	0·30	8000	10	0·32	0·36
		0·050			2·41	2·47			2·90	2·70
		0·0051			24·0	26·5			28·8	31·0
		0·00055			220	252			266	295
	500 000	0·46	14 000	12	1·07	1·15	8000	10	1·31	1·26
		0·050			9·85	9·50			12·1	10·9
		0·0051			96·7	108			118	123
		0·00055			898	1010			1096	1210
	50 000	0·46	13 400	20	10·1	11·1	8000	10	13·0	13·5
		0·050			93·5	104			120	125
		0·0051			916	1090			1180	1290
	10 000	0·46	11 000	24	48·3	53·0	7800	12	70·5	66·2
		0·050			448	500			650	605
		0·0051			4380	5050			6380	6150
		0·00055			40300	39050			59000	53000
50 000	2050 000	0·46	18 400	−5	0·20	0·23	10 400	−3	0·24	0·25
		0·050			1·77	1·85			2·24	2·06
		0·0051			17·5	20·4			22·0	23·9
		0·00055			162	185			204	232

Resistors have been inserted in the grid circuits of the tubes to ensure that the grid during the conductive period of the tubes is nearly at cathode potential, thus approximating the condition $V_g=0$ stipulated for the V_a/I_a-characteristic in fig. 3 and fig. 9. The resistance value R_3 of the grid circuit is the combination of the resistance of this external grid resistor and the resistance of the grid-to-cathode path which is varying but has been taken to be 1000 ohms in the case of the tubes VR 37 and VR 38.

The frequency values obtained by experiment were measured, respectively, with a stop-watch by direct observation or observation with a scaling-down circuit combined with a mechanical counter, and in the higher frequency range with a standard signal generator in combination with two cathode-ray oscillographs operated by a common time-base circuit. (Two cathode-ray oscillographs were used to reduce coupling between the oscillator circuit and the signal-generator, thus avoiding locking-in.)

Fig. 9.



Determination of the approximately equivalent circuit values of a tube.

The circuit components used were ordinary radio components the values of which were measured with an accuracy of about 1 per cent. The frequencies were observed with about 1 per cent. accuracy and their predicted values calculated with slide-rule accuracy.

The predicted values of frequency differ from the values of experiment by amounts which are within 1 to 15 per cent.

Conclusions.

The frequency of the symmetrical push-pull relaxation oscillator is governed by the time-constant of the circuit path during the discharging period of the capacitor and the natural logarithm of a factor k whose magnitude is determined by the values of the circuit components and the characteristic of the tubes and to some degree by the voltage of the direct current supply. The frequency can be modified by applying a bias voltage in the grid circuits of the tubes.

The oscillator will operate only if the values of the circuit components result in the condition $k>1$.

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XXXVI. *The Electrostatic Field Distribution near a Circular Aperture or Short Cylinder.*

By G. Liebmann *.

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Summary.

A method is given for computing the electrostatic field near a circular aperture in a conducting sheet of finite thickness or near and within a short cylinder with plane end faces. The potential $\phi(r, z)$ is the sum of a potential $\phi_1(r, z)$ calculated previously by Bertram, and of a correction potential $\phi_c(r, z)$. In many cases of interest in electron optics or electronic tube design the correction potential $\phi_c(r, z)$ is the more important term. Tables for the easy determination of $\phi_1(r, z)$ and $\phi_c(r, z)$ are presented.

1. *Introduction.*

Electrostatic fields of rotational symmetry, where the potential is a function of the co-ordinates r and z, play a considerable part in electron optics, and a fairly accurate knowledge of such potential distributions is necessary for the design of many electronic devices using electron beams of circular cross-section, *e. g.* cathode-ray tubes and velocity modulated amplifiers. The potential distribution must satisfy Laplace's equation $\nabla^2\phi(r, z)=0$. The general solution of Laplace's equation for the case of rotational symmetry of the electrodes is known to be a combination of linear and exponential functions and of Bessel functions ⁽¹⁾, ⁽²⁾. Yet to satisfy a given set of boundary conditions by specializing the general solution is very difficult even for simple electrode shapes. Indeed, the only completely and rigorously solved problem involving such rotationally symmetrical fields appears to be the field near a circular aperture in an infinitely thin conducting plane. This exact solution was given by Ollendorff ⁽³⁾, and by Fry ⁽⁴⁾. Diaphragms used in

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* Communicated by the Author.

electronic devices are, however, by no means "infinitely thin" and the exact solution cannot, therefore, as a rule be applied. For instance, in the case of the amplification factor of a modulating grid with circular apertures, which comes nearest to the rigorously solved problem, measurements show that the amplification factor found in practical cases is considerably greater than would be predicted from this solution.

Although it is always possible to compute the configuration of the electrostatic field for given boundary conditions by numerical methods (5),(6), such computations become rather tedious for the degree of accuracy usually needed, as it is inherent in the numerical methods that the potential must be calculated for a great number of points forming a net, even if a knowledge of the potential values is only required for a few prescribed points. Moreover, the convergence of the numerical approximation towards the correct potential distribution is mostly rather slow; if the net is not chosen fine enough, the numerical solution may even converge towards a wrong value.

The field distribution *within* cylinders of unit radius was calculated by Bertram ( ) for the case that the potential distribution *along* the cylinder of unit radius is known, and is represented by a power series in z. Bertram already pointed out that only the constant and linear terms of this power series in z satisfy Laplace's equation, and the numerical example given in his paper is based on zero or constant potential gradient along the boundary ($\phi(1, z)=A$ or $\phi(1, z)=Bz$). It is this restriction imposed on the boundary condition which turns Bertram's otherwise exact solution into an approximate one.

It will be shown in the next section that rotationally symmetrical fields which are due to hollow electrodes with circular cross-section and plane end faces can be calculated with considerable accuracy by combining Bertram's method with Ollendorff's rigorous solution. Such electrodes form the main structural elements of many electronic devices.

Our solution will involve some convenient approximations, but the accuracy obtainable will still be considerably better than that of the known experimental methods for mapping potential distributions; the computing time necessary is only a very small fraction of that required by numerical methods for the same degree of accuracy.

2. *Calculation of Potential near Diaphragm.*

If the potential $\phi(1, \zeta)$ for all points $z=\zeta$ along a cylinder of unit radius is given, the potential $\phi(r, z)$ for any point (r, z) within this cylinder is determined, according to Bertram (7), by the equation

$$\phi(r, z)=\sum_{+\mu}\frac{J_0(\mu r)}{J_1(\mu)}\int_{-\infty}^{\infty}\phi(1, \zeta)e^{-\mu|z-\zeta|}d\zeta, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where J_0 and J_1 are the Bessel functions of zero and first order and the summation is taken over all positive roots μ of the equation $J_0(\mu)=0$. This equation is rigorous, but the integral cannot be evaluated except for certain simple functions $\phi(1, z)$.

Bertram evaluated the integral for boundary functions of the type

$$\phi_n(1, z)=\begin{Bmatrix} 0, & z<0 \\ z^n, & z>0 \end{Bmatrix}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

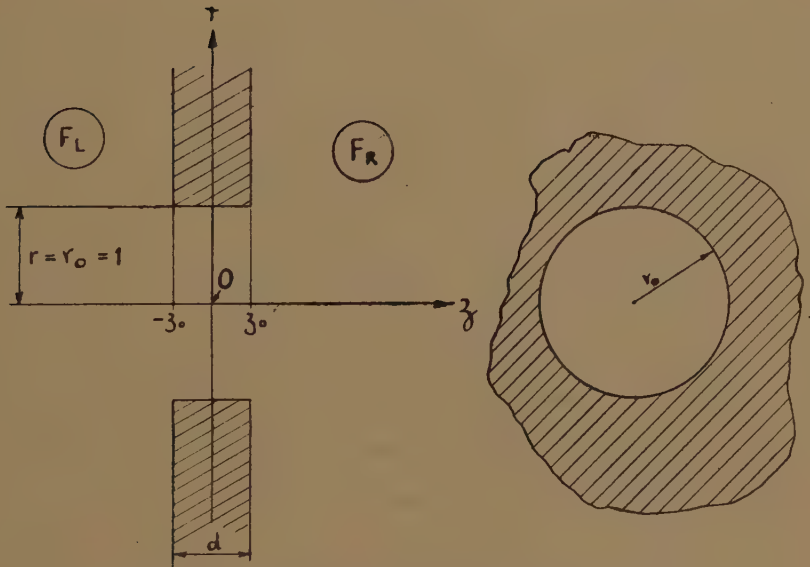
and obtained for $n=0$ and $n=1$ (the only values of n satisfying Laplace's equation, and therefore the only admissible values of n):

$$\phi_0(r, z)=\begin{Bmatrix} 1-G_0(r, z), & z>0 \\ G_0(r, -z), & z<0 \end{Bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$\phi_1(r, z)=\begin{Bmatrix} z+G_1(r, z), & z>0 \\ G_1(r, z), & z<0 \end{Bmatrix}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Fig. 1.



Diaphragm with circular hole (radius of hole $r=r_0=1$, thickness of diaphragm d.

The functions

$$G_0(r, z)=\sum_{+\mu} \frac{J_0(\mu r)}{\mu J_1(\mu)} e^{-\mu z}$$

and

$$G_1(r, z)=\sum_{+\mu} \frac{J_0(\mu r)}{\mu^2 J_1(\mu)} e^{-\mu z}$$

were tabulated by Bertram. The function $G_1(r, z)$ is reproduced as our Table I. because it will be required in actual calculations of the potential distribution, and a collection in this paper of all tables necessary for this purpose appeared advisable.

We consider now a conducting diaphragm D of thickness d, taking the radius r_0 of the hole as unit of measurement and placing the origin 0 at the centre of the hole (fig. 1).

TABLE I.

Potential Function $G_1(r, z)$.

r z	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
0·00	0·2674	0·2658	0·2607	0·2522	0·2398	0·2232	0·2016	0·1739	0·1377	0·0883
0·05	0·2432	0·2416	0·2366	0·2281	0·2158	0·1993	0·1779	0·1505	0·1150	0·0674
0·10	0·2207	0·2191	0·2141	0·2058	0·1937	0·1775	0·1567	0·1301	0·0964	0·0531
0·15	0·1998	0·1982	0·1934	0·1852	0·1735	0·1579	0·1378	0·1126	0·0813	0·0431
0·20	0·1805	0·1790	0·1743	0·1664	0·1551	0·1402	0·1212	0·0976	0·0691	0·0358
0·25	0·1627	0·1613	0·1568	0·1492	0·1385	0·1243	0·1065	0·0848	0·0591	0·0301
0·30	0·1465	0·1450	0·1408	0·1336	0·1235	0·1102	0·0937	0·0738	0·0509	0·0256
0·35	0·1316	0·1302	0·1262	0·1195	0·1100	0·0976	0·0824	0·0644	0·0440	0·0220
0·40	0·1180	0·1168	0·1130	0·1067	0·0979	0·0865	0·0726	0·0564	0·0382	0·0190
0·45	0·1057	0·1045	0·1011	0·0952	0·0870	0·0766	0·0640	0·0494	0·0333	0·0165
0·50	0·0946	0·0935	0·0903	0·0849	0·0774	0·0678	0·0564	0·0433	0·0291	0·0143
0·55	0·0845	0·0835	0·0805	0·0756	0·0687	0·0601	0·0498	0·0381	0·0255	0·0125
0·60	0·0754	0·0745	0·0718	0·0673	0·0611	0·0532	0·0439	0·0335	0·0224	0·0110
0·65	0·0673	0·0664	0·0640	0·0598	0·0542	0·0471	0·0388	0·0295	0·0196	0·0690
0·70	0·0600	0·0592	0·0569	0·0532	0·0481	0·0417	0·0343	0·0261	0·0173	0·0085
0·75	0·0534	0·0527	0·0507	0·0473	0·0427	0·0370	0·0303	0·0230	0·0152	0·0075
0·80	0·0475	0·0469	0·0451	0·0420	0·0379	0·0328	0·0268	0·0203	0·0134	0·0066
0·85	0·0423	0·0417	0·0401	0·0373	0·0336	0·0290	0·0237	0·0179	0·0119	0·0058
0·90	0·0376	0·0371	0·0356	0·0332	0·0298	0·0257	0·0210	0·0159	0·0105	0·0051
0·95	0·0334	0·0330	0·0316	0·0294	0·0265	0·0228	0·0186	0·0140	0·0093	0·0045
1·00	0·0297	0·0293	0·0281	0·0261	0·0235	0·0202	0·0165	0·0124	0·0082	0·0040
1·05	0·0264	0·0260	0·0249	0·0232	0·0208	0·0179	0·0146	0·0110	0·0072	0·0035
1·10	0·0234	0·0231	0·0221	0·0206	0·0185	0·0159	0·0129	0·0097	0·0064	0·0031
1·15	0·0208	0·0205	0·0196	0·0183	0·0164	0·0141	0·0115	0·0086	0·0057	0·0028
1·20	0·0185	0·0182	0·0174	0·0162	0·0145	0·0125	0·0101	0·0076	0·0050	0·0024
1·25	0·0164	0·0162	0·0155	0·0144	0·0129	0·0111	0·0090	0·0068	0·0044	0·0022
1·30	0·0145	0·0143	0·0137	0·0127	0·0114	0·0098	0·0080	0·0060	0·0039	0·0019
1·35	0·0129	0·0127	0·0122	0·0113	0·0101	0·0087	0·0071	0·0053	0·0035	0·0017
1·40	0·0114	0·0113	0·0108	0·0100	0·0090	0·0077	0·0063	0·0047	0·0031	0·0015
1·45	0·0102	0·0100	0·0096	0·0089	0·0080	0·0068	0·0055	0·0042	0·0027	0·0013
1·50	0·0090	0·0089	0·0085	0·0079	0·0071	0·0060	0·0049	0·0037	0·0024	0·0012
1·60	0·0070	0·0069	0·0066	0·0061	0·0055	0·0047	0·0038	0·0028	0·0019	0·0009
1·70	0·0056	0·0055	0·0053	0·0049	0·0044	0·0037	0·0030	0·0023	0·0015	0·0007
1·80	0·0045	0·0044	0·0042	0·0039	0·0035	0·0030	0·0024	0·0018	0·0012	0·0006
1·90	0·0035	0·0034	0·0033	0·0030	0·0027	0·0023	0·0019	0·0014	0·0009	0·0005
2·00	0·0027	0·0027	0·0026	0·0024	0·0021	0·0018	0·0015	0·0011	0·0007	0·0004
2·10	0·0021	0·0021	0·0020	0·0019	0·0017	0·0014	0·0012	0·0009	0·0006	0·0003
2·20	0·0017	0·0017	0·0016	0·0015	0·0013	0·0011	0·0009	0·0007	0·0005	0·0002
2·30	0·0013	0·0013	0·0012	0·0012	0·0010	0·0009	0·0007	0·0005	0·0004	0·0002
2·40	0·0010	0·0010	0·0010	0·0009	0·0008	0·0007	0·0006	0·0004	0·0003	0·0001
2·50	0·0008	0·0008	0·0008	0·0007	0·0006	0·0005	0·0004	0·0003	0·0002	0·0001
2·60	0·0006	0·0006	0·0006	0·0006	0·0005	0·0004	0·0003	0·0003	0·0002	0·0001
2·80	0·0004	0·0004	0·0004	0·0003	0·0003	0·0003	0·0002	0·0002	0·0001	0·0001
3·00	0·0002	0·0002	0·0002	0·0002	0·0002	0·0002	0·0001	0·0001	0·0001	0·0000
3·50	0·0001	0·0001	0·0001	0·0001	0·0001	0·0001	0·0000	0·0000	0·0000	0·0000
4·00	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000

A potential ϕ_D may be applied to this diaphragm, which may be immersed in an electrostatic field that, in the absence of the circular aperture, would have the field gradient F_R to its right and the field gradient F_L to its left.

If we could disregard the field disturbance caused by the edges of the hole, and the penetration of the field through the hole, the potential $\phi(1, z)$ would be $\left(\text{with } z_0=\frac{d}{2}\right)$:

$$\phi_1(1, z)=\left\{\begin{array}{ll} \phi_D+(z-z_0)F_R, & z>z_0 \\ \phi_D, & -z_0 \leqq z \leqq z_0 \\ \phi_D+(z+z_0)F_L, & z<-z_0 \end{array}\right\} . \quad . \quad . \quad . \quad (5)$$

As this boundary potential has the form of equation (2), the potential for any point (r, z) could be taken immediately from Bertram's set of tables. This would give an approximation to the correct potential which will be valuable in many cases, but as these boundary conditions neglect the effect of the hole itself, the result will be more or less in error. This error may become considerable if the potential is required in a region where the potential gradient is small, whereas the potential gradient is great on the other side of the diaphragm. An example of such a case is the potential distribution near a control grid, where the field in the cathode space is much weaker than the field in the anode space, but has to be known much more accurately.

To find the correct boundary values which should be used to represent the influence of the hole, we turn to the rigorous solution for an infinitely thin diaphragm ($d=0$, $z_0=0$), dividing a field-free space ($F_L=0$ for $z<0$) from a space with constant field gradient $F_R(z>0)$. This solution was given by Ollendorff (3) as:

$$\phi(r,z)=F_R\left[z-\frac{1}{\pi}\left\{(u^2-1)^{\frac{1}{2}} \tan^{-1}(u^2-1)^{-\frac{1}{2}}-1\right\}(1-v)^{\frac{1}{2}}\right], \quad z\geqq 0, \quad . \quad (6)$$

where

$$r=uv$$

and

$$z=(u^2-1)^{\frac{1}{2}}(1-v^2)^{\frac{1}{2}}.$$

The amount, $\delta\phi$, by which this exact potential along the cylinder $r=1$ differs from the previously assumed potential ϕ_D+zF_R (equation (5) with $z_0=0$) is represented in fig. 2, where $\frac{\delta\phi}{F_R}$ is plotted as a function of z.

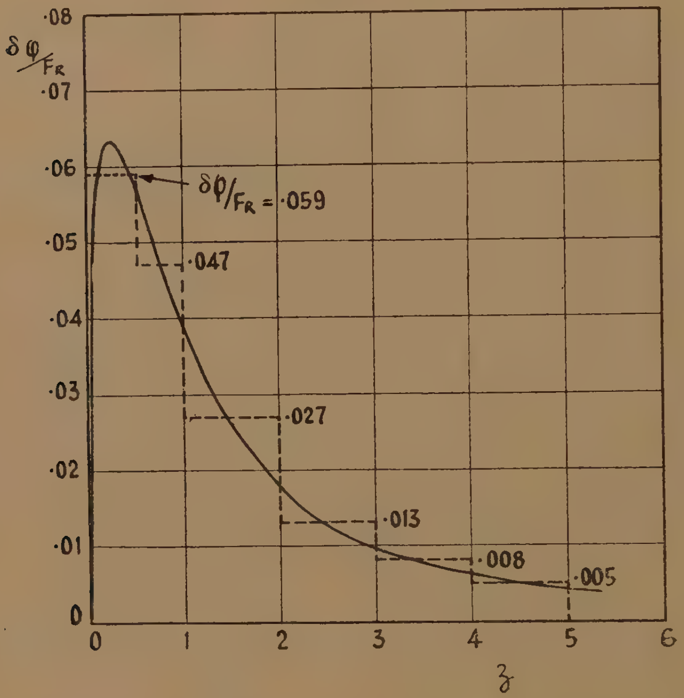
It is the boundary potential $\phi(1, \zeta)=\phi_D+F_R\left(z+\frac{\delta\phi}{F_R}\right)$ which must be inserted into equation (1), the term $\frac{\delta\phi}{F_R}$ giving now rise to a correction potential $\phi_c(r, z)$.

The required integration cannot be carried out directly. We adopt, therefore, the simple expedient of approximating the function $\frac{\delta\phi}{F_R}$ by the step-function shown by the dashed line in fig. 2. A step-function involving a greater number of steps, and therefore approaching the function $\frac{\delta\phi}{F_R}$ more closely, would give a greater accuracy in the values of the correction potential $\phi_c(r, z)$. However, it appears from computing the values of the correction potential ϕ_c for several "step-functions" approximating to the function $\frac{\delta\phi}{F_R}$ more closely than the step-approximation shown on fig. 2, that this fairly rough approximation is already good enough to calculate the final potential

$$\phi(r, z) = \phi_1(r, z) + \phi_c(r, z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

to almost four-figure accuracy.

Fig. 2.



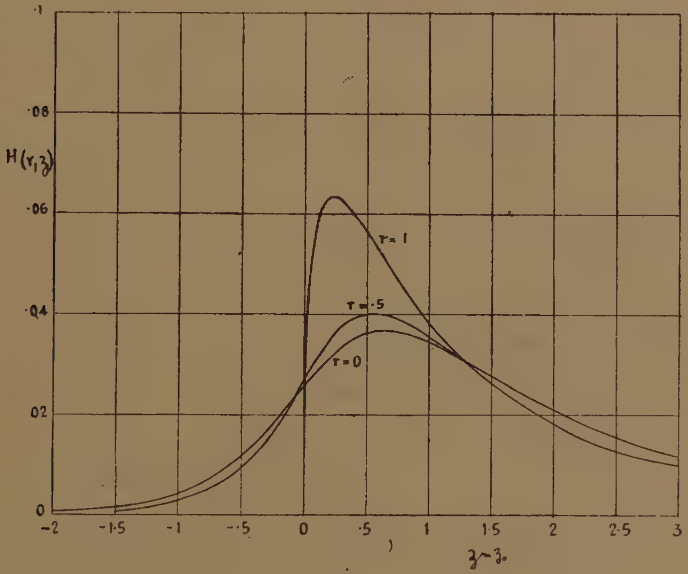
Relative potential deviation $\frac{\delta\phi}{F_R}$ from linear potential along cylinder $r=1$ for infinitely thin diaphragm ($r_0=1$), according to Ollendorff, and approximating "step-function."

Writing $\frac{\delta\phi(z_2, z_1)}{F_R}$ for the constant value of the step function between the step co-ordinates $z=z_2$ and $z=z_1$ $(z_2>z_1)$, the contribution to the correction potential $\phi(r, z)$, due to the field F_R acting at the right edge of the hole, is therefore given by

$$H(r, z)=\sum_{z_2,\, z_1} \frac{\delta\phi(z_2, z_1)}{F_R}\,[G_0(r, z-z_1)-G_0(r, z_2-z)]. \quad . \quad . \quad (8)$$

In the more general case, the edge of the hole is located at $z_0>0$, and $(z-z_0)$ takes the place of z. The correction functions $H(r, z-z_0)$ can be easily computed for any value $(z-z_0)$. Fig. 3 gives a plot of $H(r, z-z_0)$ as a function of $z-z_0$ for the values $r=0$, $r=0{\cdot}5$ and $r=1$. In order to allow more accurate calculations, $H(r, z-z_0)$ is tabulated in Table II. for closer intervals of r; irregularities in the fourth digit are due to the relative coarseness of the step approximation used.

Fig. 3.



Potential correction function $H(r, z)$.

So far we have dealt only with the contribution to the correction potential $\phi(r, z)$ of the field F_R acting at the right edge of the hole. We have now to take into account the penetration of the field F_R through the hole into the space to the left of the diaphragm. The field F_R penetrating through the hole, although being attenuated, produces a field gradient to the left of the diaphragm and therefore sets up a modified boundary potential $\phi(1, \zeta)$ for $\zeta<-z_0$. This boundary potential is of the same shape as the function $\frac{\delta\phi}{F_R}$, but its value is multiplied by the field attenuation factor α ($\alpha \leq 1$). The exact value of α could only be found if the problem were already solved which we intend to solve. As a first approximation we assume, therefore, that α is given as the ratio of the field strengths at the points on the axis located under the two edges of the hole, where the attenuation of the field within the hole is given by the decay of the field as described by the function $G_1(0, z)$. This assumption disregards therefore the effect of $\frac{\delta\phi}{F_R}$ on α, but the error, being a correction of a correction term, should not be appreciable.

TABLE II.

Potential Correction Function H(r, z).

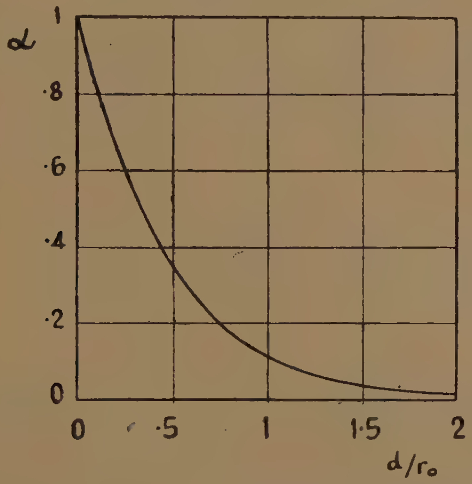
z \ r	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1
−2·5	0·0001	0·0001	0·0001	0·0001	0·0001	0·0001	0·0001	0·0000	0·0000	0·000	0
−2·0	0·0004	0·0003	0·0003	0·0003	0·0003	0·0002	0·0001	0·0001	0·0001	0·000	0
−1·5	0·0012	0·0011	0·0011	0·0010	0·0009	0·0008	0·0006	0·0005	0·0003	0·000	0
−1·2	0·0023	0·0023	0·0022	0·0021	0·0019	0·0016	0·0013	0·0010	0·0007	0·000	0
−1·0	0·0037	0·0037	0·0036	0·0033	0·0030	0·0026	0·0022	0·0016	0·0011	0·001	0
−0·8	0·0059	0·0058	0·0056	0·0053	0·0048	0·0042	0·0035	0·0027	0·0018	0·001	0
−0·6	0·0091	0·0090	0·0088	0·0084	0·0077	0·0069	0·0059	0·0046	0·0031	0·002	0
−0·5	0·0112	0·0111	0·0109	0·0104	0·0097	0·0088	0·0076	0·0061	0·0042	0·002	0
−0·4	0·0136	0·0135	0·0133	0·0128	0·0122	0·0112	0·0099	0·0081	0·0058	0·003	0
−0·3	0·0163	0·0162	0·0160	0·0157	0·0151	0·0143	0·0130	0·0110	0·0082	0·004	0
−0·2	0·0193	0·0192	0·0191	0·0189	0·0185	0·0179	0·0169	0·0152	0·0122	0·007	0
−0·1	0·0224	0·0224	0·0224	0·0223	0·0222	0·0221	0·0217	0·0208	0·0187	0·013	0
0·0	0·0255	0·0255	0·0256	0·0258	0·0261	0·0265	0·0269	0·0275	0·0281	0·029	0
0·1	0·0284	0·0285	0·0287	0·0291	0·0298	0·0307	0·0321	0·0341	0·0374	0·044	0·0582
0·2	0·0310	0·0311	0·0315	0·0321	0·0330	0·0344	0·0363	0·0392	0·0435	0·050	0·0634
0·3	0·0332	0·0333	0·0338	0·0345	0·0356	0·0371	0·0393	0·0423	0·0466	0·052	0·0626
0·4	0·0349	0·0350	0·0354	0·0362	0·0373	0·0389	0·0410	0·0438	0·0475	0·052	0·0599
0·5	0·0360	0·0361	0·0365	0·0373	0·0383	0·0398	0·0416	0·0440	0·0468	0·050	0·0563
0·6	0·0365	0·0366	0·0370	0·0377	0·0386	0·0399	0·0415	0·0433	0·0453	0·047	0·0524
0·7	0·0365	0·0366	0·0370	0·0375	0·0383	0·0394	0·0407	0·0422	0·0439	0·046	0·0486
0·8	0·0361	0·0362	0·0365	0·0369	0·0376	0·0384	0·0394	0·0407	0·0422	0·044	0·0449
0·9	0·0354	0·0355	0·0357	0·0360	0·0364	0·0370	0·0377	0·0386	0·0398	0·042	0·0414
1·0	0·0344	0·0344	0·0346	0·0348	0·0351	0·0354	0·0358	0·0361	0·0365	0·037	0·0382
1·1	0·0332	0·0332	0·0333	0·0334	0·0335	0·0337	0·0337	0·0336	0·0331	0·033	0·0352
1·2	0·0319	0·0319	0·0319	0·0320	0·0320	0·0320	0·0319	0·0315	0·0310	0·030	0·0325
1·3	0·0305	0·0305	0·0305	0·0305	0·0305	0·0304	0·0302	0·0299	0·0292	0·029	0·0300
1·4	0·0291	0·0291	0·0291	0·0291	0·0291	0·0290	0·0288	0·0286	0·0282	0·028	0·0277
1·5	0·0277	0·0277	0·0277	0·0277	0·0277	0·0276	0·0276	0·0275	0·0273	0·028	0·0257
1·6	0·0263	0·0263	0·0263	0·0263	0·0263	0·0263	0·0264	0·0264	0·0265	0·027	0·0238
1·8	0·0236	0·0236	0·0236	0·0236	0·0236	0·0236	0·0237	0·0239	0·0245	0·025	0·0205
2·0	0·0209	0·0209	0·0209	0·0208	0·0207	0·0206	0·0205	0·0204	0·0203	0·020	0·0178
2·2	0·0183	0·0184	0·0183	0·0182	0·0180	0·0178	0·0175	0·0169	0·0161	0·015	0·0156
2·5	0·0152	0·0152	0·0151	0·0150	0·0149	0·0147	0·0144	0·0141	0·0138	0·013	0·0123
3·0	0·0114	0·0113	0·0113	0·0113	0·0112	0·0111	0·0110	0·0109	0·0107	0·011	0·0096
3·5	0·0087	0·0086	0·0086	0·0086	0·0085	0·0085	0·0084	0·0083	0·0082	0·008	0·0074
4·0	0·0066	0·0066	0·0066	0·0066	0·0066	0·0066	0·0066	0·0065	0·0065	0·006	0·0059
4·5	0·0047	0·0047	0·0047	0·0047	0·0047	0·0048	0·0048	0·0048	0·0049	0·005	0·0048
5·0	0·0028	0·0028	0·0027	0·0027	0·0027	0·0027	0·0026	0·0026	0·0026	0·003	0·0039

Hence we have :

$$\alpha = \frac{G_1(0, d)}{G_1(0, 0)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The attenuation factor α, as a function of diaphragm thickness $d=2z_0$, is plotted in fig. 4 and tabulated in Table III.

Fig. 4.



Attenuation factor α of circular aperture as function of relative diaphragm thickness d/r_0.

TABLE III.

Attenuation Factor α of Circular Aperture as a function of relative Diaphragm Thickness d/r_0.

d	α	d	α	d	α	d	α
0·00	1·000	0·50	0·354	1·00	0·111	1·50	0·034
0·05	0·910	0·55	0·316	1·05	0·099	1·60	0·027
0·10	0·825	0·60	0·282	1·10	0·088	1·70	0·021
0·15	0·747	0·65	0·252	1·15	0·078	1·80	0·017
0·20	0·675	0·70	0·224	1·20	0·069	1·90	0·013
0·25	0·609	0·75	0·200	1·25	0·061	2·00	0·010
0·30	0·548	0·80	0·178	1·30	0·054	2·10	0·008
0·35	0·492	0·85	0·158	1·35	0·048	2·20	0·006
0·40	0·441	0·90	0·141	1·40	0·043	2·30	0·005
0·45	0·395	0·95	0·125	1·45	0·038	2·40	0·004
						2·50	0·003

The total contribution of the field F_R to the correction potential $\phi_c(r, z)$ is therefore $[H(r, z-z_0)+\alpha H(r, -z-z_0)F_R$. If the diaphragm has to its left not a space field-free except for the penetration of F_R but a uniform field (except for the influence of the hole in the diaphragm) of gradient F_L, the term $-[H(r, -z-z_0)+\alpha H(r, z-z_0)]F_L$ has to be added, representing the contribution of the field F_L to the correction potential (the minus sign is entering owing to the change of sign in z when passing through the origin).

Collecting now all terms we can write :

$$\phi_1(r, z)=\left\{\begin{array}{ll} \phi_D+F_R[(z-z_0)+G_1(r, z-z_0)]-F_LG_1(r, z+z_0), & z\geqq z_0 \\ \phi_D+F_RG_1(r, z-z_0)-F_LG_1(r, z+z_0), & -z_0\leqq z\leqq z_0 \\ \phi_D+F_RG_1(r, z-z_0)+F_L[(z+z_0)-G_1(r, z+z_0)], & z\leqq -z_0 \end{array}\right\}, \quad . \quad . \quad . \quad (10)$$

and

$$\phi_c(r, z)=[F_R-\alpha F_L]H(r, z-z_0)+[\alpha F_R-F_L]H(r, -z-z_0), \quad . \quad . \quad (11)$$

which must be added to give the complete potential

$$\phi(r, z)=\phi_1(r, z)+\phi_c(r, z).$$

Taking $G_1(r, z\pm z_0)$ from Table I., $H(r, z-z_0)$ and $H(r, -z-z_0)$ from Table II. and α from Table III., a numerical evaluation of the potential for any desired point (r, z) can be quickly effected, provided $r\leq 1$. This restriction $r\leq 1$ is usually not very important. However, it is frequently possible to extend the range of the solution to points $r>1$, either by extrapolation, or by developing $\phi(r, z)$ into a series in r after $\phi(0, z)$ has been determined (8), or by inversion at the cylinder $r=1$.

3. *Discussion of Results for some Special Cases.*

A frequently encountered case is that of a symmetrical field distribution around the diaphragm, $F_L=-F_R$. Then our equations (10) and (11) become simplified to :

$$\phi_1(r,z)=\left\{\begin{array}{ll} \phi_D+F_R[(z-z_0)+G_1(r, z-z_0)+G_1(r, z+z_0)], & z\geqq -z_0 \\ \phi_D+F_R[G_1(r, z-z_0)+G_1(r, +z+z_0)], & -z_0\leqq z\leqq z_0 \\ \phi_D+F_R[-(z+z_0)+G_1(r, z+z_0)+G_1(r, z-z_0)], & z\leqq -z_0 \end{array}\right\}. \quad (12)$$

and

$$\phi_c(r, z)=(1+\alpha)F_R[H(r, z-z_0)+H(r, -z-z_0)].$$

It is interesting to see, in the case of the infinitely thin diaphragm, $z_0=0$, how far the potential distribution calculated from equation (11) agrees with that calculated from the rigorous solution, equation (5).

Taking $\phi_D=0$, $F_R=1$ and $F_L=0$, we have :

$$\phi_1(r, z)=\begin{cases} z+G_1(r, z), & z\geq 0, \\ G_1(r, z), & z\leq 0, \end{cases}$$

$$\phi_c(r, z)=H(r, z)+H(r, -z).$$

The result of this calculation for points along the axis, $r=0$, is given in Table IV.

The agreement of the potential values computed from our equations (10) and (11) with those calculated rigorously is quite good, in spite of the coarse approximation used to determine the function $H(r, z)$ of the correction potential. The accuracy is certainly far better than could be hoped for from experimental field plots. To solve the same problem by numerical methods up to the same amount of agreement would be very laborious.

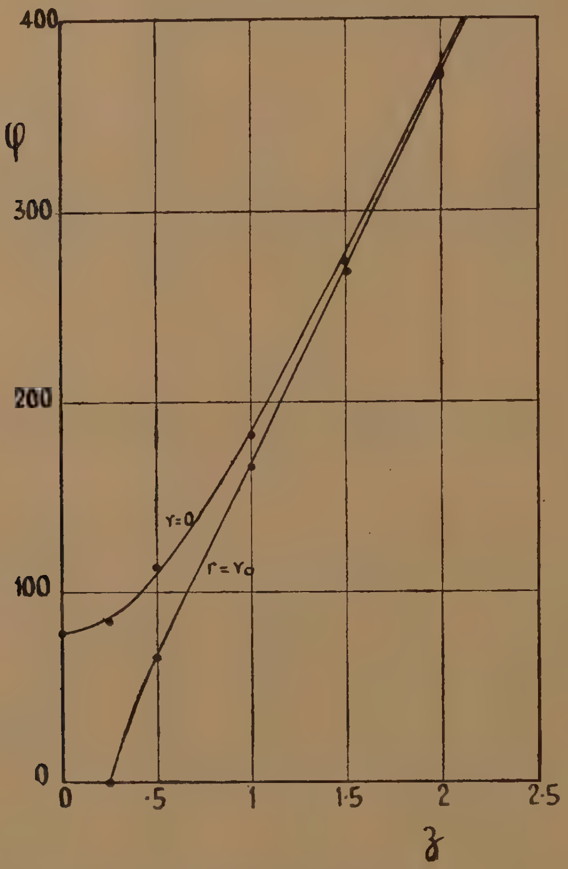
TABLE IV.

Potential along Axis of Infinitely Thin Diaphragm located at $z=0$.

z	$\phi(0, z)$	$\phi_c(0, z)$	$\phi(0, z)$ $=\phi_1(0, z)$ $+\phi_c(0, z)$	$\phi(0, z)$ rigorous
−4·0	0·0000	0·0066	0·0066	0·0065
−3·0	0·0002	0·0114	0·0116	0·0112
−2·5	0·0008	0·0153	0·0161	0·0156
−2·0	0·0027	0·0212	0·0239	0·0232
−1·5	0·0090	0·0289	0·0379	0·0376
−1·0	0·0297	0·0381	0·0678	0·0683
−0·5	0·0946	0·0472	0·1418	0·1421
0·0	0·2674	0·0509	0·3183	0·3183
+0·5	0·5946	0·0472	0·6418	0·6421
+1·0	1·0297	0·0381	1·0678	1·0683
+1·5	1·5090	0·0289	1·5379	1·5376
+2·0	2·0027	0·0212	2·0239	2·0232
+2·5	2·5008	0·0153	2·5161	2·5156
+3·0	3·0002	0·0114	3·0116	3·0112
+4·0	4·0000	0·0066	4·0066	4·0065

It may be noted that $\phi_1(r, z)$ is a reasonably good approximation some distance away from the hole on the side of the *strong* field gradient. At the side where the field is due only to the field penetration through the hole, the correction potential contributes a considerable fraction of the space potential, and at a distance of half the hole diameter, the correction term $\phi_c(r, z)$ is already greater than the uncorrected potential $\phi_1(r, z)$, increasing in relative importance with greater distance from the hole.

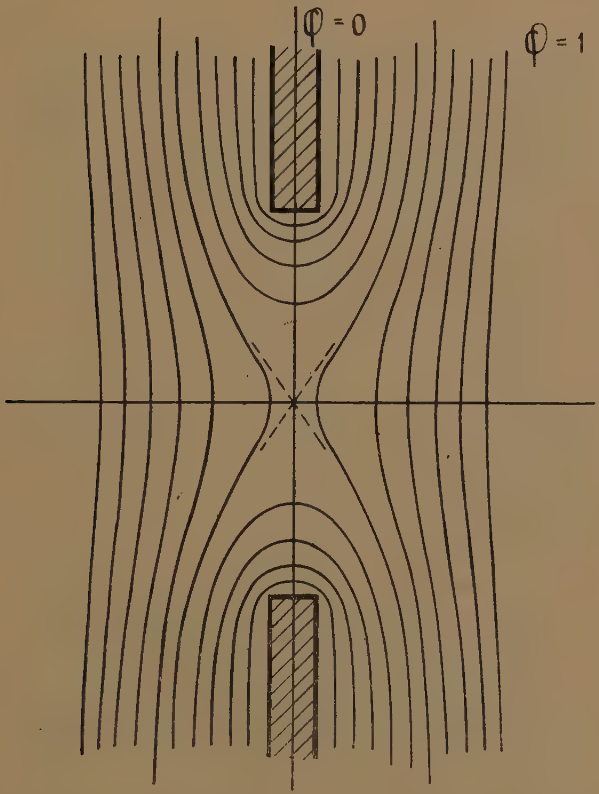
Fig. 5.



Potential distribution near diaphragm, thickness $d=0{\cdot}5r_0$, placed between two plane electrodes at distance $10r_0$ from centre of diaphragm. Full lines : computed by method of this paper. Circles : values computed by numerical method (ref. (6)).

As a further check of the reliability of the computing method, the potential distribution near a diaphragm, $d=0{\cdot}5r_0$, $\phi_D=0$, placed between two parallel electrodes at potential $\phi=1{,}000$ at a distance $10r_0$, was computed by the method of this paper and by a numerical method, carried to an estimated accuracy of approximately 0·3 per cent. of the value of the potential applied to the plane electrodes. As can be seen from the plot of the potential for $r=0$ and $r=r_0$ (fig. 5), the potential values computed numerically are scattered around the potential calculated by the method of this paper; there can be little doubt that carrying the numerical approximation further would bring the potential values computed by the different methods into still closer agreement.

Fig. 6*a*.

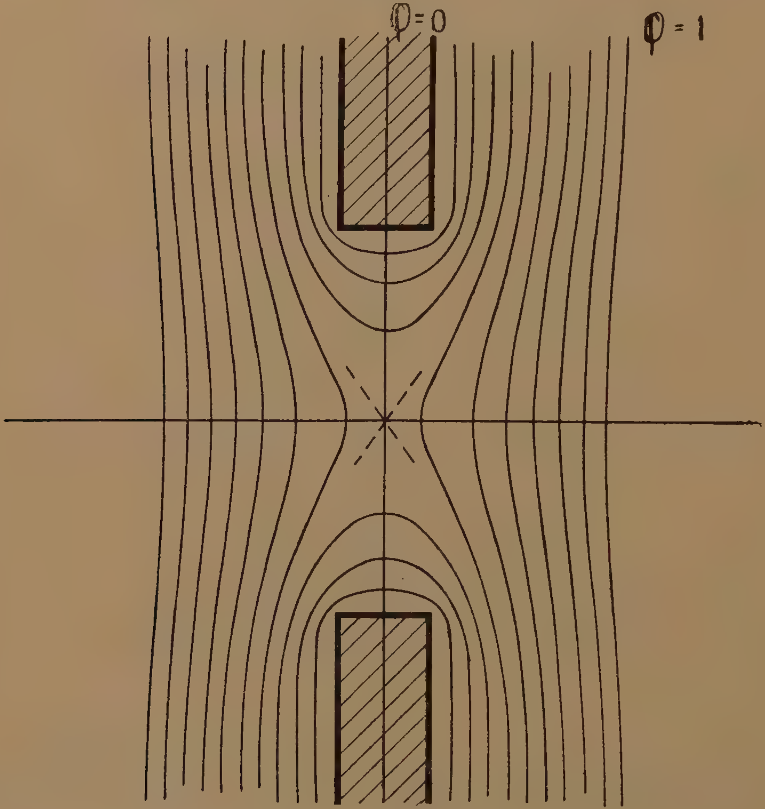


Figs. 6 *a–c*.—Potential distribution near three diaphragms of relative thickness $d/r_0=0{\cdot}25, d/r_0=0{\cdot}5$, and $d/r_0=1$. Field gradient $\frac{\partial\phi}{\partial z}=F=-F_L=1$, potential of diaphragm $\phi_D=0$.

As a further example, the potential distributions near three diaphragms (potential $\phi_D=0$) of increasing thickness, $d=0{\cdot}25r_0$, $d=0{\cdot}5r_0$ and $d=r_0$, with uniform fields $F_R=-F_L=1$ on either side, were calculated. The resulting potential distributions are shown on figs. 6 *a*, 6 *b*, and 6 *c*, the potential between successive equipotential lines increasing by 0·1 in each case.

Finally, the potential in the centre of a diaphragm of thickness *d*, having a symmetrical field gradient F on either side, was computed.

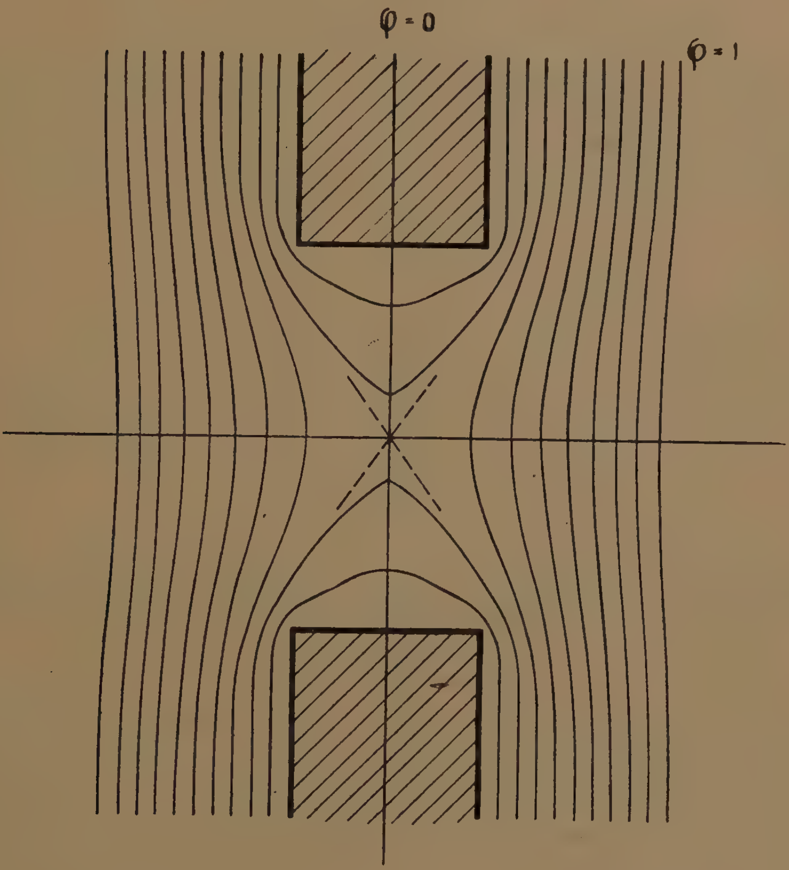
Fig. 6*b*.



In fig. 7, the relative potential $\frac{\phi}{F}$ at the centre of the hole ($z=0$, $r=0$) is plotted as a function of relative thickness d/r_0. Fig. 7 shows also a graph of the potential at a relative distance d/r_0 from the open end of a semi-infinitely long cylinder with plane end face. At a distance from the open end equal to the cylinder diameter, $d=2r_0$, the potential is just 1/100 of that found at the open end $d=0$.

An application of the method described to the theory of modulating grids with circular apertures, and to cathode-ray tube gun design in particular, will be given in a later paper.

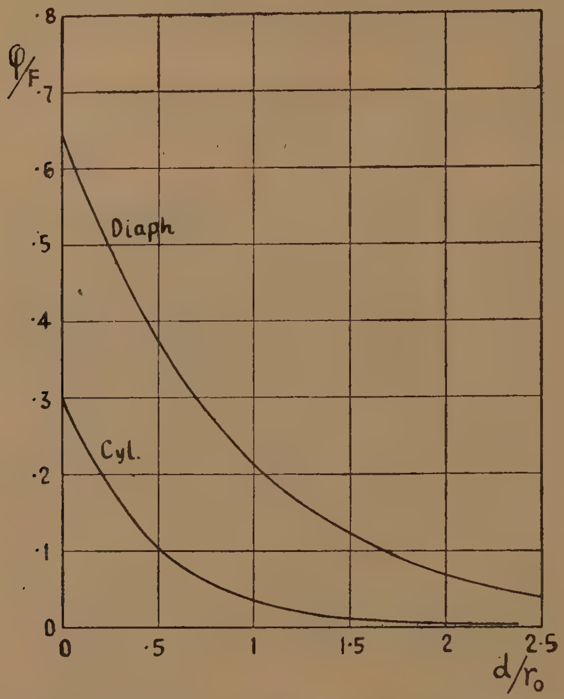
Fig. 6*c*.



4. *Acknowledgements.*

This paper was written whilst the author was on the Staff of Cathodeon, Limited, and he wishes to thank the company's directors for permitting its publication. The author is indebted to Mr. S. Bertram and to the editor of the Journal of Applied Physics for permission to reprint Table I. from the Journal of Applied Physics, vol. xiii. p. 499, and to Mr. G. Archard for the numerical computation used in the field plot, fig. 5.

Fig. 7.



Relative potential ϕ/F in centre of diaphragm of relative thickness d/r_0, and at relative distance d/r_0 from open end of semi-infinitely long cylinder.

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XXXVII. *The Dirichlet Problem for a Ring Space.*

By A. C. STEVENSON *.

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THE "internal problems of two-dimensional potential theory" recently treated by R. Morris ⁽¹⁾ admit of a somewhat more simple treatment if the boundaries are mapped into circles, as the present writer has shown ⁽²⁾ in the case of a domain S in the z-plane which maps into the interior of unit circle in a complex σ-plane, via a conformal transformation of type

$$z=z(\sigma)=\sum_{0}^{\infty} a_r\sigma^r,\ z'(\sigma)\neq 0 \text{ or } \infty \text{ within S.}$$

The object of the present paper is to establish the same point and develop a similar technique in the case of such doubly connected domains S in the z-plane which map into the interior of a concentric ring-space in the σ-plane.

1. *The Harmonic Boundary Problem.*

Take a doubly connected region in the z-plane which maps into the region $r_1 \leqslant |\sigma| \leqslant r_2$ in the σ-plane, bounded by concentric circles γ_1, γ_2 of radii r_1, r_2 via a conformal transformation of type

$$z=z(\sigma)=\sum_{-\infty}^{\infty} a_n\sigma^n,\ z'(\sigma)\neq 0 \text{ or } \infty \text{ within S.} \quad . \quad . \quad . \quad (1.1)$$

The typical Dirichlet problem here is a boundary problem in which an analytic function or complex potential $\omega_1(z)=\phi+i\psi$ has to be determined from given values of ψ on the boundaries C of the domain S in the z'-plane.

If we write $\omega_1(z)=\omega_1\{z(\sigma)\}=\omega(\sigma)$,

then the problem is to determine $\omega(\sigma)$ from given values of ψ on the boundaries γ of the transformed domain in the σ-plane, since

$$\omega_1(z)-\overline{\omega}_1(\overline{z})=2i\psi(x, y)=2i\text{F}(z, \overline{z}) \text{ along C}$$

becomes

$$\omega(\sigma)-\overline{\omega}(\overline{\sigma})=2if(\sigma, \overline{\sigma})=2if(\sigma, r^2\sigma^{-1}) \text{ along } \gamma,$$

since $\sigma\overline{\sigma}=r^2$, where γ is each of the boundary circles γ_1, γ_2 of the domain in the σ-plane in turn.

If the function f is expanded in a Laurent series, we can write this

$$\omega(\sigma)-\overline{\omega}(\overline{\sigma})=i\sum_{-\infty}^{\infty} \text{B}_n(r)\sigma^n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.2)$$

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* Communicated by the Author.

and if we imagine this done for the two circular boundaries γ_1, γ_2 given by $r=r_1$, $r=r_2$, we may accordingly regard the boundary coefficients $B_n(r_1)$, $B_n(r_2)$ as the given data in a problem of this type. Notice, however, that since $\psi(x, y)$ is real,

$$\sum_{-\infty}^{\infty} B_n(r)\sigma^n=\sum_{-\infty}^{\infty} \overline{B}_n(r)\overline{\sigma}^n=\sum_{-\infty}^{\infty} \overline{B}_{-n}(r)\overline{\sigma}^{-n},$$

or

$$\sum_{-\infty}^{\infty} \{B_n(r)-\overline{B}_{-n}(r)r^{-2n}\}\sigma^n=0,$$

has to hold for all values of σ on γ, so that the coefficients $B_n(r)$ will always be found to satisfy the relations

$$B_n(r)=r^{-2n}\overline{B}_{-n}(r), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.3)$$

$$B_0(r)=\overline{B}_0(r), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.4)$$

so that $B_0(r)$ is necessarily real.

To solve for $\omega(\sigma)$ we assume tentatively

$$\omega(\sigma)=i\sum_{-\infty}^{\infty} A_n\sigma^n+i\kappa\log\sigma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.5)$$

where κ is a real constant, and A_n complex, so that the problem is to determine A_n, κ explicitly in terms of the coefficients B_n. The boundary conditions become

$$\omega(\sigma)-\overline{\omega}(\overline{\sigma})=i\sum_{-\infty}^{\infty} \{A_n\sigma^n+\overline{A}_n r^{2n}\sigma^{-n}\}+i\kappa\log\sigma\overline{\sigma}$$

$$=i\sum_{-\infty}^{\infty}{}' \{A_n+\overline{A}_{-n}r^{-2n}\}\sigma^n+2i\kappa\log r \quad . \quad . \quad . \quad (1.6)$$

along $r=r_1$, r_2.

Comparison of (1.6) with the given boundary conditions (1.2) allows us to identify

$$B_n(r)=A_n+\overline{A}_{-n}r^{-2n} \qquad (n\neq 0) \quad . \quad . \quad . \quad . \quad (1.7)$$

and

$$B_0(r)=A_0+\overline{A}_0+2\kappa\log r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.8)$$

for $r=r_1$, r_2.

Note that the solution for $B_0(r)$ is satisfactorily real, as (1.4) demands, whilst

$$B_n(r)=r^{-2n}\{\overline{A}_{-n}+A_n r^{2n}\}=r^{-2n}\overline{B}_{-n}(r)$$

is in agreement with (1.3).

Solving (1.7) for A_n, we find, using (1.3),

$$A_n=\frac{\overline{B}_{-n}(r_2)-\overline{B}_{-n}(r_1)}{r_2^{2n}-r_1^{2n}} \qquad (n\neq 0), \quad . \quad . \quad . \quad . \quad (1.9)$$

and solving (1.8) for κ and the real part A_{01} of A_0, we find

$$\kappa=\frac{B_0(r_2)-B_0(r_1)}{2\log(r_2/r_1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.10)$$

$$A_{01}=-\frac{B_0(r_2)\log r_1-B_0(r_1)\log r_2}{2\log(r_2/r_1)}, \quad . \quad . \quad . \quad . \quad . \qquad (1.11)$$

so that the function $\omega(\sigma)$ is determinate, apart from a real constant, which is quite immaterial in the usual physical applications of the theory.

2. *The Torsion of Hollow Tubes.*

We illustrate by a consideration of the Saint–Venant torsion of a long hollow cylinder of cross-section S in the z-plane. Here

$$\psi=\tfrac{1}{2}z\bar{z}+C(r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.1)$$

along the boundaries of C. The constant must be different along the two boundaries in this elastic problem, so we write it $C(r)$, which is to become C_1, C_2 along $r=r_1$, r_2 respectively. The main object of the investigation for $\omega(\sigma)$ is usually to find the twisting couples N oppositely applied to the ends of such a long cylinder. This is given by (2) the real part of

$$N+iN^*=\mu\tau(I-J), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.2)$$

where I is the polar moment of inertia $\int z\bar{z}\,dS$ for the cross-section, or

$$I=-\tfrac{1}{4}i\int_C z\bar{z}^2 z'(\sigma)d\sigma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.3)$$

and

$$J=-\tfrac{1}{2}\int_C z\bar{z}\omega'(\sigma)d\sigma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.4)$$

where C consists of the two circles γ_1, γ_2 described in opposite senses, so as to keep the ring-space area on the left.

With the transformation (1.1) we have, along $\sigma\bar{\sigma}=r^2$,

$$z(\sigma)\bar{z}(\bar{\sigma})=\sum_{-\infty}^{\infty}a_m\sigma^m\sum_{-\infty}^{\infty}\bar{a}_s\bar{\sigma}^s=\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}a_m\bar{a}_s r^{2s}\sigma^{m-s}$$

$$=\sum_{-\infty}^{\infty}b_n(r)\sigma^n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.5)$$

where

$$b_n(r)=\sum_{s=-\infty}^{\infty}a_{n+s}\bar{a}_s r^{2s}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.6)$$

Hence the boundary condition is

$$\omega(\sigma)-\bar{\omega}(\bar{\sigma})=i\sum_{-\infty}^{\infty}b_n(r)\sigma^n+2iC(r),$$

and comparison with (1.2) shows that the given boundary coefficients of the problem are here

$$B_n(r)=b_n(r), \quad n\neq 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.7)$$

$$B_0(r)=b_0(r)+2C(r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.8)$$

Since the real part ϕ of the complex potential in this problem gives the longitudinal displacement and is single-valued in this problem, $\kappa=0$, so that (1.10) gives $B_0(r_1)=B_0(r_2)$, and (1.11) then leads to

$$A_{01}=\tfrac{1}{2}B_0(r)=\tfrac{1}{2}b_0(r)+C(r). \quad \ldots \ldots \ldots \quad (2.9)$$

We can, in fact, take C_1 to be any constant, but this fixes C_2. If we take $C_1=-\tfrac{1}{2}b_0(r_1)$, then $C_2=-\tfrac{1}{2}b_0(r_2)$ and $A_{01}=0$, so that finally our solution is

$$\omega(\sigma)=i\sum_{-\infty}^{\infty}{}' \frac{\bar{b}_{-n}(r_2)-\bar{b}_{-n}(r_1)}{r_2^{\,2n}-r_1^{\,2n}}\sigma^n, \quad \ldots \ldots \quad (2.10)$$

where the coefficients are determined from (2.5) and Σ' omits the term $n=0$.

Now along the boundary $\sigma\bar{\sigma}=r^2$,

$$\bar{z}(\bar{\sigma})z'(\sigma)=\sum_{-\infty}^{\infty}\bar{a}_s r^{2s}\sigma^{-s}\sum_{-\infty}^{\infty} m a_m \sigma^{m-1}=\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}(n+s)a_{n+s}\bar{a}_s r^{2s}$$

$$=\sum_{-\infty}^{\infty} c_n(r)\sigma^{n-1},$$

where
$$c_n(r)=\sum_{s=-\infty}^{\infty}(n+s)a_{n+s}\bar{a}_s r^{2s} \quad \ldots \ldots \ldots \ldots \quad (2.11)$$

are coefficients which can be determined from the transformation coefficients of (1.1).

Hence
$$I=-\tfrac{1}{4}i\int_C \sum_{-\infty}^{\infty} c_n(r)\sigma^n \sum_{-\infty}^{\infty} b_s(r)\sigma^{s-1}d\sigma,$$

or
$$I=\tfrac{1}{2}\pi\sum_{-\infty}^{\infty}\{c_n(r_2)b_{-n}(r_2)-c_n(r_1)b_{-n}(r_1)\}. \quad \ldots \quad (2.12)$$

Also
$$J=-\tfrac{1}{2}i\int_C \sum_{-\infty}^{\infty} b_s(r)\sigma^s \sum_{-\infty}^{\infty} nA_n\sigma^{n-1}d\sigma$$

$$=\pi\sum_{-\infty}^{\infty} nA_n\{b_{-n}(r_2)-b_{-n}(r_1)\},$$

or
$$J=\pi\sum_{-\infty}^{\infty}\frac{n\{b_{-n}(r_2)-b_{-n}(r_1)\}\{\bar{b}_{-n}(r_2)-\bar{b}_{-n}(r_1)\}}{r_2^{\,2n}-r_1^{\,2n}}. \quad (2.13)$$

Hence (2.2), (2.12), (2.13) constitute the general solution for N in terms of constants b_n and c_n defined by (2.5) and (2.11).

3. *Cross-section Bounded by Confocal Ellipses.*

For the cross-section bounded by confocal ellipses of semi-axes (a, b), (a', b'), $a>a'$, we use

$$z=c(\sigma+\sigma^{-1}), \quad \ldots \ldots \ldots \quad (3.1)$$

where
$$2cr_2=a+b,\ 2cr_1=a'+b',\ 4c^2=a^2-b^2=a'^2-b'^2. \quad \ldots \quad (3.2)$$

Then the transformation coefficients are

$$a_n=0,\ n\neq 1,\ -1\ ;\ \text{but } a_1=a_{-1}=c,$$

so that (2.5) gives

$$b_n=0,\ n\neq 0,\ 2,\ -2\ ;\ \text{but } b_2=c^2r^{-2},\ b_{-2}=c^2r^2,\ b_0=c^2(r^2+r^{-2}),$$

whence

$$\omega(\sigma)=ic^2\left\{\frac{\sigma^2}{r_1^{\,2}+r_2^{\,2}}+\frac{\sigma^{-2}}{r_1^{\,-2}+r_2^{\,-2}}\right\}. \quad . \quad . \quad . \quad . \qquad (3.3)$$

Also, from (2.10),

$$c_n=0,\ n\neq 0,\ 2,\ -2\ ;\ \text{but } c_2=c^2r^{-2},\ c_{-2}=c^2r^2,\ c_0=c^2(r^2-r^{-2}),$$

so that we find easily

$$\mathrm{N}=\frac{1}{2}\pi\mu\tau c^4\left\{(r_2^{\,4}-r_2^{\,-4})-(r_1^{\,4}-r_1^{\,-4})-8\,\frac{r_2^{\,2}-r_1^{\,2}}{r_2^{\,2}+r_1^{\,2}}\right\},$$

or

$$\mathrm{N}=\frac{1}{4}\pi\mu\tau\left\{ab(a^2+b^2)-a'b'(a'^2+b'^2)\right.$$
$$\left.-\frac{(a+b)^4(a-b)^2-(a'+b')^4(a'-b')^2}{(a+b)^2+(a'+b')^2}\right\}, \quad . \quad . \quad . \qquad (3.4)$$

is the explicit form of the torsion moment here.

4. *Cross-sections Bounded by Eccentric Circles.*

For cross-sections bounded by two eccentric circles the simplest transformation is

$$z=c/(1-\sigma), \qquad c \text{ real}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4.1)$$

so that the ring-space must not include the critical point I, $\sigma=1$, corresponding to the point at infinity in the z-plane.

Now $\qquad z\sigma=z-c, \qquad \bar{z}\bar{\sigma}=\bar{z}-c,$

so that $\qquad (x^2+y^2)\sigma\bar{\sigma}=x^2+y^2-2cx+c^2$

and it is clear that $|\sigma|=r$ corresponds to a circle in the z-plane whose equation is

$$(x-h)^2+y^2=k^2,$$

where $\qquad h(r)=c/(1-r^2),\ k(r)=cr/(1-r^2),\ r=k/h. \quad . \quad . \quad . \quad . \quad . \qquad (4.2)$

The whole of the z-plane to the right of the line $x=c/2$ is mapped conformally on the space interior to unit circle in the σ-plane, so that $r<1$. The origin is one of the limiting points of the family of circles $r=$constant, the other being internal to these circles.

From (4.1) the transformation coefficients are

$$a_n=c,\ n\geqslant 0\ ;\ a_{-n}=0,\ n>0\ ;$$

so that (2.5) gives

$$b_n=c^2\sum_{0}^{\infty}r^{2s}=c^2/(1-r^2)=ch, \quad n\geqslant 0, \quad . \quad . \quad . \quad . \qquad (4.3)$$

whilst from (1.3)

$$b_{-n}=r^{2n}\bar{b}_n=r^{2n}ch, \quad n\geqslant 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.4)$$

Accordingly we find the solution for the complex potential to be

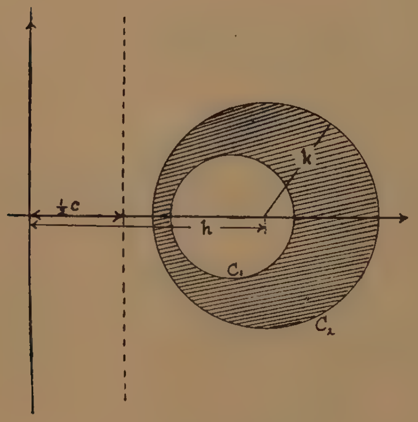
$$\omega=ic\sum_{-\infty}^{\infty}{}' \frac{h_2r_2{}^{2n}-h_1r_1{}^{2n}}{r_2{}^{2n}-r_1{}^{2n}}\sigma^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.5)$$

We do not really need c_n for the polar moment of inertia I, since this can be written down from elementary rules as

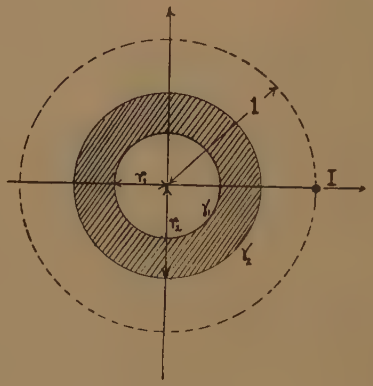
$$\mathrm{I}=\tfrac{1}{2}\pi(a^4-b^4)+\pi(a^2h_2{}^2-b^2h_1{}^2), \quad . \quad . \quad . \quad . \quad . \quad (4.6)$$

where we have written $h(r_2)=h_2$, $h(r_2)=h_1$, $k(r_2)=a$, $k(r_1)=b_1(a>b)$. For completeness, however, from (2.1),

$$c_n=c^2\sum_{s=0}^{\infty}(n+s)r^{2s}=nb_n+c^2\sum_0^{\infty}sr^{2s}, \quad n\geqslant 0,$$



z-plane.



σ-plane.

and
$$c_{-n}=c^2\sum_{s=n}^{\infty}(s-n)r^{2s}=c^2r^{2n}\sum_{m=0}^{\infty}mr^{2m}, \quad n\geqslant 0.$$

But
$$\sum_0^{\infty}sr^{2s}=\tfrac{1}{2}r\frac{d}{dr}\sum_0^{\infty}r^{2s}=\tfrac{1}{2}r\frac{d}{dr}\frac{1}{1-r^2}=\frac{r^2}{(1-r^2)^2}=k^2/c^2, \quad . \quad . \quad . \quad (4.7)$$

so that
$$c_n=nch+k^2, \quad c_{-n}=r^{2n}k^2=k^{2n+2}/h^2, \quad n\geqslant 0. \quad . \quad . \quad . \quad (4.8)$$

Since the b_n's are real, we can write

$$\mathrm{J}=\pi\sum_0^{\infty}n\frac{\{b_{-n}(r_2)-b_{-n}(r_1)\}^2}{r_2{}^{2n}-r_1{}^{2n}}-\pi\sum_0^{\infty}n\frac{\{b_n(r_2)-b_n(r_1)\}^2}{r_2{}^{-2n}-r_1{}^{-2n}}$$

$$=\pi c^2\sum_0^{\infty}\frac{n}{r_2{}^{2n}-r_1{}^{2n}}\{(h_2r_2{}^{2n}-h_1r_1{}^{2n})^2+r_1{}^{2n}r_2{}^{2n}(h_2-h_1)^2\}.$$

But $$(h_2 r_2^{2n} - h_1 r_1^{2n})^2 = (h_2^2 r_2^{2n} - h_1^2 r_1^{2n})(r_2^{2n} - r_1^{2n}) + r_1^{2n} r_2^{2n}(h_2 - h_1)^2,$$
so that
$$J = \pi c^2 \sum_0^\infty n(h_2^2 r_2^{2n} - h_1^2 r_1^{2n}) + 2\pi c^2 d^2 \sum_0^\infty \frac{n r_1^{2n} r_2^{2n}}{r_2^{2n} - r_1^{2n}},$$
where $d = h_2 - h_1$ = distance between the centres of the eccentric circles. This reduces still further on using (4.7) and (4.2) to
$$J = \pi(h_2^2 a^2 - h_1^2 b^2) + 2\pi c^2 d^2 \sum_0^\infty \frac{n}{(h_1/b)^{2n} - (h_2/a)^{2n}},$$
so that N simplifies to
$$N = \tfrac{1}{2}\mu\pi\tau(a^4 - b^4) - 2\pi\mu\tau c^2 d^2 \sum_0^\infty \frac{n}{(h_1/b)^{2n} - (h_2/a)^{2n}}. \quad . \quad . \quad (4.8)$$

If we set $r_2 = e^{-\alpha}$, $r_1 = e^{-\beta}$, we get
$$N = \tfrac{1}{2}\mu\pi\tau(a^4 - b^4) - \pi\mu\tau c^2 d^2 \sum_0^\infty n e^{-n(\alpha+\beta)} \operatorname{cosech} n(\beta - \alpha), \quad . \quad (4.9)$$
in agreement with Macdonald's result for this problem(3), when it is remarked that his c is twice that used here.

Since the Saint–Venant flexure with shear problem has been broken up into relatively simple Dirichlet problems(4), the usefulness of the method for hollow tubes is at once apparent and the writer hopes to give the corresponding solutions for the canonical flexure functions, the associated twist and the centres of flexure of such tubes in a further communication.

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XXXVIII. *On the Ionization Potentials of the K and L electrons.*

By Anthony Felix Richter *.

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In the year 1931 a general theory of conjugation was outlined by the author. When a certain group (of electrons or of atoms or molecules) is in a condition of unsaturation, it strives to alter its state by conjugating with another similar group. According to the nature of the unsaturated group, whose conjugation we are considering, three different effects of conjugation have been described (1), (2).

The intra-atomic conjugation is effected by the mutual neutralization of the residual momenta of the electronic groups, which are not sufficiently stable, the result being their closer joining together, or, respectively, a closer joining of valency electrons with the group concerned. The experimental evidence for this conjugation is given by the deviations of the physico-chemical properties from their systematic course, as expected from the systematic increase of the atomic number Z(3). It is conclusive to assume "disjugation," the opposite phenomenon to conjugation; in the first paper on this subject(4) I assumed the disjugation of the conjugated (coupled) duplet when the nuclear charge increased by at least two units in the subsequent elements of the given period. ("Stabilisierung der (zuvor) gekoppelten Zweirschale.")

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* Communicated by the Author.

TABLE I.

$\mathfrak{J}$ of the members of the monoisoelectronic row, the values of the first differential row Δ and those of the second one $\Delta\Delta$.

Symbol	$\mathfrak{J}_I$	Δ	$\Delta\Delta$
H	15·53	—	—
He	54·14	40·61	—
Li	121·84	67·70	27·09
Be	216·63	94·77	27·07
B	338·50	121·87	27·10
C	487·55	149·05	27·18
N	663·73	176·28	27·23
O	*c* 867·08	203·35	27·07
F	1097·66	230·58	27·23
Ne	1355·48	257·82	27·24
Σ	5216·14	1341·93	217·21

c=calculated.

In the effort to bring further experimental evidence for the conjugation concept, the aim of which is to elucidate the irregularities mentioned above, the iscelectronic rows have been investigated. This work, the first part of which we are dealing with in this paper, resulted in the recognition *that the ionization potentials $\mathfrak{J}$ of the iso-rows form arithmetical progressions of the second degree.* The figures arrived at on the basis of the known empirical values of $\mathfrak{J}$ of the first two iso-rows demonstrate this, as shown in Tables I. and II. As far as the data for $\mathfrak{J}$ are available, it could be established in further 20 iso-rows that the law of the arithmetical progression is valid with reasonable accuracy.

Hence on purely empirical grounds it is possible to assume the general validity of this principle and to form equations which permit the calculation of $\mathfrak{J}$ for any member of the isoelectronic row i in question when $\mathfrak{J}$ values for some members are known. The general formula is as follows :—

$$\mathfrak{J}_i=A_i n+\tfrac{1}{2}B_i n(n-1)+D_i, \quad \dots\dots \quad (1)$$

or

$$\mathfrak{J}_i=\tfrac{1}{2}B_i n^2+(A_i-\tfrac{1}{2}\,.\,B_i)n+D_i, \quad \dots\dots \quad (1.2)$$

where A_i, B_i, D_i are constants of the corresponding iso-row i and n is the ordinal number of the element in it. The constant B * of the second term is the mean value of the second differential row :

Table II.

$\mathfrak{J}$ of the seven members of the di-isoelectronic row, the values of the first differential row $\varDelta$ and those of the second one $\varDelta\varDelta$.

Symbol	$\mathfrak{J}_{II}$	$\varDelta$	$\varDelta\varDelta$
He	24·47	—	—
Li	75·26	50·79	—
Be	153·11	77·85	27·06
B	258·10	104·99	27·14
C	390·02	131·92	26·93
N	549·08	159·06	27·14
O	735·22	186·14	27·08
Σ	2185·26	710·75	175·35

$$B=(\varDelta\varDelta_1+\varDelta\varDelta_2+\dots\varDelta\varDelta_n)\frac{1}{n}=\overline{\varDelta\varDelta}. \quad \dots\dots \quad (2)$$

It is called the quantum constant because of its probable relation to the quantum number. The mean value of the row constant A of the first term of the equation is evaluated as follows :

$$A=[(\varDelta_1+\varDelta_2+\dots\varDelta_n)-Bn(n-1)\,.\,\tfrac{1}{2}]\,.\,\frac{1}{n}. \quad \dots\dots \quad (3)$$

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* The index i is omitted in the case where there is no danger of confusion. In a special case the symbol i is substituted for by roman figures (i=I, II, . . . XCII) and is put into brackets when arabic figures are added to denote the member of the iso-row in question, *e. g.* $\mathfrak{J}_{(II)3}$ signifies $\mathfrak{J}$ Be III.

The arithmetical ionization potentials $\mathfrak{J}_{\text{ar}}$ are determined by these two constants:

$$\mathfrak{J}_{\text{ar}}=\text{A}n+\tfrac{1}{2}\text{B}n(n-1). \quad \dots \quad (4)$$

The correction constant D is given by the difference between A and $\mathfrak{J}_1$ of the first member of the isoelectronic row

$$\text{D}_1=\text{A}-\mathfrak{J}_1, \quad \dots \quad (5)$$

or, better, its mean value is evaluated as follows

$$\text{D}=\left[\sum_1^n\mathfrak{J}-\sum_1^n\mathfrak{J}_{\text{ar}}\right]\frac{1}{n}$$

$$=\left\{\sum_1^n\mathfrak{J}-\left[\tfrac{1}{2}\text{A}n(n+1)+\tfrac{1}{6}\text{B}(n-1)n(n+1)\right]\right\}\frac{1}{n}. \quad \dots \quad (5.2)$$

The theoretical equalizing of the quantum constants as well as the experimental inaccuracy are reflected in the oscillation of the individual values of the correction constant D_i, about its average value. Thus the average value D_{I} amounts to $-0{\cdot}25$; its single values lie between $-0{\cdot}02$ and $-0{\cdot}40$. This difference, as great as twenty-fold, which occurs between the single values of D_{I}, has not been encountered in other rows inquired into and we prefer to neglect it entirely.

Thus the *calculated* ionization potentials $\mathfrak{J}_{\text{calc}}$:

$$\mathfrak{J}_{\text{calc}}=\mathfrak{J}_{\text{ar}}+\text{D}. \quad \dots \quad (6)$$

The figuring of the quantum constant of the monoisoelectronic row gives two different quantities due to the arbitrary selection of $\mathfrak{J}$, *i. e.* $\text{B}_{\text{I}}=27{\cdot}134$ or $27{\cdot}15$, that of $\text{B}_{\text{II}}=27{\cdot}07$, as recorded in Table III. We assume, however, the common quantum constant for the K electrons

$$\text{B}_{\text{I}}=\text{B}_{\text{II}}=27{\cdot}11\,;\ \text{mean error } \sigma=0{\cdot}02, \quad \dots \quad (7)$$

as the difference between the individual values of the quantum constants in the rows referred to will exceed sometimes the difference between B_{I} and B_{II}.

The study of the eight L isoelectronic rows resulted in the conclusion that it is possible to assume a common quantum constant for all the L electrons too. Its mean value has been calculated on the basis of 46 empirical values of $\mathfrak{J}$:

$$\text{B}_{\text{L}}=6{\cdot}895\,;\ \sigma=0{\cdot}033. \quad \dots \quad (8)$$

The corresponding row constants A_{III} to A_{X} are increasing from $5{\cdot}58$ to $18{\cdot}80$. They are to be found in Table III.

The sum of the known empirical $\mathfrak{J}$ does not differ from the value calculated by means of the common quantum constants according to the formula

$$\sum_1^n\mathfrak{J}_{\text{calc}}=\sum_1^n\mathfrak{J}_{\text{ar}}+\text{D}\,.\,n. \quad \dots \quad (8.2)$$

This agreement (pointed out numerically in last two columns of Table III.) seems to yield a criterion justifying especially the equalization of the L quantum constants.

TABLE III.

Row-, quantum- and correction constants of the equations for $\mathfrak{J}$ of the K and L isoelectronic rows.
Abbreviations : emp.=empirical, int=interpolated (calculated) $\mathfrak{J}$.

Iso-electronic row	The number of members the $\mathfrak{J}$ of which are known and which have been used for the calculation	Constants with individual quantum constants B_i			Constants with common quantum constants B_K and B_L			Sum of the known empirical and interpolated $\mathfrak{J}$ in the iso-row	Sum of the $\mathfrak{J}$ according to the formula (6·2) with common quantum constants
		A	B	D	A	B	D		
K 1. *s*	7 emp	13·43	27·134	0·0					
	7 emp+1 int+2 emp	13·43	27·15	0·18	13·55	27·11	−0·25	5216·14	5215·9
2. *s*	7 emp	23·71	27·07	0·78	23·57		1·02	2185·26	2185·26
L 1. *s*	7 emp	5·94	6·804	−0·33	5·58	6·90	−0·20	544·62	541·24
	7 emp+1 int+1 emp	5·96	6·816	−0·57				1080·93	1077·30
2. *s*	6 emp+1 int+3 emp	8·76	6·866	0·51	8·60				
	6 emp+3 emp	8·35	6·954	1·52			0·91	1621·03	1620·60
1. *p*	5 emp+1 int+3 emp	8·87	6·98	−0·45	9·23		−1·25	1232·45	1232·10
2. *p*	8 emp	11·45	6·87	−0·27	11·33		0·02	987·34	987·64
3. *p*	3 emp+1 int+4 emp	12·83	7·05	2·58	13·50		1·14	1074·84	1074·72
4. *p*	7 emp	14·75	6·81	−1·48	14·36		−0·60	784·41	784·28
5. *p*	6 emp	16·90	6·80	0·40	16·60		0·86	595·28	595·26
6. *p*	5 emp	18·82	6·89	2·63	18·80		2·97	433·33	434·85

A further criterion of the common quantum constants lies in the theory of the arithmetical progression itself. The differences between the $\mathfrak{J}$ of the corresponding members of two iso-rows compared, so called *inter-potentials* (*e. g.* $\mathfrak{J}_{II,\,I}$ and $\mathfrak{J}_{I,\,III}$, recorded in Table IV.), again form an arithmetical progression.

If the quantum constants of two iso-rows X and Y equal one another, the inter-row potentials form a progression of the first degree ; in this instance the constant value of the first differential row, called the inter-row constant, is

$$A_{X,\,Y}=A_X-A_Y. \quad \ldots\ldots\ldots \quad (9)$$

If the quantum constants of two isorows differ one from another, the inter-row potentials form an arithmetical progression of the second degree ; in this instance the constant value of the second differential row is called the inter-quantum constant.

Applied to our case of the first three iso-rows, the theory requires

$$A_{II,\,I}=A_{II}-A_I=10{\cdot}02,\ \text{when}\ B_I=B_{II}=27{\cdot}11, \quad \ldots \quad (9.2)$$

or

$$B_{I,\,II}=B_I-B_{II}=0{\cdot}06,\ \text{when}\ B_I\neq B_{II}. \quad \ldots\ldots \quad (10)$$

The quantity 0·06 is beyond the possibility of empirical determination ; from that point of view the equalizing of the quantum constants for K electrons as well as for L electrons appears to be justified. A further proof is brought forward by the comparison of the theoretical quantities $B_{I,\,L}$, $B_{II,\,L}$ and $B_{K,\,L}$ with empirical ones :

$$B_{I,\,L}=B_I-B_L=27{\cdot}13-6{\cdot}90=20{\cdot}23, \quad \ldots \quad (10.2)$$

$$B_{II,\,L}=B_{II}-B_L=27{\cdot}07-6{\cdot}90=20{\cdot}17, \quad \ldots \quad (10.3)$$

$$B_{K,\,L}=B_K-B_L=27{\cdot}11-6{\cdot}90=20{\cdot}21. \quad \ldots \quad (10.4)$$

On the basis of the empirical $\mathfrak{J}$ values which are known from the literature the following mean value has been calculated :

$$B_{I,\,L}=20{\cdot}20 \quad \text{and} \quad B_{II,\,L}=20{\cdot}27.$$

The illustrative computations of the inter-constants on the ground of the available $\mathfrak{J}$ values in the first three iso-rows are recorded in Table IV. When substituting the Moseley's number Z for the ordinal number n of the element in the iso-row i according to the relation

$$Z+1=i+n, \quad \ldots\ldots\ldots\ldots \quad (11)$$

one arrives at the modified equation for $\mathfrak{J}$ for the corresponding iso-row. Thus the equation for the monoiso-row is as follows :

$$\mathfrak{J}_I=13{\cdot}55Z+27{\cdot}11Z(Z-1)\tfrac{1}{2}-0{\cdot}25. \quad \ldots \quad (1,\ \text{K}1.s)$$

When neglecting the correction term D and simplifying the equation, we obtain the already known expression for $\mathfrak{J}$ of the monoiso-row : $\mathfrak{J}=13{\cdot}55\,.\,Z^2$.

TABLE IV.

The inter-row potentials $\mathfrak{J}_{\mathrm{II,\,I}}$ and $\mathfrak{J}_{\mathrm{I,\,III}}$ and their differential rows. n is the ordinal number of the element in the isoelectronic row. $\mathfrak{J}_c$ are calculated with common quantum constants (Table III.).

n	$\mathfrak{J}_{\mathrm{II}}$	${}_1\mathfrak{J}_{\mathrm{II,\,I}}$	$\Delta_{\mathrm{II,\,I}}$	$\Delta\Delta_{\mathrm{II,\,I}}$	$\mathfrak{J}_{\mathrm{I}}$	${}_2\mathfrak{J}_{\mathrm{I,\,III}}$	$\Delta_{\mathrm{I,\,III}}$	$\Delta\Delta_{\mathrm{I,\,III}}$	$\mathfrak{J}_{\mathrm{III}}$
1	24·47	10·94	—	—	13·53	8·17	—	—	5·36
2	75·26	21·12	10·18	—	54·14	36·02	27·85	—	18·12
3	153·11	31·27	10·15	—0·03	121·84	84·09	48·07	20·22	37·75
4	258·10	41·47	10·20	0·05	216·63	152·41	68·32	20·25	64·22
5	390·02	51·48	10·01	—0·19	338·50	241·07	88·66	20·34	97·43
6	549·08	61·53	10·05	0·04	487·55	350·07	109·00	20·34	137·48
7	735·22	71·49	9·96	—0·09	663·73	479·47	129·40	20·40	184·26
8	c 948·66	81·58	10·09	0·13	c 867·08	629·04	149·57	20·17	c 238·04
9	c1189·11	91·45	9·87	—0·22	1097·66	799·27	170·23	20·66	298·39
10	c1456·67	101·19	9·74	—0·13	1355·48	989·43	190·16	19·93	c 366·05

c=calculated.

The following are the equations of the di- and tri-iso-rows :

$$\mathfrak{J}_{II}=23{\cdot}57(Z-1)+27{\cdot}11(Z-1)(Z-2)\tfrac{1}{2}+1{\cdot}02$$
$$=13{\cdot}55(Z-1)^2+10{\cdot}02(Z-1)+1{\cdot}02. \quad . \quad . \quad . \quad . \quad (1, K2.s)$$
$$\mathfrak{J}_{III}=5{\cdot}94(Z-2)+6{\cdot}80(Z-2)(Z-3)\tfrac{1}{2}-0{\cdot}33$$
$$=3{\cdot}40(Z-2)^2+2{\cdot}54(Z-2)-0{\cdot}33. \quad . \quad . \quad . \quad . \quad (1, L1.s)$$

As the quantum constant 13·55 of the first term represents the energy of the K (monoquantum) level of the hydrogen atom, the following quantum constants are to be postulated with increasing quantum number :

K 13·55, L 3·39, M 1·50, N 0·85 O 0·54,

when assuming the ideal screening power of the electrons (electronic shells) and the close approaching to the energy conditions of the hydrogen atom. Consequently the constant of the linear term is related to the screening power S, the value of which can be ascertained by the formula

$$\mathfrak{J}_i=\tfrac{1}{2}B_in^2+(A_i-\tfrac{1}{2}B_i)n+D_i=\tfrac{1}{2}B_i(n-S)^2. \quad . \quad . \quad . \quad (12)$$

Thus, applied to the investigated two iso-rows :

$$13{\cdot}55(Z-1)^2+10{\cdot}02(Z-1)+1{\cdot}02=13{\cdot}55(Z-S_I)^2, \quad . \quad (12, K2.s)$$
$$3{\cdot}40(Z-2)^2+2{\cdot}54(Z-2)-0{\cdot}33=3{\cdot}39(Z-S_{II})^2, \quad . \quad (12, L1.s)$$

$$\left.\begin{array}{ll}\text{the screening power of the first K electron} & S_I=0{\cdot}6527\\ \text{and that of both K electrons} & S_{II}=1{\cdot}715\end{array}\right\}\text{for He.}$$

For the $\mathfrak{J}$ value of the di-iso-row, the following equation has been derived by C. A. Hylleraas ⁽⁵⁾.

$$\mathfrak{J}_{II}=R\,.\,hc\left(Z^2-1{\cdot}25Z+0{\cdot}31488-\frac{0{\cdot}01752}{Z}+\frac{0{\cdot}00548}{Z^2}\right)=E_H\,.\,Z^* \quad . \quad . \quad . \quad (13)$$

(the factor having regard to the reduced mass of electron has been left out in the equation).

Accordingly, the value of the Z* for He amounts to 1·807.

R. D. Hartree's ⁽⁵⁾ value resulting from his own method is 1·835 ; that of mine 1·815, when putting $Z^*=(2-S)^2$.

The screening number of the two K electrons S_{II} resulting from (12, L1.*s*) agrees with J. C. Slaters ⁽⁵⁾ value $\Delta Z=2{\cdot}085$. In this connection it may be emphasized that the screening numbers of the subsequent L electrons are not identical, but increase as we see from the data of Table III. A more detailed study of these matters is contemplated later on.

Evidently, by means of the "inter"-constants, it is possible to *convert the equation of any higher isoelectronic row to the chosen fundamental equation of the mono-iso-row* and to obtain in this way further information as to the screening power of the electronic shells. It seems to be more convenient and of greater interest to compare the isoelectronic rows in such a way that the *relations between the* $\mathfrak{J}$ *values of the electrons within* the same atom are elucidated. We calculate the "inter-row" potentials between the nth member of iso-row i and $(i-1+n)$th member of the fundamental mono-iso-row. The number of shiftings $\hat{s}$ of the mono-row needed for the comparison of the isoatomic terms

$$\hat{s}=i-1=Z-n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

is written as an index on the left below the symbols of the inter-constants.

Their values follow from the theory :

$${}_{\hat{s}}B_{I,\,i}=B_K-B_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$${}_{\hat{s}}A_{I,\,i}=A_I-A_i+\hat{s}\,.\,B_K=A_I(2\hat{s}+1)-A_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$${}_{\hat{s}}D_{I,\,i}=A_I\,.\,\hat{s}+\tfrac{1}{2}B_K\,.\,\hat{s}(\hat{s}-1)+D_I-D_i=A_I\,.\,\hat{s}^2+D_I-D_i. \quad . \quad (17)$$

Then the general converting formula becomes

$$\mathfrak{J}=A_I\,.\,Z^2-[{}_{\hat{s}}A_{I,\,i}(Z-\hat{s})+\tfrac{1}{2}B_{I,\,i}(Z-\hat{s})(Z-\hat{s}-1)+{}_{\hat{s}}D_{I,\,i}]. \quad . \quad (18)$$

Inserting the preceding expressions in it, we obtain

$$\mathfrak{J}_i=(A_I-\tfrac{1}{2}B_{I,\,i})(Z-\hat{s})(Z-\hat{s}-1)+A_i(Z-\hat{s})+D_i. \quad . \quad . \quad (18.2)$$

Let us calculate as an example the value of $\mathfrak{J}$ of L$p6\,.\,e^-$ in silicon: $Z=14$, $i=10$ (deka-iso-electronic row), $\hat{s}=9$.

$$\mathfrak{J}\,Si\;V=(13{\cdot}55-10{\cdot}11)\,.\,5\,.\,4+18{\cdot}80\,.\,5+2{\cdot}97=165{\cdot}97\,;$$

the empirical value is 165·66.

The illustrative calculations of the inter-constants with one and two shiftings are recorded in Table V.; these empirical values are in good agreement with the theoretical ones, as it is shown in Table VI. Besides, in this table theoretical and some empirical inter-constants of the K and L electrons are summarized. In the last two columns the sum of the inter-row constants (equation (16)) and of the inter-correction constants (equation (17)) of the same order of the shifting ($\hat{s}=1$ to 9) is recorded and expressed as whole multiples of values oscillating about the average value 29·48.

$${}_{\hat{s}}M_{I,\,i}=A_I\,.\,i^2-A_i-D_i+D_I. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

At the end of this communication the value of $\mathfrak{J}$ CoXVIII calculated by E. Tyrén (6), ca 14,000 eV, is compared with our figure 1395·6 calculated with individual quantum constant, or 1397·0 calculated with common quantum constant B_L.

The conjugation with subsequent disjugation is characterized by the decreasing individual values of the quantum constant. This phenomenon can be investigated with greater lucidity in the M isoelectronic rows than in any other iso-row of the L electrons, and therefore we prefer to discuss this problem in the next paper on the M electrons.

TABLE V.

Showing the inter-row potentials of one shifting occurring in the first three iso-rows. $_{1}\mathfrak{J}_{\mathrm{I, II}}$ and $_{1}\mathfrak{J}_{\mathrm{II, III}}$ and those of two shiftings $_{2}\mathfrak{J}_{\mathrm{I, III}}$.

Z	$\mathfrak{J}_{\mathrm{I}}$	$_{1}\mathfrak{J}_{\mathrm{I, II}}$	Δ	$\mathfrak{J}_{\mathrm{II}}$	$_{1}\mathfrak{J}_{\mathrm{II, III}}$	Δ	$\Delta\Delta$	$\mathfrak{J}_{\mathrm{III}}$	$_{2}\mathrm{I, III}$	Δ	$\Delta\Delta$
1	13·53	—	—	—	—	—	—	—	—	—	—
2	54·14	29·67	—	24·47	—	—	—	—	—	—	—
3	121·84	46·58	16·91	75·26	69·90	—	—	5·36	116·48	—	—
4	216·63	63·52	16·94	153·11	135·00	65·10	—	18·12	198·51	82·03	—
5	338·50	80·40	16·88	258·10	226·35	85·35	20·25	37·75	300·75	102·24	20·21
6	487·55	97·65	17·25	389·90	325·68	105·33	19·98	64·22	423·33	122·58	20·34
7	663·73	114·65	17·00	549·08	451·65	125·97	20·64	97·43	566·30	142·97	20·39
8	*c* 867·08	131·86	17·21	735·22	597·74	146·09	20·12	137·48	729·60	163·30	20·33
9	1097·66	—	—	—	—	—	—	184·26	913·40	183·80	20·50
10	1355·48	—	—	—	—	—	—	237·92	1117·56	204·16	20·36
11	*c*1640·10	—	—	—	—	—	—	298·39	1341·71	224·15	20·00
Mean value of the empirical values of inter-constants			17·03				20·25				20·30

c=calculated.

TABLE VI.

The inter-constants with increasing shifting. The real function of the M (equation (19)) of L electrons may be expressed quadratically as a parabola.

	Inter-row constants ${}_{\hat{s}}A_{\mathrm{I},\,i}=A_{\mathrm{I}}(2\hat{s}+1)-A_i$		Inter-correction constants ${}_{\hat{s}}D_{\mathrm{I},\,i}=A_{\mathrm{I}}\,.\,\hat{s}^2+D_{\mathrm{I}}-D_i$		${}_{\hat{s}}M_{\mathrm{I},\,i}={}_{\hat{s}}A_{\mathrm{I},\,i}+{}_{\hat{s}}D_{\mathrm{I},\,i}$ $=A_{\mathrm{I}}\,.\,i^2-A_i-D_i+D_{\mathrm{I}}$	
	Theor.	Emp.	Theor.	Emp.		
${}_1A_{\mathrm{I},\,\mathrm{II}}$	17·08	17·03	12·28	12·50	29·35	1. 29·36
${}_2A_{\mathrm{I},\,\mathrm{III}}$	62·19	61·8	54·15	54·52	116·34	4. 29·08
${}_3A_{\mathrm{I},\,\mathrm{IV}}$	86·28	86·07	120·82	121·01	207·29	7. 29·58
${}_4A_{\mathrm{I},\,\mathrm{V}}$	112·76	—	215·36	—	328·12	11. 29·83
${}_5A_{\mathrm{I},\,\mathrm{VI}}$	137·77	—	338·58	—	476·35	16. 29·77
${}_6A_{\mathrm{I},\,\mathrm{VII}}$	162·71	—	486·61	—	649·32	22. 29·51
${}_7A_{\mathrm{I},\,\mathrm{VIII}}$	188·96	—	664·01	—	852·97	29. 29·41
${}_8A_{\mathrm{I},\,\mathrm{IX}}$	213·85	—	866·37	—	1080·22	37. 29·20
${}_9A_{\mathrm{I},\,\mathrm{X}}$	238·76	—	1094·69	—	1333·45	45. 29·63 46. 28·94

Summary.

The ionization potentials $\mathfrak{J}$ of the K and L isoelectronic rows (mono- to deka-iso-rows) form arithmetical progressions of second degree. The equations basing upon this recognition make it possible to figure out the $\mathfrak{J}$ values of K and L electrons of any element Z in the given iso-row. There are three constants in the equations: the row constant A_i, the quantum constant B_i and the correction constant D_i, the theoretical significance of which has been explained.

The deductions from the theory are as follows:—The differences between $\mathfrak{J}$ of two corresponding members in two isoelectronic rows, so-called inter-row potentials, again form arithmetical progressions; these are the progressions of the first degree when the quantum constants of the two compared equations are identical, or, they are of the second degree when the quantum constants are different. By means of the inter-row potentials and of the inter-constants it is possible to convert any equation of the higher iso-row to the fundamental equation of the mono-iso-row. Further, when treating the isoatomic members of the two iso-rows inquired into, one can establish the relations between $\mathfrak{J}$ of the same atom.

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Caroline University, Prague,
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XXXIX. *A Note on M. de Broglie's Theory of the Photon.*

By Marjorie Williamson *.

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M. de Broglie, in developing his theory of the photon †, endeavours to establish an analogy between polarization and spin by employing the theory of Dirac, which is so satisfactory with respect to the spin of the electron.

Since interaction with matter results always in the disappearance of the photon, the theory must include the possibility of annihilation, and this leads M. de Broglie to suggest that the photon may be composed of two complementary particles, " demi-photons," corresponding to the positron and the electron in the Dirac theory. By giving each half photon a spin of $\frac{1}{2}\frac{h}{2\pi}$, as for an electron, the integral spin of the complete photon is obtained.

It seems possible, however, to present the theory rather more simply, considering the photon as a single particle of mass μ_0 which may be vanishingly small. Its behaviour can be described by the Dirac equation if the wave function ψ is regarded as a complete matrix with 16 components, and not merely 4 as in the case of the charged particle.

The Photon Equations.

de Broglie's equations are :—

For one " half-photon "

$$\frac{1}{c}\frac{\partial\phi_{ik}}{\partial t}=(A_1)_{iklm}\frac{\partial\phi_{lm}}{\partial x}+(A_2)_{iklm}\frac{\partial\phi_{lm}}{\partial y}+(A_3)_{iklm}\frac{\partial\phi_{lm}}{\partial z}+\frac{2\pi i}{h}\mu_0 c(A_4)_{iklm}\phi_{lm}, \quad . \quad . \quad . \quad . \quad (1)$$

and for the other

$$\frac{1}{c}\frac{\partial\phi_{ik}}{\partial t}=(B_1)_{iklm}\frac{\partial\phi_{lm}}{\partial x}+(B_2)_{iklm}\frac{\partial\phi_{lm}}{\partial y}+(B_3)_{iklm}\frac{\partial\phi_{lm}}{\partial z}+\frac{2\pi i}{h}\mu_0 c(B_4)_{iklm}\phi_{lm}, \quad . \quad . \quad . \quad . \quad (2)$$

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* Communicated by Prof. H. T. Flint.

† 'Une Nouvelle Théorie de la Lumière' (1940).

where
$$(\mathrm{A}_r)_{iklm}=(\alpha_r)_{il}\delta_{km}$$
$$(\mathrm{B}_r)_{iklm}=-(-1)^r(\alpha_r)_{km}\delta_{il},$$

the α's being the matrices of Dirac's electron theory

$$\alpha_1=\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix},\qquad \alpha_2=\begin{bmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{bmatrix},$$

$$\alpha_3=\begin{bmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{bmatrix},\qquad \alpha_4=\begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$

These equations may, therefore, be written

$$\frac{1}{c}\,\frac{\partial\phi_{ik}}{\partial t}=(\alpha_1)_{il}\frac{\partial\phi_{lk}}{\partial x}+(\alpha_2)_{il}\frac{\partial\phi_{lk}}{\partial y}+(\alpha_3)_{il}\frac{\partial\phi_{lk}}{\partial z}+\frac{2\pi i}{h}\mu_0c(\alpha_4)_{il}\phi_{lk}, \quad . \quad (3)$$

and

$$\frac{1}{c}\,\frac{\partial\phi_{ik}}{\partial t}=(\alpha_1)_{km}\frac{\partial\phi_{im}}{\partial x}+(\alpha_2)_{km}\frac{\partial\phi_{im}}{\partial y}+(\alpha_3)_{km}\frac{\partial\phi_{im}}{\partial z}-\frac{2\pi i}{h}\mu_0c(\alpha_4)_{km}\phi_{im}$$
$$=\frac{\partial\phi_{im}}{\partial x}(\widetilde{\alpha}_1)_{mk}-\frac{\partial\phi_{im}}{\partial y}(\widetilde{\alpha}_2)_{mk}+\frac{\partial\phi_{im}}{\partial z}(\widetilde{\alpha}_3)_{mk}-\frac{2\pi i}{h}\mu_0c\phi_{im}(\widetilde{\alpha}_4)_{mk}$$
$$=\frac{\partial\phi_{im}}{\partial x}(\alpha_1)_{mk}+\frac{\partial\phi_{im}}{\partial y}(\alpha_2)_{mk}+\frac{\partial\phi_{im}}{\partial z}(\alpha_3)_{mk}-\frac{2\pi i}{h}\mu_0c\phi_{im}(\alpha_4)_{mk}, \quad . \ . \ . \ . \quad (4)$$

where $\widetilde{\alpha}_{mn}=\alpha_{nm}$ and α_2 is antisymmetric, the other α's being symmetric.

The conjugate equations associated with (3) and (4) are :—

$$\frac{1}{c}\,\frac{\partial\phi^+}{\partial t}=\frac{\partial\phi^+}{\partial x}\alpha_1+\frac{\partial\phi^+}{\partial y}\alpha_2+\frac{\partial\phi^+}{\partial z}\alpha_3-\frac{2\pi i}{h}\mu_0c\phi^+\alpha_4 \quad . \ . \ . \quad (5)$$

and

$$\frac{1}{c}\,\frac{\partial\phi^+}{\partial t}=\alpha_1\frac{\partial\phi^+}{\partial x}+\alpha_2\frac{\partial\phi^+}{\partial y}+\alpha_3\frac{\partial\phi^+}{\partial z}+\frac{2\pi i}{h}\mu_0c\alpha_4\phi^+, \quad . \ . \ . \quad (6)$$

since the α's are Hermitian.

Now, by adding (3) and (6),

$$\frac{1}{c}\,\frac{\partial}{\partial t}(\phi+\phi^+)=\alpha_1\frac{\partial}{\partial x}(\phi+\phi^+)+\alpha_2\frac{\partial}{\partial y}(\phi+\phi^+)+\alpha_3\frac{\partial}{\partial z}(\phi+\phi^+)$$
$$+\frac{2\pi i}{h}\mu_0c\alpha_4(\phi+\phi^+). \quad . \ . \quad (7)$$

and by adding (4) and (5),

$$\frac{1}{c}\frac{\partial}{\partial t}(\phi+\phi^+)=\frac{\partial}{\partial x}(\phi+\phi^+)\alpha_1+\frac{\partial}{\partial y}(\phi+\phi^+)\alpha_2+\frac{\partial}{\partial z}(\phi+\phi^+)\alpha_3 - \frac{2\pi i}{h}\mu_0 c(\phi+\phi^+)\alpha_4, \quad . \quad . \quad (8)$$

and this is simply the conjugate equation of (7).

This suggests that $\psi=\phi+\phi^+$ might be considered as the wave function which represents the complete photon, and

$$\frac{1}{c}\frac{\partial\psi}{\partial t}=\alpha_1\frac{\partial\psi}{\partial x}+\alpha_2\frac{\partial\psi}{\partial y}+\alpha_3\frac{\partial\psi}{\partial z}+\frac{2\pi i}{h}\mu_0 c\alpha_4\psi \quad . \quad . \quad . \quad (9)$$

and its conjugate equation

$$\frac{1}{c}\frac{\partial\psi}{\partial t}=\frac{\partial\psi}{\partial x}\alpha_1+\frac{\partial\psi}{\partial y}\alpha_2+\frac{\partial\psi}{\partial z}\alpha_3-\frac{2\pi i}{h}\mu_0 c\psi\alpha_4 \quad . \quad . \quad . \quad (10)$$

as the photon equations.

ψ is then a Hermitian matrix.

The Field Quantities.

In order to obtain an expression for the electromagnetic field of the photon the operators α are associated with the vector potentials.

However, with this new interpretation of the wave function, it is no longer possible to obtain the value of an operator F by forming the product $\psi^+F\psi$ as in the theory of the electron, for such a product is a matrix still. Instead, the value associated with an operator F which represents a field quantity is defined as the inner product of two matrix quantities, *i. e.*, as

$$\overline{F}=F:\psi=F_{mn}\psi_{nm}=\psi:F.$$

The vector potential A of electromagnetic theory is represented by the operators

$$-K\alpha_1,\ -K\alpha_2,\ -K\alpha_3 \text{ and } +iKI, \quad . \quad . \quad . \quad . \quad (11)$$

where K is some constant, and I is the unit matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

i. e.,

$$\left.\begin{aligned} \overline{A}_x&=-K\alpha_1:\psi=-K(\psi_{41}+\psi_{32}+\psi_{23}+\psi_{14}),\\ \overline{A}_y&=-K\alpha_2:\psi=-Ki(\psi_{41}-\psi_{32}+\psi_{23}-\psi_{14}),\\ \overline{A}_z&=-K\alpha_3:\psi=-K(\psi_{31}-\psi_{42}+\psi_{13}-\psi_{24}),\\ \overline{V}&=-iA_4=+K(\psi_{11}+\psi_{22}+\psi_{33}+\psi_{44}). \end{aligned}\right\} \quad . \quad . \quad . \quad (12)$$

Then the field vectors E and H are derived from A in the usual way by means of the relations

$$H_{nm}=\frac{\partial A_m}{\partial x^n}-\frac{\partial A_n}{\partial x^m}, \quad \text{where } x^4=ict, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$H_{23}=H_x, \qquad H_{31}=H_y, \qquad H_{12}=H_z$$

and $$H_{14}=-iE_x, \qquad H_{24}=-iE_y, \qquad H_{34}=-iE_z,$$

so that the operator

$$\begin{aligned} E_x &= \frac{1}{c}\frac{\partial A_1}{\partial t}-i\frac{\partial A_4}{\partial x} \\ &= K\left(\frac{1}{c}\frac{\partial}{\partial t}\alpha_1-\frac{\partial}{\partial x}I\right), \quad . \quad . \quad . \quad . \quad . \quad (14) \end{aligned}$$

and the corresponding value of E_x will be

$$\overline{E}_x=E_x:\psi.$$

Now multiplying equation (9) by α_1 from the left, equation (10) by α_1 from the right, adding and taking the inner product,

$$\begin{aligned} 2\left(\frac{1}{c}\cdot\frac{\partial}{\partial t}\alpha_1-\frac{\partial}{\partial x}I\right):\psi &= \left(\alpha_1\alpha_2:\frac{\partial\psi}{\partial y}+\frac{\partial\psi}{\partial y}:\alpha_2\alpha_1\right) \\ &+\left(\alpha_1\alpha_3:\frac{\partial\psi}{\partial z}+\frac{\partial\psi}{\partial z}:\alpha_3\alpha_1\right)+\frac{2\pi i}{h}\mu_0 c(\alpha_1\alpha_4:\psi-\psi:\alpha_4\alpha_1). \end{aligned}$$

The first term of the R.H.S. $=(\alpha_1\alpha_2+\alpha_2\alpha_1)_{mn}\dfrac{\partial\psi_{nm}}{\partial y}$,

$$=0,$$

and similarly the second term $=0$.

The third term $=\dfrac{2\pi i}{h}\mu_0 c(\alpha_1\alpha_4-\alpha_4\alpha_1)_{mn}\psi_{nm}$

$$=2\frac{2\pi i}{h}\mu_0 c(\alpha_1\alpha_4)_{mn}\psi_{nm}.$$

Hence $$\left(\frac{1}{c}\cdot\frac{\partial}{\partial t}\alpha_1-\frac{\partial}{\partial x}I\right):\psi=\frac{2\pi i}{h}\mu_0 c(\alpha_1\alpha_4):\psi,$$

so that the value of E_x is

$$\left.\begin{aligned} \overline{E}_x &= K\frac{2\pi i}{h}\mu_0 c(\alpha_1\alpha_4):\psi. \\ \text{Similarly} \quad \overline{E}_y &= K\frac{2\pi i}{h}\mu_0 c(\alpha_2\alpha_4):\psi, \\ \overline{E}_z &= K\frac{2\pi i}{h}\mu_0 c(\alpha_3\alpha_4):\psi \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (15)$$

so that since

$$\alpha_1\alpha_4=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},\quad \alpha_2\alpha_4=\begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix},\quad \alpha_3\alpha_4=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\left.\begin{aligned} \overline{\mathrm{E}}_x&=\mathrm{K}\frac{2\pi i}{h}\mu_0 c(\psi_{41}+\psi_{32}-\psi_{23}-\psi_{14}), \\ \overline{\mathrm{E}}_y&=\mathrm{K}\frac{2\pi i}{h}\mu_0 ci(\psi_{41}-\psi_{32}-\psi_{23}+\psi_{14}), \\ \overline{\mathrm{E}}_z&=\mathrm{K}\frac{2\pi i}{h}\mu_0 c(\psi_{31}-\psi_{42}-\psi_{13}+\psi_{24}) \end{aligned}\right\} \quad . \quad . \quad . \quad (16)$$

The operator
$$\mathrm{H}_x=\frac{\partial \mathrm{A}_2}{\partial z}-\frac{\partial \mathrm{A}_3}{\partial y},$$

$$=\mathrm{K}\left(\frac{\partial}{\partial z}\alpha_2-\frac{\partial}{\partial y}\alpha_3\right). \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Multiplying equation (9) by $\alpha_2\alpha_3$ from the left

$$\frac{1}{c}\alpha_2\alpha_3\frac{\partial\psi}{\partial t}=\alpha_2\alpha_3\alpha_1\frac{\partial\psi}{\partial x}-\alpha_3\frac{\partial\psi}{\partial y}+\alpha_2\frac{\partial\psi}{\partial z}+\frac{2\pi i}{h}\mu_0 c\alpha_2\alpha_3\alpha_4\psi,$$

and multiplying equation (10) by $\alpha_2\alpha_3$ from the right

$$\frac{1}{c}\frac{\partial\psi}{\partial t}\alpha_2\alpha_3=\frac{\partial\psi}{\partial x}\alpha_1\alpha_2\alpha_3+\frac{\partial\psi}{\partial y}\alpha_3-\frac{\partial\psi}{\partial z}\alpha_2-\frac{2\pi i}{h}\mu_0 c\psi\alpha_4\alpha_2\alpha_3.$$

Subtracting, taking the inner products and re-arranging,

$$2\left(\frac{\partial}{\partial z}\alpha_2-\frac{\partial}{\partial y}\alpha_3\right):\psi=\frac{1}{c}\left(\alpha_2\alpha_3:\frac{\partial\psi}{\partial t}-\frac{\partial\psi}{\partial t}:\alpha_2\alpha_3\right)$$
$$-\left(\alpha_2\alpha_3\alpha_1:\frac{\partial\psi}{\partial x}-\frac{\partial\psi}{\partial x}:\alpha_1\alpha_2\alpha_3\right)-\frac{2\pi i}{h}\mu_0 c(\alpha_2\alpha_3\alpha_4:\psi+\psi:\alpha_4\alpha_2\alpha_3).$$

The first term on the R.H.S.=0.

Since $\alpha_2\alpha_3\alpha_1=\alpha_1\alpha_2\alpha_3$ the second term also $=0$, and since $\alpha_2\alpha_3\alpha_4=\alpha_4\alpha_2\alpha_3$ the third term$=-2\dfrac{2\pi i}{h}\mu_0 c(\alpha_2\alpha_3\alpha_4):\psi$.

Hence
$$\left(\frac{\partial}{\partial z}\alpha_2-\frac{\partial}{\partial y}\alpha_3\right):\psi=-\frac{2\pi i}{h}\mu_0 c(\alpha_2\alpha_3\alpha_4):\psi, \quad . \quad . \quad . \quad (18)$$

so that the value of H_x is :—

and similarly

$$\left.\begin{aligned}\overline{\mathrm{H}}_x&=-\mathrm{K}\frac{2\pi i}{h}\mu_0c(\alpha_2\alpha_3\alpha_4):\psi,\\ \overline{\mathrm{H}}_y&=-\mathrm{K}\frac{2\pi i}{h}\mu_0c(\alpha_3\alpha_1\alpha_4):\psi,\\ \overline{\mathrm{H}}_z&=-\mathrm{K}\frac{2\pi i}{h}\mu_0c(\alpha_1\alpha_2\alpha_4):\psi\end{aligned}\right\}\quad\ldots\ldots\quad(19)$$

so that since

$$\alpha_2\alpha_3\alpha_4=\begin{bmatrix}0&i&0&0\\ i&0&0&0\\ 0&0&0&-i\\ 0&0&-i&0\end{bmatrix},\qquad \alpha_3\alpha_1\alpha_4=\begin{bmatrix}0&-1&0&0\\ 1&0&0&0\\ 0&0&0&-1\\ 0&0&1&0\end{bmatrix},$$

$$\alpha_1\alpha_2\alpha_4=\begin{bmatrix}i&0&0&0\\ 0&-i&0&0\\ 0&0&-i&0\\ 0&0&0&i\end{bmatrix},$$

$$\left.\begin{aligned}\overline{\mathrm{H}}_x&=-\mathrm{K}\frac{2\pi i}{h}\mu_0ci(\psi_{21}+\psi_{12}-\psi_{43}-\psi_{34}),\\ \overline{\mathrm{H}}_y&=-\mathrm{K}\frac{2\pi i}{h}\mu_0c(\psi_{21}-\psi_{12}-\psi_{43}+\psi_{34}),\\ \overline{\mathrm{H}}_z&=-\mathrm{K}\frac{2\pi i}{h}\mu_0ci(\psi_{11}-\psi_{22}-\psi_{33}+\psi_{44})\end{aligned}\right\}\quad\ldots\quad(20)$$

The values of the electromagnetic field quantities A, E and H given by equations (12), (16) and (20) agree with those obtained by M. de Broglie *.

He finds, subsequently, from energy considerations, that the value of K is $\dfrac{h}{4\pi\sqrt{\mu_0}}$.

These equations are then all dimensionally correct if $|\psi|^2$ has the physical dimensions of a volume, *i. e.*, if ψ is normalized in the same way as the wave function of the electron, $|\psi|^2dv$ being of the nature of a probability.

The Hamiltonian Operator of the Photon.

This will be given by

$$\mathrm{H}=\frac{h}{2\pi i}\frac{\partial}{\partial t}.$$

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* 'Une Nouvelle Théorie de la Lumière,' p. 160.

As is usual in the theory, for reasons of symmetry, both the photon equation (9) and its conjugate equation (10) must be taken into account when expressing this operator in terms of the α's, and the Hamiltonian is obtained in the following way :—

Adding equations (9) and (10)

$$\left.\begin{aligned} 2\frac{1}{c}\frac{\partial\psi}{\partial t} &= \left(\alpha_1\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial x}\alpha_1\right)+\left(\alpha_2\frac{\partial\psi}{\partial y}+\frac{\partial\psi}{\partial y}\alpha_2\right) \\ &\quad+\left(\alpha_3\frac{\partial\psi}{\partial z}+\frac{\partial\psi}{\partial z}\alpha_3\right)+\frac{2\pi i}{h}\mu_0 c(\alpha_4\psi-\psi\alpha_4), \\ \text{so that} \\ \mathrm{H} &= \frac{hc}{2\pi i}\left\{\frac{\partial}{\partial x}\frac{\alpha_1 o+o\alpha_1}{2}+\frac{\partial}{\partial y}\frac{\alpha_2 o+o\alpha_2}{2}\right. \\ &\quad\left.+\frac{\partial}{\partial z}\frac{\alpha_3 o+o\alpha_3}{2}+\frac{2\pi i}{h}\mu_0 c\frac{\alpha_4 o-o\alpha_4}{2}\right\} \end{aligned}\right\}, \quad . \quad (21)$$

where o simply indicates the position of the operand.

The Angular Momentum Operators.

These are expressed in the usual way

$$\mathrm{M}_x=-\frac{h}{2\pi i}\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right), \text{ etc.,} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and as they do not involve the α's need not be written in the special form above.

As might be expected, these do not commute with H for, *e. g.*,

$$\left.\begin{aligned} \mathrm{HM}_x &= \frac{h^2}{4\pi^2}c\left\{\frac{\partial}{\partial x}\frac{\alpha_1\mathrm{M}_x o+o\mathrm{M}_x\alpha_1}{2}+\ldots\text{ etc.}\right\} \\ &= \frac{h^2}{4\pi^2}c\,\tfrac{1}{2}\left\{\left(\frac{\partial}{\partial x}\alpha_1+\frac{\partial}{\partial y}\alpha_2+\frac{\partial}{\partial z}\alpha_3+\frac{2\pi i}{h}\mu_0 c\alpha_4\right)\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)o\right. \\ &\quad\left.+o\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}\alpha_1+\frac{\partial}{\partial y}\alpha_2+\frac{\partial}{\partial z}\alpha_3-\frac{2\pi i}{h}\mu_0 c\alpha_4\right)\right\}, \\ \text{and} \\ \mathrm{M}_x\mathrm{H} &= \frac{h^2}{4\pi^2}c\left\{\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}\frac{\alpha_1 o+o\alpha_1}{2}+\frac{\partial}{\partial y}\frac{\alpha_2 o+o\alpha_2}{2}\right.\right. \\ &\quad\left.\left.+\frac{\partial}{\partial z}\frac{\alpha_3 o+o\alpha_3}{2}+\frac{2\pi i}{h}\mu_0 c\frac{\alpha_4 o-o\alpha_4}{2}\right)\right\}, \\ \text{so that} \\ \mathrm{HM}_x-\mathrm{M}_x\mathrm{H} &= \frac{h^2}{4\pi^2}c\left\{\frac{\partial}{\partial z}\frac{\alpha_2 o+o\alpha_2}{2}-\frac{\partial}{\partial y}\frac{\alpha_3 o+o\alpha_3}{2}\right\}\neq 0 \end{aligned}\right\}, \quad (23)$$

and therefore M_x is not a constant of motion.

The Spin of the Photon.

By analogy with the theory of the electron it is expected that (M+S) should be a constant of motion, where S represents spin angular momentum. Guided by this analogy, the spin operators S_x, S_y, S_z are associated with the Hermitian matrices

$$\frac{h}{2\pi}i\alpha_2\alpha_3, \quad \frac{h}{2\pi}i\alpha_3\alpha_1 \quad \text{and} \quad \frac{h}{2\pi}i\alpha_1\alpha_2$$

respectively.

Here also the operator must be expressed in a form similar to that of H,

i. e.,
$$S_x = \frac{h}{2\pi}i\frac{\alpha_2\alpha_3 o - o\alpha_2\alpha_3}{2}, \text{ , etc.} \quad . \; . \; . \; . \; . \; . \quad (24)$$

but with a negative sign, because the spin, in general, is an antisymmetric quantity.

Now

$$HS_x = \frac{h^2}{4\pi^2}c\left\{\tfrac{1}{2}\frac{\partial}{\partial x}\left(\alpha_1\frac{\alpha_2\alpha_3 o - o\alpha_2\alpha_3}{2} + \frac{\alpha_2\alpha_3 o - o\alpha_2\alpha_3}{2}\alpha_1\right) + \ldots, \text{etc.}\right\}$$

and

$$\begin{aligned}S_xH = \frac{h^2}{4\pi^2}c\,\tfrac{1}{2}\Big\{\alpha_2\alpha_3\Big(&\frac{\partial}{\partial x}\frac{\alpha_1 o + o\alpha_1}{2} + \frac{\partial}{\partial y}\frac{\alpha_2 o + o\alpha_2}{2} + \frac{\partial}{\partial z}\frac{\alpha_3 o + o\alpha_3}{2} \\ &+ \frac{2\pi i}{h}\mu_0 c\frac{\alpha_4 o - o\alpha_4}{2}\Big) - \Big(\frac{\partial}{\partial x}\frac{\alpha_1 o + o\alpha_1}{2} + \frac{\partial}{\partial y}\frac{\alpha_2 o + o\alpha_2}{2} \\ &+ \frac{\partial}{\partial z}\frac{\alpha_3 o + o\alpha_3}{2} + \frac{2\pi i}{h}\mu_0 c\frac{\alpha_4 o - o\alpha_4}{2}\Big)\alpha_2\alpha_3\Big\}.\end{aligned}$$

Subtracting, the terms involving $\frac{\partial}{\partial x}$

$$\begin{aligned}= \frac{h^2}{4\pi^2}c\,\tfrac{1}{4}\frac{\partial}{\partial x}\{&\alpha_1\alpha_2\alpha_3 o - \alpha_4 o\alpha_2\alpha_3 + \alpha_2\alpha_3 o\alpha_1 - o\alpha_2\alpha_3\alpha_1 - \alpha_2\alpha_3\alpha_1 o - \alpha_2\alpha_3 o\alpha_1 \\ &+ \alpha_1 o\alpha_2\alpha_3 + o\alpha_1\alpha_2\alpha_3\}\end{aligned}$$

$$=0,$$

the terms involving $\frac{\partial}{\partial y}$

$$\begin{aligned}= \frac{h^2}{4\pi^2}c\,\tfrac{1}{4}\frac{\partial}{\partial y}\{&\alpha_2\alpha_2\alpha_3 o - \alpha_2 o\alpha_2\alpha_3 + \alpha_2\alpha_3 o\alpha_2 - o\alpha_2\alpha_3\alpha_2 - \alpha_2\alpha_3\alpha_2 o - \alpha_2\alpha_3 o\alpha_2 \\ &+ \alpha_2 o\alpha_2\alpha_3 - o\alpha_2\alpha_2\alpha_3\}\end{aligned}$$

$$= \frac{h^2}{4\pi^2}c\frac{\partial}{\partial y}\frac{\alpha_3 o + o\alpha_3}{2},$$

and the terms involving $\frac{\partial}{\partial z}$

$$\begin{aligned}= \frac{h^2}{4\pi^2}c\,\tfrac{1}{4}\frac{\partial}{\partial z}\{&\alpha_3\alpha_2\alpha_3 o - \alpha_3 o\alpha_2\alpha_3 + \alpha_2\alpha_3 o\alpha_3 - o\alpha_2\alpha_3\alpha_3 - \alpha_2\alpha_3\alpha_3 o - \alpha_2\alpha_3 o\alpha_3 \\ &+ \alpha_3 o\alpha_2\alpha_3 + o\alpha_3\alpha_2\alpha_3\}\end{aligned}$$

$$= -\frac{h^2}{4\pi^2}c\frac{\partial}{\partial z}\frac{\alpha_2 o + o\alpha_2}{2}.$$

The terms remaining

$$=\frac{h}{2\pi}ic\,\tfrac{1}{4}\{\alpha_4\alpha_2\alpha_3 o-o\alpha_2\alpha_3\alpha_4-\alpha_2\alpha_3 o\alpha_4+o\alpha_2\alpha_3\alpha_4-\alpha_2\alpha_3\alpha_4 o+\alpha_2\alpha_3 o\alpha_4$$
$$+\alpha_4 o\alpha_2\alpha_3-o\alpha_4\alpha_2\alpha_3\}$$

$$=0.$$

Hence $$\mathrm{HS}_x-\mathrm{S}_x\mathrm{H}=\frac{h^2}{4\pi^2}c\left\{\frac{\partial}{\partial y}\frac{\alpha_3 o+o\alpha_3}{2}-\frac{\partial}{\partial z}\frac{\alpha_2 o+o\alpha_2}{2}\right\} \quad . \quad . \quad (25)$$

Combining this with equation (22)

$$\mathrm{H}(\mathrm{M}_x+\mathrm{S}_x)-(\mathrm{M}_x+\mathrm{S}_x)\mathrm{H}=0,$$

i. e., $(\mathrm{M}_x+\mathrm{S}_x)$ is a constant of motion.

This may be shown in a similar way for the y and z components.

Thus the photon, unlike the electron, has a spin angular motion which is an *integral* multiple of $\frac{h}{2\pi}$.

The Spin of the Photon.

This is a quantity of the nature of a density, and to make it dimensionally correct, the spin operator must be multiplied on either side by ψ.

The electron theory would suggest forming a product $\psi^+\mathrm{S}\psi$, but for the photon this is still a matrix. To obtain a scalar quantity the product $(\psi^+\,.\,\mathrm{S}):\psi$ may be taken, or equally well the product $\psi^+:(\mathrm{S}\,.\,\psi)$.

Since ψ is Hermitian these are equal for

$$(\psi\,.\,\mathrm{S}):\psi=(\psi\,.\,\mathrm{S})_{mn}\psi_{nm}$$
$$=\psi_{ml}\mathrm{S}_{ln}\psi_{nm}$$
$$=\psi_{ml}(\mathrm{S}\,.\,\psi)_{lm}$$
$$=\psi:(\mathrm{S}\,.\,\psi),$$

so that we may take the spin density to be $\psi\,.\,\mathrm{S}:\psi$.

The values of the components of the spin density are

$$\overline{\mathrm{S}}_x=\psi\,.\,\mathrm{S}_x:\psi=\frac{h}{2\pi}\psi_{mn}(i\alpha_2\alpha_3)_{nl}\psi_{lm},$$

$$\overline{\mathrm{S}}_y=\psi\,.\,\mathrm{S}_y:\psi=\frac{h}{2\pi}\psi_{mn}(i\alpha_3\alpha_1)_{nl}\psi_{lm},$$

and

$$\overline{\mathrm{S}}_z=\psi\,.\,\mathrm{S}_z:\psi=\frac{h}{2\pi}\psi_{mn}(i\alpha_1\alpha_2)_{nl}\psi_{lm}.$$

The Magnetic Moment.

The form of equation (25) is of particular interest for, according to the usual convention,

$$\frac{d}{dt}\mathrm{S}_x = \frac{2\pi i}{h}(\mathrm{HS}_x - \mathrm{S}_x\mathrm{H})$$

$$= \frac{h}{2\pi} ic\left\{\frac{\partial}{\partial y}\,\frac{\alpha_3 o + o\alpha_3}{2} - \frac{\partial}{\partial z}\,\frac{\alpha_2 o + o\alpha_2}{2}\right\}.$$

This is the operator corresponding to the rate of change of the x-component of the spin angular momentum.

The value of this will be $\psi \,.\, \dot{\mathrm{S}} : \psi$, *i. e.*, a double scalar multiplication is involved and so, since $\alpha_3 : \psi = \psi : \alpha_3$, etc.,

$$\dot{\mathrm{S}}_x = \frac{h}{2\pi} ic\left(\frac{\partial}{\partial y}\alpha_3 - \frac{\partial}{\partial z}\alpha_2\right)$$

$$= -\mu_0 c^2 \alpha_2\alpha_3\alpha_4 \text{ from equation (18)}$$

$$= \tfrac{1}{2}\mu_0 c^2(\alpha_3\alpha_1 \,.\, \alpha_1\alpha_2\alpha_4 - \alpha_1\alpha_2 \,.\, \alpha_3\alpha_1\alpha_4)$$

$$= -\frac{1}{2\mathrm{K}}\,\frac{hc}{2\pi i}\,(\alpha_3\alpha_1 \,.\, \mathrm{H}_z - \alpha_1\alpha_2 \,.\, \mathrm{H}_y) \text{ from equation (19).}$$

Now let

$$\mu_x = -\frac{1}{2\mathrm{K}}\,\frac{hc}{2\pi i}\,\alpha_2\alpha_3$$

$$= \sqrt{\mu_0}\, ci\alpha_2\alpha_3,$$

$$\mu_y = \sqrt{\mu_0}\, ci\alpha_3\alpha_1$$

and

$$\mu_z = \sqrt{\mu_0}\, ci\alpha_1\alpha_2.$$

Then

$$\dot{\mathrm{S}}_x = \mu_y\mathrm{H}_z - \mu_z\mathrm{H}_y.$$

There will be similar expressions for $\dot{\mathrm{S}}_y$ and $\dot{\mathrm{S}}_z$, and combining all three

$$\dot{\mathrm{S}} = \mu \times \mathrm{H},$$

where H is the magnetic field associated with the photon.

Thus the operator representing the rate of change of spin angular momentum is equal to the vector product of the quantity μ and the magnetic field operator.

This would seem to suggest that there is something of the nature of a magnetic moment associated with the photon, this magnetic moment being represented by the operator whose componenents are

$$\sqrt{\mu_0}\, ci\alpha_2\alpha_3,\ \sqrt{\mu_0}\, ci\alpha_3\alpha_1 \text{ and } \sqrt{\mu_0}\, ci\alpha_1\alpha_2.$$

The actual value of $\dot{\mathrm{S}}$, since it is of the nature of a density, will be

$$\bar{\dot{\mathrm{S}}} = \psi \,.\, \dot{\mathrm{S}} : \psi = \psi \,.\, (\mu \times \mathrm{H}) : \psi.$$

The actual value of the field quantity H is $\mathrm{H} : \psi$, and therefore it appears that to obtain a value for the quantity μ the operator must be multiplied by ψ.

Single scalar multiplication would give a matrix quantity, so it is suggested that the value of the magnetic moment may be given by

$$\bar{\mu}=\mu : \psi.$$

Then, for instance,

$$\bar{\mu}_x=\sqrt{\mu_0}ci\alpha_2\alpha_3 : \psi.$$

This has the dimensions of a magnetic moment divided by a volume, so that μ must represent the density of the magnetic moment associated with the photon. The operators μ_x, μ_y, μ_z are Hermitian, so that μ will represent a real quantity. However, according to de Broglie, the photon mass μ_0 is a very small quantity, so that the magnetic moment of the photon will be vanishingly small.

Finally, I should like to thank Professor Flint for many helpful discussions upon this subject.

XL. *A Feature of Milne's Universe.*

By W. Gleissberg*.

[Received December 23, 1946.]

In this paper I prove that in Milne's universe the following theorem holds good, if the local density is 5×10^{-28} gm.cm.$^{-3}$:—

Theorem.—The sum of potential and kinetic energy as computed by any observer for all particles in his neighbourhood is zero.

The neighbourhood of any observer may be represented by a sphere of radius R, mass M and density ρ, where R is small in comparison with the radius of the universe. Since, according to Milne †, particles can be considered as homogeneously distributed in the neighbourhood of any observer, ρ is constant within the sphere of radius R. Moreover, Newtonian mechanics can be applied within this sphere, since in the neighbourhood of any observer speeds (relative to him) are much less than the velocity of light. Accordingly, the potential energy U of the sphere in consideration is given by the well-known formula of classical mechanics :

$$U=-\frac{3GM^2}{5R},$$

G being the constant of gravitation. Introducing here the density ρ instead of mass M, from

$$M=\frac{4}{3}\pi\rho R^3,$$

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* Communicated by Prof. E. A. Milne, F.R.S.

† E. A. Milne, 'Relativity, Gravitation, and World-Structure,' p. 105 (Oxford 1935).

we obtain

$$U=-\frac{16}{15}G\pi^2\rho^2R^5. \qquad \ldots \ldots \quad (1)$$

The kinetic energy of a spherical shell of radius r, thickness dr and density ρ is $2\pi\rho v^2r^2dr$, where v denotes the speed of recession at distance r. Thus, the kinetic energy T of the whole sphere of radius R is given by

$$T=2\pi\rho\int_0^R v^2r^2dr. \qquad \ldots \ldots \quad (2)$$

According to Milne *

$$v=\frac{r}{t}, \qquad \ldots \ldots \quad (3)$$

where t denotes the age of the universe. Substituting (3) in (2) we get

$$T=\frac{2\pi\rho}{t^2}\int_0^R r^4dr\,;$$

integration yields

$$T=\frac{2\pi\rho R^5}{5t^2}. \qquad \ldots \ldots \quad (4)$$

Adding (4) to (1) we obtain

$$T+U=\frac{2\pi\rho R^5}{5t^2}\left(1-\frac{8}{3}G\pi\rho t^2\right). \qquad \ldots \ldots \quad (5)$$

By hypothesis, $\rho=5\times10^{-28}$ gm. cm.$^{-3}$, and, according to Milne *,

$$t=2\times10^9 \text{ years}=6\times10^{16} \text{ sec.};$$

furthermore, we have $G=7\times10^{-8}$ cm.^{3} gm.$^{-1}$ sec.$^{-2}$.

With these numerical values we find (very approximately)

$$\frac{3}{8}G\pi\rho t^2=1. \qquad \ldots \ldots \quad (6)$$

Hence, (5) reduces to

$$T+U=0. \qquad \ldots \ldots \quad (7)$$

This proves the theorem.

The above calculation was made only for the present age of the universe. That the theorem as proved above will remain valid also for other values of t, can easily be shown as follows :—Since all distances increase proportionally with t, density ρ decreases inversely proportionally with t^3. On the other hand G, the constant of gravitation, increases proportionally with t in Milne's universe †. Thus, the product $G\rho t^2$ is independent of t. Hence, it follows that (6), and therefore (7), remain unchanged for ever.

University Observatory,
Bayazit, Istanbul, Turkey.

November, 1946.

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* *Op. cit.*, p. 76.

† *Op. cit.*, p. 292.

XLI. *A Form of Laplace's Equation.*

By A. C. BARTLETT *.

[Received January 22, 1947].

THE purpose of this note is to put forward a form of Laplace's equation which, if not new, seems little known.

If we have a set of surfaces $V(x, y, z)=\text{const.}$, then it is known that at any point x, y, z, the sum of the principal curvatures of the surface through that point is

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{2\Sigma\frac{\partial V}{\partial x}\cdot\frac{\partial V}{\partial y}\cdot\frac{\partial^2 V}{\partial x\partial y}-\Sigma\left(\frac{\partial V}{\partial x}\right)^2\left(\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}\right)}{K^3},$$

where

$$K=\sqrt{\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2}.$$

If, now, $\frac{\partial}{\partial n}$ represents differentiation in the direction of the normal,

i. e.,

$$\frac{\partial}{\partial n}=\frac{1}{K}\left\{\frac{\partial V}{\partial x}\cdot\frac{\partial}{\partial x}+\frac{\partial V}{\partial y}\cdot\frac{\partial}{\partial y}+\frac{\partial V}{\partial z}\cdot\frac{\partial}{\partial z}\right\},$$

it at once follows that

$$K=\frac{\partial V}{\partial n}.$$

With a little manipulation it further follows that

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}-\frac{\partial^2 V}{\partial n^2}}{\frac{\partial V}{\partial n}}.$$

Hence

$$\nabla^2 V=\frac{\partial^2 V}{\partial n^2}+\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\frac{\partial V}{\partial n},$$

and we have

$$\frac{\partial^2 V}{\partial n^2}+\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\frac{\partial V}{\partial n}=0$$

as an alternative form of Laplace's equation.

This intrinsic form of the equation has one immediate application. Consider the case when the surfaces V are parallel planes and r is the distance of a plane from some origin; both the curvatures are zero and Laplace's equation becomes

$$\frac{\partial^2 V}{\partial r^2}=0.$$

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* Communicated by the Author.

Next consider the symmetrical cylindrical case ; if r is the radius of a cylinder then one of the curvatures is zero and the other is $1/r$.

Laplace's equation is thus :—

$$\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}=0.$$

Finally, in the symmetrical spherical case both curvatures are equal to $1/r$, and therefore

$$\frac{\partial^2 V}{\partial r^2}+\frac{2}{r}\frac{\partial V}{\partial r}=0.$$

Thus a simple explanation is given of why, in the cartesian, cylindrical and spherical forms of Laplace's (and of course, the wave) equation, the coefficients of the first differential-coefficient are 0, $1/r$ and $2/r$ respectively.

XLII. *Notices respecting New Books.*

Five-Figure Tables of Natural Trigonometrical Functions. Prepared by H.M. Nautical Almanac Office. [Pp. iv+123.] (London: His Majesty's Stationery Office, 1947.) Price 15*s.* 0*d.* net.

This is a reprint of a table issued, but not put on sale to the public, in 1941. At that time it was prepared for use of the War Office, with a view to facilitating survey calculations. It gives the sine, cosine, tangent and cotangent of angles throughout the quadrant at intervals of 10″ of arc, each value being correct to five significant figures. To make interpolation easy, there is an auxiliary table of cotangents at intervals of 1″ up to 7° 30′ and therefore, of tangents from 82° 30′ to 90°, so that every angle is well defined if the tangent is given.

There are many tables of the trigonometric functions to five figures, and the merit claimed for this set over its precursors is that owing to the fineness of the tabulation, there can never be difficulty in interpolation. This is, of course, true, but it may be noted that there are already six-figure tables at the same interval, and that to extract five-figure values from them requires no more interpolation than to use five-figure tables directly. Nevertheless, it may be conceded that a five-figure set with one page to each degree is more convenient to use if its accuracy can be relied on.

On the question of accuracy there is little means of judging. The Introduction does not describe how the copy was prepared, except to make clear that the tables are not the result of a new calculation. All that the reader is told is that the proofs were checked against Andoyer's and some other, unspecified, tables. (Andoyer gives 14 decimals, at the same interval as the book under review.) Nothing is said about checking by differences or otherwise, and confidence is perhaps a little undermined when we learn that the first edition contained two numerical errors not corrected in a reprint which was issued in 1945, and that in the new, published edition three misprints have been noticed as it passed through the press, one at least of which might lead to numerical errors by a computer. The printing is satisfactory. Long-tailed figures are used, of rather small size to meet the need for 60 lines to a page, but some broken type appears. It seems that the type has not yet been put on stereo plates, so that in future editions there is the possibility of errors creeping in which were not present in the first. J. H. Awbery.

Methods of Algebraic Geometry. By W. V. D. HODGE and D. PEDOE. [Pp. viii+440.] (Cambridge University Press, 1947.) Price 30*s*. 0*d*.

IN this work the algebraic method in algebraic geometry is heavily emphasized. For this purpose, Book I., about half of the present volume, is devoted entirely to algebra, and in it we find a carefully presented account of such topics as rings and fields, linear algebra, matrices, algebraic equations and the theory of resultant systems. Much of the material is already available in German texts, notably Van der Waerden's *Moderne Algebra*, but this is the first account in English, and there is some new material too ; for example, the theory of linear dependence over a non-commutative field is developed and the algebraic theory of Jacobians is presented.

Book II. is concerned with the properties of a projective space. These are developed firstly from the algebraic definition and then from the synthetic definition. The Grassmannian coordinates of a sub-space are described in a separate chapter, and the treatment is more comprehensive than any that has previously appeared. The book concludes with purely algebraic accounts of collineations and correlations.

The present work is not a textbook. No exercises are given and, indeed, the authors have rigidly maintained their principle of treating only general methods. But the book will be of value to those students contemplating a course of research in algebraic geometry, using algebraic methods. L. S. G.

Applied Bessel Functions. By Professor F. E. RELTON. [Pp. 191.] (Blackie.) Price 17*s*. 6*d*.

BASED on lectures given to graduates in physics, soil mechanics and engineering, this book provides an admirable introduction to Bessel functions and their applications to practical problems.

The first two chapters deal with the error function, beta and gamma functions and linear differential equations, a knowledge of equations with constant coefficients being assumed. An interesting feature here is the discussion of the zeros of a solution of an equation of the second order. Then the cylinder functions are defined by the recurrence relations

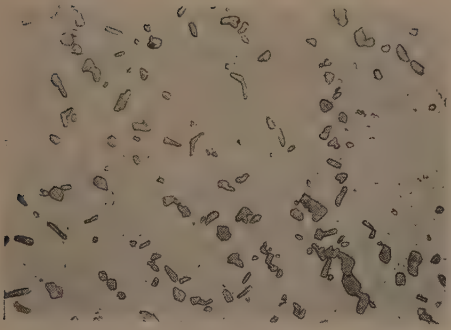
$$C_{n-1}+C_{n+1}=(2nC_n)/x, \qquad C_{n-1}-C_{n+1}=2C_n',$$

whence Bessel's equation is derived and zeros of its solutions are discussed. This unusual procedure has the moral support of E. B. Wilson and is certainly justified by the result in this book. The series solution is then obtained and J_n defined. Chapter V gives applications to the lengthening pendulum, vibrations of a string, instability of the deep cantilever, etc. Next, the second solution for integral orders is obtained and applied to the longitudinal vibrations of a tapered bar, the vibrating membrane, etc. Functions with imaginary argument are considered and then come applications to hydrodynamics, heat-conduction and elasticity. Chapter IX deals with Bessel coefficients, the integrals of Bessel, Poisson, Lipschitz and Sonine, and the Fourier-Bessel and Schlömilch expansions. Allied functions, including ber and bei, solutions by definite integrals, and asymptotic expansions appear in the closing chapter.

There are numerous worked examples as well as exercises for the reader. The style is lively and stimulating. Without using contour integration the author has written a valuable introduction which may be confidently recommended to all who are interested in Bessel functions and not only to those whose main interest is in their applications. S. L. GREEN.

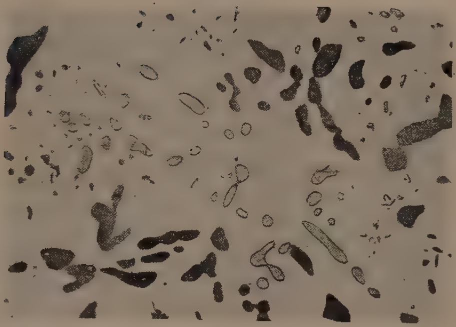
[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

Fig. 9.



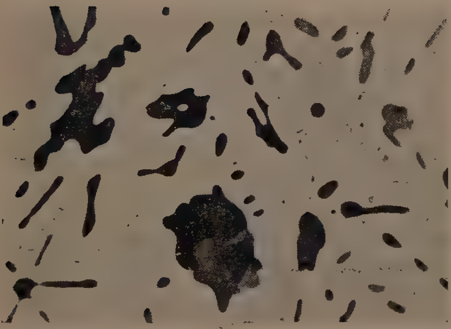
Alloy 4·0/6·0 $\alpha+MnAl_6$. ×500

Fig. 10.



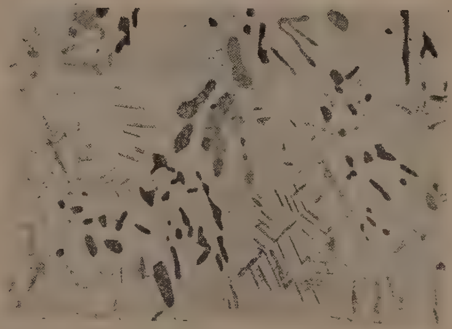
Alloy 3·75/26·5 $\alpha+MnAl_6+T_1$. ×500

Fig. 11.



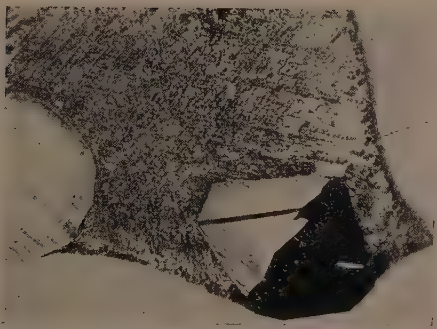
Alloy 3·60/29·5 $\alpha+T_1$. ×500

Fig. 12.



Alloy 3·42/33·0 $\alpha+T_1+MnAl_4$. ×500

Fig. 18.



Alloy 2·0/32 annealed 26 days at 528°C. showing α, chilled liquid, and large crystals of T_1, one being twinned.

XLIII. *Excitation of the Balmer Series Lines in Hydrogen Spark Channels.*

By J. D. Craggs, G. C. Williams and W. Hopwood,
Research Department, Metropolitan-Vickers Elec. Co. Ltd., Manchester *.

[Received October 17, 1947.]

Summary.

Experiments on the determination of the Balmer series decrements, *i. e.* the intensity ratios for certain Balmer lines, for short sparks in hydrogen at 1 atmosphere pressure, are discussed. From the results, deductions concerning the excitation conditions in the spark channels are made and it is shown that, whilst the results are complex, a pure recombination or thermal excitation mechanism is unlikely to explain them and that electron excitation appears to be the operative mechanism for the conditions studied.

1. *Introduction.*

Previous papers have described various experiments made with square current pulse sparks such as, for example, a preliminary study of the afterglows at high pressures in various gases (Craggs and Meek, 1946) and measurements of ion concentrations in spark channels in hydrogen (Craggs and Hopwood, 1947 *a*). Attempts are being made in various directions to measure electron, ion, gas and excitation temperatures in sparks, using techniques whose applications to the parallel study of arc discharges are well known (Ornstein and Brinkman, 1931, etc.). Part of this programme was carried out on hydrogen sparks to determine, if possible, the mode of excitation of the Balmer lines (H_α, H_β and H_γ particularly) in sparks and in the subsequent afterglows. The measurements are described in this paper, and were made by observation of the light emitted from the spark, using a photo-multiplier tube, amplifiers and cathode-ray oscillograph.

It is hoped in a later paper to connect the results with electrical and other measurements made on the sparks, as has been done by Flowers (1943) and other workers.

Very little quantitative work appears to have been carried out on the spectral properties of sparks; notable exceptions are the work of Langstroth and McRae (1938 *a* and *b*). A great many spectral measurements of an exact nature have been made on arcs, but the widespread use of spark discharges for spectrographic analysis has resulted in little of fundamental value.

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* Communicated by Professor Sir George Thomson, F.R.S.

2. *Experimental Techniques.*

As in previous experiments on hydrogen spark discharges (Craggs and Hopwood, 1947 *a* and *b*), the sparks were passed between pointed tungsten electrodes of the order of 5 mm. separation in hydrogen at a pressure of 10 cm. Hg above atmospheric pressure. The circuit used for the generation of the square current pulses applied to the test spark-gap was

Fig. 1*a*.

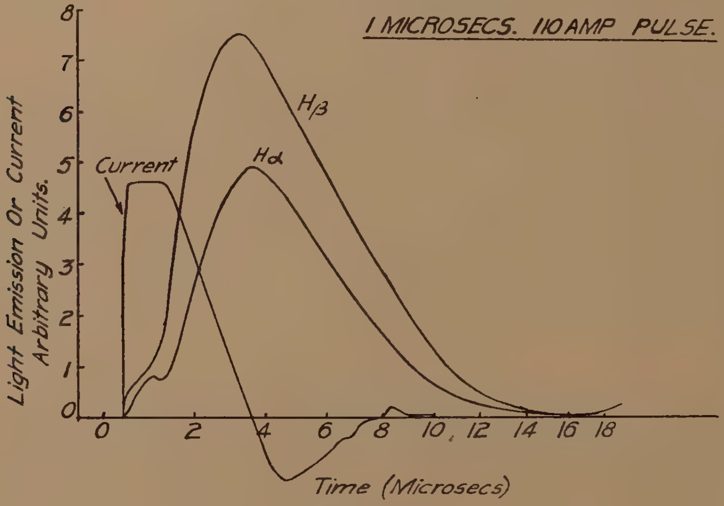
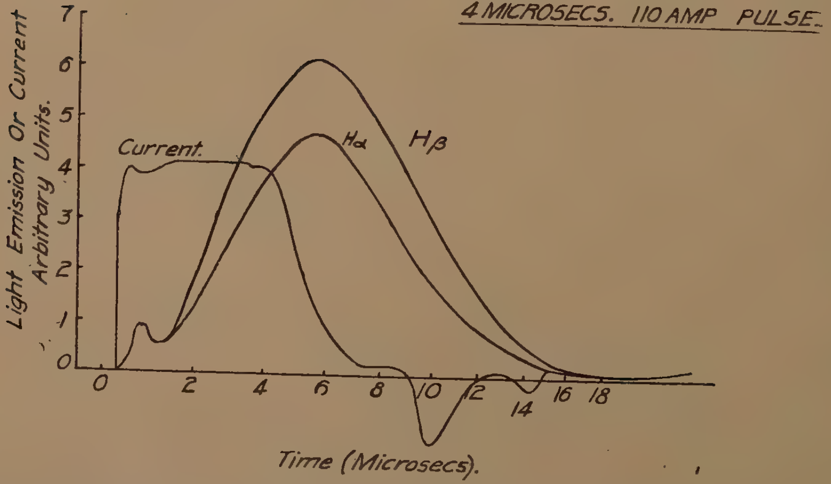


Fig. 1*b*.



briefly described in previous publications by Craggs and Hopwood (*loc. cit.*) and in detail by Craggs, Haine and Meek (1946). Current pulses of from 1–6 microseconds duration of 110 amp. peak current and 1–4 microseconds duration of 170 amp. peak current were applied to the spark-gap. Typical records of the 4 microseconds 110 amp. peak current and of the 1 microsecond 170 amp. peak current are shown in figs. 1*b* and 1*a* respectively. The small peaks at the beginning of the light emission/time oscillograms are due to electrical pick-up; it is easy to show that such disturbances are due to the tripping circuits and do not affect the oscillograms beyond $t=2$ microseconds.

Fig. 1*c*.

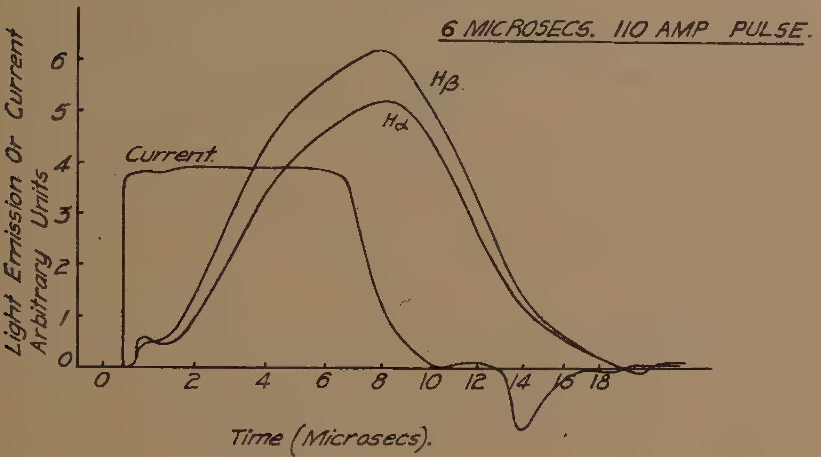
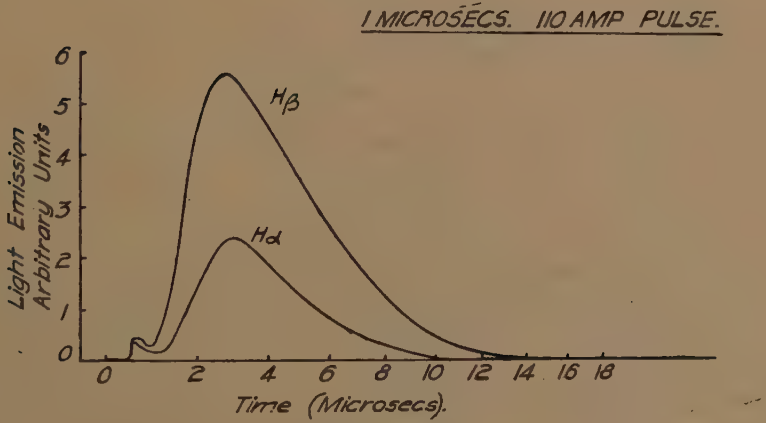


Fig. 1*d*.



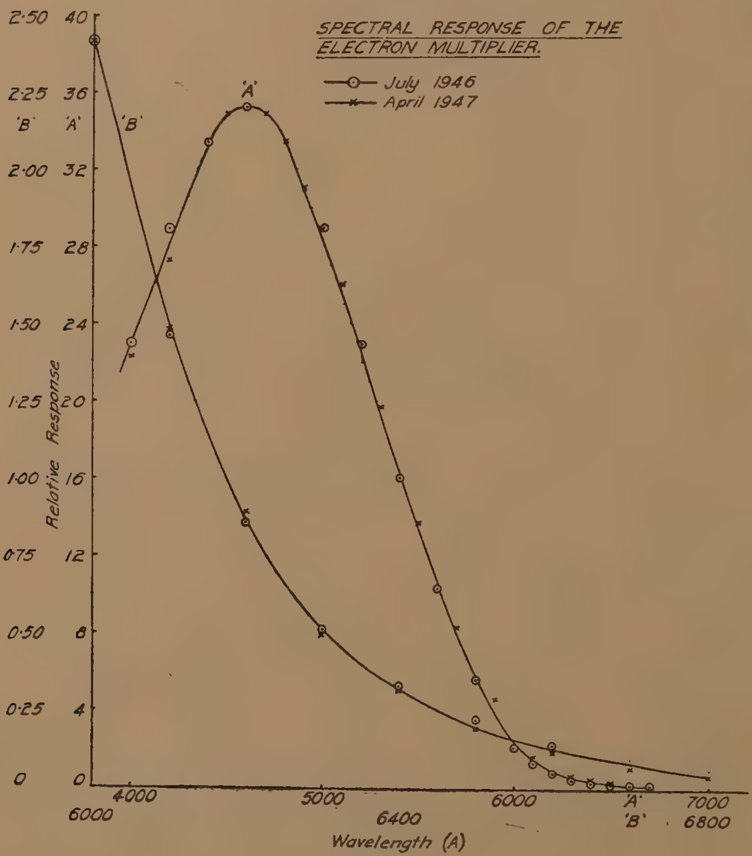
Figs. 1 (*a*–*d*).

Light emission/time and current/time oscillograms for the hydrogen lines H_α and H_β (6563 A and 4861 A) for various spark currents and times of duration, obtained with spectrometer and multiplier tube.

The relative intensities of H_α, H_β and H_γ for various pulse lengths were determined by means of the electron multiplier technique, and the relative intensity of H_α and H_β confirmed, using Wratten filters in conjunction with the electron multiplier and cathode-ray oscillograph.

The observation of the intensities of spectral lines with electron multiplier tubes has been in a previous publication (*e. g.* Williams, Craggs and Hopwood, 1947). The colour response of the multiplier was measured in a subsidiary series of experiments and found to be exceedingly low in the spectral region beyond $\lambda=6000$ A (see curve "A," fig. 2). In the determination of the Balmer decrements an accurate evaluation of the tube sensitivity to H_α was essential, and a more accurate plot of the relative spectral response, suitably corrected for the spectrometer dispersion and the emission characteristics of the standard tungsten filament lamp, for the region $\lambda=6000$ A to $\lambda=6800$ A, was obtained by increasing the exit slit width (curve "B," fig. 2). A curve showing the relative spectral response of the multipliers, which has remained sensibly constant over a period of approximately 10 months, is given in fig. 2.

Fig. 2.



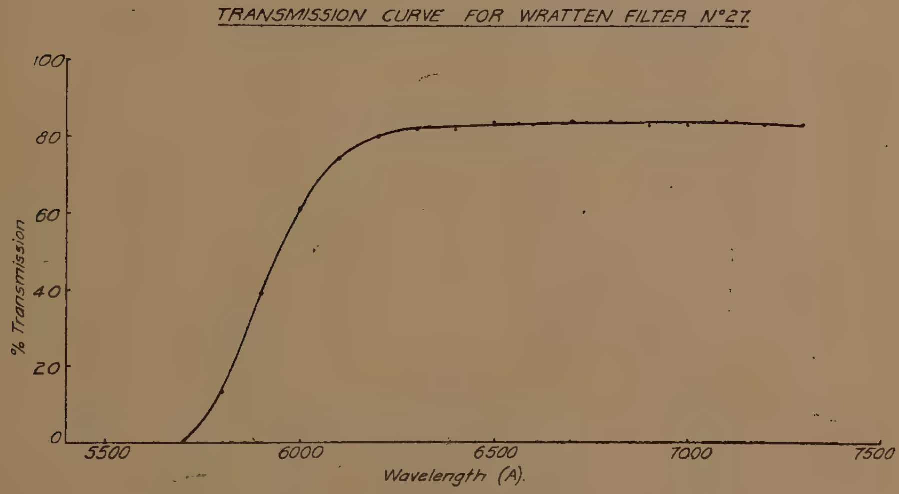
The optical system used in this series of experiments was briefly described by Hopwood and Craggs (1947) and Williams, Craggs and

Hopwood (*loc. cit.*) and has the advantage that only an area equal to that of the image of the spark source on the collimator lens is effective in the illumination of each point of the spectral line. As the image of the spark is smaller than the whole area of the lens, the lens is in effect stopped down and undesirable aberrations minimized.

With a discharge between pointed tungsten electrodes in hydrogen no large variations in light intensity from spark to spark were observed as in the case of the observations carried out on metal lines in air at atmospheric pressure using electrodes of Cd, Ba, Mg, etc. (Williams, Craggs and Hopwood, *loc. cit.*).

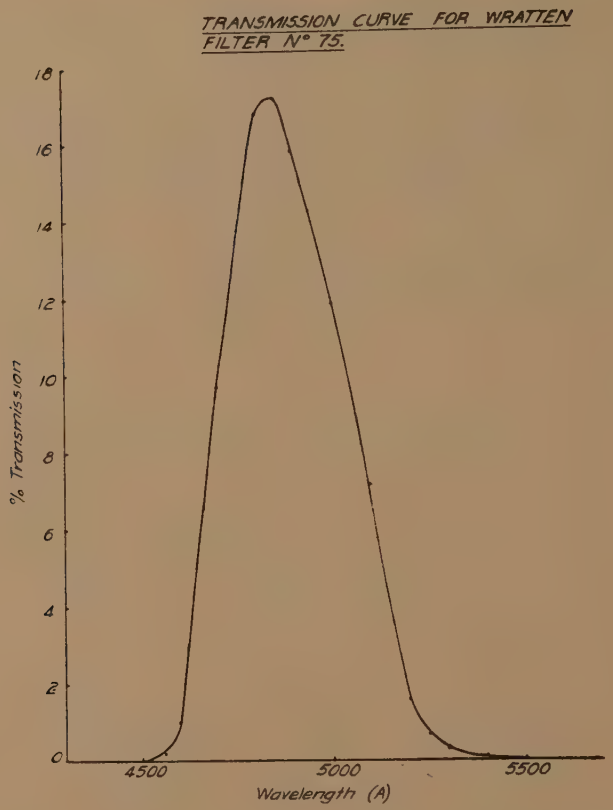
The use of various filters for the isolation of certain spectral regions and the determination of the light intensity-time variation for those regions during the discharge was mentioned by Craggs and Meek (1946) and has, of course, been used in many previous investigations such as those cited by Llewellyn Jones (1936).

Fig. 3.



As the background between the various components of the Balmer series was negligible (see fig. 4, Craggs and Hopwood, 1947 *a*), it was possible to isolate H_α and H_β by inserting Wratten filters Nos. 27 and 75 in turn between the spark source and the multiplier window. In order to obtain a value for the relative intensity of the components an accurate determination of the spectral transmission curves was required. The calibration of the above mentioned filters along with many others was undertaken by the Research Laboratories of the British Thomson-Houston Co., Rugby, with the aid of a Hilger-Müller quartz double monochromator. Spectral transmission curve; for Wratten filters Nos. 27 and 75 are shown in figs. 3 and 4 respectively.

Fig. 4.

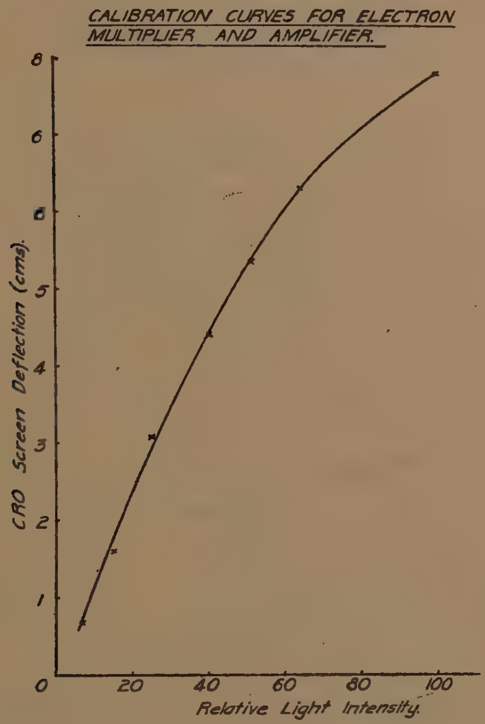


3. *Experimental Results.*

Before discussing the experimental results it is of interest to note that determinations of the Balmer decrement by the photographic-photometric method proved unsuccessful due to the spectral variations of the intermittency effect (Hopwood and Craggs, 1947). The line widths, which are very great owing to the strong inter-atomic Stark effect, were, however, obtained for H_α, H_β and H_γ by the photographic method, and those for H_β and H_γ confirmed by means of the electron multiplier technique (Craggs and Hopwood, 1947 *a*) in order to determine the ion concentrations in the spark channels. As the spectral response of the electron multiplier was exceedingly low in the region $\lambda=6000$ to $\lambda=6800$ A, no reliable profile for the narrow H_α was obtained.

In the present determinations of the Balmer decrements the telescope slit was opened so that the whole line profile was transmitted on to the multiplier cathode in each case, and because of the great line widths mentioned above, wide telescope slits were necessary. A run of photographs was taken for H_α, H_β and H_γ in turn, keeping the collimator slit width constant while varying the telescope slit width. Above a certain critical value no further increase in the magnitude of the light intensity-time diagram was observed. The final telescope slit width selected was that for which H_α, H_β and H_γ were transmitted in turn on to the multiplier cathode. For a collimator slit width of about 125 microns, H_β and H_γ were respectively fully transmitted without appreciable contributions from background through telescope slits 1000 and 1250 microns wide. H_α is, of course, narrower than H_β. In the experiments, therefore, a telescope slit width of 1250 microns was generally used; many check tests were made throughout the course of the work to ensure that the whole line widths were transmitted and that the backgrounds for H_α and H_β were negligible.

Fig. 5.



Typical light intensity-time oscillograms for H_α and H_β for various pulse lengths are reproduced in fig. 1.

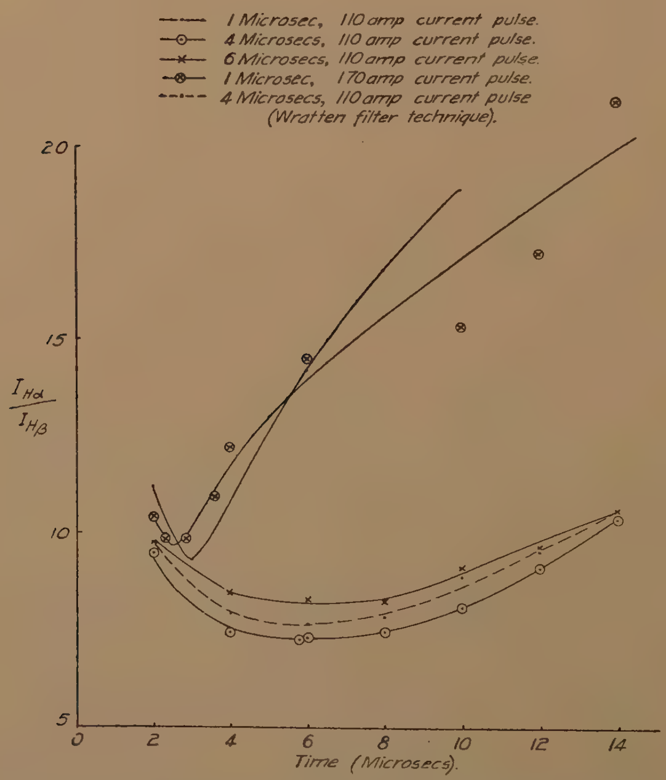
The procedure adopted in the evaluation of the decrements from the oscillograms consisted of the measurement of the peak light intensity and correction for demagnification of the camera and the non-linearity of the electron multiplier and amplifier system. (See fig. 5, which was taken with hydrogen sparks and attenuating gauzes.) The variation of decrements with pulse length is tabulated in Table I.

From Table I. it is seen that the H_α/H_β decrement is very sensitive to changes in the duration of the current pulse but very insensitive to a

TABLE I.

Pulse length microsecs.	Peak current 110 amps.		Peak current 170 amps.	
	$\frac{H_\alpha}{H_\beta}$	$\frac{H_\beta}{H_\gamma}$	$\frac{H_\alpha}{H_\beta}$	$\frac{H_\beta}{H_\gamma}$
1	10·1	2·28	10·9	1·9
2	8·1	2·1	7·8	2·0
4	7·2	2·3	7·3	2·2
6	8·2	2·2		

Fig. 6.

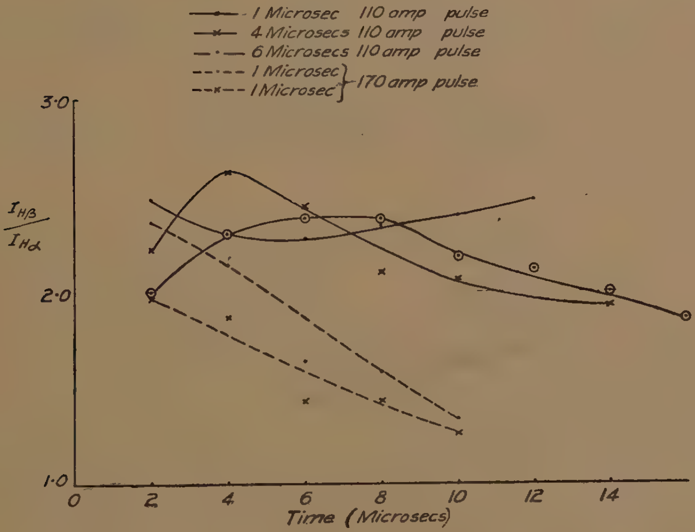


Variation of I_{H_α}/I_{H_β} ratio during and after the passage of current in spark discharges, with various current pulse lengths, obtained with spectrometer and multiplier tube.

60 amp. peak current change in the magnitude of the current pulse. On the other hand, the H_β/H_γ decrement is very insensitive both to pulse length and to current changes within the same range.

The variation of the H_α/H_β decrements was determined as a function of time for a number of pulse lengths, figs. 6 and 7. The decrement steadily decreases to a minimum value at a time which is dependent on the pulse length. On the extinction of the current pulse the decrement increases at a rate which is also dependent on the duration of the current pulse being very much greater for a 1 microsecond pulse than for a 4 and 6 microsecond pulse. The significance of this increase in the H_α/H_β decrement after the extinction of the current will be discussed in section 4.

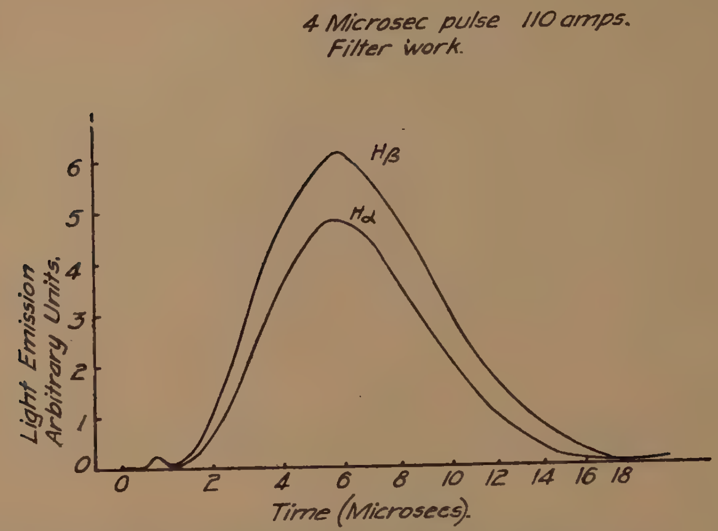
Fig. 7.



Variation of I_{H_α}/I_{H_β} ratio during and after the passage of current in spark discharges, with various current pulse lengths, obtained with spectrometer and multiplier tube.

Typical light intensity-time oscillograms taken with Wratten filters Nos. 27 and 75 inserted in turn between the spark and the multiplier window are reproduced in fig. 8. As stated in section 2, the transmission characteristics of the filters mentioned are such that it is possible to isolate and thus compare the intensities of the H_α and H_β. Values of from 7·0 to 7·6 were obtained with a 4 microsecond 110 amp. peak current pulse. Agreement between this value and the H_α/H_β decrement as determined by the electron multiplier-spectrometer technique is thus very close.

Fig. 8.



Light emission/time oscillograms for the hydrogen lines H_α and H_β obtained with filters and multiplier tube.

4. *Discussion of Results.*

It is necessary first to discuss the most probable means by which the Balmer lines may be excited in a spark discharge, since it is hoped to elucidate the excitation processes occurring there by measurements of Balmer decrements, *i. e.* by the ratios I_{H_α}/I_{H_β} and I_{H_β}/I_{H_γ}.

A recombination spectrum due to electron capture by positive ions has been observed (Craggs and Hopwood, 1947 *a* and *b*) with the present spark discharges. The spectrum in question is the continuum beyond the Balmer series limit. The spectrum due to free-free transitions (Finkelnburg and Hahn, 1942, and references there cited) should be of a different nature, and apparently becomes manifest at higher current densities than those obtained in the present sparks. The Finkelnburg continua extend to wave-lengths longer than 3647 A (the Balmer series limit).

Electron capture spectra in hydrogen have been extensively studied by astrophysicists, *e. g.* Carroll (1930), Menzel and Pekeris (1935), Cillié (1932, 1936) and others. Pioneer work was carried out by Zanstra (1927) and Plaskett (1928). The following data are due to Cillié (1936) :—

$$N_iN_e \,.\, C(n, T_e) = N_iN_e \frac{2^9\pi^5}{(6\pi)^{3/2}} \frac{\epsilon^{10}}{m^2c^3h^3}\left(\frac{m}{kT_e}\right)^{3/2}$$

$$\times \frac{x^1}{n^3} e^{\chi n/nT_e}\{E_i(-\chi n/kT_e)\}$$

where T_e=electron temperature,
χ_n=ionization potential of nth state,
N_i and N_e are ion and electron concentrations/cc.,
m and e are electronic mass and charge,
k=Boltzmann's content.

Line excitation from subordinate levels is assumed to be negligible and self-reversal is also, after discussion, ignored by Cillié (1936). The captured electron is responsible for the cascaded emission of radiation (controlled by the Einstein A coefficients) until the ground state is reached and the emitting gas layer is assumed to be opaque to the ensuing subordinate series radiation. It is also assumed that the Lyman radiation is strongly absorbed so that the numbers of transitions up to and downwards from the nth level to the ground state are equal. These assumptions appear to be reasonable for spark channels. The absorption of Balmer radiation by atoms already excited by absorption of Lyman α-radiation outside the channel is unlikely, since the emitted lines are very broad and have strong centres (Craggs and Hopwood, 1947 *a*). If the effect exists it would relatively strengthen H_α. The number of atoms/cc./sec. leaving state n for lower levels is $N_n \sum_{n=2}^{n-1} A_{n,s}$ and the number arriving there per sec. per cc. from higher states is $\sum_{s=n+1}^{n=\infty} N_s A_{s,n}$. Ignoring stimulated transitions, then

$$N_n \sum_{s=2}^{n-1} A_{n,s} - \sum_{s=n+1}^{n=\infty} N_s A_{s,n} = N_i N_e C(_n, T_e).$$

From this equation and the emission in ergs/cc./sec. for a line H_n which is given by

$$E_n = N_n A_{n,2} h\nu_{n,2}.$$

Cillié then deduced, for various electron temperatures the Balmer line intensities (H_β=1) given in Table II. Only the first four members are given here.

TABLE II.

H_n	T_e=5000°	10,000	20,000	50,000
H_α	2·70	2·78	2·88	3·01
H_β	1·00	1·00	1·00	1·00
H_γ	0·51	0·50	0·48	0·47
H_δ	0·30	0·29	0·27	0·25

The decrement is low and varies relatively little with temperature.

The light emission observed with the sparks might also be caused at least partly by purely thermal excitation. If we therefore assume a

Boltzmann distribution relative to the level $n=2$, the use of well-known formulæ (see, *e. g.*, Taylor and Glasstone, 1942, p. 354) leads to the relation :

$$\log_e\left[\frac{I_\alpha A_{4,2} E_\beta g_4}{I_\beta A_{3,2} E_\alpha g_3}\right]=\frac{E_\beta-E_\alpha}{kT},$$

where $I_{\alpha,\beta}$ are the intensities of $H_{\alpha,\beta}$. $E_\beta=h\nu_\beta\equiv 4861$ÅU, k is Boltzmann's constant and T is the absolute temperature. $A_{4,2}$ and $A_{3,2}$ are the appropriate Einstein coefficients which have been tabulated, for example, by Menzel and Pekeris (1935). The data of Table III. were then calculated ; the ratios are given only to two significant figures, although greater accuracy is possible.

The decrements so observed are greater than the recombination decrements and vary more rapidly with temperature. The use of such data for the determination of steady arc temperatures, where thermal equilibrium can be shown to apply, is well known. It is also widely recognized that about 1 millisecond is required for the establishment of such conditions in arcs. Langstroth and McRae (1938 *a*) also show that thermal equilibrium is practically attained in sparks lasting $\sim 10^{-4}$ secs., using low-current oscillatory discharges. Conditions in 1–6 microsecond sparks may be expected, however, to be different and to be governed only partly by thermal considerations (see Levy, 1940). Flowers (1943) and Craggs and Meek (1946) have considered the energy balance in spark channels, but reached no conclusion as to the value of gas temperature developed during the discharge, although there is evidence that the gas temperature rise may be large, due to the large number of electronic collisions in spark channels even in times <1 microsecond (Levy, 1945 ; Craggs and Meek, 1946). It is, for example, rather surprising that the questionable application of Saha's equation (Craggs and Meek, 1946) to spark channel conditions should in a preliminary survey give results (*e. g.* ion concentration values) agreeing fairly well with those obtained by other methods (Craggs and Hopwood, 1947 *a*).

Table III.

Temp. °K.	I_{H_α}/I_{H_β}	I_{H_β}/I_{H_γ}
5000	9·5	4·0
8000	5·5	3·0
10000	4·5	2·75
12000	4·0	2·6
15000	3·6	2·4

Electron excitation should now be considered. It seems, firstly, reasonable to suppose that most of the hydrogen molecules are dissociated probably by electron impact, but possibly by thermal means also ; this process needs 4·4 eV energy ; dissociation with excitation or ionization requires, of course, much higher electron energies. Further, Wrede (1929), Bay and Steiner (1930) and Willey (1935) have conclusively demonstrated this dissociation in condensed spark discharges. Wrede (*loc. cit.*) reported 80 per cent. dissociation in hydrogen in discharges comparable with those described in this paper. The electron excitation problem in hydrogen then is simplified to a critical survey of previous work, due to Bethe (1930), Elsasser (1927), Ornstein and Lindeman (1931) and others.

An extreme form of statistical excitation by electrons is obtained when the electron temperature (T_e) tends to infinity (Ruark and Urey, 1930, p. 699 ; and Condon and Shortley, 1935, p. 97). Now the intensity I_{nm} of a line, emitted when a transition between levels m and n occurs, is given by :—

$$I_{n,\,m}=N_n A_{nm} h\nu.$$

The population ratio $$\frac{N_n}{N_m}=\frac{g_n}{g_m}\frac{c^{-E_n/kT_e}}{e^{-E_m/kT_e}}$$

for $T_e \rightarrow \infty$ $$=\frac{N_n}{N_m}\frac{g_n}{g_m}.$$

TABLE IV.

Energy (*ev*) (E)	Population N(E)*d*E
12·1	0·00006
12·75	0·000034
13·1	0·000024

For this case $H_\alpha/H_\beta/H_\gamma$ intensity ratios are as 4 . 2/1 . 9/1. Ornstein and Lindeman (1931) have used Bethe's calculations to show that for 30 eV. electrons the decrements are the values of Table V., line 5. The decrement could be expected to be steeper for electrons of lower energy, especially if the latter be very near the excitation potential of the Balmer series (H_α, H_β and H_γ respectively correspond to about 12·1, 12·75 and 13·1 eV respectively). The variation in electron population with energy is very rapid when the distribution is Maxwellian or of a simular nature, for energies appreciably greater (factor >4, say) than the mean energy (Llewellyn Jones, 1936). The latter is probably $\sim$10,000–15,000°, say 12,000° for the present discharges (Williams, Craggs and Hopwood, 1947), hence the mean electron energy ($3kT/2$) is about 1·5 eV (most probably only $\sim$0·5 eV), or about one-eigth of the excitation potentials of the Balmer lines and so the steep decrements of Table I. may be expected. The relative numbers of electrons having energies corresponding to the different Balmer lines, in terms of the most probable energy, are shown in Table IV. for T_e=12,000°.

It is seen (from the data given in Table IV.) that the decrements for excitation by electrons (Table VI.) would need to be corrected particularly for H_α/H_β, when the factor would be ∼2 in favour of strengthening H_α. For non-Maxwellian distributions the above argument might need modification, but the field/pressure quotient X/p in spark channels after the short initial breakdown period is over is low, *i. e.* about 0·2 v./cm./mm., and is practically zero after the current in the square pulse has fallen (see fig. 1 for the shape of current pulses), so that a Maxwellian distribution in the afterglow, if not during current flow, is likely to be quickly reached.

Balmer decrements have been measured by many workers, generally for discharges whose electrical characteristics are ill defined, and representative results are summarized in Table V.

TABLE V.

Line intensities			Discharge conditions etc.	Reference
H_α	H_β	H_γ		
104	7·7	1	0·62 mm. Hg pressure, 800 milliamps. 1 m. tube, ∼ kv.	Jackson, 1936
12	3·1	1	Electrodeless discharge at low pressure. Decrements vary with position in discharge-tube.	Herzberg, 1927
46	5·8	1	Electron excitation (energy unknown).	Bongers, 1927
17	2·9	1	Spiral nebulæ.	Plaskett, 1928
5·2	1	–	L. p. arc (440 volts, 400 milliamps. 0·022 mm. Hg pressure).	Ornstein and Lindeman, 1933

Astronomical data have been obtained by Menzel, Cillié (references given above) and others. No data for Balmer decrements in spark discharges appear to have been published recently, but Merton and Nicholson (1918) obtained such information in somewhat ill-defined electrical conditions.

The decrement calculations are conveniently summarized in Table VI., for comparison with the experimental results of Table I.

The results of the present experiments are shown in Table I. and figs. 6 and 7. Fig. 6 shows that I_{H_α}/I_{H_β} increases after the cessation of current flow and the effect of lengthening the pulse is apparent both in delaying the rise in I_{H_α}/I_{H_β} and also in giving a decreasing intensity ratio during current flow, presumably because of an increasing electron temperature. The rise in intensity ratio during the afterglow may be explicable, since the electron temperature should decrease rapidly (Craggs, Hopwood and Meek, 1947, and references there cited). The variation of Ramsauer cross-sections with energy will hardly alter the population variation with energy for electrons capable of exciting the

Balmer series. The recombination certainly taking place during the afterglow involves electrons of much lower energy, *i. e.* <1 eV (Craggs and Hopwood, 1947 *b*).

It seems unlikely that thermal excitation contributes significantly to the Balmer radiation either during or after current flow, since I_{H_β}/I_{H_γ} is found experimentally to be too low (fig. 7). It is more difficult to decide the relative importance of electron excitation and electron/ion recombination, since it seems likely that both forms of spectra are simultaneously emitted. Excitation by electrons with energies not much greater than that required for excitation of the Balmer lines would probably account for the results, particularly, as shown above, a Maxwellian distribution with $T_e \sim 12{,}000°$ would strongly favour excitation of the lower Balmer lines, particularly H_α. The rising decrement in the afterglow indicates that recombination is not the major mechanism of light emission, although other experiments cited above have demonstrated its presence.

TABLE VI.

Excitation mechanism	Intensities			Intensity ratios	
	H_α	H_β	H_γ	H_α/H_β	H_β/H_γ
Electron (30 eV) excitation	24·1	3·6	1·0	6·7	3·6
Chaotic excitation (electron temp. $\to\infty$)	4·2	1·9	1·0	2·2	1·9
Electron/ion recombination :					
10000°	5·6	2·0	1·0	2·8	2·0
20000°	6·0	2·1	1·0	2·9	2·1
Thermal excitation :					
2500°	340	7·7	1·0	44	7·7
10000°	12·3	2·75	1·0	4·5	2·75
15000°	8·7	2·4	1·0	3·6	2·4
See Table III.					

Later experiments to be performed on these sparks, to measure with accuracy their electrical and geometrical characteristics (voltage gradient, channel radius, etc.), will, it is hoped, help to elucidate these results further.

5. *Conclusions.*

The relative intensities of the first three Balmer lines have been measured in condensed spark discharges of a simplified type. The excitation of the Balmer series appears to be due largely to electron excitation and probably thermal excitation is of minor importance.

6. *Acknowledgments.*

The filter calibrations of figs. 3 and 4 were obtained for us at the B.T.H. research laboratories, Rugby, and we are greatly indebted to Mr. L. J. Davies, director of those laboratories, for this invaluable help.

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XLIV. *The Response of a Resonant System to a Gliding Tone.*

By N. F. **Barber** and F. **Ursell** *.

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Summary.

A solution is given for the response of an oscillatory system to a tone whose frequency slowly increases or decreases. The result is of importance in the construction of apparatus to perform frequency analysis. It is shown that the form of the response is very complicated, but that the variation of amplitude near resonance depends upon a single parameter involving the constants of the apparatus. There is an optimum value of this parameter which leads to the highest resolving power of the frequency analyser.

The converse problem, in which a tone of fixed frequency acts upon a resonant system whose natural frequency is slowly changed is also considered. The results are similar.

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I. An automatic apparatus to perform a frequency analysis of a recorded trace has recently been developed and used in the Admiralty Research Laboratory, Teddington. The apparatus is described more fully in 'Nature '(1). It is very suitable for examining the spectrum of ocean waves and has been in continuous use on this work since February 1944. The spectrum can be examined as far as the four-hundredth harmonic of the length of the record. It must be noted that beyond the sixtieth harmonic individual harmonics are not resolved, but that belts of frequency differing by more than 3 per cent. can be resolved in all parts of the spectrum. The time taken for an analysis is about five minutes per octave.

The record in the form of a variable-area black trace on a white background is wrapped round the periphery of a wheel which is made to revolve, and an electrical output is obtained from an optical device which views the record through a narrow slit close to the wheel. This electrical output is, therefore, a continued repetition of the recorded trace, and it is made to drive a sharply resonant vibration galvanometer. If the galvanometer has a natural resonance to a frequency of p cycles per second and the record is sinusoidal with N complete waves on the periphery of the wheel, then it is necessary for the wheel to revolve at p/N cycles per second if the galvanometer is to be forced into resonance.

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* Communicated by Admiralty Research Laboratory.

We can imagine the record, whatever its form, to be compounded of individual sine waves having 1, 2 . . . N . . . complete waves in the periphery of the wheel. The galvanometer can be made to vibrate in sympathy with each of these Fourier components by driving the wheel successively at speeds of p, $p/2$ . . . p/N . . . cycles per second.

This method has been used by Dietsch and Fricke (2) but the process of adjusting the wheel to these various speeds implies continual attention on the part of the operator and takes a considerable time. In the apparatus described here the wheel is made to revolve at a gradually decreasing speed so that its speed passes in turn through all the desired values, and the successive resonances of the galvanometer are recorded by a pen on moving paper. The action of the system is, therefore, quite automatic, and the pen record consists of a series of peaks, each of which corresponds to the amplitude of a Fourier component of the trace on the original record.

It is believed that this process of sweeping slowly through the spectrum to be analysed would be commonly used if the nature of the resonance of the vibration system under these conditions were more fully understood. The difficulty is that the resonant system must not be heavily damped if it is to have high resolving power between different frequencies. This condition, however, implies that the resonant system will take a considerable time to build up to a steady amplitude in response to a given frequency. In the present apparatus it is clear that each Fourier component of the record gives rise to a tone whose frequency slowly glides from high to low values as the speed of the wheel decreases. If this gliding is too rapid the tone will have passed out of the resonance-frequency band of the galvanometer before the latter has had time to attain its steady state.

Experimentally it can be shown that the time-response curve of the vibration galvanometer presents a curious appearance in this case. The amplitude builds up smoothly to a maximum value and then decreases in a series of beats which become more and more rapid as time goes on. There may be no visible beats when the rate of gliding is slow enough or a large number when the frequency glides very quickly. One of the minima of the beats may be zero and this again depends on the conditions. An experimental curve is shown in fig. 4. The phenomenon is known as "ringing."

It has been found possible to make a theoretical description of the response of a resonant system to a gliding tone, and to specify an optimum value of damping for the resonant system when a given range of the spectrum is to be explored in a given time. This optimum damping, which depends also on the natural frequency of the resonant system, gives the best frequency discrimination which is obtainable in the time allowed. The findings will later be applied to the converse case in which a resonant system is gradually tuned so as to explore a spectrum of steady frequencies.

(*a*) *A resonant system of constant natural frequency.*

The mechanical equation of motion of the resonant system will be assumed to have the simple form

$$\frac{d^2x}{dt^2}+2n\omega\frac{dx}{dt}+\omega^2x=\frac{\mathrm{E}(t)}{\mathrm{M}}, \quad \ldots \ldots \quad (1)$$

where $\mathrm{E}(t)$ is the applied force at time t and M is the inertia of the moving part. The natural period of motion of the system is $2\pi/\omega\sqrt{(1-n^2)}$ and its coefficient of damping is n. These are related to the inertia M, the damping force K per unit velocity of the moving part, and the restoring force C per unit displacement of the moving part, by the equations

$$\omega^2=\mathrm{C/M}, \qquad 2n\omega=\mathrm{K/M}.$$

The free motion of such a system under no applied force is a damped oscillatory one of the form

$$x=\mathrm{X}e^{-n\omega t}\sin(\omega t\sqrt{(1-n^2)}+\mathrm{Y}), \quad \ldots \ldots \quad (2)$$

where X and Y can be chosen to fit any given initial condition.

The gliding tone.

A tone whose frequency decreases exponentially with time at a rate a is given by

$$\sin\left(\frac{p}{a}(1-e^{-at})+\epsilon\right), \quad \ldots \ldots \quad (2)$$

for the rate of change of phase angle with time is

$$\frac{d}{dt}\left(\frac{p}{a}(1-e^{-at})+\epsilon\right)=pe^{-at}.$$

This quantity replaces the pulsatance, p, of a steady tone. The frequency is $\frac{1}{2\pi}pe^{-at}$. Over a short time interval about $t=0$ this tone is approximately represented by

$$\sin(\epsilon+pt-\tfrac{1}{2}apt^2). \quad \ldots \ldots \quad (4)$$

It will therefore be assumed that the force on the resonant system due to the gliding tone is

$$\mathrm{E}(t)=\mathrm{E}_0\sin(\epsilon+pt-\tfrac{1}{2}apt^2) \quad \ldots \ldots \quad (5)$$

(*b*) *The response to a gliding tone.*

In order to find the response of the resonant system to a force $\mathrm{E}(t)$ varying with time it is convenient to replace the force by a series of impulses $\mathrm{E}(\tau)d\tau$. The free motions of the system following each of these impulses may be added together to give the total deflection at some later time.

The expression for the deflection x at time t is obtained in this way as

$$x=\frac{1}{\omega\sqrt{(1-n^2)}}\int_{-\infty}^{t}\frac{\mathrm{E}(\tau)}{\mathrm{M}}\,.\,e^{n\omega(\tau-t)}\,.\,\sin[\omega(1-n^2)^{\frac{1}{2}}(\tau-t)]\,.\,d\tau. \quad (6)$$

It will be supposed that the coefficient of damping is much smaller than unity, so that it is often sufficient to write $\surd(1-n^2)^{\frac{1}{2}}\equiv m$ as unity. Substituting for $E(\tau)$ from expression (5), we obtain for the response to a gliding tone

$$x=\frac{E_0}{\omega m M}\int_{-\infty}^{t} e^{n\omega(\tau-t)}\,.\,\sin m\omega(\tau-t)\,.\,\sin(\epsilon+pt-\tfrac{1}{2}apt^2)\,d\tau. \quad . \quad (7)$$

(*c*) *Evaluation of the integral.*

Two methods will be given for evaluating this integral. The first is exhaustive and determines an upper limit to the errors involved in the approximations. The second is more superficial, but suggests a physical picture of the response.

Method 1.

By a change of the variable, expression (6) is written as

$$x(t)=\frac{1}{\omega m M}\int_{0}^{\infty} e^{-n\omega u}\,.\,\sin m\omega u\,.\,E(t-u)\,.\,du, \quad . \quad . \quad . \quad (8)$$

in which $\surd(1-n^2)^{\frac{1}{2}}$ has been written as m.

The applied force is written more generally as

$$E(t)=E_0\sin\phi(t),$$

and the phase angle $\phi(t)$ may be expanded by Taylor's theorem to give

$$\begin{aligned}E(t-u)=&\ E_0\sin[\phi(t-u)]\\ =&\ E_0\sin[\phi(t)-u\phi'(t)+\tfrac{1}{2}u^2\phi''(t)-\ldots]\\ =&+E_0\sin\phi(t)\,.\,\cos[u\phi'(t)-\tfrac{1}{2}u^2\phi''(t)+\ldots]\\ &-E_0\cos\phi(t)\,.\,\sin[u\phi'(t)-\tfrac{1}{2}u^2\phi''(t)+\ldots].\end{aligned}$$

Substituting in the expression (8), we obtain

$$\begin{aligned}x(t)=&\frac{E_0}{\omega m M}\sin\phi(t)\int_0^\infty e^{-n\omega u}\,.\,\sin m\omega u\,.\,\cos[u\phi'(t)-\tfrac{1}{2}u^2\phi''(t)+\ldots]du\\ &-\frac{E_0}{\omega m M}\cos\phi(t)\int_0^\infty e^{-n\omega u}\cos m\omega u\,.\,\sin[u\phi'(t)-\tfrac{1}{2}u^2\phi''(t)+\ldots]du\\ =&\frac{E_0}{\omega M}(A\sin\phi(t)-B\cos\phi(t))\\ =&\frac{E_0}{\omega M}R\sin[\phi(t)-g(t)],\end{aligned}$$

where

$$\cos g(t)\,.\,R=A,$$

$$\sin g(t)\,.\,R=B.$$

The quantities A, B, R, $g(t)$ are all functions of t, but it will be shown that they vary only slowly with t near $t=0$. The quantity R is, therefore, the amplitude of the oscillatory response. It is the envelope of the family of x curves obtained by varying $\phi(t)$ by an additive constant. In order to determine R it is sufficient to investigate one case; we shall take $\phi(0)=0$.

The integral expressions for A and B have quantities under the integral sign which oscillate rapidly with time and a computation of these integrals would be laborious. In the present case, however, measuring time from the instant at which $d\phi/dt=\omega m$, so that $p=\omega m$, we have from (5),

$$\phi(t)=\omega mt-\tfrac{1}{2}\omega mat^2\,;$$

so $$\phi'(t)=\omega m(1-at), \qquad \phi''(t)=-\omega ma,$$

and the expression for A can be written

$$\begin{aligned}\mathrm{A}(t)&=\frac{1}{m}\int_0^\infty e^{-n\omega u}\,.\,\sin m\omega u\,.\,\cos m\omega u(1-at+\tfrac{1}{2}au)\,.\,du\\ &=\frac{1}{2m}\int_0^\infty e^{-n\omega u}\,.\,\sin\,(2k\omega^2tu-k\omega^2u^2)\,.\,du\\ &\quad+\frac{1}{2m}\int_0^\infty e^{-n\omega u}\,.\,\sin\,(2m\omega u-2k\omega^2tu+k\omega^2u^2)\,.\,du\\ &=a_1(t)+a_2(t).\end{aligned}$$

In these expressions k is written for $\dfrac{ma}{2\omega}$. It will be shown that $a_2(t)$ is negligible compared with $a_1(t)$ and that $a_1(t)$ is convenient for computation.

By the same means we have

$$\begin{aligned}\mathrm{B}(t)&=\frac{1}{2m}\int_0^\infty e^{-n\omega u}\,.\,\cos\,(2k\omega^2tu-k\omega^2u^2)\,.\,du\\ &\quad-\frac{1}{2m}\int_0^\infty e^{-n\omega u}\,.\,\cos\,(2\omega mu-2k\omega^2tu+k\omega^2u^2)\,.\,du\\ &=b_1(t)+b_2(t).\end{aligned}$$

Approximation.

As $k\rightarrow 0$, $b_1(t)\rightarrow 1/2n\omega m$ near $t=0$. If we can prove that $(a_2{}^2+b_2{}^2)$ is less than $1/2m^4\omega^2$, it will follow that a_2 and b_2 can be neglected provided that n can be neglected compared with unity.

If $$2m\omega-2k\omega^2t\equiv c,$$

then

$$b_2-ia_2=\frac{1}{2m}\int_0^\infty \exp\,[-(n\omega+ic)u-ik\omega^2u^2]du.$$

Consider $\int\exp[-(n\omega+ic)z-ik\omega^2z^2]\,dz$ taken round the sector formed by the lines $\theta=0$, $\theta=-\pi/4$, $r=\mathrm{R}$. This integral is zero by Cauchy's theorem (*cf.* Whittaker and Watson, 'Modern Analysis,' 5.2). As $R \to \infty$ the part of the integral along the arc of the circle tends to zero by Jordan's Lemma ('Modern Analysis,' 6.222), so that

$$\int_0^\infty \exp\,[-(n\omega+ic)u-ik\omega^2u^2]du$$

$$=\frac{1-i}{\sqrt{2}}\int_0^\infty \exp\,[-(n\omega+ic)\,.\,\frac{1-i}{\sqrt{2}}\,.\,u-k\omega^2u^2]du.$$

The modulus of this second integral is less than

$$\int_0^\infty \left|\exp\left[-(n\omega+ic)\,.\,\frac{1-i}{\sqrt{2}}\,.\,u-k\omega^2u^2\right]\right|du$$

$$=\int_0^\infty \exp\left[-\frac{n\omega+c}{\sqrt{2}}\,.\,u-k\omega^2u^2\right]du$$

$$<\int_0^\infty \exp\left[-\frac{n\omega+c}{\sqrt{2}}\,.\,u\right]du$$

$$=\sqrt{2}/(n\omega+c)<\sqrt{2}/c<\sqrt{2}/m\omega$$

so long as t is less than $\frac{m}{2k\omega}$, which is true in most applications.

So

$$b_2{}^2+a_2{}^2<\frac{1}{4m^2}\,.\,\frac{2}{m^2\omega^2}=\frac{1}{2m^4\omega^2}\,.$$

Thus, if we neglect errors of the order n, we have approximately for the amplitudes of the response, writing $m=1$,

$$\left.\begin{aligned}A=a_1=\tfrac{1}{2}\int_0^\infty e^{-n\alpha u}\sin\,(a\omega tu-\tfrac{1}{2}a\omega u^2)du,\\ B=b_1=\tfrac{1}{2}\int_0^\infty e^{-n\alpha u}\cos\,(a\omega tu-\tfrac{1}{2}a\omega u^2)du\end{aligned}\right\}\quad . \quad . \quad . \quad (9)$$

The expression for the deflection of the resonant system is, in the general case,

$$x=\frac{E_0}{M\omega}\,(A\sin\,(\epsilon+\omega t-\tfrac{1}{2}a\omega t^2)-B\cos\,(\epsilon+\omega t-\tfrac{1}{2}a\omega t^2)).\quad . \quad (10)$$

The response is therefore expressed as the sum of two oscillations whose frequencies glide like the frequency of the force E and whose amplitudes vary slowly with time.

We have expressed the result in the form

$$x(t)=\frac{E_0}{M\omega}\,[A\sin\phi(t)-B\cos\phi(t)],$$

so that the rate of change of $\frac{1}{2\pi}\tan^{-1}\frac{B}{A}$ will indicate the difference between the input frequency $\phi'(t)$ and the instantaneous frequency of oscillation.

We may also write the output in the form

$$x(t)=\frac{\mathrm{E}_0}{\mathrm{M}\omega}[\mathrm{A}_1 \sin (\epsilon+\omega t)+\mathrm{B}_1 \cos (\epsilon+\omega t)], \quad . \quad . \quad . \quad (11)$$

where the rate of change of $\frac{1}{2\pi} \tan^{-1} \frac{\mathrm{B}_1}{\mathrm{A}_1}$ is a measure of the difference between the resonant frequency and the instantaneous frequency of oscillation. The connection between the coefficients is

$$\left.\begin{aligned}\mathrm{A}_1&=\mathrm{A} \cos \tfrac{1}{2}a\omega t^2-\mathrm{B} \sin \tfrac{1}{2}a\omega t^2,\\ \mathrm{B}_1&=-\mathrm{A} \sin \tfrac{1}{2}a\omega t^2-\mathrm{B} \cos \tfrac{1}{2}a\omega t^2\end{aligned}\right\}. \quad . \quad . \quad . \quad (12)$$

Both these representations will be used later in discussing the physical significance of the response.

Method 2.

The signal E(t), whose frequency is slowly changing, may be regarded near $t=0$ as the sum of two tones of constant frequency $\omega/2\pi$, whose amplitudes slowly vary. From equation (5), writing $p=\omega$ and $\sqrt{(1-n^2)}=1$ approximately,

$$\begin{aligned}\mathrm{E}(t)&=\mathrm{E}_0 \sin (\epsilon+\omega t-\tfrac{1}{2}a\omega t^2)\\ &=\mathrm{E}_0 \cos \tfrac{1}{2}a\omega t^2 \,.\, \sin (\epsilon+\omega t)-\mathrm{E}_0 \sin \tfrac{1}{2}a\omega t^2 \,.\, \cos (\epsilon+\omega t). \quad . \quad (13)\end{aligned}$$

The effect of these tones may be considered separately by using the integral form (6) for each in turn. The first gives a response having deflection at time t:

$$x=\frac{\mathrm{E}_0}{\omega\mathrm{M}}\int_{-\infty}^{t} e^{n\omega(\tau-t)} \,.\, \cos \tfrac{1}{2}a\omega\tau^2 \,.\, \sin (\epsilon+\omega\tau) \,.\, \sin \omega(\tau-t) \,.\, d\tau$$

or

$$x=\frac{\mathrm{E}_0}{2\omega\mathrm{M}}\int_{-\infty}^{t} e^{n\omega(\tau-t)} \,.\, \cos \tfrac{1}{2}a\omega\tau^2 \,.\, [\cos (\epsilon+\omega t)-\cos (\epsilon-\omega t+2\omega\tau)]d\tau.$$

In this form the only factor which varies rapidly with τ is the second term in the square brackets, and if we write for this term its mean value of zero the integral becomes

$$\begin{aligned}x&=\frac{\mathrm{E}_0}{2\omega\mathrm{M}} \,.\, \cos (\epsilon+\omega t) \,.\, e^{-n\omega t}\int_{-\infty}^{t} e^{n\omega\tau} \cos \tfrac{1}{2}a\omega\tau^2 \, d\tau\\ &=\frac{\mathrm{E}_0}{\omega\mathrm{M}} \,.\, \cos (\epsilon+\omega t) \,.\, \mathrm{B}_1,\end{aligned}$$

where B_1 is a function which varies slowly with t and is in the nature of an amplitude. By the same means we obtain the response to the second tone in (13) and arrive at

$$x=\frac{\mathrm{E}_0}{\omega\mathrm{M}}[\mathrm{A}_1 \sin (\epsilon+\omega t)+\mathrm{B}_1 \cos (\epsilon+\omega t)], \quad . \quad . \quad . \quad (14)$$

in which

$$\left.\begin{matrix}A_1\\B_1\end{matrix}\right\}=\tfrac{1}{2}e^{-n\omega t}\int_{-\infty}^{t}e^{n\omega\tau}\left.\begin{matrix}\sin\\\cos\end{matrix}\right\}\tfrac{1}{2}a\omega\tau^2\,.\,d\tau. \quad . \; . \; . \; . \; . \quad (15)$$

The response of the system is therefore expressed as the sum of two tones of frequency $\omega/2\pi$ whose amplitudes vary slowly with time. This expression for x is the same as expression (11) obtained by method 1.

(*d*) *Discussion of the response.*

1. *The parameter.*

The two expressions (10) and (11) are equivalent and lead to the same value for the amplitude

$$R=[A^2+B^2]^{\frac{1}{2}}=[A_1{}^2+B_1{}^2]^{\frac{1}{2}}.$$

Over a range of values of the characteristics a, ω and n the form of the response amplitude depends only on the value of a parameter $a/2n^2\omega$, apart from changes in the scale of amplitude and time. The Admiralty Computing Service has given a Table of values of the functions

$$\left.\begin{matrix}C(T)\\S(T)\end{matrix}\right\}=\sqrt{k}\int_{0}^{\infty}e^{-u}\left\{\begin{matrix}\cos\\\sin\end{matrix}\right.(2Tu\sqrt{k}-ku^2)du \quad . \; . \; . \quad (16)$$

and of $[C^2+S^2]^{\frac{1}{2}}$, over a range of T somewhat greater than ±5 and for values of the parameter k equal to $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 4 and 8.

The quantities A, B are related to these in the manner

$$A(t)=\frac{1}{\sqrt{(2a\omega)}}\,.\,S(t\sqrt{(\tfrac{1}{2}a\omega)}),$$

$$B(t)=\frac{1}{\sqrt{(2a\omega)}}\,.\,C(t\sqrt{(\tfrac{1}{2}a\omega)}).$$

The curves of R are shown in fig. 1 and have been drawn to represent the envelopes when the damping coefficient of the resonant system assumes different values, the resonant frequency and the rate of glide being unaltered.

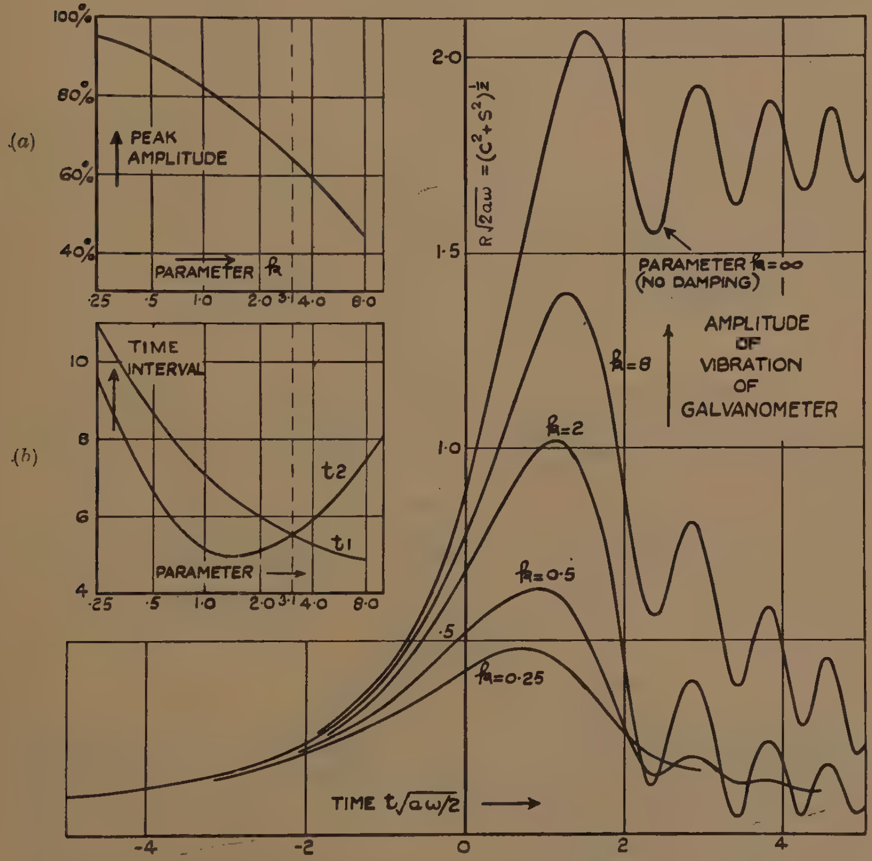
The curves show that the amplitude of the oscillatory motion increases smoothly to a peak value at a time a little after $t=0$ and thereafter decays in a series of beats. The number and relative size of the beats depend on the value of the parameter. There are no visible beats when the damping is large, but a large number of beats develop when the damping is small.

The peak amplitude of the response is always less than would be the amplitude of response to a steady tone of frequency $\omega/2\pi$. The factor of reduction for various values of parameter is shown in fig. 1 (*a*). It illustrates the remark made previously that when the damping is low the amplitude has not time to build up to its steady value during the time that the frequency of the gliding tone is near $\omega/2\pi$.

2. *Optimum conditions.*

If there are two tones in the input whose frequencies are in the ratio $(1+\alpha)$ and the frequencies are gliding at a rate $(1+at)$, then they produce two resonances of the vibrating system separated by a time interval α/a. The two resonance curves will both be of the shape of one of the curves of fig. 1 and will differ only by a factor of amplitude. We may consider them to be sufficiently well resolved if, at the moments when each individual response is at its maximum, the activity of the system due to the other response is less than 1/10 of the peak value of the other response. To find which of the various curves on fig. 1 leads to the best

Fig. 1.



The envelopes of a transient resonance computed for various values of the parameter. The relative magnitudes of the various curves illustrate the effect of using resonant systems which have different damping coefficients but are similar in all other respects, and are subjected to the same rate of change of frequency. Insert (*a*) expresses the peak amplitude in a transient resonance as a percentage of the response that the resonant system would show if the rate of change of frequency were very small. Insert (*b*) illustrates the existence of an optimum damping in a given resonant system, leading to the greatest frequency discrimination of the spectrum of a given gliding frequency.

resolution, we decide for each curve the instant $-t_1$ prior to which the response never attains more than 0·1 of its peak value, and the instant $+t_2$, after which the response always has a value less than 0·1 of its peak value. These times $-t_1$, t_2 are measured relative to the instant at which the peak value is attained. Since the interference on one resonance may be due either to the following resonance when t_1 is large or to the preceding resonance when t_2 is large, the possible resolution is determined by the greater of these two. Two frequencies in the ratio $(1+\alpha)$, producing resonances at a time interval α/a, will be resolved, therefore, if α/a is greater than both t_1 and t_2.

Fig. 1 (*b*) shows curves of t_1 and t_2 against the parameter. The parameter which gives best resolution is

$$k=a/2n^2\omega=3{\cdot}1.$$

Here

$$t_1=t_2=5{\cdot}5\sqrt{(2/a\omega)},$$

so

$$\alpha=5{\cdot}5\sqrt{(2/a\omega)}.$$

When we are dealing with a series of harmonics of a recorded trace in the manner described earlier, the frequencies of the Nth and (N+1)th harmonic are in the ratio (1+1/N). Harmonics up to the Nth will, therefore, be resolved, where N is given by

$$1/\mathrm{N}=5{\cdot}5\sqrt{(2a/\omega)}$$

or

$$\mathrm{N}=0{\cdot}13\sqrt{(\omega/a)}, \qquad (17)$$

on the understanding that, whatever values we assign in the apparatus to ω and a, the damping n is adjusted to give the parameter K its optimum value.

The time taken by an analysis is of importance in practice. It is clear that one octave of the spectrum is analysed in the time T_0 taken for the speed of the wheel to fall by a factor of 2.

$$\mathrm{T}_0=(\log_e{}^2)/a.$$

Using equation (17),

$$\mathrm{T}_0=41{\cdot}6\mathrm{N}^2/\omega. \qquad (18)$$

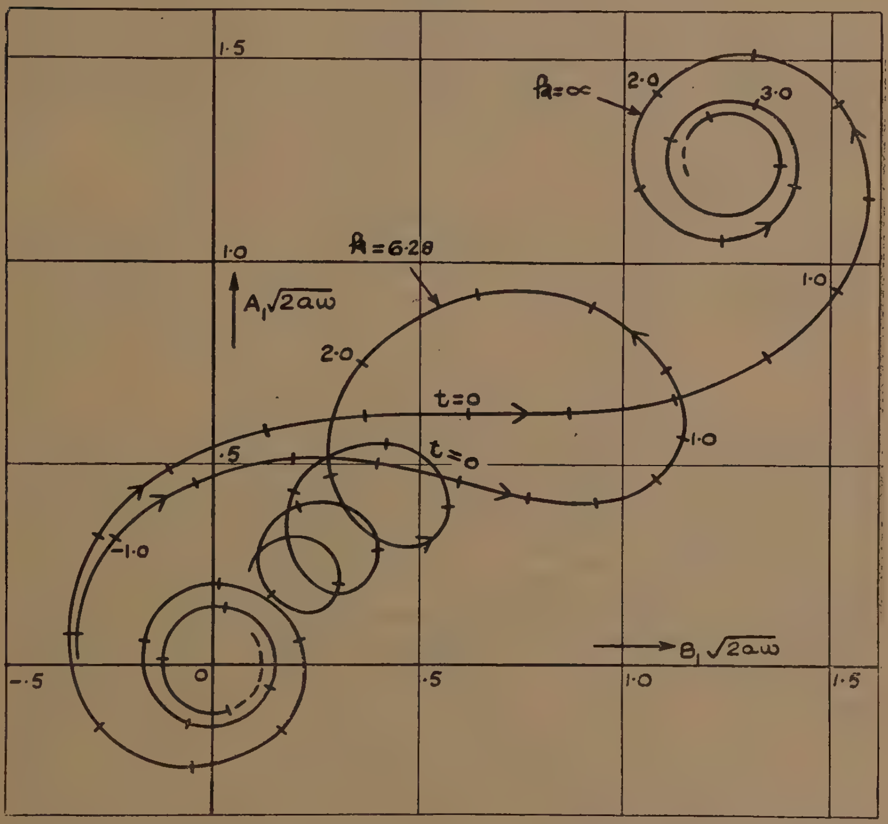
The time taken to make an analysis is therefore proportional to the square of the order of harmonic which is resolved and to the natural period of motion of the resonant system. To obtain an analysis at high resolving power in a short time it is necessary to employ a high-frequency resonant system. The apparatus used so far has employed a vibration galvanometer of about 100 c.p.s. In future an attempt is being made to use a 1000-c.p.s. tuning-fork and obtain analyses more quickly.

3. *The form of the response.*

The complicated forms of the curve of R in fig. 1 are best examined by considering the component amplitudes A_1 and B_1 used in formula (15).

In fig. 2 these are used as rectangular co-ordinates to define the position of a point on a plane. The total amplitude R is the distance of the point from the origin. The curve described by the point is marked at equal time intervals to indicate the rate of progress of the point along the curve. In the limiting case when $n=0$ the curve is a Cornu spiral. Prior to $t=0$ the point is moving around the origin and R increases smoothly.

Fig. 2.



Curves traced by the end point of the vector amplitude whose components are A_1 and B . The motion of the vector along the curve is shown in units of time $t\sqrt{(aw/2)}$. The fluctuations in length of the vector correspond to the beats in the decay of a transient resonance and the angular velocity of the vector in cycles per second measures the difference between the frequency of the motion and the natural frequency of the resonant system. This difference is small during the important part of the transient resonance.

The curves are for parametric values of 6·28 and infinity.

Beyond $t=0$ the point begins to move round the second part of the spiral and the amplitude R oscillates about a mean value. In this special case the vibrations of the system continue indefinitely. There are no frictional losses to destroy the energy acquired near $t=0$.

Fig. 2 also shows the curve, when n is not zero. The later parts of the spiral collapse towards the origin and with increasing time the amplitude R finally becomes zero. The rapidity with which it collapses towards the origin depends on the value of k. The collapse may be interpreted as showing the decay, due to friction, of the energy acquired near $t=0$. When k is large the first few turns of the spiral after $t=0$ do not include the origin and the beats of R are large. The later turns pass round the origin and the beats of R are smaller. One of the turns of the spiral may pass through the origin and the corresponding minimum of R is then zero. The way in which R decays with time is, therefore, a very complicated function of the parameter, and it would be very difficult to produce purely by experiment a series of response curves to illustrate the rules which govern the response.

4. *The frequency of motion in the response.*

It has been pointed out in Section I (c) that the instantaneous frequency of the motion is

$$\frac{1}{2\pi}(\omega+d\beta_1/dt),$$

where β_1 is the angle between the radius R and the A_1 axis in fig. 2. The quantity is not large except when the point on the spiral is moving round the origin. From $t=0$ to the time when the spiral has so far collapsed as to enclose the origin the frequency is on the average equal to the natural frequency of motion.

The frequency in the early and late parts of the motion is best shown by using the quantities A, B, of formula 11, to define the position of a point. Such a curve for $k=2$ is given in fig. 3. The instantaneous frequency is the

$$\frac{1}{2\pi}(\omega(1-at)-d\beta/dt),$$

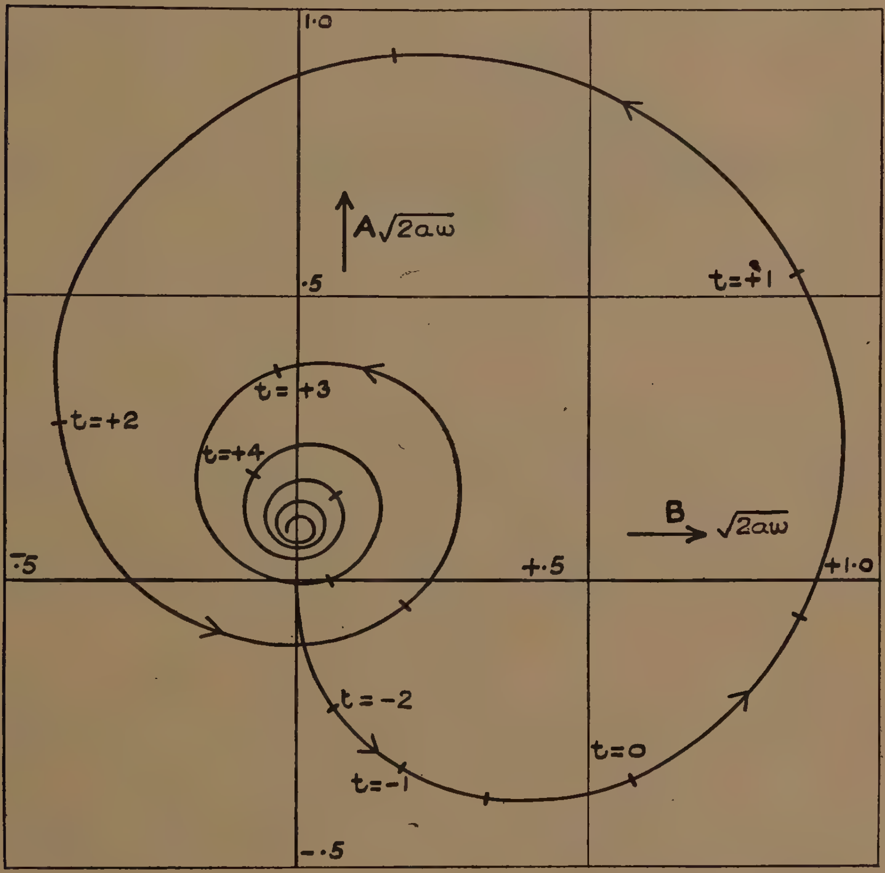
where β is the angle between the radius R and the A axis. In the early part of the motion the angle β changes very slowly and the frequency of motion is nearly

$$\frac{1}{2\pi}\,\omega(1-at).$$

This is the same as the frequency of the gliding tone. In the late parts of the curve the spiral does not enclose the origin and β does not change by a large amount. The average frequency is, therefore, that of the gliding tone.

The physical interpretation is that the motion after $t=0$ is partly a free motion due to energy acquired near $t=0$ and is partly a forced motion. Until the frictional forces have destroyed the energy of the free motion, the frequency is not far from $\omega/2\pi$. After this time the frequency is that of the forced motion due to the gliding tone. The beat on the R curves of fig. 1 should be regarded as a beating interference between the free and forced motions.

Fig. 3.



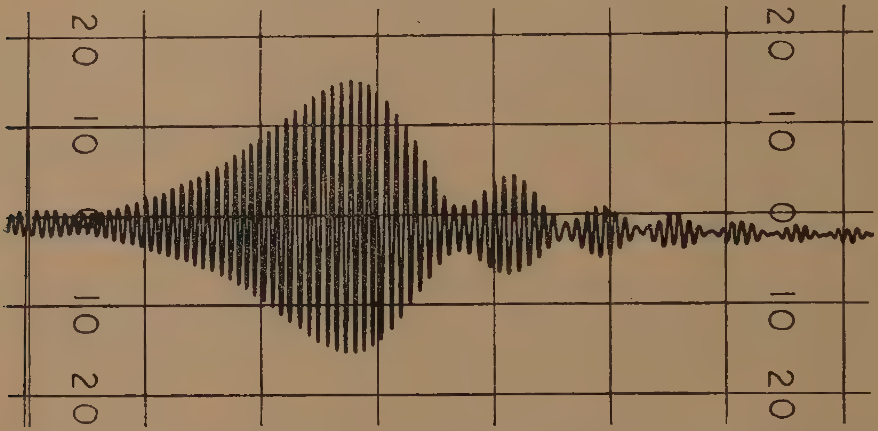
Curve traced by the end point of the vector amplitude whose components are A and B. The motion of the vectors along the curve is shown in units of time $t\sqrt{(a\omega/2)}$. The fluctuations in length of the vector correspond to beats in the decay of a transient resonance, and the angular velocity of the vector in cycles per second measures the difference between the frequency of the motion and the frequency of the impressed gliding tone. This difference is small in the early and late parts of the motion, when the amplitude itself is small.

The curve is drawn for a parametric value of 2.

(e) *Experimental Confirmation.*

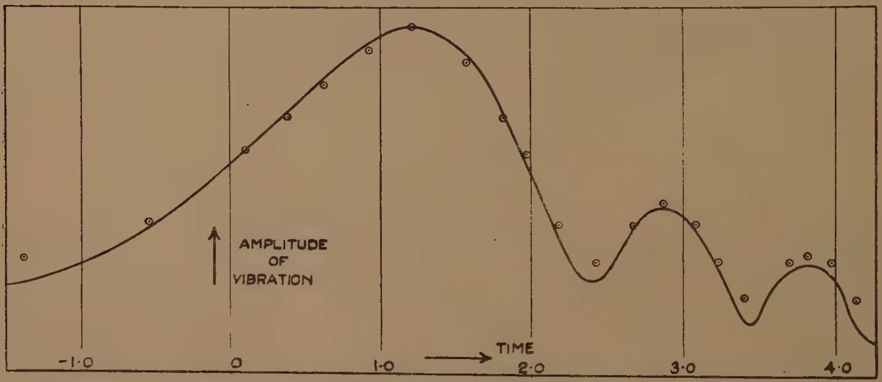
Fig. 4 shows a typical response of a resonant system to a gliding tone. This example is provided by G. Collins, of this laboratory, and shows clearly the gradual increase and fluctuating decay of the amplitude. The value of the parameter k, given by the experimental constants was about $2\frac{1}{2}$.

Fig. 4.



A typical transient resonance. In this case the natural frequency of a glavanometer system was slowly changed and the electrical signal supplied to the system was of constant frequency. The slight asymmetry is due to the nature of the recording system. The parametric value was about $2\frac{1}{2}$.

Fig. 5.



A comparison between a calculated curve of amplitude in a transient resonance and values observed by experiment. The parametric value is 4.

Fig. 5 is an experimental confirmation due to M. J. Tucker, of this laboratory. The constants of the resonant system (a tuning-fork) and the gliding tone were adjusted to give a value as near as possible to $k=4$. The amplitude of vibration of the fork was recorded against time and is indicated by the experimental points in the figure. The theoretical curve for $k=4$ is shown as the full line drawn so that its peak value coincides with the peak experimental value. The agreement appears to be good.

II. A Resonant System with a Slowly Varying Natural Period.

A converse case to that already examined is when we explore a spectrum of fixed frequencies by means of a resonant system whose natural period of oscillation is slowly changed. This problem has arisen in practice in an instrument devised by G. Collins to pick out the various frequency components in ocean waves and to reveal the presence of long-period swell when other waves of much shorter period are present. It is proposed to take the electrical output from a subsurface instrument measuring waves and to supply it to the coils of a long-period galvanometer situated ashore. If there are any wave components whose periods are near to the period of the galvanometer, the galvanometer will resonate. In the course of two hours the period of the galvanometer is slowly changed from 5 seconds to 25 seconds by changing the strength of a magnetic controlling force, and the varying amplitude of resonance of the galvanometer is recorded on moving paper. It therefore presents a picture of the spectrum every two hours.

It is natural to expect that the response of such a system as it passes through the frequency of a steady tone is very similar to the response curves already obtained. The details of the response will depend on the method by which the natural period is made to change, but one example is sufficient to illustrate the result. We will suppose that the coefficient of the controlling force increases slightly with time.

Equation (1) for the resonant system then becomes

$$\frac{d^2x}{dt^2}+2n\omega\frac{dx}{dt}+\omega^2(1+\lambda t)x=\frac{\mathrm{E}(t)}{\mathrm{M}}. \quad . \quad . \quad . \quad . \quad (19)$$

Here λt is always small compared with unity.

It will be assumed that the free vibration of this system can be represented with sufficient accuracy as being of the form

$$x=\exp\left[-n\omega t(1+a+bt+ct^2)\pm i\omega t(1+\mathrm{A}+\mathrm{B}t+\mathrm{C}t^2)\right], \quad . \quad . \quad (20)$$

which is similar to equation (2) except for the introduction of the quantities a, bt, etc., which will be comparable with λt and whose squares may be neglected compared with unity. We have assumed again that n^2 is negligible compared with unity.

Substituting this value of x in the equation of free motion, where $\mathrm{E}(t)=0$, we obtain, by equating to zero the real and imaginary coefficients of powers of x, six equations leading to the result

$$\mathrm{A}=b=c=\mathrm{C}=\text{zero},$$

$$a=\lambda/4\omega n, \qquad \mathrm{B}=\lambda/4,$$

so that for free motion is

$$x=\mathrm{X}e^{-(n\omega+\lambda/4)t}\,.\,\sin(\omega t+\tfrac{1}{4}\lambda\omega t^2+\mathrm{Y}). \quad . \; . \; . \; . \quad (21)$$

It appears, therefore, that the free motion is similar to that when λ is zero except for the following changes :—

(1) The damping is slightly different,
(2) the frequency at time t is $\omega(1+\frac{1}{2}\lambda t)$.

Regarding the first point, it is instructive to consider the case of no frictional damping, n zero.

The solution then becomes

$$x=\mathrm{X}e^{-\lambda t/4}\,.\,\sin(\omega t+\tfrac{1}{4}\lambda\omega t^2+\mathrm{Y}), \quad . \; . \; . \; . \; . \quad (22)$$

with a velocity

$$\dot{x}=\mathrm{X}\omega(1+\tfrac{1}{2}\lambda t)e^{-\lambda t/4}\cos(\omega t+\tfrac{1}{4}\lambda\omega t^2+\mathrm{Y}).$$

Since the expression is only correct to the first power of λt, this is equivalent to

$$x=\mathrm{X}\omega(1+\tfrac{1}{4}\lambda t)\cos(\omega t+\tfrac{1}{4}\lambda\omega t^2+\mathrm{Y}). \quad . \; . \; . \; . \; . \quad (23)$$

The maximum velocity and, therefore, the total energy, increases as time increases in the manner

$$\tfrac{1}{2}\mathrm{M}\dot{x}^2_{(\max)}=\tfrac{1}{2}\mathrm{MX}^2(1+\tfrac{1}{2}\lambda t)\omega^2. \quad . \; . \; . \; . \; . \; . \; . \quad (24)$$

This increase in energy is presumably due to work done by the elastic member on the moving mass. An estimate of this is the product of the decrease in amplitude $\mathrm{X}\lambda t/4$ and the maximum restoring force $\mathrm{MX}\omega^2$. This product is $\mathrm{MX}^2\omega^2\lambda t/4$, and can be seen to be equal to the increase in energy of the moving system in time t as given by (24).

Thus, although the amplitude decreases with time, the energy increases. It is the increasing frequency which makes this possible.

The forced motion.

In order to calculate the response of this system to a steady tone it is necessary to construct an integral similar to (6). A force $\mathrm{E}(\tau)$ applied to the system at time τ produces an acceleration $\mathrm{E}(\tau)/\mathrm{M}$ and the impulse $\mathrm{E}(\tau)d\tau$ gives the system a velocity $(\mathrm{E}(\tau)d\tau)/\mathrm{M}$. The subsequent free motion is given by equation (6), with the constants X, Y adjusted to fit the initial conditions

$$x(\tau)=0, \qquad \dot{x}(\tau)=(\mathrm{E}(\tau)d\tau)/\mathrm{M}.$$

This is

$$x=-\frac{\mathrm{E}(\tau)d\tau}{\mathrm{M}\omega(1+\frac{1}{2}\lambda\tau)}\,.\,e^{(n\omega+\lambda/4)(\tau-t)}\,.\,\sin\omega(\tau-t+\lambda(\tau^2-t^2)/4).$$

Since the equation of motion is linear, it is correct to sum up to free motions from a succession of impulses so that for a continued force

$$\mathrm{E}(t)=\mathrm{E}_0\sin(\omega t+\epsilon)$$

the deflection of the system at time t is

$$x=-\frac{\mathrm{E}_0}{\mathrm{M}\omega}\,e^{-(n\omega+\lambda/4)t}\int_{-\infty}^{t} e^{(n\omega-\lambda/4)\tau}\,.\,\sin\omega(\tau-t+\lambda(\tau^2-t^2)/4)\,.\,\sin(\omega\tau+\epsilon)d\tau. \quad . \quad . \quad . \quad (25$$

The sine terms under the integral can be expanded as

$$[\sin\omega\tau\,.\,\cos\omega(t+\lambda t^2/4-\lambda\tau^2/4)+\cos\omega\tau\,.\,\sin\omega(t+\lambda t^2/4-\lambda\tau^2/4)]$$
$$\times[\sin\omega\tau\,.\,\cos\epsilon+\cos\omega\tau\,.\,\sin\epsilon].$$

On removing the brackets there are four terms, and the factors in them which vary rapidly with τ are

$$\sin^2\omega\tau, \qquad \cos^2\omega\tau \qquad \text{and} \qquad \sin\omega\tau\,.\,\cos\omega\tau.$$

It is sufficiently accurate to give these factors their mean values $\frac{1}{2}$, $\frac{1}{2}$ and 0 respectively, and after doing this the two terms which remain in the expression may be combined to give

$$\cos(\epsilon+\omega t+\omega\lambda/4t^2-\omega\lambda/4\tau^2)=\cos(\epsilon+\omega t+\omega\lambda/4t^2)\,.\,\cos\lambda/4\omega\tau^2$$
$$+\sin(\epsilon+\omega t+\omega\lambda/4t^2)\,.\,\sin\lambda/4\omega\tau^2.$$

The integral (25) may, therefore, be written as

$$x=-\mathrm{E}_0/\omega\mathrm{M}\,.\,e^{-\lambda/2t}.\cos(\epsilon+\omega t+\lambda/4\omega t^2)\,.\,e^{-(n\omega-\lambda t/4)t}\int_{-\infty}^{t} e^{+(n\omega-\lambda/4)\tau}\cos\lambda/4\omega\tau^2\,.\,d\tau$$
$$-\mathrm{E}_0/\omega\mathrm{M}\,.\,e^{-\lambda/2t}.\sin(\epsilon+\omega t+\lambda/4\omega t^2)\,.\,e^{-(n\omega-\lambda t/4)t}\int_{-\infty}^{t} e^{(n\omega-\lambda/4)\tau}\sin\lambda/4\omega\tau^2.\,d\tau.$$

In these two terms the second parts are quantities which vary slowly with and may be regarded as the amplitudes of the oscillatory factors written first. In form they are precisely the same as the coefficients A, B, already calculated for the case of a gliding tone. The effective value of the damping coefficient is $(n-\lambda/4\omega)$. The oscillatory factors have a pulsatance

$$\omega(1+\lambda t/2),$$

equal to the natural pulsatance of the resonant system at time t. Over a considerable number of cycles the frequency is sensibly constant and it is correct to regard the quantity

$$\mathrm{E}_0/\omega\mathrm{M}e^{-\lambda/t2}(\mathrm{A}^2+\mathrm{B}^2)^{\frac{1}{2}}$$

as the amplitude of the oscillatory response at time t. The factor $e^{-\lambda t/2}$ has no significant effect on the form of the curves, which are, therefore, the same as those in fig. 1. The arguments developed previously for the optimum value of the damping when a given frequency range is to be explored in a given time apply equally well here.

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XLV. *A Diffusion Problem in which the Amount of Diffusing Substance is Finite.*—IV. *Solutions for Small Values of the Time.*

By J. Crank,

Courtaulds, Ltd., Maidenhead *.

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§ 1. *Introduction.*

This paper is one of a series concerned with a diffusion problem in which the amount of diffusing substance is finite. Diffusion is considered to take place along capillaries within a plane sheet or a cylinder, and to be accompanied by adsorption on the walls of the capillaries. In the first paper of the series (1) (subsequently referred to as Paper I.) the concentration of adsorbed solute was taken to be a linear function of the concentration of solute free to diffuse, and formal solutions of the appropriate diffusion equations were obtained. In the second paper (2) (Paper II.) the relation between the concentrations of adsorbed and free solute was assumed non-linear and approximate numerical solutions were evaluated. Paper III. (3) dealt with similar problems in a composite circular cylinder having an outer " skin " in which the diffusion coefficient differed from that of the central core.

The solutions obtained present M_t, the total mass of solute in the sheet or cylinder at time t, as a function of t. The series solutions of Paper I. converge very slowly for small values of the time and become inconvenient for numerical evalution. The numerical solutions of Paper II. were obtained by a method which involves replacing derivatives by finite difference ratios. As there is a singularity in the solution at $t=0$, the finite difference approximation is a poor one in the region of small t. The purpose of the present paper is to develop solutions, for both linear and non-linear adsorption, which are particularly suitable for small t. It is found that not only can the new formal solutions be used for values of t so small that the solutions of Paper I. are inconvenient, but also in some cases they provide useful alternatives to the earlier solutions in the intermediate range between small and large t. The range in time for which each solution is applicable in practice is considered more carefully later in the paper (§ 4.1).

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* Communicated by A. H. Wilson, F.R.S.

Diffusion into a Lamina.

§ 2.1. *Linear adsorption.*

It is convenient to repeat the statement of the problem and the relevant equations as set out in Paper I. Suppose that an infinite sheet of uniform material of thickness $2a$ is placed in a solution and that the solute is allowed to diffuse into the sheet and is deposited in the capillaries of the sheet. The sheet occupies the space $-a \leqslant x \leqslant a$, while the solution is of limited extent and occupies the spaces $-l-a \leqslant x \leqslant -a$, $a \leqslant x \leqslant l+a$. The concentration of the solute in the solution is always uniform and is initially C_B, while initially the sheet is free from solute.

The diffusion equation is

$$\frac{\partial C}{\partial t}=\frac{K}{R+1}\frac{\partial^2 C}{\partial x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

with C the concentration of diffusing solute, R the ratio of the concentrations of adsorbed and free solute in a given volume, and K the diffusion coefficient. A new variable $y(x, t)$ is introduced, where

$$y(x, t)=\int_0^x C(x', t)dx', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and equation (1) reduces to

$$\frac{\partial y}{\partial t}=\frac{K}{R+1}\frac{\partial^2 y}{\partial x^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The boundary condition on $x=a$ expressing the conservation of the solute is

$$\frac{\partial y}{\partial x}=C_B-\frac{R+1}{l}y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Retaining the nomenclature of the earlier paper it is convenient to write

$$X=\frac{x}{a}, \quad T=\frac{Kt}{(R+1)a^2}=\beta t, \quad \alpha=\frac{l}{(R+1)a}, \quad . \quad . \quad . \quad . \quad (5)$$

so that it is required to find solutions of

$$\frac{\partial y}{\partial T}=\frac{\partial^2 y}{\partial X^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

with the boundary condition

$$\frac{\partial y}{\partial X}=aC_B-\frac{y}{\alpha} \quad (X=1), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and the initial condition at $T=0$

$$y=0 \quad (-1 \leqslant X \leqslant 1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The Fourier type of solution developed in Paper I. is

$$\frac{M_t}{M_\infty}=1-\sum_n \frac{2\alpha(1+\alpha)}{1+\alpha+\alpha^2 q_n^2}e^{-\beta q_n^2 t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where the q_n's are the non-zero positive roots of

$$\tan q_n = -\alpha q_n. \quad \ldots \ldots \ldots \quad (10)$$

M_t is the total amount of solute in the sheet at time t and M_∞ the quilibrium value attained theoretically after infinite time.

In the case of an infinite amount of solute ($\alpha=\infty$) the roots of (10) are $q_n=(n+\frac{1}{2})\pi$ and we then have

$$\frac{M_t}{M_\infty}=1-\sum_{n=0}^{\infty}\frac{8}{(2n+1)^2\pi^2}e^{-\beta(n+\frac{1}{2})^2\pi^2 t}. \quad \ldots \quad (11)$$

The solutions (9) and (11) are convenient for moderate and large values of βt, but for small values of βt the series are only slowly convergent and so are inconvenient for numerical evaluation. If a solution of the equations (6), (7), (8) is obtained by means of Heaviside operators or Laplace transforms the solution of type (9) arises by expanding the operator solution in partial fractions. An alternative is to expand the operator solution in exponential functions. The expansion in partial fractions leads to a solution suitable for large βt, and in exponentials to one suitablef or small βt (4). Thus, following Goldstein (5), the required solution of the present problem for small βt may be obtained as follows.

Writing
$$\bar{y}(p)=\int_0^\infty y(\mathrm{T})e^{-p\mathrm{T}}\,d\mathrm{T},$$

the operational solution of (6) with (7) and (8), which is anti-symmetric about $\mathrm{X}=0$, is

$$\bar{y}=\frac{a\mathrm{C_B}\sinh p^{\frac{1}{2}}\mathrm{X}}{p\{p^{\frac{1}{2}}\cosh p^{\frac{1}{2}}+(1/\alpha)\sinh p^{\frac{1}{2}}\}}. \quad \ldots \ldots \quad (12)$$

Putting $q=p^{\frac{1}{2}}$ and $h=1/\alpha$ for ease in presentation (12) may be written

$$\left.\begin{aligned}\bar{y}&=\frac{a\mathrm{C_B}(e^{q\mathrm{X}}-e^{-q\mathrm{X}})}{q^2\{q(e^q+e^{-q})+h(e^q-e^{-q})\}}\\&=\frac{a\mathrm{C_B}e^{-q(1-\mathrm{X})}}{q^2(q+h)}(1-e^{-2q\mathrm{X}})\left(1+\frac{h-q}{h+q}e^{-2q}+\ldots\right),\\ \text{i. e.}\quad \bar{y}&=\frac{a\mathrm{C_B}}{q^2(q+h)}\{e^{-q(1-\mathrm{X})}-e^{-q(1+\mathrm{X})}\}+\frac{a\mathrm{C_B}(h-q)}{q^2(q+h)^2}\{e^{-q(3-\mathrm{X})}-e^{-q(3+\mathrm{X})}\}.\\&\qquad+\ldots\end{aligned}\right\} \quad \ldots \ldots \quad (13)$$

Taking the first term only and interpreting the operator solution by the known rules (see for example ref. (6), Appendix V. 14) we find

$$y(1,\mathrm{T})=\frac{l\mathrm{C_B}}{\mathrm{R}+1}\left\{1-e^{\mathrm{T}/\alpha^2}\operatorname{erfc}\frac{\mathrm{T}^{\frac{1}{2}}}{\alpha}\right\}-\left\{\operatorname{erfc}\frac{1}{\mathrm{T}}-e^{(2/\alpha)+(\mathrm{T}/\alpha^2)}\operatorname{erfc}\left(\frac{1}{\mathrm{T}}+\frac{\mathrm{T}^{\frac{1}{2}}}{\alpha}\right)\right.. \quad \ldots \ldots \quad (14)$$

The main purpose of the present work is to obtain a solution convenient in the range of βt for which (9) is cumbersome. In this range it is found that the second term in the right hand side of (14) is negligible compared with the first and so, remembering that for half the sheet $M_t=(R+1)y(1, T)$ and $M_\infty=lC_B/(1+\alpha)$, the required solution is

$$\frac{M_t}{M_\infty}=(1+\alpha)\left\{1-e^{\beta t/\alpha^2}\ \operatorname{erfc}\left(\frac{\beta t}{\alpha^2}\right)^{\frac{1}{2}}\right\}. \qquad (15)$$

For $\alpha=\infty$, *i.e.* $h=0$, the complete operator solution (12) may be expressed in the form of a simple series (ref. (6), p. 252) namely,

$$\bar{y}=\frac{aC_B}{q^3}+\frac{2aC_B}{q^3}\sum_{n=1}^{\infty}(-1)^n e^{-2nq}, \qquad (16)$$

and the corresponding solution is

$$\frac{M_t}{M_\infty}=\frac{2}{\pi^{\frac{1}{2}}}(\beta t)^{\frac{1}{2}}+4\sum_{n=1}^{\infty}(-1)^n\ \operatorname{ierfc}\frac{n}{(\beta t)^{\frac{1}{2}}}. \qquad (17)$$

The special case of (15) for $\alpha=\infty$, which may be obtained by expanding the exponential function and the error function complement in powers of $(\beta t)^{\frac{1}{2}}$ is

$$\frac{M_t}{M_\infty}=\frac{2}{\pi^{\frac{1}{2}}}(\beta t)^{\frac{1}{2}}, \qquad (18)$$

which is seen to be the first term of the complete solution (17). The notation adopted above is that

$$\operatorname{erf} x=\frac{2}{\pi^{\frac{1}{2}}}\int_0^x e^{-z^2}dz,\quad \operatorname{erfc} x=1-\operatorname{erf} x,\quad \operatorname{ierfc} x=\int_x^{\infty}\operatorname{erfc} z\,dz.$$

It is useful to note that MacKay⁽⁷⁾ has tabulated M_t/M_∞ using (11), with (18) for small βt, for $0\leqslant\pi^2\beta t/4\leqslant 3{\cdot}50$ in intervals of $0{\cdot}01$ in $\pi^2\beta t/4$, to four figures.

For small α, equation (15) is still valid, but is not very convenient for numerical evaluation for moderate βt because the exponential term becomes large and the error function complement small so that good tables and considerable care are needed to avoid errors in the product. In such a situation it is preferable to substitute the asymptotic expansion (see ref. (6) Appendix II. (5))

$$\operatorname{erfc} x=\frac{e^{-x^2}}{x\pi^{\frac{1}{2}}}\left(1-\frac{1}{2x^2}+\frac{1\,.\,3}{4x^4}-\frac{1\,.\,3\,.\,5}{2^3x^6}+\ldots.\right). \qquad (19)$$

Substitution of (19) in (15) with $x=(\beta t)^{\frac{1}{2}}/\alpha$ gives

$$\frac{M_t}{M_\infty}=(1+\alpha)\left\{1-\frac{\alpha(\beta t)^{-\frac{1}{2}}}{\pi^{\frac{1}{2}}}+\frac{\alpha^3(\beta t)^{-\frac{3}{2}}}{2\pi^{\frac{1}{2}}}-\frac{3\alpha^5(\beta t)^{-\frac{5}{2}}}{4\pi^{\frac{1}{2}}}+\ldots\right\} \qquad (20)$$

which is convenient for small α provided βt is not too small.

§ 2.2. *Non-linear adsorption.*

The equation for diffusion with non-linear adsorption is

$$\frac{\partial C}{\partial t}=K\frac{\partial^2 C}{\partial x^2}-\frac{\partial S}{\partial t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where S, the concentration of adsorbed solute, is a known function of C. A convenient method of evaluating approximate numerical solutions of this type of equation was described in Paper II. This method involves replacing the derivatives with respect to x and t by finite difference ratios and it is clearly undesirable to make such a replacement in the region of $t=0$, where there is a singularity in the solution for the present boundary conditions. On interpreting the operator solution (13), before substituting the value $X=1$ to give (14), the solution for small t when the sheet is effectively semi-infinite, is found to be a function of the variables $t^{\frac{1}{2}}$ and $x/t^{\frac{1}{2}}$. (Note that x is now measured from the surface of the sheet instead of from the centre.) This suggests the use of the variables

$$\tau=(Kt)^{\frac{1}{2}}/a, \quad \xi=x/(Kt)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

in the numerical treatment of the non-linear problem. If the sheet is considered to be semi-infinite the boundary conditions are

$$lC+\int_0^\infty C+S)dx=lC_B, \quad x=0, \quad t>0, \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$C=0, \quad x=\infty, \quad t\geqslant 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

with the initial conditions at $t=0$,

$$C=C_B, \quad x=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$C=0, \quad x\geqslant 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

On using (22) and writing $c=C/C_B$, $s=S/C_B$ equations (21) and (23–26) reduce to

$$2\frac{\partial^2 c}{\partial \xi^2}=\tau\frac{\partial}{\partial \tau}(c+s)-\xi\frac{\partial}{\partial \xi}(c+s), \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$lc+a\int_0^\infty (c+s)\tau d\xi=l, \quad \xi=0, \quad \tau>0, \quad . \quad . \quad . \quad . \quad (28)$$

$$c=0, \quad \xi=\infty, \quad \tau\geqslant 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

$$c=1, \quad \xi=0, \quad \tau=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

The condition at $\tau=0$ for finite $\xi>0$ is given by putting $\tau=0$ in (27), *i. e.*

$$2\frac{\partial^2 c}{\partial \xi^2}=-\xi\frac{\partial}{\partial \xi}(c+s), \quad \tau=0, \quad \xi>0. \quad . \quad . \quad . \quad . \quad . \quad (31)$$

This is an ordinary differential equation in ξ, the solution of which, for the conditions

$$c=1, \quad \xi=0, \quad c=0, \quad \xi=\infty, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

gives the initial condition at $\tau=0$ for the partial differential equation (27).

If s is a function of c such that no formal solution of (31) exists, a numerical solution can readily be evaluated. In this case a trial and error process is necessary to fit the boundary conditions at the two ends of the range of integration. Such a process can be performed by step-by-step integration using any suitable integration formula, or a differential analyser can be used provided the accuracy of this machine is suitable for the particular problem considered.

Numerical solutions of (27) may be evaluated by the method described in Paper II., the derivatives with respect to both ξ and τ being replaced in a particular way by finite difference ratios. If c_{m-1}, c_m, c_{m+1} denote the values of c at $\xi_m-\delta\xi$, ξ_m, $\xi_m+\delta\xi$ respectively at time τ and c'_{m-1}, c'_m, c'_{m+1} are the corresponding values at $\tau+\delta\tau$ then, with a similar notation for s, the required finite difference form of (27) for the point ξ_m is

$$(c'_m+s'_m)-(c_m+s_m)=\frac{\delta\tau}{(\tau+\frac{1}{2}\delta\tau)(\delta\xi)^2}\{(c'_{m+1}+c_{m+1})-2(c'_m+c_m)+(c'_{m-1}+c_{m-1})\}$$
$$+\frac{m\delta\tau}{4(\tau+\frac{1}{2}\delta\tau)}\{(c'_{m+1}+s'_{m+1}+c_{m+1}+s_{m+1})$$
$$-(c'_{m-1}+s'_{m-1}+c_{m-1}+s_{m-1})\} \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Equations (33) for different $m(m=1, 2, \ldots)$ form a set of simultaneous algebraic equations for c'_m as a function of m, given c_m and s_m for each value of m. c_0 is given by the boundary condition (28). The equations are conveniently solved by the iterative process described in Paper II. applied at successive times so that the solution proceeds in steps $\delta\tau$, starting from the values of c_m, s_m at $\tau=0$ given by previous solution of (31).

Diffusion into a Circular Cylinder of Infinite Length and Radius "a".

§ 3.1. *Linear adsorption.*

In this section diffusion is considered to take place into a circular cylinder of infinite length. The cylinder occupies the space $r \leqslant a$ (using cylindrical coordinates) while the cross-section of the bath of solution in which it is immersed is A (excluding the space occupied by the cylinder). The concentration of solute in the solution is always uniform and is initially C_B. The cylinder is initially free from solute.

In order to obtain solutions suitable for small values of the time in the plane case, the hyperbolic functions occurring in the operator solution were expanded in a series of exponentials. In the cylindrical case, the corresponding procedure, suggested by Goldstein (8), is to use the asymptotic expansions of the Bessel functions which occur in the operator solution.

The problem is formulated in Paper I. in terms of the new variable

$$y(r, t)=2\pi\int_0^r r'C(r', t)dr'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Solutions are required of the equation

$$\frac{\partial y}{\partial t}=\frac{K}{R+1}r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial y}{\partial r}\right), \qquad (35)$$

with the boundary condition

$$\frac{A}{2\pi a}\frac{\partial y}{\partial r}=C_B-(R+1)y, \quad (r=a), \qquad (36)$$

and the initial condition at $t=0$,

$$y=0, \quad 0\leqslant r\leqslant a. \qquad (37)$$

The Fourier-Dini type of solution developed in Paper I. is

$$\frac{M_t}{M_\infty}=1-\sum_n\frac{4\alpha(1+\alpha)}{4+4\alpha+\alpha^2q_n{}^2}e^{-\beta q_n{}^2t}, \qquad (38)$$

where the q_n's are the positive, non-zero roots of

$$\alpha q_nJ_0(q_n)+2J_1(q_n)=0 \qquad (39)$$

and

$$\alpha=\frac{A}{\pi a^2(R+1)}, \quad \beta=\frac{K}{(R+1)a^2}. \qquad (40)$$

For $\alpha=\infty$, (38) reduces to

$$\frac{M_t}{M_\infty}=1-\sum_n\frac{4}{q_n{}^2}e^{-\beta q_n{}^2t}, \qquad (41)$$

where

$$J_0(q_n)=0, \quad q_n>0. \qquad (42)$$

The series (38) and (41) are convenient for large βt but converge very slowly for small βt. Alternative solutions may be developed in the following way.

Writing $\bar{y}$ for the Laplace transform of y as before, the subsidiary equations are

$$\frac{K}{R+1}r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \bar{y}}{\partial r}\right)-p\bar{y}=0, \qquad (43)$$

$$\frac{A}{2\pi a}\frac{\partial \bar{y}}{\partial r}=\frac{AC_B}{p}-(R+1)\bar{y}, \quad r=a. \qquad (44)$$

The required solution of (43) is

$$\bar{y}=F(p)rI_1\left(\frac{p^{\frac{1}{2}}}{\beta^{\frac{1}{2}}}\frac{r}{a}\right) \qquad (45)$$

with F(p) given by the boundary condition (44), so that finally, putting $q=p^{\frac{1}{2}}/(\beta^{\frac{1}{2}}a)$, we have

$$\bar{y}=\frac{2AC_BrI_1(qr)}{(R+1)ap\{\alpha qaI_0(qa)+2I_1(qa)\}}. \qquad (46)$$

Provided r/a is not small, the asymptotic expansions of the Bessel functions $I_1(qr)$, $I_1(qa)$, $I_0(qa)$ may be used and subsequent expansion of (46) in ascending powers of $1/q$ gives

$$\bar{y}=\frac{2AC_Br^{\frac{1}{2}}\beta^{\frac{1}{2}}e^{-q(a-r)}}{(R+1)a^{\frac{1}{2}}\alpha p^{\frac{3}{2}}}\left\{1-\frac{1}{q}\left(\frac{3}{8r}+\frac{1}{8a}+\frac{2}{\alpha a}\right)-\frac{1}{q^2}\left(\frac{15}{128r^2}+\frac{7}{128a^2}-\frac{5}{4\alpha a}-\frac{3}{4\alpha r}-\frac{3}{64ar}\right)+\dots\right\}. \quad (47)$$

The successive terms may be interpreted using the following relations (9)

$$\int_0^\infty e^{-pt}(4t)^{\frac{1}{2}n}i^n \operatorname{erfc}\left(\tfrac{1}{2}xt^{-\frac{1}{2}}\right)dt=p^{-\frac{1}{2}n-1}e^{-xp^{\frac{1}{2}}}, \quad (48)$$

where

$$i^n \operatorname{erfc} x=\int_x^\infty i^{n-1}\operatorname{erfc}\xi d\xi, \quad (49)$$

and

$$\operatorname{erfc} x=\frac{2}{\pi^{\frac{1}{2}}}\int_x^\infty e^{-\xi^2}d\xi. \quad (50)$$

Now $M_t=(R+1)y(a, t)$ and $M_\infty=AC_B/(1+\alpha)$, and so if $y(a, t)$ is derived from (47) by using (48)–(50) and substituting $r=a$, it follows readily that

$$\frac{M_t}{M_\infty}=\frac{2}{\alpha}(1+\alpha)\left\{\frac{2}{\pi^{\frac{1}{2}}}(\beta t)^{\frac{1}{2}}-\left(\frac{1}{2}+\frac{2}{\alpha}\right)\beta t-\frac{4}{3\pi^{\frac{1}{2}}}\left(\frac{1}{8}-\frac{2}{\alpha}-\frac{4}{\alpha^2}\right)(\beta t)^{\frac{3}{2}}+\dots\right\}. \quad (51)$$

In the special case $\alpha=\infty$, (51) becomes

$$\frac{M_t}{M_\infty}=2\left\{\frac{2}{\pi^{\frac{1}{2}}}(\beta t)^{\frac{1}{2}}-\frac{1}{2}\beta t-\frac{1}{6\pi^{\frac{1}{2}}}(\beta t)^{\frac{3}{2}}+\dots\right\}. \quad (52)$$

It is found that (51) is suitable for large α but it is clear that for small α the series will converge only for very small βt. In this case it is necessary to develop an alternative solution.

The operator solution (46) may be re-written

$$\bar{y}=\frac{AC_BrI_1(qr)}{(R+1)apI_1(qa)[1+\{\alpha qaI_0(qa)\}/\{2I_1(qa)\}]}. \quad (53)$$

Using the asymptotic expansions for $I_0(qa)$ and $I_1(qa)$ as above we have

$$1+\frac{\alpha qaI_0(qa)}{2I_1(qa)}=1+\frac{1}{2}\alpha qa+\frac{1}{4}\alpha+\frac{3}{16}\frac{\alpha}{qa}+\dots \quad (54)$$

In these expansions $1/qa$ has been assumed small and hence the term $\frac{1}{2}\alpha qa$ will be large compared with $\frac{3}{16}\alpha/qa$ and higher terms. Unless α is very small $\frac{1}{2}\alpha qa$ may be of the order of one for small t and hence it is desirable to avoid the binomial expansion of the denominator of (53) and to write (53), for $r=a$, in the form

$$\bar{y}=\frac{AC_B}{(R+1)p}\frac{1}{(1+\frac{1}{4}\alpha+\frac{1}{2}\alpha qa)}=\frac{2AC_B\beta^{\frac{1}{2}}}{(R+1)\alpha}\frac{1}{p\{p^{\frac{1}{2}}+(1+\frac{1}{4}\alpha/a)2\beta^{\frac{1}{2}}/\alpha\}}. \quad (55)$$

This operational solution may be interpreted immediately (see for example ref. (6), Appendix V. 14) and the solution in the desired form is

$$\frac{\mathrm{M}_t}{\mathrm{M}_\infty}=\frac{1+\alpha}{1+\frac{1}{4}\alpha}[1-e^{4(1+\frac{1}{4}\alpha)^2\beta t/\alpha^2}\operatorname{erfc}\{2(1+\tfrac{1}{4}\alpha)(\beta t)^{\frac{1}{2}}/\alpha\}]. \quad . \quad . \quad (56)$$

This solution is closely comparable with (15) and is convenient for use when α is moderate. For α very small, so that $\beta t/\alpha^2$ is large for moderate βt, it is convenient to use the asymptotic expansion (19) as in the plane case and hence to write

$$\frac{\mathrm{M}_t}{\mathrm{M}_\infty}=\frac{1+\alpha}{1+\frac{1}{4}\alpha}\left\{1-\frac{\alpha(\beta t)^{-\frac{1}{2}}}{2\pi^{\frac{1}{2}}(1+\frac{1}{4}\alpha)}+\frac{\alpha^3(\beta t)^{-\frac{3}{2}}}{16\pi^{\frac{1}{2}}(1+\frac{1}{4}\alpha)^3}-\frac{3\alpha^5(\beta t)^{-\frac{5}{2}}}{128\pi^{\frac{1}{2}}(1+\frac{1}{4}\alpha)^5}+\ldots\right\}. \quad . \quad . \quad . \quad . \quad . \quad (57)$$

§ 3.2. *Non-linear adsorption.*

The treatment here is closely similar to that of section 2.2, and the equations may be presented with little explanation.

In terms of the variables

$$\tau=(\mathrm{K}t)^{\frac{1}{2}}/a, \quad \xi=(a-r)/(\mathrm{K}t)^{\frac{1}{2}}, \quad c=\mathrm{C}/\mathrm{C}_\mathrm{B}, \quad s=\mathrm{S}/\mathrm{C}_\mathrm{B}, \quad . \quad . \quad (58)$$

the relevant equations are

$$\tau=\frac{\partial}{\partial\tau}(c+s)-\xi\frac{\partial}{\partial\xi}(c+s)=2\frac{\partial^2 c}{\partial\xi^2}-\frac{2\tau}{1-\tau\xi}\frac{\partial c}{\partial\xi}, \quad . \quad . \quad . \quad (59)$$

$$c=0, \quad \xi=\infty, \quad \tau\geqslant 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

$$c=1, \quad \xi=0, \quad \tau=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

$$1=c+\frac{2\pi a^2}{\mathrm{A}}\int_0^\infty(c+s)(1-\tau\xi)\tau d\xi, \quad \xi=0, \quad \tau>0, \quad . \quad . \quad . \quad (62)$$

since for small τ the upper limit of ξ may be taken as infinite. For intermediate ξ the initial condition at $\tau=0$ is the solution of the ordinary differential equation (31) with the end conditions (32) as in the plane case. The desired finite difference form of (59) is

$$\begin{aligned}(c'_m+s'_m)-(c_m+s_m)=&\frac{\delta\tau}{(\tau+\frac{1}{2}\delta\tau)(\delta\xi)^2}\{(c'_{m+1}+c_{m+1})-2(c'_m+c_m)+(c'_{m-1}+c_{m-1})\}\\&+\frac{m\delta\tau}{4(\tau+\frac{1}{2}\delta\tau)}\{(c'_{m+1}+s'_{m+1}+c_{m+1}+s_{m+1})\\&\qquad-(c'_{m-1}+s'_{m-1}+c_{m-1}+s_{m-1})\}\\&-\frac{\delta\tau\{(c'_{m+1}+c_{m+1})-(c'_{m-1}+c_{m-1})\}}{2\delta\xi\{1-\xi_m(\tau+\frac{1}{2}\delta\tau)\}}, \quad . \quad . \quad . \quad (63)\end{aligned}$$

The equations (63) for different m may be solved iteratively as in the plane case, the process being only slightly more laborious because of the extra term $\frac{2\tau}{1-\tau\xi}\frac{\partial c}{\partial\xi}$ in (59). c_0 is given by the boundary condition (62).

In carrying out a solution of this kind it is found that the values of c become vanishingly small for $\xi > 3{\cdot}0$ approximately. Thus for small τ we have $\tau\xi \ll 1$, in which case (59) may be simplified to

$$\tau\frac{\partial}{\partial\tau}(c+s)-\xi\frac{\partial}{\partial\xi}(c+s)=2\frac{\partial^2 c}{\partial\xi^2}-2\tau\frac{\partial c}{\partial\xi}, \quad . \quad . \quad . \quad . \quad (64)$$

and (62) to

$$1=c+\frac{2\pi a^2}{\mathrm{A}}\int_0^\infty (c+s)\tau d\xi, \quad \xi=0, \quad \tau>0, \quad . \quad . \quad . \quad . \quad (65)$$

with corresponding simplification of the finite difference equations and their solution. The range for which this approximation may be used depends on the accuracy required in a given case.

§ 4.1. *Numerical examples.*

Examples of diffusion with linear adsorption into a plane sheet and into a cylinder have been examined numerically for three values of the parameter α, namely $\alpha=\infty$, $\frac{1}{9}$, $\frac{1}{99}$. $\alpha=\infty$ implies an infinite amount of solute, while for $\alpha=\frac{1}{9}$ and $\alpha=\frac{1}{99}$ in the final equilibrium state 90 and 99 per cent. respectively of the solution is in the sheet or cylinder.

Table I. shows for each α and for several values of βt, values of $\mathrm{M}_t/\mathrm{M}_\infty$ calculated for the plane sheet. Solutions (15), (18), (20), (52), (56), (57), derived by the use of error functions or by the asymptotic expansions for the Bessel functions, are referred to as "small t solutions." Those derived from Fourier, Fourier-Bessel, or Dini series, *i. e.* (9), (11), (41), (38) are called "large t solutions" for convenience. The details of the source of each coloumn of values are given at the head of the column concerned. The values calculated from the small t solutions are presented not only for small βt where they are correct, but also for large βt to show how they deviate there from the correct values indicated by an asterisk and derived from the large t solutions. There is an intermediate range of βt for which the two types of solution overlap and are seen to agree.

For the plane sheet the small t solutions give 3-figure accuracy in the range $0<\beta t \leqslant 0{\cdot}10$ for each value of α examined. These solutions can be used for larger βt if a lower order of accuracy suffices. When α is very small, *e. g.* $\alpha=\frac{1}{99}$, the small t solution can be used over the whole range of βt of practical interest, since in this case it still gives the correct value when $\mathrm{M}_t/\mathrm{M}_\infty$ is very near to its equilibrium value of $1{\cdot}000$, *e. g.* $\mathrm{M}_t/\mathrm{M}_\infty=0{\cdot}997$ when $\beta t=0{\cdot}20$. Where they overlap, the small t solution is slightly more convenient for numerical evaluation, since there is no need to obtain the roots of an equation such as (10). It is useful to note however, that Carslaw and Jaeger have obtained the first 6 roots of (10) for a number of values of α between 0 and 100 and for $\alpha=\infty$ (see ref. (6), Appendix IV., Table II.). The number of terms of the series (9) and (11) used for each value of βt is given in Table I. to indicate the increasing difficulty of using these series as βt is decreased.

Table II. presents the corresponding information for diffusion into a cylinder. For $\alpha=\infty$ the small t solution is only slightly in error at $\beta t=0{\cdot}10$, but for α small and moderate the range of validity is much less, being approximately $0\leqslant\beta t\leqslant 0{\cdot}02$ if 3-figure accuracy is required. However, only 4 or 5 terms of the series (38) are needed for $\beta t=0{\cdot}02$ and so this solution is convenient for higher values of βt. It is clear, however, that calculations for the cylinder tend to be more cumbersome than for the plane sheet, a point noted previously by Jaeger (10) in a similar context.

Table I.

Diffusion into a Plane Sheet.

βt	M_t/M_∞					
	$\alpha=\infty$		$\alpha=\frac{1}{9}$		$\alpha=\frac{1}{99}$	
	Large t solution. Equation (11)	Large α, small t solution. Equation (18)	Large t solution. Equation (9)	Moderate α, small t. Equation (15)	Large t solution. Equation (9)	Small α, small t solution. Equation (20)
0·0000391				0·067		0·445†
0·000156				0·128		0·637†
0·000625		0·0282		0·234		0·794
0·00250	0·0564 (12 terms)	0·0564		0·398		0·895
0·010	0·113 (5 terms)	0·113	0·604 (6 terms)	0·604	0·953 (7 terms)	0·953
0·050	0·252 (3 terms)	0·252	0·829 (4 terms)	0·829	0·984 (4 terms)	0·984
0·100	0·357 (2 terms)	0·357	0·902 (3 terms)	0·902	0·992 (3 terms)	0·992
0·200	0·504* (2 terms)	0·505	0·959* (2 terms)	0·960	0·997 (2 terms)	0·997

* Indicates the correct value where the two types of solution differ.

† Calculated from equation (15) because the series 20 converges slowly for very small βt.

The step-by-step method of solution (§ 2.2) has been used to evaluate approximate numerical solutions when the adsorption is linear. Some of the details of the process of numerical solution for the plane sheet in this special case may be helpful.

When $s=\mathrm{R}c$ the diffusion coefficient K is to be replaced by K/(R+1) and the transformations (22) become

$$\tau=\left(\frac{\mathrm{K}t}{\mathrm{R}+1}\right)^{\frac{1}{2}}\frac{1}{a}, \quad \xi=\left(\frac{\mathrm{R}+1}{\mathrm{K}t}\right)^{\frac{1}{2}}x, \quad \ldots \ldots \quad (66)$$

i. e.

$$\tau=(\beta t)^{\frac{1}{2}}, \quad \xi=\mathrm{X}/(\beta t)^{\frac{1}{2}}, \quad \ldots \ldots \quad (67)$$

for comparison with (5).

Table II.

Diffusion into a Cylinder.

βt	$\mathrm{M}_t/\mathrm{M}_\infty$					
	$\alpha=\infty$		$\alpha=\frac{1}{9}$		$\alpha=\frac{1}{99}$	
	Large *t* solution. Equation (41)	Large α, small *t* solution. Equation (52)	Large *t* solution. Equation (38)	Moderate α, small *t* solution. Equation (56)	Large *t* solution. Equation (38)	Small α, small *t* solution. Equation (57)
0·0000391				0·128		0·634†
0·000156				0·233		0·796 †
0·000625		0·056		0·394		0·895
0·00250		0·110		0·596		0·951
0·010	0·215 (7 terms)	0·215	0·786 (6 terms)	0·786	0·979 (6 terms)	0·979
0·020	0·299 (6 terms)	0·299	0·860* (5 terms)	0·862	0·987 (5 terms)	0·987
0·050	0·452 (4 terms)	0·452	0·933* (4 terms)	0·938	0·994* (3 terms)	0·995
0·100	0·606* (2 terms)	0·608	0·970* (2 terms)	0·978	0·998* (2 terms)	0·999

* Indicates the correct value where the two types of solution differ.

† Calculated from equation (56) because the series (57) converges slowly for very small βt.

Equations (27) to (31) inclusive now reduce to

$$2\frac{\partial^2 c}{\partial \xi^2}=\tau\frac{\partial c}{\partial \tau}-\xi\frac{\partial c}{\partial \xi}, \quad \ldots \ldots \quad (68)$$

$$c+\frac{1}{\alpha}\int_0^\infty c\tau\, d\xi=1, \quad \ldots \ldots \quad (69)$$

$$c=0, \quad \xi=\infty, \quad \tau\geqslant 0, \quad \ldots \ldots \ldots \quad (70)$$

$$c=1, \quad \xi=0, \quad \tau=0, \quad \ldots \ldots \ldots \quad (71)$$

$$2\frac{\partial^2 c}{\partial \xi^2}=-\xi\frac{\partial c}{\partial \xi}, \quad \tau=0, \quad \xi>0. \quad \ldots \ldots \quad (72)$$

Equation (72) has the formal solution

$$c=\operatorname{erfc}\left(\tfrac{1}{2}\xi\right) \quad \ldots \ldots \ldots \quad (73)$$

satisfying the end conditions (32). The presence of the error function complement in (73) suggests that the range of $\frac{1}{2}\xi$ over which c might be expected to vary appreciably is approximately $0\leqslant\frac{1}{2}\xi\leqslant 3{\cdot}0$, since the error function changes very slowly for higher values of $\frac{1}{2}\xi$. This expectation is confirmed in practice; for the range of βt considered and working to 3-figure accuracy c is found to remain zero for all $\xi\geqslant 6{\cdot}0$. The results presented in Table III. were evaluated by taking $\delta\xi=\frac{1}{2}$, $\delta\tau=\frac{1}{160}$, so that taking the upper limit of ξ as 6·0, 12 intervals $\delta\xi$ were used. With these values the form of the finite difference equation most convenient for numerical evaluation is

$$(\tau_{\mathrm{M}}+2)c'_m=(\tau_{\mathrm{M}}-2)c_m+(c'_{m+1}+c_{m+1})\left(1+\frac{m}{16}\right)+(c'_{m-1}+c_{m-1})\left(1-\frac{m}{16}\right), \quad \ldots \ldots (74)$$

where $\tau_{\mathrm{M}}=(\delta\xi)^2(\tau+\frac{1}{2}\delta\tau)/\delta\tau$, so that writing $\tau=n\delta\tau$ we have

$$\tau_{\mathrm{M}}+2=(2n+17)/8, \quad \tau_{\mathrm{M}}-2=(2n-15)/8, \quad \ldots \ldots (75)$$

which factors are readily evaluated as functions of n before the step-by-step process of solution commences. Using Simpson's one-third formula for numerical integration, (69) becomes

$$c_0=1-\frac{\tau\delta\xi}{3\alpha}(c_0+4c_1+2c_2+4c_3+\ldots+4c_{11}+c_{12}), \quad \ldots (76)$$

and in this example $\tau\delta\xi/(3\alpha)=\frac{3}{2}\tau$. The solution proceeds as described in Paper II.

Several values of $\mathrm{M}_t/\mathrm{M}_\infty$ obtained in this way for the plane sheet and for the cylinder are compared in Table III. with the values given by the formal solutions for small t, in the range $0<\tau\leqslant 0{\cdot}05$, *i. e.* $0<\beta t\leqslant 0{\cdot}0025$. The values for the cylinder are obtained using $\delta\xi=\frac{1}{2}$, $\delta\tau=\frac{1}{320}$, but otherwise the details of solution are very similar to those for the plane sheet and do not merit further discussion. The comparisons with the formal solutions in Table III. show that the transformations (22) for the plane and (58) for the cylinder are successful in removing the singularity at $t=0$. They also give some indication of the sizes of the intervals $\delta\xi$, $\delta\tau$ required to obtain a desired degree of accuracy. The values shown in Table III. are sufficiently accurate for most practical purposes.

Better accuracy can be obtained if necessary by taking smaller intervals $\delta\xi$ and $\delta\tau$.

If solutions are to be evaluated by the step-by-step numerical method for moderate and large βt, the variables ξ and τ are used for small βt as indicated in this paper. At some suitable value of βt the solution is contained in terms of the original variables x and t following the method of Paper II. Two considerations determine the value of βt at which the change-over is made. They are the same for the plane sheet and the cylinder and so only the former need be discussed. First, it is inconvenient to use the variables ξ, τ beyond the time at which the concentration at the mid-section of the sheet ceases to be effectively zero, because a boundary condition must then be imposed at a finite, constant, but non-zero value of x. This means imposing a condition at a boundary variable in ξ, and the upper limit of the integral in (28) is no longer infinite but varies with τ, conditions which are inconvenient to handle numerically. The second consideration fixes a lower limit to the range of βt for which the variables ξ and τ are used in practice. The solution in terms of ξ and τ provides values of the concentration c at discrete values of ξ, *e. g.* $\xi=0, \frac{1}{2}, 1, \frac{3}{2}$, etc., for successive values of τ increasing in steps $\delta\tau$. The last line of the solution in ξ and τ provides the starting values for the first line of the solution in x and t. For this purpose it is desirable to carry the ξ, τ-solution to such a value of τ that the solute has penetrated sufficiently far into the sheet to give appreciable values of the concentration at 3 or 4 values of x, when the whole range in x is divided into a reasonable number of intervals, *e. g.* 12 for the x, t-solution.

TABLE III.

Comparison of Numerical and Formal Solutions for $\alpha=\frac{1}{9}$.

	M_t/M_∞			
βt	Plane sheet		Cylinder	
	Moderate α, small t solution. Equation (15)	Approximate numerical solution.	Moderate α, small t solution. Equation (56)	Approximate numerical solution.
0·00000977			0·067	0·068
0·0000391	0·067	0·068	0·128	0·128
0·000156	0·128	0·128	0·233	0·231
0·000625	0·234	0·231	0·394	0·390
0·00250	0·398	0·394	0·596	0·591

Thus in the present example for $\alpha=\frac{1}{9}$ at $\tau=0{\cdot}05$ the values of c for successive values of ξ are shown in Table IV. Now, from (22), $x/a=\xi\tau$ so that c is known at the values of x/a shown in Table IV.

TABLE IV.

Concentration-Distance Relation at $\tau=0{\cdot}05$ for the Plane Sheet.

ξ	0	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0	4·5	5·0	5·5	6·0
c	0·643	0·494	0·340	0·213	0·118	0·058	0·028	0·013	0·007	0·004	0·002	0	0
x/a	0	0·025	0·050	0·075	0·100	0·125	0·150	0·175	0·200	0·225	0·250	0·275	0·300

If it is desired to carry out the x, t-solution with steps $\delta x=\frac{1}{12}$, c may readily be obtained by interpolation at $x=0$, $\frac{1}{12}$, $\frac{1}{6}$, $\frac{1}{4}$, etc. and in this case there are 4 non-zero values of c, which are suitable as starting values for the x, t-solution, *i. e.* 0·643, 0·183, 0·018, 0·002 respectively.

The author is indebted to Miss S. M. Godson who carried out the numerical calculations, and to the directors of Courtaulds, Ltd. for permission to publish the paper.

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XLVI. *A Comparison of the Rayleigh Disk and the Acoustic Radiometer Methods for the Measurement of Sound-Wave Energy.*

By JUL. HARTMANN and TAGE MORTENSEN *.

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[Plate II.]

1. INTRODUCTION.

ABSOLUTE measurement of sound-wave energy is based on the well-known Rayleigh Disk and Acoustic Radiometer Methods. The theory of the former has been developed by W. König (1). The theory of the acoustic radiometer was given by Lord Rayleigh (2). He showed that a train of sound waves impinging perpendicularly on a plane wall reflecting the waves totally will act on the wall with a pressure S dyne/cm.^{2}, which is equal to twice the density of the wave energy E erg/cm.^{3} in the impinging wave train, thus

$$S=2E. \quad \ldots\ldots\ldots\ldots (1)$$

This expression is completely analogous with the Maxwell expression for the pressure of light on a totally reflecting wall. Now, however, Rayleigh, on further consideration, modified his first theory and arrived at the expression

$$S=(\kappa+1)E, \quad \ldots\ldots\ldots\ldots (2)$$

where $\kappa=c_p/c_v$ is the ratio of the two specific heats for the gas in question. The latter formula was put to the test by W. Zernov (3), who compared measurements with the acoustic radiometer and the so-called Wien Membrane Manometer, applying the theory (2) to the former apparatus. He found that the results agreed within 2 per cent. This was taken as a proof of the reliability of Lord Rayleigh's second formula and so this formula (2) was generally adopted, although the evidence established by Zernov was certainly not very strong. Lord Rayleigh's second formula was in fact adopted in practically all publications and handbooks up to our days, thus, for instance, in 'Lehrbuch der praktischen Physik', by F. Kohlrausch, fünfzehnte Auflage 1927 †, and in, 'Handbuch der Experimentalphysik', Bd. 17, 1934. In the monograph, 'The Acoustic Air-Jet Generator' (4) ‡, by one of the authors of the present paper, the second Rayleigh formula was also utilized, which gave rise to a divergence of about 20 per cent. between the figures found for the sound-wave intensity

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[*] Communicated by the Authors.

[†] Also in the 18th edition of this Handbuch published in New York in 1947, this formula is stated.

[‡] In the following this monograph is referred to by the letters A J G.

by the Rayleigh disk and the radiometer methods. Unfortunately, it was not until the monograph indicated above had been published that the attention of its author was drawn to the doubtfulness of Lord Rayleigh's second formula by Egon Hiedemann's book on Super Sonics (5). From this book he learned that Lord Rayleigh's first formula, $S=2E$, had during the last two decades been derived in various ways different from that taken by Lord Rayleigh, thus by L. Brillouin (6), by P. Langevin and by P. Biquard (7), and also by Larmor (8). Prompted by these considerations he first of all subjected the observations in the monograph mentioned above to calculations based on the first of the two Rayleigh formulæ. The new treatment resulted in three average values for the ratio E_B/E_R of the figures found for the same wave field by the radiometer (E_B) and by the Rayleigh disk indicator (E_R), viz.,

$$E_B/E_R: \quad 1{\cdot}05, \quad 1{\cdot}01 \quad \text{and} \quad 0{\cdot}93,$$

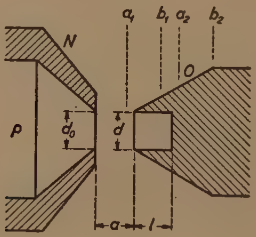
the average being accidentally 1·00. So there could be no doubt that the agreement between the two absolute methods becomes much closer if Rayleigh's first formula (1) is used than if the second hitherto accepted formula (2) is adopted.

After the review of the older measurements had been completed by the author of the monograph A J G he found it a matter of considerable practical and theoretical importance to repeat the comparison with as high a degree of precision as could be attained. The present paper describes how the problem was tackled and gives the results of the new comparison.

2. General Accessories of the Comparison.

The chief means for the comparison was the acoustic air-jet generator, an apparatus which is particularly suited for the production of waves

Fig. 1.



The acoustic air-jet generator.

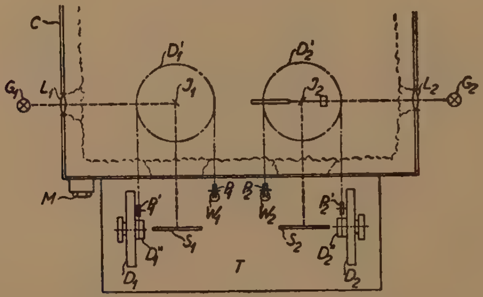
in air of high frequencies and of relatively large intensities. Fig. 1 shows the generator. N is a nozzle from which an air-jet of a velocity exceeding that of sound is emitted. It is known that a jet of this description is produced if the excess pressure in the container from which the jet is fed exceeds approximately 1 atm. The jet has a periodic structure, *i. e.*, it consists of a series of equal sections. Each section comprises a domain, the interval of instability. In fig. 1, a_1b_1 and a_2b_2 represent the first and second interval. The interval is the condition for the sound-wave production. The latter is effected by means of a cylindrical oscillator O, *i. e.*, a cylindrical bore with a sharpened edge at the end of a cylindrical piece of brass. If the aperture of O is arranged within one of the intervals of instability, in practical use the first from the nozzle, the air in the oscillator will vibrate with a frequency of the order of size of the natural frequency of the oscillator, and spherical waves are emitted from the generator. By varying the depth l of the oscillator, which is as a rule chosen equal to the diameter d of the bore and of d_0 of the nozzle ($d_0=d$), all frequencies from, say, 10,000 Hz up to, say, 100,000 Hz may readily be produced. In the case of the former frequency the power radiated is of the order of 100 watts. Generally it is in inverse proportion to the square of the frequency.

The air-jet generator—or the so-called Hartmann generator—lends itself particularly well to the comparison in question. This fact may be explained thus. The Rayleigh disk indicator was used in a modified shape in which the disk was replaced by a narrow strip. It is required that the width b of the strip shall be small compared to the wave-length λ, *i. e.*, less than say 1/10–1/5 of λ. As to the radiometer, the diameter d of its disk must be relatively large compared to λ, say $d>5\lambda$. With waves which may be produced by the air-jet generator both of these conditions may be directly fulfilled. The comparisons reported in the sequel were made with frequencies between 10,000 and 30,000 Hz, *i. e.*, with waves between 34 and 12 mm. A Rayleigh strip of a width of 2 mm. and a radiometer disk of a diameter of 200 mm. will be respectively small and large enough to cover the entire range, and at the same time the intensities of radiation will amply suffice to ensure the necessary sensibility of both of the indicators. In the case of sound-waves of the ordinary frequencies, say a wave of 500 Hz or wave-length 70 cm., it would prove practically impossible to produce a radiometer with a sufficiently large disk. The difficulty might, one would think, be overcome by the use of a smaller disk embedded in a plane wall. Measurement with such an arrangement is, however, beset with considerable encumbrances as was experienced by one of the authors (Comp. A J G). The difficulties were in fact so great that the idea of using an embedded disk was definitely relinquished.

The air for the generator was supplied from a piston compressor which would compress 3 m.^{3}/min. up to 8 atm. In the conduit to the generator an air chamber of 3 m.^{3} was inserted together with a reduction valve which reduced the excess pressure to 3 atm. A smaller air chamber was arranged immediately before the generator. The excess pressure behind the nozzle was measured by a mercury manometer. Precision regulation of the excess pressure was achieved by an artificial leak controlled by a needle valve. In this way the pressure could be kept constant within 1–2 mm. Hg.

The comparison was made in the so-called acoustic cabinet constituted by a wooden box, the interior dimensions of which were: length 4·50 m., height 2·00 m., and width 2·00 m. The walls, ceiling, and floor were covered with cotton wool in order to render them as far as possible absorbent to the waves. The two indicators were suspended at a distance of 1 m. from one end of the walls. They were arranged 40 cm. apart symmetrically with regard to the plane of symmetry of the cabinet. A pair of rails were carried along the cabinet, destined to take a carriage. On this carriage another pair of rails was mounted cross-wise to the above-named. A carriage could be displaced along the cross-rails. On this carriage was a table which could be rotated about a vertical journal. The angular position of the table could be read on a circular scale. The table carried a vertical column coaxial with the journal. On top of this column the generator shown in fig. 2 (Plate II.), was clamped with its centre in the axis of the column. In the comparison the cross-wise motion was used for displacing the generator from the one indicator to the other, the generator being always set so that the line from the generator to the indicator in question was parallel to the plane of symmetry of the cabinet and at the same time horizontal.

Fig. 3.



Means for setting and reading the indicators from outside the cabinet.

The two indicators could be set and read from outside the cabinet. This will be understood from fig. 3, showing a horizontal section of the end of the cabinet with the indicators I_1 and I_2 and in addition the table T from which the indicators are controlled. I_1 is the Rayleigh strip, I_2 the radiometer. Both indicators were based on the torsion balance principle. When exposed to the action of the wave field they are deflected from their zero positions read by light ray indexes on the scales S_1 and S_2 on the table T. The light is supplied by the two incandescent lamps G_1 and G_2; it is collected by the lenses L_1 and L_2 and reflected from the silvered surface of the Rayleigh strip and from a small mirror attached to the axis of the radiometer. The light rays pass out of the cabinet through a narrow obliquely set strip of glass in the end wall of the cabinet. When, now, one of the indicators is deflected from the zero position as indicated, it is taken back by turning the torsion head of the balance. The angle of rotation required is a measure of the intensity of the wave. The angle, or rather a figure proportional to it, is read directly by the observer seated in front of the table T. This is achieved in the following way.

A disk D_1' (D_2') is fixed on the torsion head. A shallow groove is cut in the cylindrical rim of the disk, and in this groove a flexible steel scale of a length of 1000 mm. is laid. It passes just once round the disk. In two narrow grooves cut in the same surface are laid two thin steel wires, each of them fixed to the disk at one point. One of the wires is passed over a pulley P_1 (P_2) and loaded with a weight W_1 (W_2), fig. 3. The other wire is carried over a similar pulley P_1' (P_2') down to the device serving for the reading of the indicator. This device consists of a disk D_1 (D_2) exactly similar to D_1' (D_2') and furnished with a steel scale like that of the latter disk. The disk D_1 (D_2) is extended by a coaxial cylinder D_1'' (D_2'') with a diameter one quarter of that of the larger disk. In a thread-shaped groove in D_1'' (D_2'') the second steel wire from D_1' (D_2') is laid and fixed at the end. It will be understood that the weight W_1 (W_2) is meant to keep the wire connecting D_1' (D_2') and D_1'' (D_2'') tight. By means of a brake acting on D_1 (D_2) the latter disk is prevented from following the pull from W_1 (W_2). When the disk D_1 (D_2)—the hand-wheel—is turned through a certain angle the torsion head or the disk D_1' (D_2') is rotated by an angle which is one quarter hereof. It will be gathered that the complete device now described is a means for taking the indicator back to its zero position from outside the cabinet and for measurement of the rotation of the torsion head. The corresponding rotation a read in millimetres on the scale of D_1 (D_2) is taken as a measure of this rotation.

From the table T the excess-pressure could, further, be read on the manometer M and the needle valve could be operated. The distance A from the generator to the indicator could be adjusted and read on a scale in front of the observer. Generally the control of the pressure was, however, carried out by the observer's assistant.

3. The Indicators.

The Rayleigh disk indicator was of the strip type; figs. 4, *a*–*b*, are to convey an idea of its design. S, the strip proper, of a length l mm. and a width of b mm. was generally made of glass of a thickness of about 0·15 mm. The length was as a rule between 30 and 40 mm. while the width was normally 2 mm. The strip acted as a mirror and so was silvered on one side. The quartz fibre Q, of a thickness of up to 0·02 mm. and a length of 50–100 mm., was not attached directly to the strip but cemented to a T-shaped connecting member T made of copper wire, 0·21 mm. thick. At the upper end, the fibre was attached to a straight piece P of the same wire which in its turn was extended by a heavier stem P′ of brass wire. To the lower part of the strip was fixed a straight copper wire W of a length of about 200 mm. and a thickness of 0·3 mm.

It terminated in a hook H destined to take up the damping member or air brake D. The latter consisted of a rectangular sheet of 0·02 mm. aluminium foil stiffened by the long side edges being bent as indicated. The sheet was enclosed in a box B of brass with heavy walls, fig. 4 *b*, with a view to protecting it from convection currents of air produced

Fig. 4(*a*).

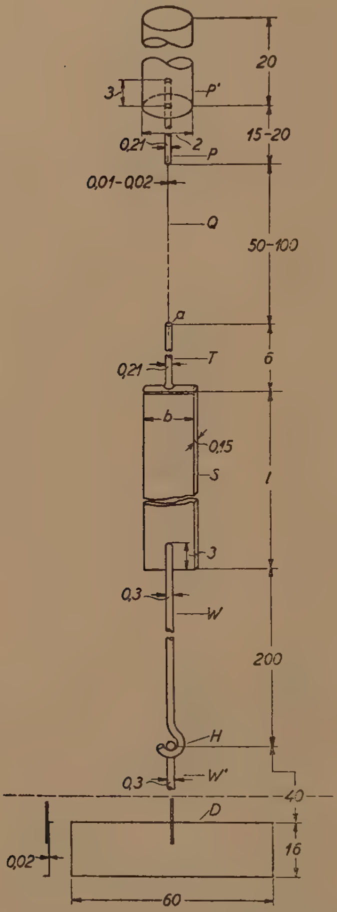
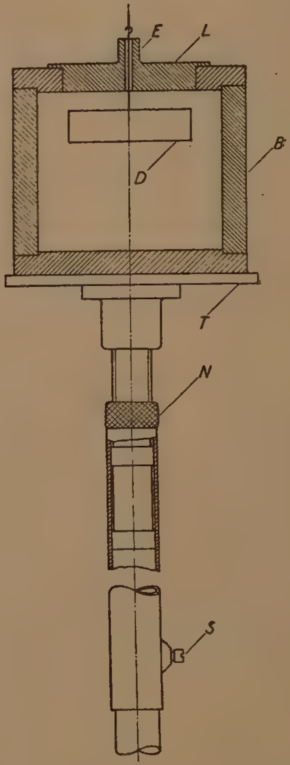


Fig. 4 (*b*).



The Rayleigh strip indicator.

by temperature differences. At the top of the box a circular opening was produced for introducing the air brake. The opening was closed by a cylindrical lid L with an upward extension, E, the lid and the extension being divided into two symmetrical pieces surrounding the stem of the air brake sheet. The box was mounted on a table T which could be raised or lowered, large displacements being effected by hand after loosening the screw S; small displacements were produced by turning the nut N.

We omit here the detailed description of the production of the indicator; it will be found in the complete report *. It will suffice to say that the production and the measuring of the strip were carried out with a tolerance of a few tenths of one per cent. Great stress was originally laid on making the suspension exactly coaxial with the strip. It turned out, however, that a comparatively large tolerance could here be allowed, as proved both by theoretical and experimental evidence.

The quartz fibre suspension of the Rayleigh strip indicator was calibrated in the ordinary way by observation of the period of oscillation of attached bodies of known moment of inertia I. A number of such bodies were used in each case resulting in a curve showing the variation of the torque H per unit angle with the load on the suspension. From this curve the value of H corresponding to the weight of the actual indicator was read. The tolerance in the determination of H was of the order of size of a few tenths of one per cent.

The formula connecting the reading a on the hand wheel D_1, fig. 3, with the energy density E of the train of waves used in the comparison is readily derived from the theory of the Rayleigh strip. According to this theory the couple M acting on the strip is

$$\mathrm{M}=\frac{\pi}{8} lb^2 \,\mathrm{E} \text{ dyne . cm.} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

It follows from this expression and from the description of the mechanical device for the setting and reading of the indicator that

$$\mathrm{E}=\frac{4\mathrm{H}}{100\, lb^2}\, a \,.\, \text{erg/cm.}^3, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

the cg.s. system of units being used exclusively. Sixteen complete Rayleigh strips were produced; the values of the coefficient of a in the formula (2) varied in these systems from about $500 \,.\, 10^{-6}$ to about $5000 \,.\, 10^{-6} \,\frac{\text{erg}}{\text{cm.}^3}$ per cm. The indefiniteness of thevalues was of the order of size of a few tenths of one per cent.

The design of the radiometer is illustrated in figs. 5 *a–c*. The central member C is shown on a larger scale in fig. 5 *b*. It consists of a cylindrical piece of brass with a cross-bore for the arm AA carrying the circular plate D exposed to the radiation pressure, and the weight B for counter balancing D. Again, two axial bores are made in C, a narrower one to take up the stem W of the liquid brake member O and a wider one for a kind of self-centring chuck C′ in which the lower end of the suspension is clamped. For this chuck the pencil holder from a case of drawing instruments was utilized. Another pencil holder C″ was employed at the other end of the suspension S. The latter was a band of phosphor bronze soldered at either end to rather stiff pins of brass P_1 and P_2. Between the uppermost pin P_1 and the suspension proper a thin piece

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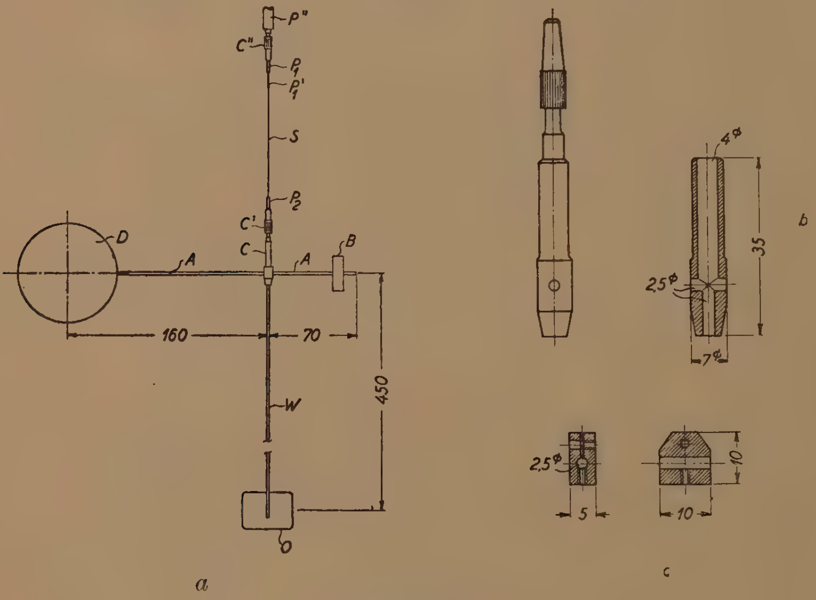
* Ingeniørvidenskabelige Skrifter 1948 Nr. 2 København. (Monograph in English published by the Academy of Technical Sciences, Copenhagen.)

P_1' of flexible copper wire was, however, inserted. The latter member was used for making the suspension coaxial with the torsion head arrangement. At its upper end the holder C″ was extended by a cylindrical pin of brass P″ which in its turn was introduced into an axial bore of the axial extension of the torsion head. The plate D was clamped to the arm A by means of a clasp shown in fig. 5 *c* fastened at the centre of the rear side of the plate by a screw through the plate. The material used for the arm A and the stem W was brass tubing of an exterior diameter of 0·25 cm. The plate D and the liquid-brake member O were generally made of sheet aluminium of a thickness of about 0·75 mm. O was fastened to W by slits crosswise to each other in the two members.

Fig. 5



The radiometer.

A number of circular plates were supplied, ranging in diameter from 40 to about 200 mm. The angular position of the radiometer was determined by a small mirror attached to the member C. The position was read on the scale S_2, fig. 3, by means of a light index produced by the lamp G_2 and the lens L_2 in the same figure.

While air damping was used in the Rayleigh strip indicator liquid damping was here employed (glycerine). In both apparatus the damping was practically aperiodic.

The torque per unit angle of the phosphor bronze suspension was determined in the same way as that of the quartz fibre suspensions, the standard bodies being only correspondingly heavier. The tolerances used in the production and calibration were again of the order of size of a few tenths of one per cent.

From the assumption that the radiation pressure acting on the radiometer disk of diameter d cm. is given by 2E, E being the density of energy in the wave train impinging on the disk, it follows that the connection between E and the reading a on the hand-wheel D_2, fig. 3, is now given by

$$E = \frac{H}{100\,d^2 l}\,a\,.\,\mathrm{erg/cm.}^3 \qquad \ldots\ldots \quad (3)$$

l being the distance from the centre of the disk to the axis of the torsion balance. Fourteen radiometers all using, however, the same suspension were employed in the course of the investigation. The values of the coefficient to a in the formula (3) varied from about $700\,.\,10^{-6}$ to about $17{,}000\,.\,10^{-6}\,\frac{\mathrm{erg}}{\mathrm{cm.}^3}$ per cm. They were determined with the same tolerance as in the case of the Rayleigh strip indicators.

4. Preparatory Work.

Numerous measures had to be taken, and several special investigations made, in order to place the comparison on a safe basis. A complete report on this preparatory work would be quite precluded within the space available. So a brief review of the more essential features must suffice, the reader being referred to the complete report of the research work.

In the first place the two indicators to be compared were produced and their suspensions calibrated with such a tolerance, *i. e.*, some few tenths of one per cent., that the indefiniteness of the figures for the energy density E due to these sources could be practically neglected. In connection with the Rayleigh strip indicator the effect of a dissymmetry of the suspension with regard to the strip was examined and the effect was practically eliminated. Similarly, the radiation pressure forces acting on the arm supporting the disk of the radiometer were determined experimentally and the readings of the radiometer corrected, if necessary, for the effect. Care was taken to otherwise protect all the parts of the radiometer from the radiation by means of layers of cotton wool. In both of the indicators the exact value of the angular transmission ratio of the gear for the setting and reading of the indicators from the seat of the observer was determined. It might deviate by a few tenths of one per cent. from the value 4·000 aimed at in the construction. The corresponding corrections were applied to the proportionality constants of each of the two indicators. The zero point of each indicator was checked by a special method before and after the performance of each comparison. It might exhibit changes due to slight relative deformations within the system of supporting points for the transmission gear. It was made certain that the changes during a comparison could either be neglected or safely corrected for.

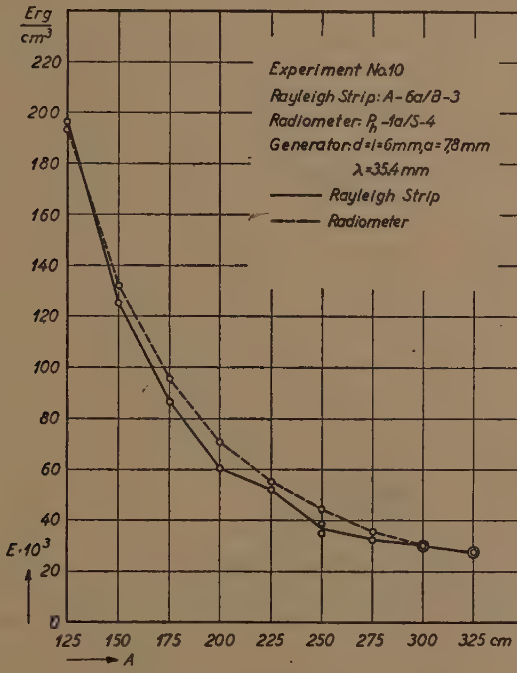
The object of measurement, *i. e.*, the acoustic wave field, was then studied. It is created by a source which may be considered as a point. In the theories of the indicators, and particularly in that of the radiometer, a plane field of waves is assumed. It was readily shown that the wave field would in practically all cases be equivalent to a plane field even with the largest radiometer disks. Again, it is assumed that the wave field is homogeneous as to the energy density. Actually the radiation from the air-jet generator varies with the "latitude", *i. e.*, with the angle between the direction of radiation and a plane through the centre of the generator perpendicular to the axis of the latter. The effect of this inhomogeneity could in extreme cases amount to an error of 1–3 per cent. in the comparison of the two indicators. Generally it was less than this. It would have been practically precluded to correct all the numerous observations of which the ultimate comparisons were built up for the effect considered. In the various comparisons the errors will fluctuate both as to size and sign, *i. e.*, the inhomogeneity will act as a source of indefiniteness the effect of which may be substantially reduced by suitable treatment of the material of observations.

The wave field is, however, not solely determined by the generator, it may be influenced by the acoustic cabinet in which the comparison was carried out. According to the Huyghens–Fresnel theory of a wave field the field intensity at the place of the indicator should be uninfluenced by the limited size of the cabinet provided the distance A from the generator to the indicator does not exceed the width or the height of the cabinet. In the case considered this condition was fulfilled with values of A up to approximately 2 m. Generally, however, distances up to 3 m. were employed without any anomalies being observed.

Again, total absorption of the waves by the cotton wool covering of the walls, ceiling, and floor of the cabinet is assumed. At the outset 0·75 kg./m.^{2} cotton wool was used; this covering proved insufficient. Eventually the amount of cotton wool was doubled, and furthermore, the rear wall and the walls round the indicators were hung with "wings", *i. e.*, strips of gauze covered with cotton wool and extending from ceiling to floor. The strips were arranged perpendicularly to the wall, 15 cm. apart. They protruded by 30 cm. from the wall. The layer of cotton wool on each strip had a thickness of 5 cm. With this covering, which was the result of comprehensive investigations, the influence of residual reflection from the walls was very effectively reduced. This was judged from the appearance of the so-called A–E-curve, *i. e.*, a curve showing the variation of the energy density at the place of the indicators with the distance A to the generator. With insufficient absorption at the walls, this curve is generally irregular. As the cover was improved the A–E-curve became more and more regular and smooth. Ultimately, a degree of regularity such as that illustrated in fig. 6 was achieved. The two curves were produced by Rayleigh strip and radiometer respectively. They represent the same field and may convey an idea of the definiteness obtained in the determination. It may here be noted that the comparison of the two indicators was ultimately based on comparisons of A–E-curves.

Finally, in the discussion of the field as an object of measurement, the influence of the radiometer disk on the field round the Rayleigh strip was made the subject of a special investigation. It should here be noted that the radiometer was for practical reasons generally left in position during measurement with the Rayleigh strip (and *vice versa*).

Fig. 6.



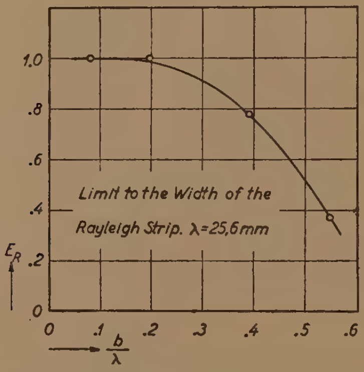
A–E-curves no. 10.

The two indicators, which were hung 400 mm. apart, were separated by a cushion of cotton wool. It was found that no influence on the readings of the Rayleigh strip could be established and this result was borne out by a series of comparisons in which the radiometer and the cushion were removed when the Rayleigh strip was used.

In the preparation of the comparison, the limitations of the theories of the two indicators had, further, to be discussed. In the production of the theory of the Rayleigh strip, as a means of measuring the intensity of sound-waves, it has been assumed that the width b of the strip is small compared to the wave-length λ. The same field of wave-length 25·6 mm. was measured with four strips of widths, 2, 5, 10 and 14 mm. From the results, in the shape of A–E-curves, the graph in fig. 7 was derived showing that if $b<0{\cdot}1\ \lambda$ the influence of the width is inappreciable. At $b=0{\cdot}2\ \lambda$ the E is measured too low by 2 per cent. The thickness of the strip was normally 0·15 mm. A strip of 0·44 mm., however, yielded the same A–E-curve as the normal strips. Three strips of 3 mm. width and 0·15 mm. thickness having the lengths 60·1, 40·9 and 10·1 mm. gave the same values for E. Normally lengths between 30 and 50 mm. were used.

The curve in fig. 7 renders it possible to estimate a source of error originating in the complex character of the waves emitted from the air-jet generator. It is known, or it is at least made highly probable, that these waves generally consist of the fundamental wave and a second harmonic. From oscillograms of the waves and from spectroscopic analysis of them, it was estimated that the ratio p_2/p_1 of the pressure amplitudes of the second harmonic and the fundamental might assume values between 0 and 1. The wave-lengths employed in the comparison were between 12·5 and 50 mm. It follows from fig. 7 that in the case of a wave of 25 mm. and $p_2/p_1=1$, *i. e.*, $E_2/E_1=(p_2/p_1)^2=1$, the total energy density E_1+E_2 is measured too low by 0·5 per cent. with a Rayleigh strip of a width of 2 mm. In the case of the shortest wave-length, 12·5mm., the corresponding error would amount to 6·5 per cent. The latter case undoubtedly represents the worst possibility; generally, the second harmonic does not give rise to any very grave error. From oscillograms of a provisional character, it appeared, however, that sometimes harmonics of a rather high order are present. The Rayleigh strip completely fails to measure the energy density of such harmonics. If this density amounts to α per cent. of the total energy density of the wave, the latter density is measured too low by α per cent. The final results of the comparison would seem to illustrate in a striking way the conclusions here drawn.

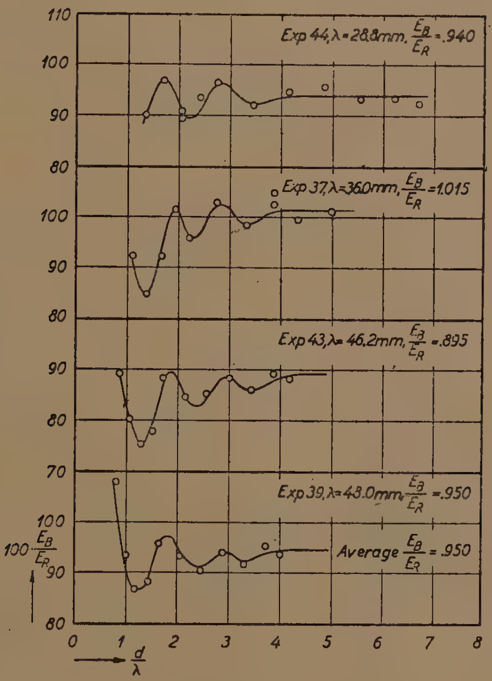
Fig. 7.



Limit to the width of the Rayleigh strip, λ=25·6 mm.

The theory of the radiometer pre-supposes that the diameter d of the disk is large compared to the wave-length. The comparison was carried out in such a way that the results would show whether this condition was in each case fulfilled; comp. figs. 8–10. It turned out that it suffices if the diameter d is 5–6 times as large as the wave-length. The theory further demands that the radiation should be completely reflected from the surface of the disk. An estimate of the amplitude of the vibrations forced on the disk, made of aluminium of a thickness 0·75 mm., showed that the loss of energy in the reflected wave was in all cases quite negligible. An estimate of the amount of energy penetrating into the material of the disk gave the same result. The final results would seem to disclose quite a new source of energy loss at the surface of the disk not foreseen in the discussion here indicated.

Fig. 8.



d/λ—E_B/E_R-curves.

5. The Comparison.

The comparison of the methods of measurements here considered, *i. e.*, of the values E_B and E_R found for the same field by the radiometer and the Rayleigh strip respectively, was carried out according to the following scheme.

Two curves—A–E-curves—representing the variation of the energy density E with the distance A from the generator were produced, one with the radiometer, the other with the Rayleigh strip. The character of such a twin experiment will appear from Table I. representing the experiment no. 33. The a-values are the readings on the hand-wheels D_1 and D_2, fig. 3, the figures stated under $E_R \cdot 10^3$ and $E_B \cdot 10^3$ are the values for E derived from the a-values. The distances A were standardized at the values stated in the table. The legend above the table gives the wave-length and the indicators utilized in the experiment, A–10/B–10 indicating a Rayleigh strip of width 2·058 mm. for which

$$E/a = 817{\cdot}1 \,.\, 10^{-6} \, \frac{\text{erg}}{\text{cm.}^3} \text{per cm.}$$

and Ph–1a/S—5 a radiometer with the constant

$$E/a = 1084 \,.\, 10^{-3} \, \frac{\text{erg}}{\text{cm.}^3} \text{per cm.}$$

and with a disk of a diameter of 16·002 cm.

TABLE I.

Exp. 33.—λ=19·2 mm. A–10/B–10. Ph–1a/S—5.

A cm.	Rayleigh strip.		Radiometer.	
	a_R cm.	$E_R \cdot 10^3$ erg/cm.^{3}	a_B cm.	$E_B \cdot 10^3$ erg/cm.^{3}
100	121·3	99·1	85·6	93·0
125	71·1	58·1	51·5	55·9
150	52·2	42·7	35·1	38·1
175	37·8	30·9	25·1	27·2
200	29·3	23·9	19·2	20·8
225	21·9	17·9	14·4	15·6
250	16·6	13·6	11·8	12·8
275	13·9	11·3	9·0	9·8
300	10·8	8·8	7·8	8·5

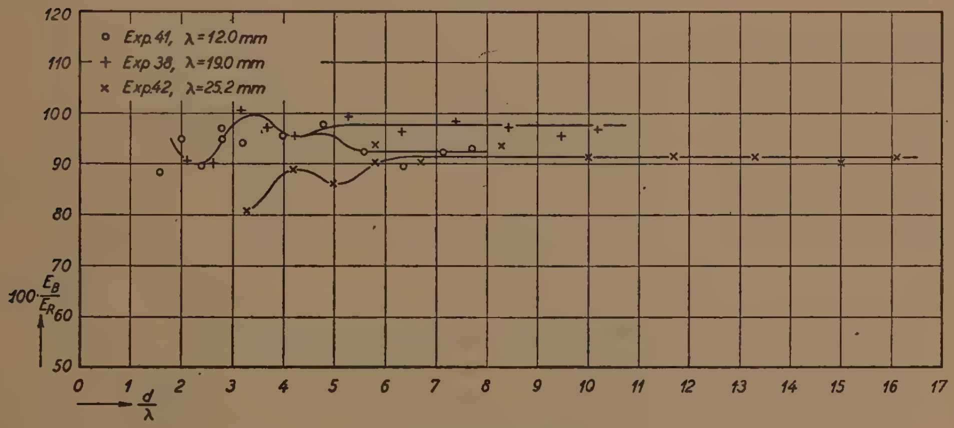
Each a-value is an average of 10 observations, thus the twin experiment in Table I. comprises 180 a-observations. In addition it involves a check of the zero points of each of the two indicators requiring about 20 observations. In the final comparisons the number of observations was, however, somewhat reduced by a reduction to five of the number of A-values employed.

Generally the observations of the twin experiment were first of all used for the plotting of an A–E_B- and an A–E_R-curve the two curves being directly compared. Otherwise the problem was to derive from the observations a single value for the ratio E_B/E_R. This could be done graphically or numerically. In the graphical treatment the a-values or the E–values of the radiometer were plotted against the corresponding values of the Rayleigh strip indicator. Generally the smoothed-out curve would turn out to be a straight line through the zero point of the system of co-ordinates when the slope of the line would give the value of E_B/E_R in question. By the numerical method the ratio was found from the expression

$$\frac{E_B}{E_R}=\frac{E_{.1}+E_{B.2}+\ldots\ldots+E_{B.n}}{E_{R.1}+E_{R.2}+\ldots\ldots+E_{R.n}},$$

i. e., as the ratio of the sums of corresponding ordinates in the A–E_B- and the A–E_R-curves respectively. The expression is a consequence of the principle of least squares when it is assumed that the indefiniteness of the E-values, $\varDelta E$ is proportional to $\sqrt{E}$ and the same for E_B and E_R. As a matter of fact, $\varDelta E$ would seem to increase slightly with E. The actual way it does so is so indefinite that the interdependence assumed in the formula above may as well be chosen as several others and so it was preferred on account of its simple consequence. Generally, it was found that the graphical and numerical treatment would yield practically the same results.

Fig. 9.

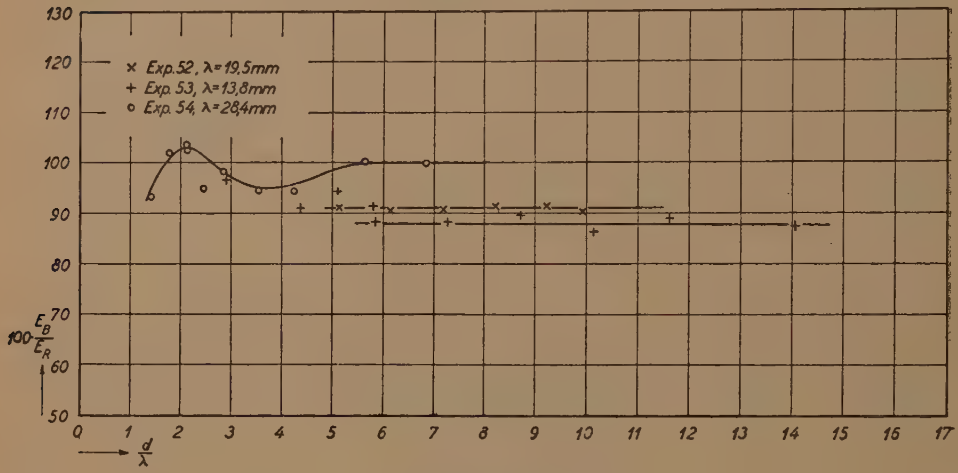


d/λ—E_B/E_R-curves.

Now, from experiments like that considered more comprehensive syntheses were built together as shown in figs. 8–10. The curves here presented show the variation of the E_B/E_R-values with the ratio d/λ, where d is the diameter of the radiometer disk employed in the comparison. Each curve in the three figures is normally the result of 13 A–E-curves, each comprising 50 a-observations, thus of 650 a-observations. Behind the 10 curves in figs. 8–10 there are thus between 6000 and 7000 a-observations. The Rayleigh strips used in the comparisons had always a width sufficiently small for measurement of the fundamental and a second harmonic, viz., a width of 2 mm. approximately. From the curves it appears that, when $d/\lambda >$ about 5 or 6, the ratio E_B/E_R no longer varies with λ. The constant value assumed by the ratio is then the ratio asked for. Fig. 8 represents determinations made with comparatively long waves. Here the highest values of d/λ are close to the limit five for d/λ and so the values derived for E_B/E_R were less definite than those found from figs. 9 and 10 corresponding to shorter wave-lengths.

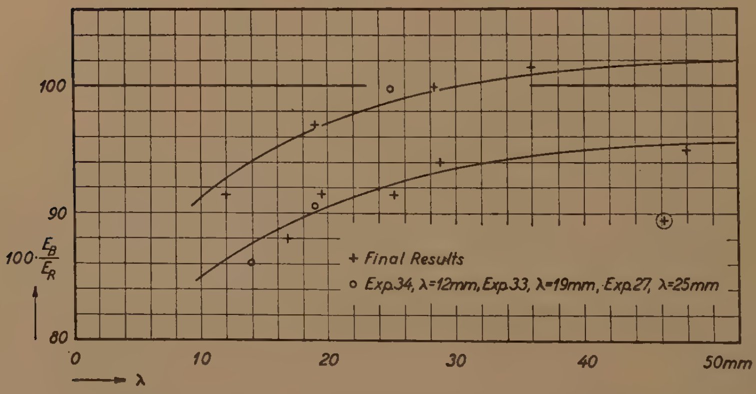
Here the horizontal part of the curves is generally long and yield a rather definite value for the ratio, the spreading of the points being astonishingly small. In the comparisons of fig. 9 both of the two indicators were suspended, in the comparisons of fig. 10 the radiometer was removed when the A–E_R-curve was produced. As will be seen, no essential influence of the radiometer could be established.

Fig. 10.



d/λ—E_B/E_R-curves.

Fig. 11.



λ—E_B/E_R-curves. Ultimate result of the comparison.

Finally, all the constant values of E_B/E_R derived from figs. 8–10 were plotted against λ in fig. 11, which represents the ultimate result of the comparison. The interdependence is rather peculiar. Apart from one single point corresponding to the wave-length 46 mm., all the E_B/E_R-values are distributed round two distinct curves, the spreading of the points being about three per cent. only in each curve. Both of the curves rise with increasing values of λ and would seem to asymptotically approach definite values of E_B/E_R. In the case of the lower curve the limit would seem to be close to 1·00. The conclusions drawn from fig. 11 are as follows. The lower curve represents the "true" value of E_B/E_R. Granted this, our comparison of the two methods of measurement agree within, say, at least 1–2 per cent., provided the wave-length is above a certain limit. This practically implies that both of the two methods measure the energy density E correctly within, say, 1–2 per cent. or less. So the question as to which of the two Rayleigh formulae for the radiation pressure is to be preferred has been answered beyond any doubt. The formula

$$S=2E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

must be adopted, because it leads to agreement between two entirely independent methods of measurement while the formula,

$$S=(\kappa+1)E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

must be rejected because with this formula the two methods disagree by about 17 per cent.

Now, however, two facts remain to be explained: the uppermost curve in fig. 11, and the downward deviation from one of the ratio E_B/E_R at shorter wave-lengths.

From the fact that two curves are found for the variation of the ratio with λ, the conclusion is drawn that the air-jet generator is able to vibrate in two different and as it would seem rather distinct ways. In the one, corresponding to the lower curve, the radiation consists in the main of the fundamental wave and a second harmonic only in which case a Rayleigh strip of a width of 2 mm. will measure E practically without error. In the other mode of vibration, corresponding to the uppermost curve, the waves must be assumed to contain higher harmonics, of an energy of seven per cent. of the total energy, not measured by the strip. This would mean that E_R is measured too low by seven per cent., with the effect that E_B/E_R is found too high by the same percentage amount, or nearly so. The conjectures here are supported, or at least not contradicted, by what is so far known about the radiation from the air-jet generator.

The decrease of the ratio E_B/E_R with decreasing wave-length is interpreted as a particular kind of absorption of wave energy at the surface of the radiometer disk. With such an absorption, the assumption $S=2E$ on which the radiometer is founded would fail and E_B would be found too small as a measure of the energy density in the impinging wave. We have formed the idea that the absorption reveals a transmission of heat energy from the stationary wave in front of the disk to the disk in the shape of heat conduction and heat radiation. Such an effect would undoubtedly be the more pronounced the shorter the wave-length.

It will be gathered that the investigation reported has furnished an answer to the question which gave rise to the research work. At the same time, it has raised fresh questions which will, it is hoped, be answered by further research.

We desire in this place to thank the Trustees of the Danish Academy of Technical Sciences and of the Carlsberg Foundation for financial support in connection with the work in question.

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The Laboratory of Technical Physics,
The Danish Technical University,
Copenhagen.
April, 1947.

XLVII. *The Curve of Error in which the Maximum Error is Defined.*

By Major W. E. Bruges, M.Sc., A.C.G.I. A.M.I.C.E. A.M.I.E.E.

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Summary.

It is shown that the curve of error deduced from the binomial theorem has a limit and that if, for example, the probable error is known, then the maximum error should not exceed 3·48 . . . times this error.

In 'Calculus of Observations' by Whittaker and Robinson, sec. 90, p. 177 (fourth edition), a method is described for deducing the normal curve of error from the binomial theorem ; but as suggested in sec. 91 (p. 179), it is not capable of exact proof. The fact is, the assumption made that the error can be infinite compels the equating of a finite error x to an infinite quantity $p/\sqrt{\mathrm{N}}$, where N is the number of trials (*e. g.*, tosses of a coin) and $(\frac{1}{2}\mathrm{N}\pm p)$ the number of positive or negative elemental errors combining to make an error or deviation x. The following is accordingly suggested.

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* Communicated by the Author.

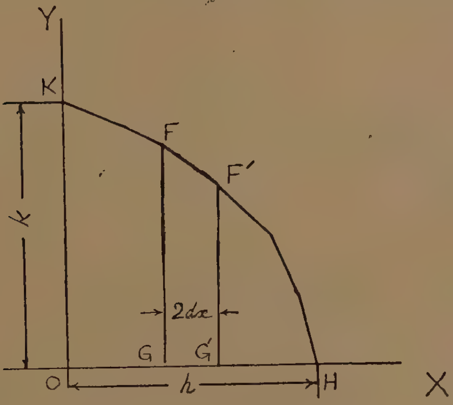
The preliminary reasoning follows that used by Hagen for the deduction of the normal curve of error.

One assumes first that an error incorporates m equal independent elemental errors $|dx,|$ each of which is equally likely to be positive or negative; hence that the chance of committing $(m-n)+$ve errors and $n-$ve errors is given by the $(n+1)$th term of the expansion of

$$(\tfrac{1}{2}+\tfrac{1}{2})^m. \qquad \ldots \ldots \ldots \ldots \qquad (1)$$

In fig. 1, OX, OY are rectangular axes, origin at O. Ordinates y, parallel to OY, represent terms of the expansion. The greatest term is represented by OK$(=k)$ measured along OY. This is also the middle term; and the length of the expansion is represented by x, measured along OX to any given ordinate y, each pair of adjacent ordinates being $2 \, . \, dx$ apart. The distance OH measured from O to the end of the expansion, is

Fig. 1.



The Binomial Expansion.

$$h=(m/2)\,.\,2\,.\,dx=m\,.\,dx. \qquad \ldots \ldots \ldots \qquad (2)$$

(As m will be large, the odd term will be neglected.)

It is to be noted that $|dx|$ is always positive, and that the equation of the curve (yet to be found) will only be applicable when x is positive. The curve for x negative will be the mirror image of it.

Thus,

$$x=(m-2\,.\,n)\,.\,dx=h-2\,.\,n\,.\,dx. \qquad \ldots \ldots \ldots \qquad (3)$$

Writing down the nth and the $(n+1)$th terms :—

The nth term $(\mathrm{F'G'}) = \dfrac{m(m-1)(m-2)\ldots(m-n+1)}{n\,!} \cdot \left(\dfrac{1}{2}\right)^m$;

the $(n+1)$th term $(\mathrm{FG}) = \dfrac{m(m-1)(m-2)\ldots(m-n)}{(n+1)\,!}\left(\dfrac{1}{2}\right)^m$,

the ratio FG/F′G′ being

$$(m-n)/(n+1). \qquad (4)$$

The difference corresponding to $2\,.\,dx$ is

$$2\,.\,dy = \mathrm{F'G'} - \mathrm{FG}. \qquad (5)$$

Now if $y=\mathrm{F}_1(n)$ and $x=\mathrm{F}_2(n)$, an equation in x and y can be obtained by eliminating the parameter n ; and this equation will be true no matter what the value or description of n, even if n approaches infinity, so long as the conditions of the two original equations hold.

If, however, $y=\mathrm{F}_1(m,n)$ and $x=\mathrm{F}_2(m.n)$, m being a quantity of the same order as n, it would be necessary to eliminate both m and n to obtain a finite equation in x and y, and a further condition would in general be required.

In the present instance, n represents the number of terms of the expansion, measured from one end, and m represents the total number of terms. Thus m is the maximum value of n.

One obtains, n large,

$$dy/dn = \mathrm{F}_1'(m,\,n) = \mathrm{F}_1(m,\,n+1) - \mathrm{F}_1(m,\,n),$$

m being constant and not varying as a function of n ; though both may vary in terms of some other variable, *e. g.*, time.

Now let

$$\mathrm{F}_1(m,\,n+1)/\mathrm{F}_1(m\,.\,n) = \mathrm{F}_3(m,\,n)\,;$$

then $$dy/dn = y\,.\,(\mathrm{F}_3(m,\,n)-1).$$

Also $$dx/dn = \mathrm{F}_2'(m,\,n),$$

therefore, $$dy/dx = y\,.\,(\mathrm{F}_3(m,\,n)-1)/\mathrm{F}_2'(m,\,n).$$

We have

$$\mathrm{F}_3(m,\,n) = (m-n)/(n+1)$$

$$\mathrm{F}_2(m,\,n) = m-2n\,;\ \ \mathrm{F}_2'(m,\,n) = -2\,;$$

hence, $$dy/dx = y\,.\,(m-2n)/(-2n)$$

and, multiplying numerator and denominator by the small quantity dx, one obtains

$$dy/dx = -xy/(h-x) \qquad (6)$$

which is the differential equation of the curve of error deduced from the binomial theorem, the maximum error being h.

Integrating, one obtains

$$\int \frac{dy}{y} = \int \left(1 - \frac{h}{h-x}\right) . dx + C$$

whence $$\log y = x + h . \log (h-x) + C.$$

When $x=0$, $y=k$; hence, $C = \log (k/h^h)$ and

$$\log y = x + \log k(1-(x/h))^h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or $$y = k . (1-(x/h))^h . e^x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7\ a)$$

This is the equation of the curve of error, the maximum error being h.

Neither h nor k are arbitrary constants, and differ in an important respect from the constants h and k of the normal curve of error

$$y = (h/\sqrt{\pi}) . e^{-(hx)^2},$$

where $k = h/\sqrt{\pi}$ and is adjusted so that the area under the curve between the limits $+\infty$ and $-\infty$ is equal to unity. The constant h is arbitrary and is generally used to represent scale. It does not affect, for example, the ratio between the probable error and any other defined error.

In the curve (7 a) described above, h and k are subject to two conditions, that is to say,

$$2 . \int_0^h y . dx = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and also to the condition that they must bear the same relationship to one another as the middle term of the expansion (1) bears to half the length of the expansion. The middle term

$$\frac{m\,!}{\left(\frac{m}{2}\right)! \left(\frac{m}{2}\right)!} . \frac{1}{2^m}$$

which, when m is large, approaches the value $\sqrt{2/m . \pi}$ as represented by k, and there are $m/2$ ordinates represented by h. To maintain this relationship for a finite curve, it is necessary that

$$k^2 . h = 1/\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and after substitution of this value of k in equation (8), one may find a value for h which will satisfy it, that is, which will make the area under the curve equal to unity. Different values of h very much affect the shape of the curve, quite apart from the scale.

Thus, from (7 a),

$$\int_0^x y . dx = k . \int_0^x (1-(x/h))^h . e^x . dx$$
$$= -k . e^x . (1-(x/h))^h . \left(\frac{h-x}{h+1} + \frac{(h-x)^2}{(h+1)(h+2)} + \frac{(h-x)^3}{(h+1)(h+2)(h+3)} + \ldots\right) \quad . \quad . \quad . \quad (10)$$

$$=F(x)$$

and

$$y\,.\,dx=F(x)-F(0):\quad\ldots\ldots\quad(10\,a)$$

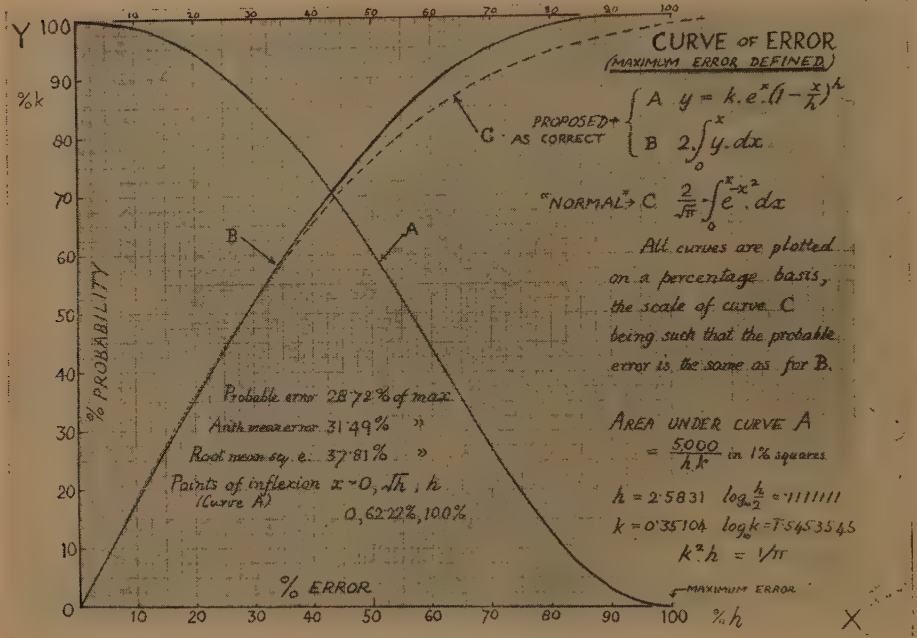
and, since $F(h)=0$, h must be chosen so that

$$-2\,.\,F(0)=1.\quad\ldots\ldots\ldots\quad(10\,b)$$

The probability of the error being between x and zero being

$$(F(x)-F(0))/(-F(0))=2\,.\,F(x)+1.\quad\ldots\ldots\quad(10\,c)$$

Fig. 2.



The Curve of Error.

It is found that $h=2{\cdot}5831$ satisfies condition (10 b) after substitution of the value of k from (9). Then k becomes $0{\cdot}351038$. These are important fixed constants, and the curve obtained with their use is shown in fig. 2. It may be noted that

$$\log_{10}(h/2)=0{\cdot}1111111$$

and

$$\log_{10}k=\bar{1}{\cdot}5453545.$$

In fig. 2 the curve is plotted on a percentage basis for quick reference purposes; probability factors are also plotted, a similar curve being plotted from the normal curve, taking the probable error as being the same in each case.

It will be observed that if both x and y represent numbers of events to scale, the ratio x/y will be infinite. This, however, does not indicate that the limiting condition, when m and n are infinite, cannot be shown in terms of x and y; but it is evident that the scale for the ordinates will differ in terms of n from that of the abscissæ when this has been done.

The elemental error is in point of fact a dimension of an "event," and at any point x the density of events is represented by y in terms of these elements. Thus $dy \, . \, dx$ is an element of probability and $y \, . \, dx$ represents the probability that the event shall be between x and $x+dx$. An event is represented by an area $dx \, . \, dx$, that is by a small square. This small square may be a shot on a target, the area occupied by the shot, or merely the abstract conception of an event such as a measurement situated at a distance x from the origin, the distance representing its error.

The expression $2 \, . \, dy/dx$ then represents the number of events in the element $2 \, . \, dy$, and the difference between two consecutive ordinates (F′G′—FG) may be equated to this, producing the same result (6).

It is evident that the existence of a maximum probable error provides a useful criterion in addition to or in place of Peirce's for doubtful readings. (This criterion is fully described in Chauvenet's 'Spherical and Practical Astronomy,' vol. ii. pp. 558–566.)

The fact that the curve of error is limited has an important bearing on astronomy and nature generally, and might enable one to estimate the normal maximum size of living organisms from a knowledge of the numbers of a given group.

It is evident that skew curves may be calculated on the same principle. In some cases a positive elemental error or difference may be more likely than a negative one, and skew curves may be deduced from the expansion of $(a+b)^m$ where $a+b=1$ and a and b are unequal to one another. This did not of course apply when an infinite error was assumed, since in the limit all curves merged into the "normal" curve. For example, in the case of chest measurements of soldiers, the recruiting serjeant would naturally show a preference for large chests.

A difference from a "mean" value might be represented by x, this value corresponding to the greatest term of the expansion. The terms increase so long as $(m-n) \, . \, a/(n+1) \, . \, b > 1$ and $n = mc/(1+c)$ gives the maximum term, where $c = a/b$ and is a "selectivity" constant.

Both portions of the curve, for $x+$ve and $x-$ve, must be calculated, k being the same for both but h being different. Thus $h_1 = (2 \, . \, mc/(1+c)) \, . \, dx$ and $h_2 = (2 \, . \, m/(1+c)) \, . \, dx$. The total area, as before, equals 1.

XLVIII. *The Variation of Beam Angle with Modulation in Electron Optical Immersion Systems.*

By L. Jacob, Ph.D., Electro-Technics Department, The University, Manchester *.

[Received July 8, 1946.]

Summary.

A mathematical proof is given of the dependence of beam angle on modulation. This is a first approximation and uses some simplifying assumptions. The theoretical implications are borne out by the experimental evidence submitted.

1. *Introduction.*

The effects of modulating the electron-optically produced beams in C.R.T. systems are fairly well understood. The modulator is not a stop in the physical sense, in that it does not, in all normal usage, trap any portion of the beam. Each alteration or variation in the potential of the modulating electrode affects the electric field established in the cathode/modulator as well as the modulator/anode regions, but the effects in the former are the more marked ; hence the more important, because here the electrons have their lowest velocities.

The major effect on the field—*i. e.*, the alteration in shape and position of the equipotential surfaces—alters the cardinal points of the first electron optical lens. This, in turn, produces at least five practical and important other effects:—(*a*) constriction of the emitting area, (*b*) change in current, (*c*) some variation in the current distribution, (*d*) variation of the focal length and magnification, (*e*) variation in beam angle.

Observations for a long time past have shown that in the case of (*e*) the relation between angle and modulation is for all practical purposes a linear one. Use of this fact has enabled the design of an electron-optical voltmeter to be partly based on it †. It is of more than passing interest to see what theoretical justification there is for expecting the linear relationship to hold at all values of modulation voltage. An analysis of the conditions in the field is made here, which, as a first approximation, indicates a linear law.

Although use of the equation to the trajectory is made in the derivation, the proof of the trajectory equation is not included ; it can be found in Myers' book ‡.

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* Communicated by the Author.

† J. I. E. E. cxi. p. 512 (1944).

‡ 'Electron Optics' : Chapman & Hall.

2. *The Trajectory Equation.*

In deriving the trajectory equation, the assumption is made that the electron is accelerated through a field whose distribution is unaffected by space charge. Further, it is assumed that the field remains constant over the region to be integrated in solving the Laplace Equation. These are reasonably true for the paraxial case, and where the space charge density is negligible.

The analysis yields the important result, that the radial gradient is everywhere proportional to the height of the ray at any instant; this condition indicates the existence of a focusing field for the electron.

The final form for the equation is

$$\frac{d^2r}{dz^2}+\frac{\mathrm{V}'}{2\mathrm{V}}\frac{dr}{dz}+\frac{\mathrm{V}''}{4\mathrm{V}}\cdot r=0,$$

where V is the potential along the axis, and V′ and V″ are its first and second derivatives.

Actually, if V was known at each point on the axis, it should be possible to solve this second-order equation for the ray path through the field. However, as every variation in modulator voltage alters the value of V, the calculation would become tedious if a large number of cases were to be investigated. It will be noted that the trajectory equation is homogeneous in V and "*r*." This enables the Principle of Similitude to be applied to the system, which is particularly helpful for design purposes. Further, it contains no term involving charge or mass, hence it defines the path of any charged particle through the field.

3. *The Principle of Superposition.*

In any immersion-objective electron optical system such as the cathode, modulator and first accelerator of a cathode-ray tube, the variation of potential of one or more of these electrodes produces a variation of potential or refractive index of the space through which the electron beam passes.

It is usual to keep the cathode and anode at fixed potentials and to vary the modulator potential; this produces changes in the focused spot characteristics as desired. The change in potential of the modulating electrode can be considered to arise from its change of charge, since the geometry of the system remains fixed. The effect on the potential of a point in space can then be derived by relating it to the separate effects produced by variations in charge on the modulator. This results from a consideration of the "Principle of Superposition" of charge, which relates to the effects on the potential, when charges are placed on conductors which may already have been charged. It can be deduced from Green's reciprocation theorem,

$$\Sigma \mathrm{Q}'\mathrm{V}=\Sigma \mathrm{Q}\mathrm{V}',$$

and is stated in the fifth edition of Jeans' 'Electricity and Magnetism,'

p. 90, in the following simple and concise form :—If charges Q_1, Q_2, etc., give rise to potentials V_1, V_2, etc., and charges Q_1', Q_2' . . . give rise to potentials V_1', V_2' . . ., then charges (Q_1+Q_1'), (Q_2+Q_2') give rise to potentials (V_1+V_1'), (V_2+V_2') . . .

It is to be understood that this statement of the principle applies to the changes on the conductors themselves. The potential produced on any of the conductors can be derived from the equations involving the Maxwell coefficients of potential in terms of the charge on it, together with that on the other conductors in the field :—

$$V_1=p_{11}Q_1+p_{21}Q_2+p_{31}Q_3+\ldots, \qquad (1)$$

$$V_2=p_{12}Q_1+p_{22}Q_2+p_{32}Q_3+\ldots, \qquad (2)$$

$$\cdots\cdots\cdots$$

$$V_n=p_{1n}Q_1+p_{2n}Q_2+p_{3n}Q_3+\ldots, \qquad (3)$$

where the coefficients p_{11}, p_{12}, etc., do not depend on either the potentials or the charges, and are purely geometrical quantities depending on the size, shape and positions of the conductors.

It is clear from (2) that if an additional charge δQ_2 be given to the second conductor, then its potential V_2 will alter. The effect on the first conductor (which, we suppose, kept at a fixed potential) will, by (1), result in a variation of Q_1 to some other value ; this will depend on Q_2 and hence on V_2.

In the case of an electrostatic field mapped out by equipotential lines, it is permissible to replace any one of the equipotential surfaces in space by a conducting infinitely thin sheet, so shaped as to conform with the shape of the equipotential surface, without affecting the field. This conducting sheet will have equal and opposite charges induced on its two surfaces, and will thus not contribute anything to the space potential. The shape of this fixed equipotential, as well as its position, will alter with each variation made in the field. If, however, instead of replacing the complete equipotential, only that part of it orthogonal to, and in the vicinity of the axis, is replaced, then variations in the field will not affect the shape of the small element on the axis, but can alter its position.

If we go further, and consider the element to degenerate into a point on the axis of the system, at which conducting material just sufficient to cover the point is situated, the presence of the conducting point will not affect the field in any way, since the conducting point remains uncharged.

This is an important conception, when dealing with the potential changes which arise at the crossover in a C/M/A system when the potential of the modulator is varied, the potentials of both cathode and anode remaining fixed.

If Q_C be the charge on the cathode,
If Q_M be the charge on the modulator,
If Q_A be the charge on the anode.

The potential of any one of these electrodes is defined in terms of the charges by the following equations :—

$$V_C=p_{CC}Q_C+p_{MC}Q_M+p_{AC}Q_A, \quad \text{(4)}$$

$$V_M=p_{CM}Q_C+p_{MM}Q_M+p_{AM}Q_A, \quad \text{(5)}$$

$$V_A=p_{CA}Q_C+p_{MA}Q_M+p_{AA}Q_A. \quad \text{(6)}$$

Now it is clear that these three simultaneous equations can be solved to give the charges in terms of the potentials. Thus

$$Q_C=r_{CC}V_C+r_{MC}V_M+r_{AC}V_A, \quad \text{(7)}$$

$$Q_M=r_{CM}V_C+r_{MM}V_M+r_{AM}V_A, \quad \text{(8)}$$

$$Q_A=r_{CA}V_C+r_{MA}V_M+r_{AA}V_A, \quad \text{(9)}$$

where r_{CC}, etc., are coefficients of capacity, r_{CM}, etc., are coefficients of induction.

From (8), if V_C and V_A are maintained constant, $V_M \propto Q_M$,
From (7), if V_C and V_A are maintained constant, $Q_C \propto V_M$,
From (9), if V_C and V_A are maintained constant, $Q_A \propto V_M$.

Hence, the effect of varying the potential (*i. e.* charge) on the modulator is to produce a variation of charge on both cathode and anode ; these separate variations are proportional to the change in modulator potential.

If we now take our imaginary point conductor as fixed at the crossover H, the effect of putting charge on the cathode, modulator and anode is to bring this conductor up to some potential intermediate between the highest and lowest in the field. The potential at H will be given in terms of the charges and coefficients already due to the electrodes, thus

$$V_H=p_{HH}Q_H+p_{CH}Q_C+p_{MH}Q_M+p_{AH}Q_A. \quad \text{(10)}$$

But as we are putting no charge on the small conductor at H, then $Q_H=0$.

$$\therefore\ V_H=p_{CH}Q_C+p_{MH}Q_M+p_{AH}Q_A. \quad \text{(11)}$$

If, now, the charge on the modulator is altered by δQ_M,

$$V_H'=(p_{CH}Q_C+k_1\delta Q_M)+p_{MH}(Q_M+\delta Q_M)+(p_{AH}Q_A+k_2\delta Q_M), \quad \text{(12)}$$

because both additional cathode and anode charges are proportional to δQ_M.

$$\therefore\ V_H'-V_H=p_{MH}\delta Q_M+k_1\delta Q_M+k_2\delta Q_M, \quad \text{(13)}$$

i. e., the change in potential at the crossover, considered as a fixed point, is proportional to the change in *charge* at the modulator.

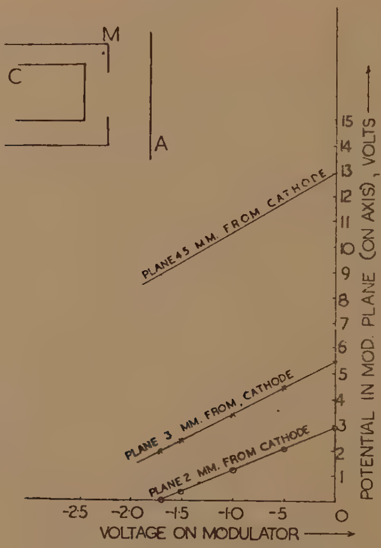
Now,
$$V_M=p_{CM}Q_C+p_{MM}Q_M+p_{AM}Q_A,$$
$$\underline{V_M'=(p_{CM}Q_C+k_1\delta Q_M)+p_{MM}(Q_M+\delta Q_M)+(p_{AM}Q_A+k_2\delta Q_M)}$$

$$\therefore\ V_M'-V_M=Kp_{MM}\delta Q_M \propto V_H'-V_H, \quad \text{(14)}$$

i. e. change in potential at the modulator is proportional to the change in potential at the crossover. Although the crossover has been taken as a specific point in the above case, because it is bound up with the beam angle, the actual position of the specimen point is quite unrestricted, since each point in the field is accompanied by its own set of coefficients, which are usually different for every point. The variations in potential of the modulator (or other electrodes) do affect the potential of this point proportionally, according to the above analysis.

It is important to point out here that this proportionality depends on the ratio of coefficients, and these will only remain constant if the position of the point with respect to the modulator remains constant. In the case of the crossover, this requirement is not strictly fulfilled in practice, as we know that the focal length of the $C/M/A_1$ lens combination varies with modulation. However, it can be taken to hold as a first approximation.

Fig. 1.



Potential in mod. plane.

At this point, then, each and every variation in modulator potential will produce a corresponding variation of potential at the crossover—to, say, $1/k$ that of the modulator. While dealing with this one point, the value of " k " will remain constant, but will vary from this value for any other point in the field.

In support of this theoretical argument, some experiments performed in the electrolytic tank are of interest. Fig. 1 shows how the potential at the centre of the modulator hole varies with the potential of the modulating electrode. This point was deliberately chosen because of its proximity to the crossover, though any other point would have given the same type of law and served to confirm the theory. The actual distribution was found from the pick-up of a probe as it was moved from cathode to anode in the tank. This was repeated for each setting of the modulation voltage (the anode and cathode voltage being kept fixed throughout, and with no alteration to the geometry of the system) and a new potential curve obtained. It will be seen that the relationship is linear, as was to be expected, *i. e.* that at the centre of the modulator (and, in fact, at any other point fixed in the field) the potential increases or decreases by a constant fraction of the modulator potential change, and thus faithfully reflects its variations. The figure also shows that at other points along the axis the relation remains linear.

In the more complicated case, where the potentials of all electrodes are made to vary, the same basic principles can be applied to determine the potential change at, say, the crossover H—because, when each is made to vary separately the potential varies proportionally. By superimposition, when they vary together, the total effect will be the sum of the separate effects,

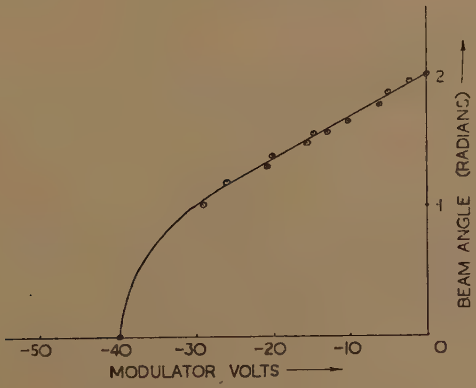
$$i.\,e.,\ \delta V_H = V_H' - V_H = K_1\delta V_C + K_2\delta V_M + K_3\delta V_A, \quad . \quad . \quad . \quad (15)$$

where the coefficients K_1, K_2, K_3 depend on the geometrical disposition of the conductors in relation to the crossover.

4. *Measurement of Beam Angle.*

Three separate experiments were made in order to establish the linear relation between beam angle and modulation voltage. In the first, a sliding engraved disc was moved along a metal line to within some 40 mm. of the modulator. At any fixed position the diameter of the fluorescent patch was read off for each modulation point, giving the beam angle.

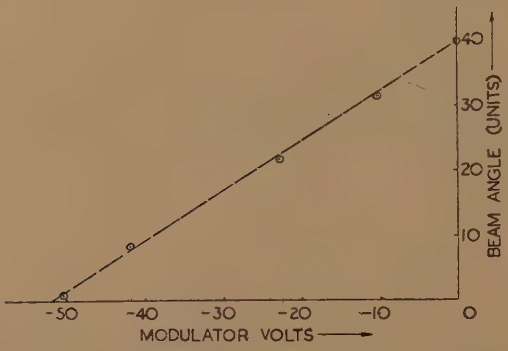
Fig. 2.



Beam angle and modulation.

The second experiment was an extension of the first and was realized in the electron optical voltmeter published in J. I. E. E. for May 1944. Here the disc is placed 220 mm. from the cathode, and beam diameters up to five times those for the small discs could be read off on the engraved anode. Fig. 2 is a reproduction of the curve connecting beam angle and modulation, and shows that the linear relation was realized, and that the error in estimating the beam angle rarely exceeded 5 per cent. One of the criticisms levelled against both these methods was that while the stated accuracy could be claimed for reading the fluorescent edge to the beam, nevertheless, owing to its indefiniteness, such readings were not reliable. To prove this point, some separate experiments were conducted in a demountable tube in which the beam was scanned by a movable Faraday cage with an 0·62 mm. diameter hole at its centre. The electron distribution in the beam was then recorded with a delicate microammeter, and the beam width measured across the tail of the curve. The measured beam angle at M=0 then agreed with the original experiments. In the case of another system, as between full and quarter-modulation, the ratio of current at the peak to that in the tail was 100/1 and 10/1 respectively, and the linear relation shown in fig. 3 was obtained. Thus, the visual method of measurement is more or less vindicated. In some ways, at any rate, marked disagreement between the two methods would not be expected, since the current in the tail was not read to better accuracy than 1/10th μA, and the fluorescent effect of this current at 5 kv. gave just such an accuracy.

Fig. 3.



Beam angle and modulation (from Faraday cage scanning).

5. *Beam Angle and Modulation.*

In order to arrive at a theoretical justification for the linear relationship between the beam angle and the modulation voltage, certain simplifying assumptions must be made and accepted. These are (*a*) that the geometrical position of the crossover does not vary with modulation—this is true to a first approximation; (*b*) that the effects of space charge are negligible. If, then, we take two trajectories each representing the beam envelope at two different values M_1 and M_2 of the modulation voltage, each will have to obey the trajectory equation for the paraxial condition. Fig. 4 shows two such rays which cross over at H, at angles θ_1 and θ_2 to the axis.

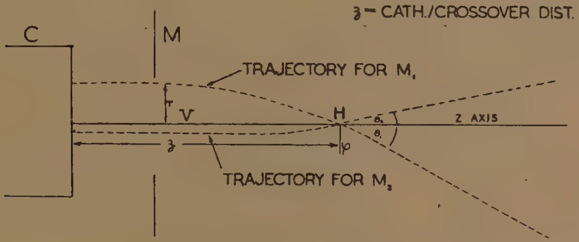
In each case
$$\frac{d^2r}{dz^2}+\frac{V'}{2V}\frac{dr}{dz}+\frac{V''}{4V}\cdot r=0, \quad \ldots \quad (1)$$

where V, V′, V″ respectively denote the potential and its first and second derivatives on the axis in each case.

At the crossover, $r=0$. $\therefore$
$$\frac{d^2r}{dz^2}+\frac{V'}{2V}\frac{dr}{dz}=0, \quad \ldots \quad (2)$$

that is, $r=Ae^{-\frac{V'}{2V}\cdot z}+B$ is the equation to the curve in the immediate vicinity of the crossover.

Fig. 4.



Immersion objective. Electron trajectories for two values of mod. volts.

The slope of the trajectory at the crossover, which defines the beam angle, is obviously (dr/dz). Hence for ray (1),

$$\left(\frac{dr}{dz}\right)_1=\tan\theta_1\doteq\theta_1=-\frac{V'}{2V}Ae^{-\frac{V'}{2V}\cdot z} \quad \ldots \quad (3)$$

$$=-\frac{1}{2}\frac{d}{dz}(\log V)Ae^{-\frac{1}{2}\frac{d}{dz}(\log V)z}. \quad \ldots \quad (4)$$

Similarly for ray (2),

$$\left(\frac{dr}{dz}\right)_2=\tan\theta_2\doteq\theta_2=-\frac{1}{2}\left(\frac{d}{dz}\log U\right)Ae^{-\frac{1}{2}\left(\frac{d}{dz}\log U\right)z}, \quad \ldots \quad (5)$$

where V=potential at the crossover in case (1)
$=\phi+kM_1$, where ϕ=potential at the crossover for zero angle condition, where $kM_1 \ll \phi$.

Similarly, U=the potential at the crossover in case (2)
$=\phi+kM_2$, where kM_2 is the potential change at the crossover due to a potential change of M_2 at the modulator, and $kM_2 \ll \phi$.

Hence, the change in beam angle, from the blackout point at which the angle $\theta=\theta_0$ is very small, to the point at which $\theta=\theta_1$, is given by

$$\theta_1-\theta_0\doteqdot-\frac{1}{2}\left(\frac{d}{dz}\log \mathrm{V}\right)\mathrm{A}e^{-\frac{\mathrm{V}'}{2\mathrm{V}}z}-\left\{-\frac{1}{2}\left(\frac{d}{dz}\log \phi\right)\mathrm{A}e^{-\frac{\phi'}{2\phi}.z}\right\} \quad . \quad . \quad . \quad (6)$$

$$\doteqdot-\frac{\mathrm{A}}{2}\left(\frac{d}{dz}\log \mathrm{V}\right)\left\{1-\frac{\mathrm{V}'}{2\mathrm{V}}z+\frac{1}{2}\left(\frac{\mathrm{V}'}{2\mathrm{V}}\right)^2z^2+\ldots\right\}$$

$$+\frac{\mathrm{A}}{2}\left(\frac{d}{dz}\log \phi\right)\left\{1-\frac{\phi'}{2\phi}.z+\frac{1}{2}\left(\frac{\phi'}{2\phi}\right)^2.z^2+\ldots\right\}. \quad . \quad (7)$$

$$\doteqdot\frac{\mathrm{A}}{2}\left[\left(\frac{d}{dz}\log \phi\right)-\left\{\frac{d}{dz}\log(\phi+k\mathrm{M}_1)\right\}\right]-\frac{\mathrm{A}z}{4}\left[\left(\frac{d}{dz}\log\phi\right)^2\right.$$

$$\left.-\left\{\frac{d}{dz}\log(\phi+k\mathrm{M}_1)\right\}^2\right]+\frac{\mathrm{A}z^2}{16}\left[\left(\frac{d}{dz}\log\phi\right)^3\right.$$

$$\left.-\left\{\frac{d}{dz}\log(\phi+k\mathrm{M}_1)\right\}^3\right]+\ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\doteqdot\frac{\mathrm{A}}{2}\left[\frac{\phi'}{\phi}-\frac{\phi'}{\phi+k\mathrm{M}_1}\right]-\frac{\mathrm{A}z}{4}\left[\frac{\phi'^2}{\phi^2}-\left(\frac{\phi'}{\phi+k\mathrm{M}_1}\right)^2\right]$$

$$+\frac{Az^2}{16}\left[\left(\frac{\phi'}{\phi}\right)^3-\left(\frac{\phi'}{\phi+k\mathrm{M}_1}\right)^3\right]+\ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\doteqdot\frac{\mathrm{A}}{2}\left[\frac{\phi'}{\phi}-\frac{\phi'}{\phi}\left(1-\frac{k\mathrm{M}_1}{\phi}\right)\right]-\frac{\mathrm{A}z}{4}\left[\frac{\phi'^2}{\phi^2}-\frac{\phi'^2}{\phi^2}\left(1-\frac{2k\mathrm{M}_1}{\phi}\right)\right]$$

$$+\frac{\mathrm{A}z^2}{16}\left[\frac{\phi'^3}{\phi^3}-\frac{\phi'^3}{\phi^3}\left(1-\frac{3k\mathrm{M}_1}{\phi}\right)\right]+\ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\doteqdot\frac{\mathrm{A}}{2}\left[\frac{\phi'k\mathrm{M}_1}{\phi^2}\right]-\frac{\mathrm{A}z}{4}\left[\frac{2\phi'^2}{\phi^3}k\mathrm{M}_1\right]+\frac{\mathrm{A}z^2}{16}\left[\frac{3\phi'^3k\mathrm{M}_1}{\phi^4}\right]+\ldots \quad . \quad (11)$$

$$\therefore\ \theta_1-\theta_0\doteqdot\frac{\mathrm{A}}{2}\frac{\phi'}{\phi^2}k\mathrm{M}_1\left[1-z\frac{\phi'}{\phi}+\frac{3}{8}z^2\left(\frac{\phi'^2}{\phi^2}\right)-\frac{1}{12}\left(\frac{\phi'}{\phi^2}\right)^3z^3+\ldots \quad . \quad . \quad (12)\right.$$

Also

$$\theta_2-\theta_0\doteqdot\frac{\mathrm{A}}{2}\frac{\phi'}{\phi^2}k\mathrm{M}_2\left[1-z\frac{\phi'}{\phi}+\frac{3}{8}z^2\left(\frac{\phi'^2}{\phi^2}\right)\ldots,\right. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where ϕ and ϕ' are the potential and its first derivative at the crossover.

$$\therefore\ \frac{\theta_1-\theta_0}{\theta_2-\theta_0}=\frac{k\mathrm{M}_1}{k\mathrm{M}_2}=\frac{\mathrm{M}_1}{\mathrm{M}_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$\therefore\ \frac{\mathrm{M}_1}{\mathrm{M}_2}=\frac{\text{change in mod. volts for } \theta_0 \text{ to } \theta_1}{\text{change in mod. volts for } \theta_0 \text{ to } \theta_2}=\frac{\theta_1}{\theta_2}. \quad . \quad . \quad . \quad (15)$$

Since θ_0 can be considered a constant (zero).

This proof need not, of course, be so general, because the potential of the crossover for the zero angle condition can be taken as the zero of potential, in which case the ϕ drops out completely. The change in potential at the crossover is then directly proportional to the change in potential at the modulator in each case.

I wish to express my thanks to the Director of the G.E.C. Laboratories, Wembley, for permission to publish this paper.

XLIX. *Flow of a Liquid from a Reservoir over a Plane—Boundary-Layer Theory.*

By Dr. M. Ray, Agra, India*.

[Received April 6, 1947.]

A liquid is standing at rest in a reservoir and is then allowed to flow down from the reservoir over a plane through a wide aperture. This problem has been discussed on the Boundary Layer theory and the growth of the layer has been found out. It is seen that at any section the layer increases rapidly at the beginning, but this growth slows down with advancing time and ultimately the liquid settles down on the plane.

Take the origin at the front edge of the plane, x-axis along its length and y-axis perpendicular to the plane. When the flow has passed over any section for some time, the pressure gradient there may be taken to be zero and then the boundary-layer equations take the forms

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \qquad \ldots \ldots \qquad (1)$$

and

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad \ldots \ldots \qquad (2)$$

where u, v are the components of velocity at any point and ν the kinematic viscosity. These equations are to be solved subject to the conditions that

$$u=0=v \text{ at } y=0,$$

and at the top of the layer

$$\frac{\partial u}{\partial y}=0=\frac{\partial^2 u}{\partial y^2}.$$

The equation of continuity is satisfied by

$$u=\frac{\partial\psi}{\partial y}, \quad v=-\frac{\partial\psi}{\partial x}. \qquad \ldots \ldots \qquad (3)$$

To solve (1) let us make the following substitutions,

$$\eta=\mathrm{A}\frac{yt^n}{x^l}, \quad \psi=\mathrm{B}f(\eta)x^kt^m. \qquad \ldots \ldots \qquad (4)$$

Then, calculating u and v from (3) and substituting in (1), we get

$$\begin{aligned}&\mathrm{AB}x^{k-l}t^{m+n-1}\{(m+n)f'(\eta)+n\eta f''(\eta)\}+\mathrm{A}^2\mathrm{B}^2x^{2k-2l-1}t^{2m+2n}f'(\eta)\times\\&\quad\times\{(k-l)f'(\eta)-l\eta f''(\eta)\}+\mathrm{A}^2\mathrm{B}^2x^{2k-2l-1}t^{2m+2n}f''(\eta)\{-kf(\eta)+l\eta f'(\eta)\}\\&\qquad=\nu\mathrm{A}^3\mathrm{B}x^{k-3l}t^{m+3n}f'''(\eta), \qquad \ldots \ldots \qquad (5)\end{aligned}$$

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* Communicated by the Author.

where dashes denote differentiations with respect to η. Equating powers of t and x from every term we find

$$m=n=-\tfrac{1}{2}, \quad l=0, \quad k=1, \qquad \ldots \ldots \quad (6)$$

and also equating the coefficients we get

$$A=\frac{1}{\sqrt{\nu}}, \quad B=\sqrt{\nu}. \qquad \ldots \ldots \quad (7)$$

Thus, finally, the equation (5) reduces to

$$f'''(\eta)+\{f(\eta)+\tfrac{1}{2}\eta\}f''(\eta)-f'^2(\eta)+f'(\eta)=0. \qquad \ldots \ldots \quad (8)$$

Now, with the help of (6) and (7), we get

$$u=\frac{x}{t}f'(\eta), \quad v=-\sqrt{\frac{\nu}{t}}\cdot f(\eta), \quad \eta=\frac{y}{\sqrt{\nu t}}, \qquad \ldots \ldots \quad (9)$$

where $f(\eta)$ is a function of η only and satisfies the differential equation (8).

The equation (8) is to be solved subject to the conditions that when

$$\eta=0, \quad f(\eta)=0, \quad f'(\eta)=0\,; \qquad \ldots \ldots \quad (10)$$

and if $\eta=\eta_0$ gives the thickness of the layer at any section, then when

$$\eta=\eta_0, \quad f''(\eta)=0, \quad f'''(\eta)=0. \qquad \ldots \ldots \quad (11)$$

The solution of (8) may be expanded in a series of the form

$$f(\eta)=a\eta^2+b\eta^3+c\eta^4+d\eta^5+\ldots,$$

where the conditions (10) at $\eta=0$ are satisfied.

Substituting this in (8) and equating the constant term and the coefficients of various powers of η to zero, we get

$$b=0, \quad c=-\frac{a}{8}, \quad d=\frac{a^2}{30},$$

hence correct up to η^5,

$$f(\eta)=a\eta^2-\frac{1}{8}a\eta^4+\frac{1}{30}a^2\eta^5.$$

The conditions (11) then give

$$2a-\frac{3}{2}a\eta_0^2+\frac{2}{3}a^2\eta_0^3=0$$

and

$$-3a\eta_0+2a^2\eta_0^2=0$$

$$\therefore \quad \eta_0=2 \text{ and } a=\frac{3}{4}.$$

Thus, from (9), we have

$$u=\frac{3xy}{4\nu^{1/2}t^{3/2}}\left(2-\frac{y^2}{2\nu t}+\frac{y^3}{8\nu^{3/2}t^{3/2}}\right)$$

$$v=-\frac{3y^2}{4\nu^{1/2}t^{3/2}}\left(1-\frac{y^2}{8\nu t}+\frac{y^3}{40\nu^{3/2}t^{3/2}}\right). \qquad \ldots \ldots \quad (12)$$

Since the thickness of the layer is small, the above gives the velocity-distribution inside the layer correct up to a sufficient approximation. The thickness of the layer corresponds to $\eta=\eta_0=2$ and is given by

$$y=2\sqrt{\nu t}, \qquad \text{. . . . .} \quad (13)$$

and at the top of this layer we have, from (12),

$$u=\frac{3}{2}\frac{x}{t}, \quad v=-\frac{21}{10}\sqrt{\frac{\nu}{t}}. \qquad \text{. . . . .} \quad (14)$$

From the solution (12) it is seen that at any instant when the liquid leaves the reservoir, it flows down and begins to spread over the plane. The equation (13) shows that the thickness of the layer at any section is of the order $\sqrt{\nu}$ and the thickness grows rapidly at the beginning, but this rate becomes slow as the flow passes on over the section for some time. The equation (12) shows that the velocity components at first rise rapidly but ultimately die down, with the result that the liquid settles down on the plane. The thickness of the layer increases as more and more liquid comes from the reservoir and flows over. The velocity of the liquid at the top cannot therefore be zero as long as the liquid flows, as is evident from equation (14).

Since ν is of the dimension $\frac{\mathrm{L}^2}{\mathrm{T}}$, η is a non-dimensional quantity, that is, a numerical constant, varying from value zero on the plane to value 2 at the top of the layer. If u', v' be the components of velocity at the top at any section, as given by (14), then

$$\frac{u}{u'}=\eta\left(1-\frac{1}{4}\eta^2+\frac{1}{16}\eta^3\right)$$

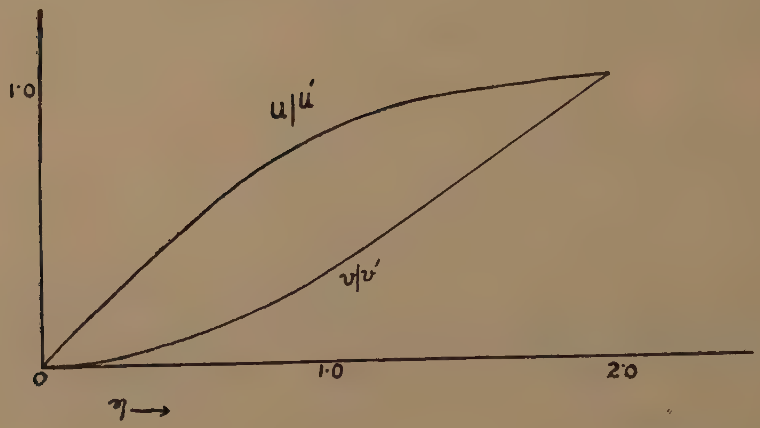
$$\frac{v}{v'}=\frac{5}{14}\eta^2\left(1-\frac{1}{8}\eta^2+\frac{1}{40}\eta^3\right). \qquad \text{. . . . . .} \quad (15)$$

The values of $\frac{u}{u'}$ and $\frac{v}{v'}$ have been tabulated for different values of η from 0 to 2, as shown in the table below, and the corresponding curves have been drawn in the figure.

TABLE.

η	0	·2	·5	1·0	1·5	1·7	1·8	1·9	2
$\frac{u}{u'}$	0	·198	·473	·813	·974	—	·997	—	1
$\frac{v}{v'}$	0	·014	·087	·322	·646	·789	·859	·93	1

Fig. 1.



The curve $\frac{u}{u'}$ rises sharply at the beginning and attends slowly the value 1 at the top. The curve $\frac{v}{v'}$ rises very slowly near the plane and attends the value 1 at the top almost linearly.

Agra College, Agra, India.

L. *Notice respecting New Book.*

An Introduction to the Theory of Seismology. By K. E. Bullen. [Pp. xiv+276.] (Cambridge University Press, 1947. Price 18*s.* 0*d.*)

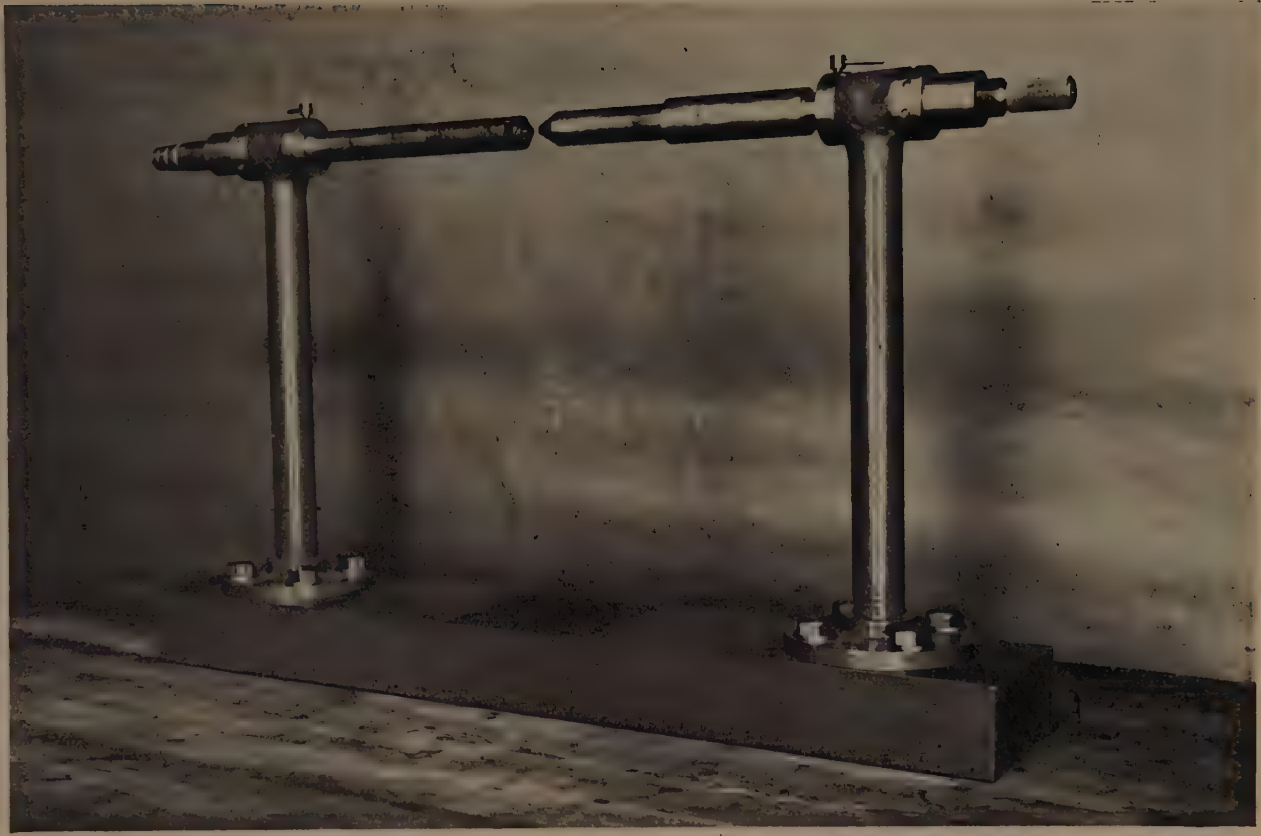
Professor Bullen has written a book of interest and value to all who have cultivated the field of seismology. Whether he be specialist or layman, the reader who studies this work in the spirit in which it was written will find himself the possessor of a clear and accurate picture of the theory of modern seismology. A mathematical mind, rather than deep mathematical knowledge, is the indispensable prerequisite for tackling the book successfully. Professor Bullen has begun at the beginning with a masterly analysis of the theory of elasticity and of wave-motion which is of interest to all students of physics. But though he may encourage the explorer "with learning put lightly like powder in jam," the author has to cope with many inherent difficulties when he comes to the application of these principles to the problems of seismology. He has accomplished this task with skill and clarity and the present reviewer owes Professor Bullen a debt of gratitude therefor. It only remains to produce a companion volume which deals in similar fashion with the experimental side of the subject.

The book is admirably printed and is produced at a reasonable price.

A. F.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

Fig. 2.



The acoustic air-jet generator.

LI. *The Conservation of Energy, Momentum and Charge in the Nuclear Field.*

By H. T. Flint and N. Symonds, Bedford College, London *.

[Received March 28, 1947.]

This contribution to the theory of the nuclear field is an amplification of a paper already published on "Energy in the Nuclear Field" ⁽¹⁾. It contains also the suggestion of a change in the sign of one component of the interaction between a nuclear particle and the nuclear field, but the principles of the theory are the same in both papers.

The discussion is limited to the type of field known as the scalar field, but a similar argument may be applied to the other kinds of field occurring in the field theories of the nucleus.

The discussion is based on the assumption that the scalar field is associated with a modified form of Dirac's equation, viz.:

$$\gamma^{\mu}\frac{\partial\psi}{\partial x^{\mu}}=g\gamma^{\mu}T_{\mu}\psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and that the field is described by the following field equations:—

$$\frac{\partial T_{\mu}}{\partial x^{\nu}}-\frac{\partial T_{\nu}}{\partial x^{\mu}}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{\partial V^{\mu}}{\partial x^{\mu}}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where (T_{μ}) and (V^{μ}) denote vectors in a five-dimensional continuum.

A vector (S^{μ}) described as an interaction vector is introduced by means of the equation

$$T^{\mu}=V^{\mu}+S^{\mu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For other details of the notation the former paper should be consulted. An account of the relations between five-dimensional and corresponding four-dimensional quantities will be found in reference ⁽²⁾.

The argument which leads to the adoption of equation (1) as the first-order equation of the nucleon leads also to the following equation ⁽²⁾ ⁽³⁾:

$$\gamma^{\mu}\frac{\partial\psi}{\partial x^{\nu}}+\Delta_{\nu}\gamma^{\mu}-\gamma^{\mu}\Delta_{\nu}=g\gamma^{\mu}H_{\nu}\psi. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The operators (Δ_{ν}) are certain matrix operators ⁽⁴⁾ which satisfy the relation

$$\Delta_{\mu}\gamma^{\mu}-\gamma^{\mu}\Delta_{\mu}=0,$$

and it is assumed that

$$\gamma^{\mu}H_{\mu}=\gamma^{\mu}T_{\mu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For the present only the degenerate form (1) of equation (5) is required.

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* Communicated by the Authors.

The Energy Tensor.

The energy tensor of the field is

$$R^{\mu}_{\nu}=(V^{*\mu}T_{\nu}-\tfrac{1}{2}\delta^{\mu}_{\nu}V^{*\sigma}T_{\sigma})+\text{complex conjugate}. \quad . \quad . \quad (7)$$

In some cases, for the sake of brevity, the complex conjugate quantity will be understood or denoted by the letters c.c.

In five-dimensions this tensor is strictly the energy momentum and charge tensor.

For the particle, which is represented by equation (1), the energy tensor is

$$Q^{\mu}_{\nu}=\frac{hc}{4\pi i}\left(\psi^{\dagger}\gamma^{\mu}\frac{\partial\psi}{\partial x^{\nu}}-\frac{\partial\psi^{\dagger}}{\partial x^{\nu}}\gamma^{\mu}\psi\right)-\tfrac{1}{2}\delta^{\mu}_{\nu}\left\{\frac{hc}{4\pi i}\left(\psi^{\dagger}\gamma^{\sigma}\frac{\partial\psi}{\partial x^{\sigma}}-\frac{\partial\psi^{\dagger}}{\partial x^{\sigma}}\gamma^{\sigma}\psi\right)\right\}. \quad . \quad (8)$$

The difference in the present case from that of the electron in Dirac's theory is the occurrence of the terms in the second bracket. These vanish for a particle which is represented by Dirac's equation, but in the case of a particle represented by equation (1) they have the value

$$\frac{hc}{4\pi i}(g\psi^{\dagger}\gamma^{\sigma}\psi T_{\sigma}-g^{*}\psi^{\dagger}\gamma^{\sigma}\psi T^{*}_{\sigma}).$$

The occurrence of the factors $\frac{1}{2}$ in equations (7) and (8) is determined by the requirement of conservation. As in the previous paper the factors of the field components $(T_{\sigma}, T^{*}_{\sigma})$ are identified with the interaction terms of the field equations.

The following suggestion for the identification of these terms is now made :—

$$\left.\begin{aligned} S^{*\sigma}&=\frac{hcg}{2\pi i}\psi^{\dagger}\gamma^{\sigma}\psi, \\ S^{\sigma}&=-\frac{hcg^{*}}{2\pi i}\psi^{\dagger}\gamma^{\sigma}\psi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

These relations differ slightly from those suggested in the previous paper, but the identification now proposed appears to be required by the form of the energy tensor of the particle and the field now adopted.

The corresponding four-dimensional quantities are denoted by (I^{m}) and $(I_{.})$.

$$\left.\begin{aligned} I^{*m}&=\frac{hcg}{2\pi i}\psi^{\dagger}\beta^{m}\psi, \\ I^{*}_{.}&=\frac{hcg}{2\pi i}\psi^{\dagger}\beta_{.}\psi. \end{aligned}\right\}.$$

In order to introduce the Dirac matrices, equation (1) is reduced by replacing the matrices (γ^{μ}) by the corresponding four-dimensional matrices (β^{m}) and $\beta_{.}$. It is then multiplied throughout by $i\beta_{4}$ and the substitutions

$$\alpha_{k}=-i\beta_{4}\beta_{k}, \qquad \beta=-i\beta_{4}\beta_{.} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

are made. The matrices (α_k) and β are the Dirac matrices. The distinction between contravariance and covariance in four-dimensions has been dropped, since the gravitational field is of no importance in the theory of the nuclear field.

The interaction terms then become

$$\left.\begin{aligned} I_k^* &= \frac{hcg}{2\pi i}\psi^\dagger\alpha_k\psi \quad (k=1,2,3), \\ I_4^* &= -\frac{hcg}{2\pi}\psi^\dagger\psi, \\ I^* &= \frac{hcg}{2\pi i}\psi^\dagger\beta\psi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

No confusion is caused by retaining the symbol I in either case, since the matrix itself denotes the way in which the interaction term is expressed.

It is more convenient to express the tensor Q_ν^μ in the form

$$Q_\nu^\mu = \frac{hc}{4\pi i}\left(\psi^\dagger\gamma^\mu\frac{\partial\psi}{\partial x^\nu} - \frac{\partial\psi^\dagger}{\partial x^\nu}\gamma^\mu\psi\right) - \delta_\nu^\mu\left\{\frac{hc}{4\pi i}\left(\psi^\dagger\gamma^\sigma\frac{\partial\psi}{\partial x^\sigma} - \frac{\partial\psi^\dagger}{\partial x^\sigma}\gamma^\sigma\psi\right)\right\}$$
$$+\tfrac{1}{2}\delta_\nu^\mu(S^{*\sigma}T_\sigma + S^\sigma T_\sigma^*). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This is possible on account of equation (1).

The use of this form is convenient in considering the energy density term $(-Q_4^4)$, for then the terms in ψ are those which arise directly from the first-order equation expressing the energy of the nucleon considered as a particle.

The expression (12) suggests that the particle acquires or loses energy by its interaction with the nuclear field, the amount of this energy being determined by the interaction terms containing S^σ and T^σ.

The total energy tensor, Π_ν^μ, of the field and particle is the sum of R_ν^μ and Q_ν^μ.

Thus

$$\Pi_\nu^\mu = T^{*\mu}T_\nu - \tfrac{1}{2}\delta_\nu^\mu T^{*\sigma}T_\sigma + \frac{hc}{4\pi i}\psi^\dagger\gamma^\mu\frac{\partial\psi}{\partial x^\nu} - \delta_\nu^\mu\frac{hc}{4\pi i}\psi^\dagger\gamma^\sigma\frac{\partial\psi}{\partial x^\sigma}$$
$$-(S^{*\mu}T_\nu - \delta_\nu^\mu S^{*\sigma}T_\sigma) + \text{c.c.}$$

Thus the total energy tensor can be regarded as made up of three parts, the first arising from the field, the second from the particle and the third from the interaction.

The Conservation of the Tensor (Π_ν^μ).

The component Π_5^m $(m\neq5)$ of this tensor is

$$\Pi_5^m = V^{*m}T_5 + V^mT_5^* + \frac{hc\kappa}{2\pi}\psi^\dagger\gamma^m\psi. \quad . \quad . \quad . \quad (13)$$

The constant κ is the reciprocal of the fundamental length $h/2\pi M_0c$ associated with the nucleon, the function ψ containing the factor $\exp(\iota\kappa x^5)$.

The current density is given by

$$J^m = \alpha \Pi^m_5,$$

where α is Kaluza's constant ($\alpha = e/M_0c^2$), where M_0 denotes the mass which occurs in the constant κ.

The latter constant for the field has the value $2\pi\mu_0 c/h$ and the reciprocal of this length is a measure of the range of action of the field. Thus for the field $\alpha = e/\mu_0 c^2$, and in all cases $\alpha\kappa = 2\pi e/hc$. With this in mind the current vector becomes

$$J^m = \alpha(V^{*m}T_5 + V^m T^*_5) + e\psi^\dagger\gamma^m\psi. \quad \ldots \quad (14)$$

By means of the field equations, equation (1) and the relations (9) it can be shown that

$$\frac{\partial J^m}{\partial x^m} = 0. \quad \ldots \quad (15)$$

It thus appears that the field contributes to the current and that this contribution together with the current of the nucleon is conserved.

This is part of the conservation of the tensor (Π^μ_ν) and the question of conservation applied to the tensor as a whole must be considered.

Let

$$L = \left(\tfrac{1}{2}T^{*\sigma}T_\sigma + \frac{hc}{4\pi i}\psi^\dagger\gamma^\sigma\frac{\partial\psi}{\partial x^\sigma} - S^{*\sigma}T_\sigma\right) + \text{c.c.} \quad \ldots \quad (16)$$

This, except for sign, is the factor of δ^μ_ν occurring in the expression for Π^μ_ν.

It will be expressed in terms of four-dimensional quantities with (β^m) and $\beta_.$, corresponding to (T^μ) and with the omission of the gravitational field.

$$L = \left\{\tfrac{1}{2}B^*_m B_m + \tfrac{1}{2}B^*_. B_. - I^*_m B_m - I^*_. B_. + \frac{hc}{4\pi i}\psi^\dagger\beta_m\frac{\partial\psi}{\partial x^m}\right.$$
$$\left. + \tfrac{1}{2}M_0c^2\psi^\dagger(\beta_. - \alpha\phi_m\beta_m)\psi\right\} + \text{c.c.} \quad \ldots \quad (17)$$

It follows from equation (2), taking $\mu = m$, $\nu = 5$, that

$$i\kappa B_m = \left(\frac{\partial}{\partial x^m} - i\alpha\kappa\phi_m\right)B_. \quad \ldots \quad (18)$$

It may be noted that

$$\frac{\partial L}{\partial\phi_m} = \{-\tfrac{1}{2}\alpha(B^*_. B_m + B_. B^*_m) + \alpha I^*_m B_. - \tfrac{1}{2}e\psi^\dagger\beta_m\psi\} + \text{c.c.}$$

Writing $V^m = A^m = B^m - I^m$, thus making (A^m) and $A_.$ the four-dimensional quantities corresponding to (V^μ),

$$\frac{\partial L}{\partial\phi_m} = -\alpha(A^{*m}B_. + A^m B^*_.) - e\psi^\dagger\beta^m\psi = -J^m. \quad \ldots \quad (19)$$

The function L depends upon $B_.$, $\frac{\partial B_.}{\partial x^m}$, ψ, $\frac{\partial\psi}{\partial x^m}$, ϕ_m and the complex conjugates of these quantities.

The following relations are equivalent to the field equations (3) and to the quantum equation (1):

$$\frac{\partial}{\partial x^m}\frac{\partial \mathrm{L}}{\partial\left(\frac{\partial \mathrm{B}.}{\partial x^m}\right)}-\frac{\partial \mathrm{L}}{\partial \mathrm{B}.}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

and

$$\frac{\partial}{\partial x^m}\frac{\partial \mathrm{L}}{\partial\left(\frac{\partial \psi}{\partial x^m}\right)}-\frac{\partial \mathrm{L}}{\partial \psi}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

L is the Lagrangian of this field and particle theory. These relations are useful in evaluating the expression $\frac{\partial \mathrm{L}}{\partial x^n}$.

For

$$\frac{\partial \mathrm{L}}{\partial x^n}=\left\{\frac{\partial \mathrm{L}}{\partial \mathrm{B}.}\frac{\partial \mathrm{B}.}{\partial x^n}+\frac{\partial \mathrm{L}}{\partial\left(\frac{\partial \mathrm{B}.}{\partial x^m}\right)}\frac{\partial}{\partial x^n}\left(\frac{\partial \mathrm{B}.}{\partial x^m}\right)+\frac{\partial \mathrm{L}}{\partial \psi}\frac{\partial \psi}{\partial x^n}+\frac{\partial \mathrm{L}}{\partial\left(\frac{\partial \psi}{\partial x^m}\right)}\frac{\partial}{\partial x^n}\left(\frac{\partial \psi}{\partial x^m}\right)\right\}$$
$$+\mathrm{c.c.}+\frac{\partial \mathrm{L}}{\partial \phi_m}\frac{\partial \phi_m}{\partial x^n}.$$

By means of equations (20) and (21) this becomes

$$\frac{\partial \mathrm{L}}{\partial x^n}=\frac{\partial}{\partial x^m}\left\{\frac{\partial \mathrm{L}}{\partial\left(\frac{\partial \mathrm{B}.}{\partial x^m}\right)}\frac{\partial \mathrm{B}.}{\partial x^n}+\frac{\partial \mathrm{L}}{\partial\left(\frac{\partial \psi}{\partial x^m}\right)}\frac{\partial \psi}{\partial x^n}+\mathrm{c.c.}\right\}+\frac{\partial \mathrm{L}}{\partial \phi_m}\frac{\partial \phi_m}{\partial x^n}$$
$$=\frac{\partial}{\partial x^m}\left\{\mathrm{A}^*_m(\mathrm{B}_n+\alpha\phi_n\mathrm{B}.)+\frac{hc}{4\pi i}\psi^\dagger\beta_m\frac{\partial \psi}{\partial x^n}+\mathrm{c.c.}\right\}+\frac{\partial \mathrm{L}}{\partial \phi_m}\frac{\partial \phi_m}{\partial x^n}.$$

Writing $\frac{\partial \mathrm{L}}{\partial x^n}=\delta^m_n\frac{\partial \mathrm{L}}{\partial x^m}$, this equation becomes

$$\frac{\partial}{\partial x^m}\left\{\left(\mathrm{V}^{*m}\mathrm{T}_n+\frac{hc}{4\pi i}\psi^\dagger\gamma^m\frac{\partial \psi}{\partial x^n}\right)+\mathrm{c.c.}-\delta^m_n\mathrm{L}\right\}=-\frac{\partial \mathrm{L}}{\partial \phi_m}\frac{\partial \phi_m}{\partial x^n}, \quad . \qquad (22)$$

where the five-dimensional quantities (V^m) and (T_n) have been introduced.

Thus within the brackets the tensor component Π^m_n has appeared.

In order to express the right-hand side of equation (22) more conveniently, let the energy tensor of the electromagnetic field, denoted by E^m_n, be introduced.

$$\mathrm{E}^m_n=\mathrm{H}^{mr}\mathrm{H}_{nr}-\tfrac{1}{4}\delta^m_n\mathrm{H}^{kl}\mathrm{H}_{kl},$$

where

$$\mathrm{H}_{nr}=\frac{\partial \phi_r}{\partial x^n}-\frac{\partial \phi_n}{\partial x^r}.$$

According to the electromagnetic theory,

$$\frac{\partial \mathrm{E}^m_n}{\partial x^m}=\mathrm{J}^m\mathrm{H}_{mn},$$

(J^m) denoting the current vector.

Thus $\dfrac{\partial E^m_n}{\partial x^m}=-\dfrac{\partial L}{\partial \phi_m}\left(\dfrac{\partial \phi_n}{\partial x^m}-\dfrac{\partial \phi_m}{\partial x^n}\right)$ and, since $\dfrac{\partial}{\partial x^m}\left(\dfrac{\partial L}{\partial \phi_m}\right)=0$,

by equation (15)

$$\frac{\partial L}{\partial \phi^m}\frac{\partial \phi^m}{\partial x^n}=\frac{\partial}{\partial x^m}\left(\phi_n\frac{\partial L}{\partial \phi_m}\right)+\frac{\partial E^m_n}{\partial x^m}.$$

Equation (22) thus becomes

$$\frac{\partial}{\partial x_m}(E^m_n+\Pi^m_n-\alpha\phi_n\Pi^m_5)=0, \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where the substitution

$$\alpha\Pi^m_5=J^m=-\frac{\partial L}{\partial \phi_m}$$

has been made.

It should be noted that $(\Pi^m_n-\alpha\phi_n\Pi^m_5)$ is the component (P^m_n) of the four-dimensional tensor corresponding to (Π^μ_ν).

Thus equation (23) represents the conservation of the four-dimensional energy tensor $(E^m_n+P^m_n)$ which is the tensor combining the electromagnetic and nuclear fields and the particle.

In the absence of an electromagnetic field the tensor (Π^m_n) is conserved.

The Total Energy of the Field and Particle.

Let the tensor be taken in its contravariant form (Π^{mn}) or (P^{mn}) so that the energy density may be obtained from the component Π^{44}. The total energy in the field is

$$W=-\int\Pi^{44}\,dv,$$

the integration being taken over three-dimensional space. The two parts R^{44} and Q^{44} from which Π^{44} is derived are given by the expressions (7) and (12).

$$R^{44}=(V^{*4}T_4-\tfrac{1}{2}V^{*\sigma}T_\sigma)+\text{c.c.}=-\tfrac{1}{2}(B^*_kA_k-B^*_4A_4+B^*_.A_.)+\text{c.c.} \qquad (24)$$

$$Q^{44}=-\frac{hc}{4\pi i}\left\{\psi^\dagger\beta_k\left(\frac{\partial}{\partial x^k}-\iota\kappa\alpha\phi_k\right)\psi-\left(\frac{\partial}{\partial x^k}+\iota\kappa\alpha\phi_k\right)\psi^\dagger\beta_k\psi\right\}-\frac{hc\kappa}{2\pi}\psi^\dagger\beta_.\psi$$
$$+(I^*_mB_m+I^*_.B_.+\text{c.c.}).\quad (k=1,2,3),\quad (m=1,2,3\;4). \quad . \quad (25)$$

The expression $(-Q^{44})$ denotes the energy density for the particle in the nuclear field, including the addition of energy arising from the interaction terms.

It is a property of R^{44} resulting from the field equations (1) that

$$\int(-R^{44})\,dv=\frac{1}{2c}\int(J^*\dot{A}_0-\dot{J}^*A_0)\,dv+\text{c.c.}; \quad . \quad . \quad . \quad . \quad (26)$$

where $A_0=\iota A_4$, $-\iota\kappa J=B_.$.

This is zero in the static case in which $\dot{A}_\iota$ and $\dot{J}$ are zero.

Thus the total energy in this case is $\int(-Q^{44})\,dv$, and the amount arising from the interaction terms is $\frac{1}{2}\int(\mathrm{I}^{*}_{m}\mathrm{B}_{m}+\mathrm{I}.^{*}\mathrm{B}.+\mathrm{c.c.})\,dv$. By a procedure similar to that which leads to equation (26) it can be shown that

$$\mathrm{N}=\frac{1}{2}\int(\mathrm{V}^{*\sigma}\mathrm{T}_{\sigma}+\mathrm{V}^{\sigma}\mathrm{T}^{*}_{\sigma})\,dv=\frac{1}{2c}\int(\mathrm{J}^{*}\dot{\mathrm{A}}_{0}+\dot{\mathrm{J}}^{*}\mathrm{A}_{0})\,dv+\mathrm{c.c.} \quad . \quad (27)$$

This again vanishes in the static case.

The sum of the expressions (24) and (25) gives the total energy density of the field and particle. By making use of equation (26) it appears that the total energy may be regarded as the energy of the nucleon with the addition of $\frac{1}{2}\int(\mathrm{I}^{*}_{m}\mathrm{B}_{m}+\mathrm{I.B.})\,dv$. This agrees with the energy calculated from equation (1). This equation suggests that this additional energy may be regarded as that acquired by the nucleon as a result of its interaction with the field.

The integral N can be divided into two parts, one containing only terms in (T_{μ}) or (B_{m}) and the other consisting of the above interaction terms which occur in the expression for the total energy.

Infinities have appeared in expressions for the total energy of the nucleon and nuclear field, and it is suggested that these arise from a procedure which is equivalent to dividing a finite quantity such as N into two parts each containing an infinity. The part of N which is required for the additional energy acquired by the nucleon must be finite if it is to have a physical meaning.

The actual mode of division must rest upon an appeal to experiment. It is justifiable to calculate the interaction part of N occurring in expression (25) by making the integration from a lower limit, not equal to zero. Experimental and theoretical considerations both suggest that this lower limit is a volume of linear dimensions, r_0, of the order of magnitude of the dimensions of the nucleon. The theoretical considerations lead to the view that the usual conceptions regarding location in space and time become illusory within these limits.

On examining the modified Dirac equation (1), as in the previous paper, it is seen that the term containing M_0c^2 in the original Dirac equation now contains $-(hcg/2\pi i)\mathrm{B}.\beta\psi$ in addition. This can be interpreted as indicating that the nucleon in a scalar field appears to change its mass by an amount equal to $-(hg\kappa/2\pi)^2e^{-\kappa r_0}/4\pi r_0$. The calculation is carried out in the same way as before.

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LII. *A New Drop-Weight Method for the Comparison of Surface Tensions.*

By R. C. Brown and H. McCormick *.

[Received November 18, 1947.]

Abstract.

It is shown by dimensional analysis that in the Harkins and Brown drop-weight method of surface-tension determination, the correction factor is a function only of the shape of the unstable drop, provided that when drops are similar in shape at the unstable stage equal fractions of the whole drop become detached. A drop-weight method has been worked out in which the usual flat tip of definite radius is replaced by an inverted cone. It is shown theoretically that all unstable drops hanging from such a cone are similar in shape, and surface-tension determinations may be carried out by this method provided that the cone is calibrated by means of a liquid of known surface tension. The conical-tip method does not require the use of correction factors, and it gives results which are in close agreement with those obtained by the method of Harkins and Brown.

1. *The method of Harkins and Brown.*

The determination of surface tension by weighing detached drops is usually carried out with the help of empirical tables compiled from experimental results by Harkins and Brown (1919). Successive drops are allowed to fall from the plane horizontal end of a circular jet or tip of radius R, and the average weight (W) of the detached drop is found either by direct weighing or by volume measurement and a knowledge of the liquid density (ρ). The relation used for the calculation of surface tension (γ) is :—

$$W=2\pi R\gamma Y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where Y is a so-called correcting factor, it being supposed that Y is unity in the case of an "ideal drop."

By determining the weights of drops of liquids of known surface tension which detached themselves from tips of various radii, Harkins and Brown showed that Y was uniquely determined by dimensionless quantities such as R/a and $R/V^{\frac{1}{3}}$, where

$$a^2=\gamma/g\rho$$

and V is the volume of the detached drop.

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* Communicated by the Authors.

Thus we can write

$$W = R\gamma f_1(R/a), \quad \ldots\ldots\ldots \quad (2)$$

or

$$W = R\gamma f_2(R/V^{\frac{1}{3}}), \quad \ldots\ldots\ldots \quad (3)$$

where

$$f_1(R/a) = f_2(R/V^{\frac{1}{3}}) = 2\pi Y.$$

When R, W and V have been determined for a liquid of unknown surface tension, γ may be calculated from equation (3), the value of $f_2(R/V^{\frac{1}{3}})$ for the appropriate value of $R/V^{\frac{1}{3}}$ being found from a table based on the experimental data of Harkins and Brown. It should be noticed that the method is not absolute, since the value of γ rests eventually upon the values of the surface tensions of the liquids which Harkins and Brown used in their determination of Y for various values of R/a. These surface tensions were determined by an independent method.

Equations (2) and (3) can be established by simple dimensional analysis supposing that W depends only upon R, γ, ρ and g. There seems little doubt that the determinations of surface tension based on the foregoing principles are satisfactory—a fact which suggests that the omission of viscosity from the factors upon which the weight of the detached drop depends is justified at any rate in those cases in which viscosity is not sufficiently high to make the experiment itself difficult. The following analysis appears to be more rigorous than any which has previously been suggested.

We can be reasonably sure that the *total* volume (V') of liquid hanging from the tip when the unstable state is reached is independent of viscosity if the drop is made to grow very slowly. Dimensional analysis shows that V' is related to R, γ, ρ and g by any of the following equations :—

$$V' = R^3 f_3(R/a), \quad \ldots\ldots\ldots \quad (4)$$

or

$$V' = aR^2 f_4(R/a), \quad \ldots\ldots\ldots \quad (5)$$

or

$$V' = a^2 R f_5(R/a), \quad \ldots\ldots\ldots \quad (6)$$

or

$$V' = a^3 f_6(R/a). \quad \ldots\ldots\ldots \quad (7)$$

Now the equation to the outline of any pendent drop referred to axes x and z, as shown in fig. 1, can be written in the dimensionless form

$$2 + \beta\frac{z}{b} = \frac{1}{r/b} + \frac{\sin\phi}{x/b}, \quad \ldots\ldots\ldots \quad (8)$$

where b is the radius of curvature at the origin O, which is the vertex of the drop, r is the radius of curvature in the plane of the paper at the point (x, z), and

$$\beta = b^2/a^2.$$

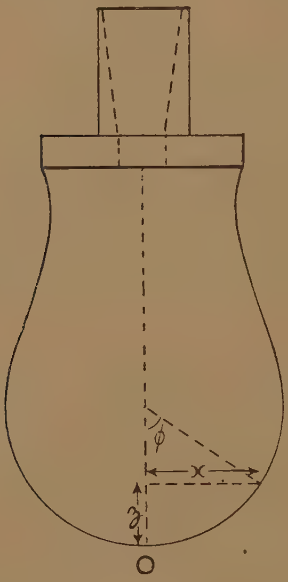
The angle ϕ is the inclination of the normal at the point (x, z) to the z-axis.

The well-known fact that the shape of the drop is determined by the value of β follows from equation (8) because, for a given value of β, if b changes its value by a certain factor, the values of x, z and r for any given value of ϕ must all change by this same factor in order that the equation shall still be satisfied.

The radius b at the vertex is a unique linear dimension of the drop and at the unstable stage its value must therefore be fixed by the independent variables R, γ, ρ and g. Dimensional analysis shows that the relation between b and these quantities can be put in the form

$$b=af_7(\mathrm{R}/a). \quad \ldots \ldots \ldots \quad (9)$$

Fig. 1.



Therefore, when drops in the unstable state have the same value of R/a, they also have equal values of b/a and therefore of β, and are consequently similar in shape. It is evident by equation (4) that such drops also have equal values of $\mathrm{R}/\mathrm{V}'^{\frac{1}{3}}$.

If W′ is the total weight of liquid hanging from the tip at the unstable stage, we have

$$\mathrm{W}'=\mathrm{V}'\rho g,$$

so that, by equation (6),

$$\begin{aligned}\mathrm{W}'&=\rho g a^2\mathrm{R}f_5(\mathrm{R}/a)\\&=\mathrm{R}\gamma f_5(\mathrm{R}/a). \quad \ldots \ldots \ldots \quad (10)\end{aligned}$$

This is of the same form as the Harkins and Brown equation for W [equation (2)], but cannot be used as a theoretical basis for the latter equation without the assumption that W/W′, the ratio of the detached drop weight to the total drop weight, is a function only of R/a, or in other words that W/W' has the same value for drops which are similar at instability. Thus if

$$W/W' = f_8(R/a),$$

then

$$W = R\gamma f_8(R/a) f_5(R/a)$$

by equation (10), so that

$$W = R\gamma f_9(R/a), \qquad (11)$$

where

$$f_9(R/a) = f_8(R/a) f_5(R/a).$$

Equation (11) is identical in form with (2). It is easily seen from equation (7) that for a given value of R/a, a is proportional to $V'^{\frac{1}{3}}$, and if for similar drops the volume of the detached drop V is proportional to the total volume V', then a is also proportional to $V^{\frac{1}{3}}$, and we can substitute a function of $(R/V^{\frac{1}{3}})$ for $f_9(R/a)$ in equation (11) and so obtain the second of Harkins and Brown's equations, viz., equation (3).

The validity of the above assumption with regard to the constancy of W/W' and V/V' for similar drops must be estimated by the success of the empirical equations of Harkins and Brown.

To sum up: the essence of the Harkins and Brown method lies in the use of tables derived from their experimental data which allow the correcting factor Y to be evaluated in any particular case when $R/V^{\frac{1}{3}}$ is known. This factor has the same value for drops which are similar in shape at the unstable stage. The occurrence of such similarity in any two cases is entirely fortuitous, since it depends upon the chance equality of values of R/a (or $R/V^{\frac{1}{3}}$).

Lord Rayleigh (1899) applied the method of dimensions to the problem of drop-detachment and derived equation (11). By weighing water drops which fell from tips of different radii he also investigated experimentally the variation of $f_9(R/a)$ with R/a in this equation. He realized that the table which he so obtained would allow the value of W for any liquid of *known* surface tension to be calculated for any given value of R. However, Rayleigh appears not to have made any suggestion with regard to the replacement of the function of R/a by a function of a variable not involving a, such as $R/V^{\frac{1}{3}}$, so that his work did not immediately make possible the use of the drop-weight method for precise surface-tension determinations. Rayleigh's statement that "for many purposes it may suffice to treat F [*i. e.* $f_9(R/a)$] as a constant, say 3·8," was for many years (and occasionally still is) taken to be a justification for the calculation of surface tension from the simple equation $W = 3{\cdot}8R\gamma$, which not only conceals the true relationship but may also lead to erroneous results.

The application of the principle of similarity to pendent drops dates back to the time of Worthington (1881), whose study of the outlines of drops established the fact that different liquids may form drops of similar shape from tips of different diameters. Iredale (1923) realized that the similarity principle was the foundation of the Harkins and Brown corrections. He used, in effect, the equations

$$\frac{a_1}{a_2}=\frac{R_1}{R_2}=\frac{V_1^{\frac{1}{3}}}{V_2^{\frac{1}{3}}},$$

in which the symbols have their present meanings and the subscripts 1 and 2 refer to similar drops of two different liquids. Using Harkins and Brown's experimental values for the weights of water drops falling from tips of different radii, Iredale plotted a graph of R/R' against R, where R' is the radius of the fallen drop considered as a sphere and is therefore proportional to $V^{\frac{1}{3}}$. The ratio R/R' is a numerical quantity which specifies the shape of the unstable drop. Thus, by assuming the value of a for water (a_1, say), Iredale determined V_2 for drops of an unknown liquid falling from a tip of radius R_2. The value of R_2/R_2' could then be found (R_2' being derived from V_2), and from the graph previously referred to it was possible to discover the value of R_1, the radius of the tip which would give water drops similar in shape at instability to those of the unknown liquid falling from the tip of radius R_2. The unknown value of a_2 could then be calculated, since a_1, R_1, and R_2 in the last equation were all known. Although this procedure can hardly be said to differ in essence from the use of Harkins and Brown's tables, it does illuminate the underlying principle of similarity.

2. *Detachment of Drops from a Conical Tip.*

We now consider the detachment of drops from a conical tip such as that shown in fig. 2. Liquid is allowed to flow slowly through the narrow holes *hh* and trickles down the sides of the inverted cone to form a drop at the tip. This drop eventually becomes unstable and falls, and the average weght of the detached drop can be found in the usual way. It is shown in what follows that, provided the angle of contact between the liquid and the surface of the cone is the same, *all* drops forming on a cone of given angle are similar in shape at the unstable stage. The condition of constant contact angle is, of course, achieved in practice by ensuring that the angle is zero, *i. e.* that the liquid wets the tip.

When the drop is unstable, let V'' be the volume of the solid of revolution below AB, which is the line of contact of the liquid surface with the cone. This volume will be completely determined by the values of γ, ρ, g and the semi-angle of the cone θ. We can therefore write

$$V''=\gamma^x\rho^y g^z\psi_1(\theta).$$

Equating powers of mass, length and time on the two sides of the equation, we obtain

$$V''=\left(\frac{\gamma}{g\rho}\right)^{\frac{3}{2}}\psi_1(\theta),$$

i. e.

$$V''=a^3\psi_1(\theta),$$

so that for a given cone V'' is proportional to a^3.

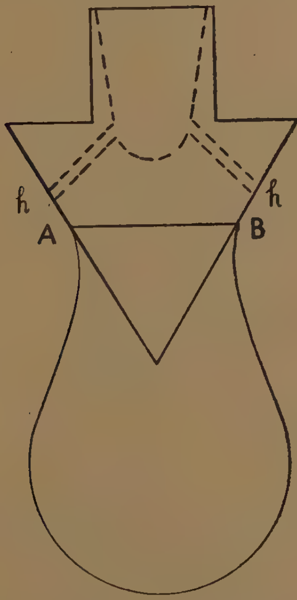
Furthermore, the value of the radius of curvature b at the vertex when the drop becomes unstable will be determined by θ, γ, ρ and g. Dimensional analysis gives

$$b=a\psi_2(\theta),$$

from which it follows that for a given value of θ all drops at the unstable state have the same value of b/a and therefore of β. Consequently all such drops are similar in shape. The condition upon which the equality of the correcting factor Y depends in the Harkins and Brown method is therefore automatically realized when the drop is free to adjust its shape and size as it is with a conical tip.

It is evident that, for similar drops, the part of the total volume below AB which consists of the solid cone itself bears a constant ratio to V''. Therefore, if V' is the actual volume of liquid, the ratio V'/V'' is a constant for all similar drops.

Fig. 2.



So far the theory is quite exact. We now make the assumption which appears to be justified in the Harkins and Brown method, viz., that the volume of the detached portion V is a constant fraction of V' when drops are similar, or in other words, that it depends only on θ, γ, ρ and g. Thus if V is proportional to V', which is proportional to V'', we can write

$$V=kV''=ka^3\psi_1(\theta),$$

where k is a constant for a given value of θ. Alternatively, we can write

$$V=a^3\psi_3(\theta),$$

where $$\psi_3(\theta)=k\psi_1(\theta).$$

If m is the mass of the detached portion, then

$$m=\mathrm{V}\rho=a^3\rho\psi_3(\theta),$$

or

$$m=\frac{\gamma^{\frac{3}{2}}}{g^{\frac{3}{2}}\rho^{\frac{1}{2}}}\cdot\psi_3(\theta),$$

so tha

$$\gamma=\mathrm{K}m^{\frac{2}{3}}\rho^{\frac{1}{3}}g, \quad \ldots \ldots \ldots \quad (12)$$

where

$$\mathrm{K}=(\psi_3(\theta))^{-\frac{2}{3}},$$

and is a constant for a given tip.

For two different liquids, 1 and 2, dropping from the same conical tip, we have, by equation (12),

$$\frac{\gamma_1}{\gamma_2}=\left(\frac{m_1}{m_2}\right)^{\frac{2}{3}}\left(\frac{\rho_1}{\rho_2}\right)^{\frac{1}{3}}. \quad \ldots \ldots \ldots \quad (13)$$

3. *Experimental Results.*

In our experiments the value of the surface tension of a certain sample of water was determined by the usual drop-weight method (Harkins and Brown) and was taken as the standard.

TABLE I.

Liquid	Surface-tension values (dyne cm.$^{-1}$)	
	Conical Tip	Harkins and Brown Method
Water	(72·78)	72·78
Propionic Acid	26·80	26·89
Methyl Alcohol	22·62	22·56
Methylated Spirit	22·49	22·44
Ethyl Alcohol	22·58	22·52
Sol. Ethyl Alc./Water	39·90	39·89
" " " "	42·51	42·54
" " " "	50·86	50·81
Sol. Methyl Alc./Water	54·42	54·49
" " " "	57·22	57·22
Sol. Ethyl Alc./Water	58·01	57·97

The mass (m) of a drop of water detached from a conical platinum tip (made in the workshop of this department) was then determined with runs of 20 or 25 drops, each drop taking at least five minutes to form and detach.

After the apparatus had been cleaned thoroughly, the operation was repeated with other liquids. In order to eliminate any loss of weight caused by evaporation, the drops were formed in a saturated atmosphere of their own vapour and the weighing-bottle was stoppered before being taken out of this atmosphere. Values of the surface tensions of the various liquids were then found by equation (13), the above mentioned value being assumed for water. Results obtained for several liquids by this method and by Harkins and Brown's method are compared in Table I.

Thermostatic temperature control was then introduced and a more precise comparison made of the values given by the two methods for a specimen of benzene. The results are given in Table II.

There appears to be a small but noticeable difference between the results obtained by Harkins and Brown's method with tips of 0·2500 and 0·2621 cm. radius. In the table given by these authors the correction factor for the latter tip falls between 0·6000 and 0·5992 and for this particular range there is a slight difference between the corrections given in the table and their own experimental data. In Table II. (page 512 of Harkins and Brown's paper) it can be seen that for a drop of water for which $R/V^{\frac{1}{3}}$ has the value of 0·82929, the correction factor is 0·59852. On calculating the corresponding correction from Table IX. (page 519), it is found to be 0·5995, which is 0·17 per cent. higher. Thus, if Harkins and Brown's experimental data are used instead of their correction tables, the value of the surface tension of benzene found with the larger tip is 28·17 dyne cm.$^{-1}$, which is very near to that found with the 0·2500 cm. tip.

Table II.

Liquid	Temperature	Surface-tension values		
		Conical Tip (platinum)	Harkins and Brown Method	
			Monel-metal Tip R=0·2500 cm.	Platinum Tip R=0·2621 cm.
Benzene	20° C.	29·03	28·97	—
,,	25° C.	28·24	28·19	28·12
Water	20° C.	(72·80)	72·80	—
,,	25° C.	(71·96)	71·96	71·96

The slight difference of about 0·05 dyne cm.$^{-1}$ (or less than 0·2 per cent.) which still exists between the conical- and cylindrical-tip methods falls within the experimental error claimed by Harkins and Brown.

It is interesting to note that if the semi-angle of a conical tip is increased, the tip eventually becomes flat when the angle reaches 90°. Since equation (13) should be valid for any one tip regardless of its angle, it was considered worth while investigating experimentally whether the relationship held good for drops falling from the lower surface of a horizontal plate.

Drops of water and benzene were allowed to form on and fall from the lower surface of a horizontal ground-glass plate and the value for benzene so obtained was 29·88 dyne cm.$^{-1}$ against 29·37 dyne cm.$^{-1}$ found by Harkins and Brown's method under approximately similar conditions. These results seem to be in fair agreement and we can conclude that, if the diameter of a flat tip is increased, there is a limit beyond which the weight of the drop of a given liquid becomes sensibly constant and is not influenced by further increase of the diameter of the tip. Drops of different liquids formed on such tips are similar in shape at instability, and their masses are related by equation (13).

4. *Discussion of the Method.*

It has already been mentioned that both the Harkins and Brown and the conical-tip methods are comparative only. The former method involves the use of tables even when different liquids are compared by means of one and the same tip. A given conical tip, however, may be used to compare surface tensions directly by means of equation (13). Such a comparison involves the determination of drop weights and densities only, and is likely to be more accurate than the Harkins and Brown method, since with a conical tip the surface tension is proportional to the two-thirds power of the drop weight, as against the first power with a cylindrical tip.

The Harkins and Brown tables are, of course, intended to make the cylindrical-tip method semi-absolute. The accuracy of a surface-tension determination by this method, however, may well be limited by the accuracy with which the radius of the tip can be determined. With the resources usually available, a drop weight of, say, 0·5 gm. can be determined to four significant figures, but it is more difficult to attain the same degree of accuracy in measuring the diameter of a tip of the order of 0·5 cm. Thus the method does not take full advantage of the high degree of precision with which weighing can be carried out.

It is our experience that it is perceptibly easier to obtain consistent drop weights with a conical tip, and it is noticeable that the quantity of liquid left adhering to the tip immediately after the detachment of the drop is smaller than it is with a cylindrical tip.

5. *Acknowledgment.*

The authors wish to thank Prof. E. N. da C. Andrade, F.R.S., for affording facilities for this research and for the helpful interest which he has displayed.

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Department of Physics,
University College,
Gower Street,
London, W.C. 1.

LIII. *The Energies of Electrons in Aluminium.*

By Z. Matyáš, Prague *.

(The 1st Physical Institute of University of Technical Sciences.)

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Abstract.

In the first part of this paper it is shown, by means of the general Bloch method, how it is possible to construct the probable band structure of aluminium. The distribution of its three valence electrons in the two Brillouin zones is the following : the first zone is *nearly* full and the Fermi surface lies near the bounding square faces of this zone in the {100}-directions. The second zone is partly filled with one electron and the Fermi surface has the shape of semi-ellipsoids surrounding the eight points $(\pm\pi/a, \pm\pi/a, \pm\pi/a)$ of the k-space. The second part of this paper contains the interpretations of some principal features of the soft X-ray emission spectra of aluminium.

Introduction

The aim of the group of papers of which the present is the first, is to find an approximate model of electronic energy bands in aluminium and on this basis to discuss some properties of aluminium and its alloys. First we shall show, using the general method of Bloch, how it is possible to calculate the band structure in the case of three valance electrons for a face-centred cubic lattice; and finally we shall try to explain some features of the x-ray emission spectra of aluminium.

§ 1. *Electronic Energy Bands in Aluminium.*

In the case of aluminium we have three valence electrons per atom; in an isolated atom two electrons are in $3s$-states and one electron is in a $3p$-state. We shall discuss these questions : how are the three electrons distributed in the first and second Brillouin zones, what is the shape of the surfaces of equal energy of different electronic states and what is the density of these states in different energy ranges ? We shall try to show that it is possible to get some semiquantitative answers by applying quite general methods for the treatment of electronic states in solids, *i. e.* by applying the approximation of nearly free electrons and the approximation of slightly bound electrons (1). Before we start with the more detailed investigations it is necessary to recapitulate briefly the well-known facts concerning the form of Brillouin zones for the face-centred cubic lattice.

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* Communicated by Professor N. F. Mott.

In the theory developed by Bloch, Brillouin and others, each state of an electron in the metal is described by a vector $\boldsymbol{k}$; the direction of $\boldsymbol{k}$ is the direction of motion of the electron, and $2\pi/k$ is its de Broglie wavelength. Bloch has shown that the wave function for each individual electron takes the form

$$\psi_{\boldsymbol{k}}(x, y, z)=u_{\boldsymbol{k}}(x, y, z)e^{2\pi i(\boldsymbol{kr})}, \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

where $u_{\boldsymbol{k}}(x, y, z)$ has the periodicity of the lattice. The energy is a function of $\boldsymbol{k}$; for the values of k_x, k_y, k_z, for which Bragg reflection takes place, a discontinuity in the energy occurs. Thus the states of an electron in a metallic lattice may be divided into zones (Brillouin zones), separated by planes in k-space across which the energy is discontinuous.

In the case of a face-centred cubic lattice there are 4 atoms in a unit cell, with the basis

$$(000,\ 0\tfrac{1}{2}\tfrac{1}{2},\ \tfrac{1}{2}0\tfrac{1}{2},\ \tfrac{1}{2}\tfrac{1}{2}0).$$

When the components of $\boldsymbol{k}$ referred to the axes $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$, of the reciprocal lattice are integers, $\boldsymbol{k}$ will coincide with a point of the reciprocal lattice. Now a face-centred cubic lattice has a body-centred lattice as its reciprocal and it is easy to show that the first Brillouin zone is a truncated octohedron, marked out by the planes $(\pm1\pm1\pm1)$ and (±20), (0 ± 20), (00 ± 2) ⁽²⁾. The second zone is bounded by the same set of planes *.

In the following section let us consider a face-centred cubic lattice of atoms and calculate the energy of the electronic levels assuming that the interaction between the atoms is small, *i. e.* from the viewpoint of the tight binding approximation. The results obtained by this method are compared later with the predictions based on the model of nearby free electrons, and in this way we shall be able to construct an approximate band-model for aluminium.

A. *Energy Levels according to Bloch's Approximation of Tightly Bound Electrons.*

In our calculation we take the ground state of an atom to be an s-state and consider the next p-state to have nearly the same energy so that the zero order wave function for the lattice will contain both s- and p-states. We denote the three independent wave functions associated with the degenerate p-state of an atom by

$$\phi_1=xf(r), \qquad \phi_2=yf(r), \qquad \phi_3=zf(r),$$

respectively, and the single spherically symmetrical wave function belonging to the s-state by $\phi_0(r)$. If the interaction between the s- and p-states is neglected, the wave functions for the electrons in the lattice can be expressed in the form given by Bloch ⁽³⁾:

$$\psi_{n\boldsymbol{k}}=\sum_{l} e^{ia(\boldsymbol{kl})}\phi_n(\boldsymbol{r}-a\boldsymbol{l}), \qquad n=0, 1, 2, 3, \ldots, \quad . \; . \; . \quad (2)$$

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* A picture of these zones is to be found in Seitz: 'Theory of Solids,' (Mcgraw-Hill, 1940) p. 289.

where a is the lattice constant and $a\boldsymbol{l}$ gives the position of any atom (the components of vector $\boldsymbol{l} \equiv (l_1, l_2, l_3)$ are integers).

The problem to be answered is how the functions ψ_{nk} derived from s- and p-states combine to form the true zero order wave functions for the electron in the lattice. We set for the wave function of an electron in the whole crystal

$$\psi_n(\boldsymbol{k}, \boldsymbol{r}) = \sum_m a_{nm}(\boldsymbol{k})\psi_{mk}(\boldsymbol{r}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If H denotes the Hamiltonian for an electron in the crystal, given by

$$\mathrm{H} = -\frac{\hbar^2}{2m}\nabla^2 + \mathrm{V}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $\mathrm{V}(\boldsymbol{r})$ is the potential of an electron inside the crystal, then the wave equation which ψ_n must satisfy is

$$(\mathrm{H} - \mathrm{E})\psi_n = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We multiply the equation (5) by each function ψ^*_{nk} (equation (2)) in turn, integrate over all $\boldsymbol{r}$ $\left(\int \ldots . d\tau\right)$ and eliminate the coefficients $a_{nm}(\boldsymbol{k})$; then we have

$$|\mathrm{H}_{nm} - \mathrm{E}\delta_{nm}| = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where

$$\left.\begin{aligned} \mathrm{H}_{nm} &= \int \psi^*_{nk}\mathrm{H}\psi_{mk}\, d\tau / \mathrm{A}_n \,.\, \mathrm{A}_m, \\ \text{and} \qquad \mathrm{A}_n{}^2 &= \int \psi^*_{nk}\psi_{mk}\, d\tau = \mathrm{N}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where N is the number of atoms.

This is the usual secular equation of quantum mechanics.

If we put the wave functions from equation (2) into the equation (7), we obtain for (7) the following formula:

$$\mathrm{H}_{nm} = \frac{1}{\mathrm{N}}\sum_l \sum_{l'} e^{ia(\boldsymbol{k}.\boldsymbol{l}-\boldsymbol{l}')} \int \phi^*_n(\boldsymbol{r} - a\boldsymbol{l}')\mathrm{H}\phi_m(\boldsymbol{r} - a\boldsymbol{l})\, d\tau. \quad . \quad . \quad . \quad (8)$$

In the last formula we may separate H into two terms, firstly the Hamiltonian of a free atom at the point $\boldsymbol{r} = a\boldsymbol{l}$,

$$\mathrm{H}_l = -\frac{\hbar^2}{2m}\nabla^2 + \mathrm{U}(\boldsymbol{r} - a\boldsymbol{l}),$$

and secondly the remaining terms in the potential, *i. e.*

$$\mathrm{H} - \mathrm{H}_l = \mathrm{V}(\boldsymbol{r}) - \mathrm{U}(\boldsymbol{r} - a\boldsymbol{l}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

We denote now by E_0 the corresponding energy of the s-state of an isolated atom and by $\mathrm{E}_1 = \mathrm{E}_2 = \mathrm{E}_3$ the corresponding energy of the triply degenerate p-state. Replacing H by two terms: $\mathrm{H}_l + \mathrm{H} - \mathrm{H}_l$ in the equation (8), the expression for matrix element H_{nm} has then this form:

$$\mathrm{H}_{nm} = \frac{1}{\mathrm{N}}\sum_l \sum_{l'} e^{ia(\boldsymbol{k}.\boldsymbol{l}-\boldsymbol{l}')} \int \phi^*_n(\boldsymbol{r} - a\boldsymbol{l}')[\mathrm{V}(\boldsymbol{r}) - \mathrm{U}(\boldsymbol{r} - a\boldsymbol{l})]\phi_m(\boldsymbol{r} - a\boldsymbol{l})\, d\tau + \mathrm{E}_n\delta_{nm},$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where $\delta_{nm} = 0$ or 1 when either $n \neq m$ or $n = m$.

We see immediately that the terms in the summation l' are all identical; it is possible therefore to transform the origin to $a\boldsymbol{l}$ in each term, to denote $a(\boldsymbol{l}-\boldsymbol{l}')$ by $-\boldsymbol{\rho}_l$ and thus write the expression (10) in more simple form:

$$\mathrm{H}_{nm}=\mathrm{E}_n\delta_{nm}+\sum_l e^{-i(\boldsymbol{k}\boldsymbol{\rho}_l)}\int\phi_n^*(\boldsymbol{r}-\boldsymbol{\rho}_l)[\mathrm{V}(\boldsymbol{r})-\mathrm{U}(\boldsymbol{r})]\phi_m(\boldsymbol{r})\,d\tau. \quad . \quad (11)$$

Now we can calculate the matrix-elements in the case of face-centred cubic lattice from this last general formula explicitly and so determine the energy levels of electrons from the secular equation (6).

(1) *The element H_{00}.*—Owing to the equation (11) this element is

$$\mathrm{H}_{00}=\mathrm{E}_0+\int\phi_0^*(\boldsymbol{r})[\mathrm{V}(\boldsymbol{r})-\mathrm{U}(\boldsymbol{r})]\phi_0(\boldsymbol{r})\,d\tau$$

$$+\sum_{l\neq0} e^{-i(\boldsymbol{k}\boldsymbol{\rho}_l)}\int\phi_0^*(\boldsymbol{r}-\boldsymbol{\rho}_l)[\mathrm{V}(\boldsymbol{r})-\mathrm{U}(\boldsymbol{r})]\phi_0(\boldsymbol{r})\,d\tau. \quad . \quad . \quad (12)$$

In the summation of equation (12) we shall neglect all integrals, except those for which the atoms are the nearest neighbours, and shall denote the types of integral occurring in this formula by the following symbols:

$$\left.\begin{aligned}-\mathrm{A}_0&=\int\phi_0^*(\boldsymbol{r})[\mathrm{V}(\boldsymbol{r})-\mathrm{U}(\boldsymbol{r})]\phi_0(\boldsymbol{r})\,d\tau,\\ -\mathrm{A}'_0&=\int\phi_0^*(\boldsymbol{r}-\boldsymbol{\rho}_l)[\mathrm{V}(\boldsymbol{r})-\mathrm{U}(\boldsymbol{r})]\phi_0(\boldsymbol{r})\,d\tau;\end{aligned}\right\} \quad . \quad . \quad . \quad (13)$$

since $\mathrm{V}(\boldsymbol{r})-\mathrm{U}(\boldsymbol{r})$ is negative (5), A_0 and A_0' have positive values.

As we know that *s*-wave function $\phi_0(\boldsymbol{r})$ is spherically symmetrical, the value of $-\mathrm{A}_0'$ must then be the same for all the nearest neighbours. Thus we obtain for H_{00} the expression:

$$\mathrm{H}_{00}=\mathrm{E}_0-\mathrm{A}_0-\mathrm{A}_0'\sum e^{-i(\boldsymbol{k}\boldsymbol{\rho}_l)}. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If we substitute the coordinates of the 12 nearest neighbours of an atom in the face-centred cubic lattice for $\boldsymbol{\rho}_l$ in the last equation we can write the matrix element H_{00} in the final form:

$$\mathrm{H}_{00}=-\mathrm{M}-4\mathrm{A}_0'(\cos\tfrac{1}{2}ak_x\cos\tfrac{1}{2}ak_y+\cos\tfrac{1}{2}ak_x\cos\tfrac{1}{2}ak_z+\cos\tfrac{1}{2}ak_y\cos\tfrac{1}{2}ak_z)$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where $-\mathrm{M}=\mathrm{E}_0-\mathrm{A}_0$.

(2) *The diagonal elements H_{11}, H_{22}, H_{33}.*—In the calculation of the other elements we can proceed in exactly the same way. In the case of H_{11} for example, the general expression (11) has the form

$$\mathrm{H}_{11}=\mathrm{E}_1+\int\phi_1^*(\boldsymbol{r})[\mathrm{V}(\boldsymbol{r})-\mathrm{U}(\boldsymbol{r})]\phi_1(\boldsymbol{r})\,d\tau$$

$$+\sum_l e^{-i(\boldsymbol{k}.\boldsymbol{\rho}_l)}\int\phi_1^*(\boldsymbol{r}-\boldsymbol{\rho}_l)[\mathrm{V}(\boldsymbol{r})-\mathrm{U}(\boldsymbol{r})]\phi_1(\boldsymbol{r})\,d\tau.$$

The exchange integrals occurring in this expression form three groups :

$$-\mathrm{A}_1=\int\phi_1^*(\boldsymbol{r})[\mathrm{V}(\boldsymbol{r})-\mathrm{U}(\boldsymbol{r})]\phi_1(\boldsymbol{r})\,d\tau$$

$$\begin{aligned}
\mathrm{A}_1&=\int\phi_1^*(x\pm\tfrac{1}{2}a,\,y\pm\tfrac{1}{2}a,\,z)[\mathrm{V}(x,y,z)-\mathrm{U}(x,y,z)]\phi_1(x,y,z)\,d\tau\\
&=\int\phi_1^*(x\pm\tfrac{1}{2}a,\,y\mp\tfrac{1}{2}a,\,z)[\mathrm{V}(x,y,z)-\mathrm{U}(x,y,z)]\phi_1(x,y,z)\,d\tau\\
&=\int\phi_1^*(x\pm\tfrac{1}{2}a,\,y,\,z\pm\tfrac{1}{2}a)[\mathrm{V}(x,y,z)-\mathrm{U}(x,y,z)]\phi_1(x,y,z)\,d\tau\\
&=\int\phi_1^*(x\pm\tfrac{1}{2}a,\,y,\,z\mp\tfrac{1}{2}a)[\mathrm{V}(x,y,z)-\mathrm{U}(x,y,z)]\phi_1(x,y,z)\,d\tau.\\
-\mathrm{A}''&=\int\phi_1^*(x,\,y\mp\tfrac{1}{2}a,\,z\pm\tfrac{1}{2}a)[\mathrm{V}(x,y,z)-\mathrm{U}(x,y,z)]\phi_1(x,y,z)\,d\tau\\
&=\int\phi_1^*(x,\,y\pm\tfrac{1}{2}a,\,z\mp\tfrac{1}{2}a)[\mathrm{V}(x,y,z)-\mathrm{U}(x,y,z)]\phi_1(x,y,z)\,d\tau.
\end{aligned}\qquad(16)$$

Since $\mathrm{V}(\boldsymbol{r})-\mathrm{U}(\boldsymbol{r})$ is negative, the first integral must be negative and therefor $\mathrm{A}_1>0$. The value of the second integral must be positive, as two wave-functions, from which it is built up, overlap in the region where they have opposite signs ($\mathrm{A}_1'>0$). In the third integral, on the other hand, $\phi_1(x, y+\frac{1}{2}a, z)$ and $\phi_1(x, y, z)$ have the same sign midway between the atoms, so that $-\mathrm{A}_1''$ is negative, *i. e.* $\mathrm{A}_1''>0$.

If we substitute again the coordinates of the 12 nearest neighbours of an atom in the face-centred cubic lattice for ρ_l into (11), we obtain the following expressions for these matrix elements :

$$\left.\begin{aligned}
\mathrm{H}_{11}&=-\mathrm{N}+4\mathrm{A}_1'\{\cos\tfrac{1}{2}ak_x\cos\tfrac{1}{2}ak_y+\cos\tfrac{1}{2}ak_x\cos\tfrac{1}{2}ak_z\}\\
&\qquad-4\mathrm{A}_1''\cos\tfrac{1}{2}ak_y\cos\tfrac{1}{2}ak_z,\\
\mathrm{H}_{22}&=-\mathrm{N}+4\mathrm{A}_1'\{\cos\tfrac{1}{2}ak_x\cos\tfrac{1}{2}ak_y+\cos\tfrac{1}{2}ak_y\cos\tfrac{1}{2}ak_z\}\\
&\qquad-4\mathrm{A}_1''\cos\tfrac{1}{2}ak_x\cos\tfrac{1}{2}ak_z,\\
\mathrm{H}_{33}&=-\mathrm{N}+4\mathrm{A}_1'\{\cos\tfrac{1}{2}ak_x\cos\tfrac{1}{2}ak_z+\cos\tfrac{1}{2}ak_y+\cos\tfrac{1}{2}ak_z\}\\
&\qquad-4\mathrm{A}_1''\cos\tfrac{1}{2}ak_x\cos\tfrac{1}{2}ak_y,\\
&\text{where } -\mathrm{N}=\mathrm{E}_1-\mathrm{A}_1.
\end{aligned}\right\}\quad.\quad(17)$$

(3) *The matrix elements H_{12}, H_{13}, H_{23}.*—From the equation (11) we have:

$$\mathrm{H}_{12}=\int\phi_1^*(\boldsymbol{r})[\mathrm{V}(\boldsymbol{r})-\mathrm{U}(\boldsymbol{r})]\phi_2(\boldsymbol{r})\,d\tau$$

$$+\sum_{l\neq 0}e^{-i(\boldsymbol{k},\boldsymbol{\rho}_l)}\int\phi_1^*(\boldsymbol{r}-\boldsymbol{\rho}_l)[\mathrm{V}(\boldsymbol{r})-\mathrm{U}(\boldsymbol{r})]\phi_2(\boldsymbol{r})\,d\tau.$$

Here again there are three types of integral :

$$0=\int \phi_1^*(\boldsymbol{r})[\mathrm{V}(\boldsymbol{r})-\mathrm{U}(\boldsymbol{r})]\phi_1(\boldsymbol{r})\,d\tau.$$

$$-\mathrm{A}=\int \phi_1^*(x\pm\tfrac{1}{2}a,\ y\pm\tfrac{1}{2}a,\ z)[\mathrm{V}(\boldsymbol{r})-\mathrm{U}(\boldsymbol{r})]\phi_2(x,y,z)\,d\tau$$

$$=-\int \phi_1^*(x\pm\tfrac{1}{2}a,\ y\mp\tfrac{1}{2}a,\ z)[\mathrm{V}(\boldsymbol{r})-\mathrm{U}(\boldsymbol{r})]\phi_2(x,y,z)\,d\tau.$$

$$0=\int \phi_1^*(x,\ y\pm\tfrac{1}{2}a,\ z\pm\tfrac{1}{2}a)[\mathrm{V}(\boldsymbol{r})-\mathrm{U}(\boldsymbol{r})]\phi_2(x,y,z)\,d\tau$$

$$=\int \phi_1^*(x,\ y\pm\tfrac{1}{2}a,\ z\mp\tfrac{1}{2}a)[\mathrm{V}(\boldsymbol{r})-\mathrm{U}(\boldsymbol{r})]\phi_2(x,y,z)\,d\tau$$

$$=\int \phi_1^*(x\pm\tfrac{1}{2}a,\ y\pm\tfrac{1}{2}a,\ z)[\mathrm{V}(\boldsymbol{r})-\mathrm{U}(\boldsymbol{r})]\phi_2(x,y,z)\,d\tau$$

$$=\int \phi_1^*(x\pm\tfrac{1}{2}a,\ y\mp\tfrac{1}{2}a,\ z)[\mathrm{V}(\boldsymbol{r})-\mathrm{U}(\boldsymbol{r})]\phi_2(x,y,z)\,d\tau.$$

We shall prove that the first integral must be equal to zero. We can write (see the beginning of this paragraph) :

$$\int \phi_1^*[\mathrm{V}-\mathrm{U}]\phi_2\,d\tau=\int xyf^2(r)[\mathrm{V}-\mathrm{U})\,d\tau.$$

If we rotate the system of coordinates about the z-axis through an angle $\frac{1}{2}\pi$, the lattice potential $\mathrm{V}(\boldsymbol{r})$ does not change. Of course the value of the potential of the isolated atom, $\mathrm{U}(\boldsymbol{r})$, remains the same. Through this rotation x transforms into y and y into $-x$; therefore our integral must change sign. On the other hand we know that its value does not change during the transformation and so it is clear that it must be equal to zero.

Similarly we prove that integrals of the third type of integral also have zero values.

Confining ourselves again to the interaction between nearest neighbours, for these matrix elements we obtain the expressions :

$$\left.\begin{aligned}\mathrm{H}_{12}&=-4\mathrm{A}\sin\tfrac{1}{2}ak_x\sin\tfrac{1}{2}ak_y,\\ \mathrm{H}_{13}&=-4\mathrm{A}\sin\tfrac{1}{2}ak_x\sin\tfrac{1}{2}ak_z,\\ \mathrm{H}_{23}&=-4\mathrm{A}\sin\tfrac{1}{2}ak_y\sin\tfrac{1}{2}ak_z.\end{aligned}\right\}\quad\ldots\ldots\quad(18)$$

Moreover, it is easy to prove that :

$$\left.\begin{aligned}\mathrm{H}_{12}&=\mathrm{H}_{21},\\ \mathrm{H}_{13}&=\mathrm{H}_{31},\\ \mathrm{H}_{23}&=\mathrm{H}_{32}.\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(19)$$

(4) *The matrix elements* H_{01}, H_{02}, H_{03}.—In the case of H_{01} the general equation (11) has the form :

$$H_{01}=\int \phi_0^*(\boldsymbol{r})[V(\boldsymbol{r})-U(\boldsymbol{r})]\phi_1(\boldsymbol{r})\,d\tau$$

$$+\sum_{l\neq 0} e^{-i(\boldsymbol{k},\boldsymbol{\rho}_l)}\int \phi_0^*(\boldsymbol{r}-\boldsymbol{\rho}_l)[V-U]\phi_1(\boldsymbol{r})\,d\tau.$$

Firstly we can easily show, as before, that the first integral and

$$\int \phi_0^*(x, y\pm\tfrac{1}{2}a, z\pm\tfrac{1}{2}a)[V(\boldsymbol{r})-U(\boldsymbol{r})]\phi_1(x, y, z)\,d\tau$$

$$=\int \phi_0^*(x, y\pm\tfrac{1}{2}a, z\mp\tfrac{1}{2}a)[V(\boldsymbol{r})-U(\boldsymbol{r})]\phi_1(x, y, z)\,d\tau$$

in the last formula are equal to zero. The remaining integrals in the summation have the same absolute value and we shall introduce the following notation :

$$\begin{aligned}-I&=\int \phi_0^*(x+\tfrac{1}{2}a, y\pm\tfrac{1}{2}a, z)[V(\boldsymbol{r})-U(\boldsymbol{r})]\phi_1(x, y, z)\,d\tau\\ &=-\int \phi_0^*(x-\tfrac{1}{2}a, y\pm\tfrac{1}{2}a, z)[V(\boldsymbol{r})-U(\boldsymbol{r})]\phi_1(x, y, z)\,d\tau\\ &=\int \phi_0^*(x+\tfrac{1}{2}a, y, z\pm\tfrac{1}{2}a)[V(\boldsymbol{r})-U(\boldsymbol{r})]\phi_1(x, y, z)\,d\tau\\ &=-\int \phi_0^*(x-\tfrac{1}{2}a, y, z\pm\tfrac{1}{2}a)[V(\boldsymbol{r})-U(\boldsymbol{r})]\phi_1(x, y, z)\,d\tau. \quad \ldots \quad (20)\end{aligned}$$

For the remaining matrix elements we thus obtain these expressions :

$$\left.\begin{aligned}H_{01}&=-H_{10}=-8iI\sin\tfrac{1}{2}ak_x\cos\tfrac{1}{4}a(k_y+k_z)\cos\tfrac{1}{4}a(k_y-k_z),\\ H_{02}&=-H_{20}=-8iI\sin\tfrac{1}{2}ak_y\cos\tfrac{1}{4}a(k_x+k_z)\cos\tfrac{1}{4}a(k_x-k_z),\\ H_{03}&=-H_{30}=-8iI\sin\tfrac{1}{2}ak_z\cos\tfrac{1}{4}a(k_x+k_y)\cos\tfrac{1}{4}a(k_x-k_y).\end{aligned}\right\} \quad . \quad . \quad . \quad (21)$$

After these calculations we see the secular equation (6) has the form :

$$\begin{vmatrix} H_{00}-E & H_{01} & H_{02} & H_{03}\\ -H_{01} & H_{11}-E & H_{12} & H_{13}\\ -H_{02} & H_{12} & H_{22}-E & H_{23}\\ -H_{03} & H_{13} & H_{23} & H_{33}-E \end{vmatrix}=0, \quad . \quad (22)$$

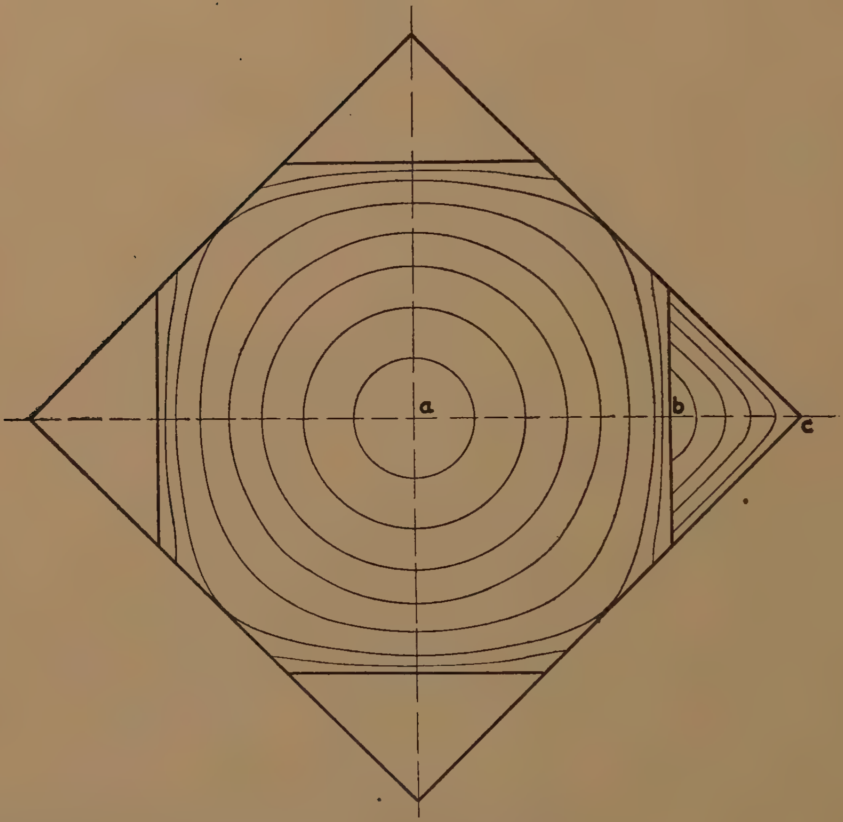
and the matrix elements are given by the expressions (15), (17), (18) and (21). This equation is of the fourth degree in E and the solutions will give the energies of the valence electrons in the *s*- and three *p*-bands as functions of $\boldsymbol{k}$.

To make the calculation manageable and yet to obtain the essential features of the band structure of aluminium, we make the plausible assumption that the exchange integrals A_1' and $-A_1''$ have the same absolute value. Now it will be easy to solve the secular equation (22) for the points in reciprocal space lying in two specific directions, the [111] and [100]-directions. Owing to the cubic symmetry we obtain, of course, the same results for the other equivalent directions.

For the points in the direction [111] the components of ***k*** have the same value $k_x=k_y=k_z=k$. The matrix elements (15), (17), (18) and (21) have then the form :

$$H_{00}=-M-12A_0' \cos^2 \tfrac{1}{2}ak,$$
$$H_{11}=H_{22}=H_{33}=-N+4A_1' \cos^2 \tfrac{1}{2}ak,$$
$$H_{12}=H_{13}=H_{23}=-4A \sin^2 \tfrac{1}{2}ak,$$
$$H_{01}=H_{02}=H_{03}=-4iI \sin ak.$$

Fig. 1.



If we put these values into the secular equation (22) we get :

$$\begin{vmatrix} H_{00}-E & H_{01} & H_{01} & H_{01} \\ -H_{01} & H_{11}-E & H_{12} & H_{12} \\ -H_{01} & H_{12} & H_{11}-E & H_{12} \\ -H_{01} & H_{12} & H_{12} & H_{11}-E \end{vmatrix}$$
$$=(H_{11}-H_{12}-E^2)\{(H_{00}-E)(H_{11}+2H_{12}-E)+3H_{01}^2\}=0.$$

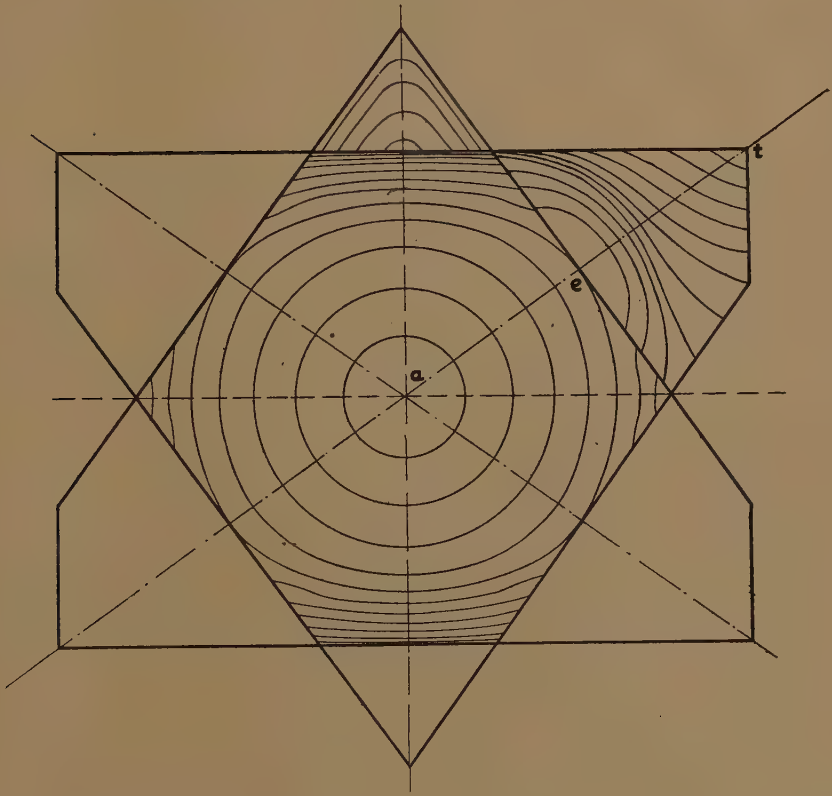
To obtain the lowest two roots of the energy E of the electrons, which correspond to the energy spectrum inside the first (E^{I}) and the second (E^{II})

Brillouin zone, we put the expression in the bracket {} equal to zero and solve for E :

$$\mathrm{E}^{\mathrm{I,\,II}}=-\frac{\mathrm{M}+\mathrm{N}}{2}+2(\mathrm{A}_1'-3\mathrm{A}_0')\cos^2\tfrac{1}{2}ak-4\mathrm{A}\sin^2\tfrac{1}{2}ak,$$
$$\pm\sqrt{\left[\frac{\mathrm{M}-\mathrm{N}}{2}+2(\mathrm{A}_1'+3\mathrm{A}_0')\cos^2\tfrac{1}{2}ak-4\mathrm{A}\sin^2\tfrac{1}{2}ak\right]^2+48\mathrm{I}^2\sin^2 ak}, \quad . \quad . \quad . \quad (23)$$

where the sign — before the square root refers to the first Brillouin zone and + to the second one.

Fig. 2.



From the last formula we shall determine the energy value of the electronic states in the special points :

1st zone ;	point (0, 0, 0) ;	$\mathrm{E}=-\mathrm{M}-12\mathrm{A}_0'$;
1st zone ;	point *e* (π/a, π/a, π/a) ;	$\mathrm{E}_e=-\mathrm{M}$;
2nd zone ;	point *t* ($2\pi/a$, $2\pi/a$, $2\pi/a$) ;	$\mathrm{E}_t=-\mathrm{N}+4\mathrm{A}_1'$;
2nd zone ;	point *e* (π/a, π/a, π/a) ;	$\mathrm{E}_e=-\mathrm{N}-8\mathrm{A}$.

The letters *e* and *t* refer to figs. 1 and 2, which represent the structures of the zones in two sections : the first in the k_x, k_y plane, the second representing the sections which contain the {111} direction and the axis k_z.

Similarly for the [100]-directions we get the following formulæ :

$$\begin{aligned}
H_{00}&=-M-4A_0'(2\cos\tfrac{1}{2}ak_x+1),\\
H_{11}&=-N+4A_1'(2\cos\tfrac{1}{2}ak_x-1),\\
H_{22}&=H_{33}=-N+4A_1',\\
H_{12}&=H_{13}=H_{02}=H_{03}=0,\\
H_{01}&=-8iI\sin\tfrac{1}{2}ak_x\,;
\end{aligned}$$

and

$$E^{I,II}=-\frac{M+N}{2}+4(A_1'-A_0')\cos\tfrac{1}{2}ak_x-2(A_0'+A_1')$$
$$\mp\sqrt{\left[\frac{M-N}{2}+4(A_1'+A_0')\cos\tfrac{1}{2}ak_x-2(A_1'-A_0')\right]^2+64I^2\sin^2\tfrac{1}{2}ak_x}. \quad . \; . \; . \quad (25)$$

The energy value of the electronic state in the 1st zone at the point $b(2\pi/a, 0, 0)$ is $-M+4A_0'$ and at the same place but in the 2nd zone is $-N-12A_1'$. In the 2nd zone at the point $c(3\pi/a, 0, 0)$ is $-N-4A_1'$.

Since we are considering here the case of aluminium, in which the valence electrons fill the first and the second Brillouin zones, we shall be interested here only in the electronic states lying in these two zones. Therefore it will be important for us to know the approximate shape of the surfaces of equal energy for the states near the mid-point $(\pi/a, \pi/a, \pi/a)$ of a hexagonal face and in the neighbourhood of a square face (*i. e.* near the plane (200)).

In the first case we can find the approximate expression for the energy of the electrons in the following way. In the secular equation (22) we can neglect the elements H_{01}, H_{02}, H_{03} (given by equation (21)), as in the neighbourhood of $(\pi a/, \pi/a, \pi/a)$, $\cos\frac{1}{4}a(k_y+k_z)=0$ and the exchange integral I is smaller than M, N or A, which determine the absolute magnitude of the other elements. It is thus clear that the energy E inside the first zone is approximately given by $E^I \doteq H_{00}$ and in the second zone by the solution of the following secular equation :

$$\begin{vmatrix}
H_{11}-E & H_{12} & H_{13}\\
H_{12} & H_{22}-E & H_{23}\\
H_{13} & H_{23} & H_{33}-E
\end{vmatrix}=0. \quad . \; . \; . \; . \quad (26)$$

The shapes of the surfaces of equal energy in the first zone, according to the equation

$$E=H_{00}=-M-4A_0'(\cos\tfrac{1}{2}ak_x\cos\tfrac{1}{2}ak_y+\cos\tfrac{1}{2}ak_x\cos\tfrac{1}{2}ak_z$$
$$+\cos\tfrac{1}{2}ak_y\cos\tfrac{1}{2}ak_z),$$

have been already determined by Bethe (*Handbuch d. Physik*, vol. xxiv. 2, p. 401) and we know that in the neighbourhood of $(\pi/a, \pi/a, \pi/a)$ they have in the first approximation the form of hyperboloids.

On the other hand, we can see from (18) that the elements H_{12}, H_{13} and H_{23} in the secular equation (26) have near the mid-point of the hexagonal face the same value and we thus assume that

$$H \doteq H_{12} \doteq H_{13} \doteq H_{23} = -\tfrac{1}{3}4A(\sin \tfrac{1}{2}ak_x \sin \tfrac{1}{2}ak_y + \sin \tfrac{1}{2}ak_x \sin \tfrac{1}{2}ak_z \\ + \sin \tfrac{1}{2}ak_y \sin \tfrac{1}{2}ak_z).$$

If we denote N+E by e and

$$h_{11} = 4A_1'\{\cos \tfrac{1}{2}ak_x \cos \tfrac{1}{2}ak_y + \cos \tfrac{1}{2}ak_x \cos \tfrac{1}{2}ak_z - \cos \tfrac{1}{2}ak_y \cos \tfrac{1}{2}ak_z\},$$

$$h_{22} = 4A_1'\{\cos \tfrac{1}{2}ak_x \cos \tfrac{1}{2}ak_y - \cos \tfrac{1}{2}ak_x \cos \tfrac{1}{2}ak_z + \cos \tfrac{1}{2}ak_y \cos \tfrac{1}{2}ak_z\},$$

$$h_{33} = 4A_1'\{-\cos \tfrac{1}{2}ak_x \cos \tfrac{1}{2}ak_y + \cos \tfrac{1}{2}ak_x \cos \tfrac{1}{2}ak_z + \cos \tfrac{1}{2}ak_y \cos \tfrac{1}{2}ak_z\},$$

the equation (26) can be written in the form

$$\begin{vmatrix} h_{11}-e & H & H \\ H & h_{22}-e & H \\ H & H & h_{33}-e \end{vmatrix} = 0.$$

Here the magnitudes of h_{11}, h_{22}, h_{33} near the point $(\pi/a, \pi/a, \pi/a)$ are very small, and therefore in the expansion of the last determinant we can neglect all terms which contain the products of h_{11}, h_{22}, h_{33}.

If we do so, we obtain :

$$-e^3 + e^2(h_{11}+h_{22}+h_{33}) + 3eH^2 - (h_{11}+h_{22}+h_{33})H^2 + 2H^3 = 0.$$

We simplify this equation if we put for e :

$$e = \epsilon + \tfrac{1}{3}(h_{11}+h_{22}+h_{33}) = \epsilon + \tfrac{1}{3}\varDelta,$$

and neglect again all terms with the products of $\varDelta$; thus we have

$$\epsilon^3 - 3H^2\epsilon - 2H^3 = 0.$$

Three roots of this equation are : $\epsilon_{1,\,2} = -H$, $\epsilon_3 = 2H$. The last one corresponds so the lowest energy and therefore to the energy in the second zone. The expression for the energy E near the point $(\pi/a, \pi/a, \pi/a)$ of the second zone is

$$e = N + E = \epsilon + \tfrac{1}{3}\varDelta = 2H + \tfrac{1}{3}\varDelta,$$

or

$$E^{II} = -N + \tfrac{4}{3}A_1'\{\cos \tfrac{1}{2}ak_x \cos \tfrac{1}{2}ak_y + \cos \tfrac{1}{2}ak_x \cos \tfrac{1}{2}ak_z + \cos \tfrac{1}{2}ak_y \cos \tfrac{1}{2}ak_z\} \\ - \tfrac{8}{3}A_1'\{\sin \tfrac{1}{2}ak_x \sin \tfrac{1}{2}ak_y + \sin \tfrac{1}{2}ak_x \sin \tfrac{1}{2}ak_z + \sin \tfrac{1}{2}ak_y \sin \tfrac{1}{2}ak_z\}. \quad (27)$$

For the points in the direction {111}, when $k_x = k_y = k_z = k$, the last formula gives the same results as equation (23), if we neglect there the term $48I^2 \sin^2 ak$ under the square root.

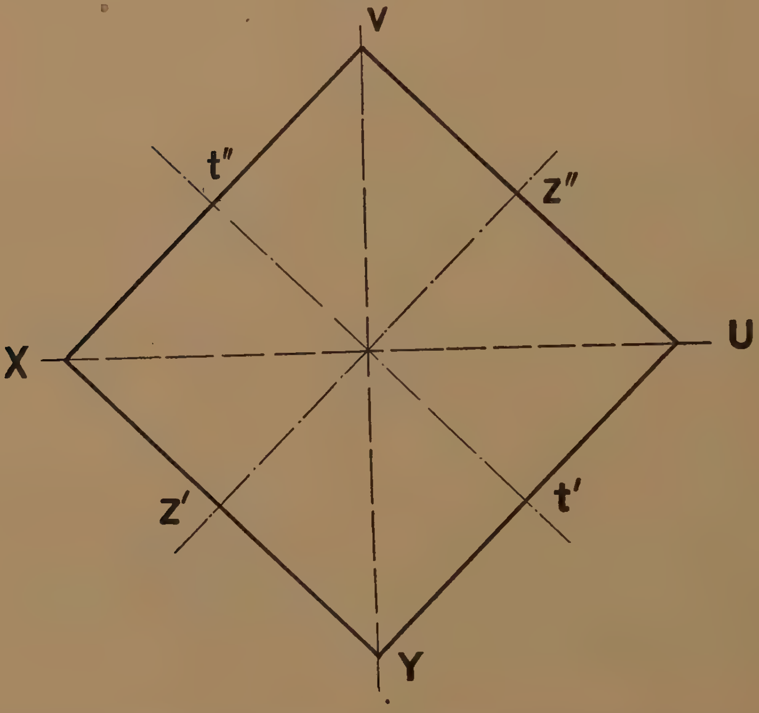
Furthermore, we can express (27) in a new coordinate system, k_x', k_y', k_z' ; we translate the origin into the point $(\pi/a, \pi/a, \pi/a)$ and turn the coordinate axis k_z' into the direction {111}, *i. e.* k_z' is perpendicular to the plane of discontinuity (111). For the small values of k_x', k_y', k_z' the energy E near the point $(\pi/a, \pi/a, \pi/a)$ is then given by :

$$E^{II} = -N - 8A + \tfrac{1}{6}(8A - A_1')a^2(k_x'^2 + k_y'^2) + \tfrac{1}{6}(2A_1' + 8A)a^2k_z'^2. \quad (28)$$

It is thus clear that in the neighbourhood of this point the surfaces of equal energy in the second zone are elipsoids.

Quite another situation occurs near the plane of discontinuity (200). We shall consider again only the states of electrons in the first Brillouin zone. First we shall show that the energy of the states lying at the diagonals (XU, VY in fig. 3) of this square face is constant. Take for example the points whose coordinates are: $2\pi/a, 0, k_z$, where $-\pi/a \leqslant k_z \leqslant \pi/a$. We see at once that in this case the elements H_{01}, H_{02} and H_{03} in equations (22) are equal to zero. Therefore, the energy in the first zone is given by

Fig. 3.



$$E^{I}=H_{00}=-M-4A_0'(\cos\tfrac{1}{2}ak_x\cos\tfrac{1}{2}ak_y+\cos\tfrac{1}{2}ak_x\cos\tfrac{1}{2}ak_z$$
$$+\cos\tfrac{1}{2}ak_y\cos\tfrac{1}{2}ak_z).$$

If we put $k_x=2\pi/a$, $k_y=0$, $k_z=k_z$, we see that for all these points the energy havs the value: $E^{I}=-M+4A_0'$.

Similarly we can show that the points along the other diagonal (*i. e.* with coordinates $2\pi/a, k_y, 0$) have the same energy.

Furthermore, we can also show that the energy of the points lying between these two diagonals on the plane of discontinuity (200) is only slightly different from the value $-M+4A_0'$. Let us take, for example, the points with coordinates $k_x=2\pi/a$, $k_y=k_z=k$. The matrix elements H_{mn}

in (22) have then the following values (equations (15), (17), (18), and (21)):

$$\left.\begin{aligned}
&H_{01}=0,\\
&\quad \epsilon=H_{02}=H_{03}=16iI\sin^3\tfrac{1}{4}ak\cos\tfrac{1}{4}ak,\\
&H_{12}=H_{13}=0,\\
&H_{23}=-4A\sin^2\tfrac{1}{2}ak,\\
&H_{11}=-N-4A_1'[2\cos\tfrac{1}{2}ak+\cos^2\tfrac{1}{2}ak],\\
&H_{22}=-N+4A_1'\cos^2\tfrac{1}{2}ak,\\
&H_{33}=-N+4A_1'\cos^2\tfrac{1}{2}ak,\\
&H_{00}=-M-4A_0'(\cos^2\tfrac{1}{2}ak-2\cos\tfrac{1}{2}ak).
\end{aligned}\right\} \quad . \;\; . \;\; . \quad (29)$$

The secular equation (22) has the form:

$$\begin{vmatrix}
H_{00}-E & 0 & \epsilon & \epsilon\\
0 & H_{11}-E & 0 & 0\\
-\epsilon & 0 & H_{22}-E & H_{23}\\
-\epsilon & 0 & H_{23} & H_{22}-E
\end{vmatrix}$$

$$=(H_{11}-E)(H_{22}-E-H_{23})\{(H_{00}-E)(H_{22}-E+H_{23})+2\epsilon^2\}=0.$$

For the energy of these points of the first zone we thus have equation:

$$E^2-E(H_{00}+H_{22}+H_{23})+H_{00}(H_{22}+H_{23})+2\epsilon^2=0. \quad . \;\; . \quad (30)$$

Now we see from (29) that the absolute value of ϵ is zero in the neighbourhood of the centre of a square face and at the point z'' (figure 3) with the coordinates $(2\pi/a, \pi/a, \pi/a)$, where the deviation of $|\epsilon|$ from zero is largest, is $\epsilon=0, 64iI$. In the first approximation we can therefore neglect the term $2\epsilon^2$ in the equation (20) and assume that the energy E along this line $z'z''$ (figure 3) is given simply by the formula:

$$E=H_{00}=-M-4A_0'(\cos^2\tfrac{1}{2}ak-2\cos\tfrac{1}{2}ak), \quad . \;\; . \;\; . \;\; . \quad (31)$$

where $-\pi/2a \leqslant k \leqslant +\pi/2a.$

From this last formula we have for the energy E in the first zone at the point $(2\pi/a, \pi/a, \pi/a)$ the value: $E^I=-M-4A_0'(-1+0{\cdot}09)$, which is nearly the same as at the centre of the square face, where $E^I=-M+4A_0'$. Exactly the same situation occurs in the other equivalent points z', t', t'' (figure 3). The energy at the other points of this plane differ even less from the value $-M+4A_0'$, as we can easily see from the symmetry conditions.

Thus we may expect that in the *neighbourhood* of this plane of discontinuity the surfaces of equal energy are practically parallel to the bounding square plane. Indeed, if we assume again that in the neighbourhood of this plane the electronic energy E is given as a first approximation by the equation:

$$E=H_{00}=-M-4A_0'(\cos\tfrac{1}{2}ak_x\cos\tfrac{1}{2}ak_y+\cos\tfrac{1}{2}ak_x\cos\tfrac{1}{2}ak_z$$
$$+\cos\tfrac{1}{2}ak_y\cos\tfrac{1}{2}ak_z),$$

we see at once that near the point $(2\pi/a, 0, 0)$ the energy has the form :

$$E=-M+4A_0'-A_0'a^2(2\pi/a-k_x)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

i. e. near their centres the surfaces of equal energy are parallel to the planes $(2\pi/a, 0, 0)$. Only near the boundaries of the first Brillouin zone the surfaces are slightly curved.

The same situation, of course, arises in the neighbourhood of the other equivalent planes : (-200), (0 ± 20) and (00 ± 2).

With this last fact is closely connected the high density of states for the energy range corresponding to these surfaces of equal energy parallel to the square faces. We know (6) that the density per unit value of the states with energy E and $E+dE$ is given by

$$N(E)=\frac{1}{8\pi^2}\iint\frac{dS}{|\operatorname{grad} E|}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where the integration is over the surface in k-space on which the energy is constant and has the value E. If we denote by $\bar{k}$ the difference : $2\pi/a-k_x$, then for these surfaces the energy has the form (equation (32)) :

$$E=-M+4A_0'-A_0'a^2\bar{k}^2,$$

and for $|\operatorname{grad} E|$ we can write :

$$|\operatorname{grad} E|=2a^2A_0'\bar{k}.$$

If we assume that the whole surface corresponding to this energy E is composed of six square faces, then the integral $\iint dS$ in (33) has the value :

$$\iint dS=6\,.\,\tfrac{1}{2}\,.\left(\bar{k}+\frac{\pi}{a}\right)^2.$$

Thus the density of these states is :

$$N(E)=\frac{1}{8\pi^2}\frac{6}{2}\left(\bar{k}+\frac{\pi}{a}\right)^2\frac{1}{2a^2A_0'\bar{k}}=\frac{3}{16\pi^2}\frac{\left(\bar{k}+\frac{\pi}{a}\right)^2}{a^2A_0'\bar{k}}. \quad . \quad . \quad (34)$$

We shall see later that a^2A_0' is a very small quantity and as was mentioned above the same is true about the quantity $\bar{k}$; thus we can conclude that the value of the fraction $(\bar{k}+\pi/a)^2/a^2A_0'\bar{k}$ is extremely high and, therefore, according to the formula (34) the density of the states corresponding to this energy is also high.

(B) *Energy Levels according to the Approximation of nearly free Electrons.*

It will be useful to compare the results just obtained with the more general predictions of the theory of nearly free electrons. First we shall consider the states lying near to the plane of discontinuity (111). We choose axes k_x, k_y, k_z so that the perpendicular from the origin coincides with the k_z axis and denote the point where this line cuts the plane of energy discontinuity by $(0, 0, k_0)$. The electronic energy E is then given by

$$E=E_0+\frac{\hbar^2}{2m}\{k_x^2+k_y^2+\alpha(k_z-k_0)^2\}, \quad . \quad . \quad . \quad . \quad (35)$$

where E_0 is the energy of the state at the point $(0, 0, k_0)$ and α is connected with the energy gap by the approximate formula : $\alpha=1-4E_0'/\Delta E$ for the states in the first zone, $\alpha=1+4E_0'/\Delta E$ for the states in the second zone, E_0' being the energy of a free electron at the point $(0, 0, k_0)$; $E_0'=\hbar^2k_0^2/2m$. We shall see later that $4E_0'/\Delta E>1$, so it follows from (35) that in the first zone the surfaces of equal energy are hyperboloids and in the second zone ellipsoids. These results are thus in agreement with the predictions obtained from the theory of tightly bound electrons in the neighbourhood of the plane of discontinuity (111).

To describe the situation near the square faces (200), we cannot use the formula (35). We know that this expression was deduced under the assumption that the energy-states near the plane of discontinuity are influenced only by this plane. The square face (200) is rather small and to determine the energy contour near this plane it is necessary to take into account the influence of the other bounding planes (111). The calculations are then rather complicated, but we can understand this situation qualitatively if we consider the following facts. It is well known that the energy surfaces near the corners of the Brillouin zones are usually spheres round these corners. In our case we can expect these spheres to combine with the hyperboloids near the centre of this plane in such a manner that the final surfaces are practically planes parallel to the square plane of discontinuity, as described in the last paragraph.

(C) *Construction of Energy Band-Structure in Aluminium.*

In the preceding sections we have deduced the expressions for the energy of the states in the first or second Brillouin zone. Now we have reached the point when combining these results with some experimental facts we can obtain the numerical values for the unknown coefficients in these formulae. Thus we shall be able to get a much more detailed picture of the band-structure of aluminium.

The electrons at the bottom of the first zone behave as if they were free and therefore the corresponding surfaces of equal energy are spherical. The energy of these electrons is thus given by the formula :

$$E=\frac{\hbar^2}{2m}k^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

when we take our zero point in such a way that the energy at the point $k_x=k_y=k_z=0$ is equal to zero.

We know also that the energy of the states near to the point $(\pi/a, \pi/a, \pi/a)$ in the first zone and in the direction [111] is approximately given by formula (23) or by (35), *i. e.*:

$$E^{I} \doteq -M - 12A_0' \cos^2 \frac{a}{2\sqrt{3}} k_0 \doteq -M - a^2 A_0' k_0^2, \quad . \quad (37\,a)$$

or

$$E^{I} \doteq E_0' + \alpha \frac{\hbar^2}{2m} k_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37\,b)$$

where k_0 means the wave vector in the [111] direction, counted from the point where this direction cuts the plane of discontinuity (111) and E_0' is the energy of the free electrons at the point $(\pi/a, \pi/a, \pi/a)$, $E_0' = \hbar^2(\sqrt{3}\pi/a)^2/2m$. The coefficient α in the equation (37 *b*) is equal to $1 - 4E_0'/\Delta E$ and to determine this value we must know the energy gap ΔE. It has, however, been pointed out by Jones ⁽⁸⁾ that we can obtain this magnitude from X-ray data. If we know the atomic structure factor f_n for the plane $(n_1 n_2 n_3)$, where $n = \sqrt{n_1^2 + n_2^2 + n_3^2}$ and denote by Z the atomic number, by $S_{n_1 n_2 n_3}$ the structure amplitude ($S_{n_1 n_2 n_3} = \left| \sum_{1}^{s} e^{2\pi i (n_1 u_t + n_2 v_t + n_3 w_t)} \right|$, $u_t\, v_t\, w_t$ are the coordinates of the *s* atoms in the unit cell), by *a* the lattice constant and by *e* the electronic charge, we can write

$$\Delta E_{n_1 n_2 n_3} = \frac{2e^2}{\pi^2 n^2 a} (Z - f_n) S_{n_1 n_2 n_3}.$$

According to James and Bradley ⁽⁹⁾ the values of the structure factors for the plane of discontinuities (111) and (200) are: $f_{111} = 8{\cdot}83$ and $f_{200} = 8{\cdot}35$. We have then for the corresponding energy gaps across these two planes the following approximate values ($S_{111} = S_{200} = 4$, $a = 4{\cdot}04 \,.\, 10^{-8}$ cm., $Z = 13$):

$$\Delta E_{111} \doteq 3{\cdot}4 \text{ eV.}, \qquad \Delta E_{200} = 3{\cdot}1 \text{ eV.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

These values have, of course, only an approximate validity, but we shall see later that they are in rather close agreement with the experimental facts derived from the soft X-ray spectra. If we assume that the electronic energy E_0' at the point $(\pi/a, \pi/a, \pi/a)$ in the first zone is approximately given by: $E_0' = \hbar^2(\sqrt{3}\pi/a)^2/2m \doteq 6{\cdot}5$ eV., we have for α in the first zone (putting $\Delta E_{111} \doteq \Delta E_{200} \doteq 3$ eV.):

$$\alpha = 1 - \frac{4E_0'}{\Delta E} = -8. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

From the comparison of the coefficient of k_0^2 in the equations (37 *a*) and (37 b) we can determine A_0' as follows:

$$A_0' = -\frac{h^2}{2m} \frac{\alpha}{a^2} \doteq 1{\cdot}8 \text{ eV.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

From the theory, developed in the preceding sections, we know that the electronic energy at the point $(2\pi/a, 0, 0)$ in the first zone is $-M + 4A_0'$ (equation (25)). The difference of the energies corresponding to the points $(\pi/a, \pi/a, /\pi a)$ and $(2\pi/a, ; 0, 0)$ is equal to $4A_0' \doteq 7{\cdot}2$ eV.

After these considerations we can already construct an approximate picture of the electronic states in the first Brillouin zone. The figures 1 and 2 represent the section through the reciprocal space in $k_z=0$ plane or in the plane containing the axis k_z and the direction [111] respectively. First we see that the energy contours near to the origin are (by equation (36)) spheres. In figure 2, we see also the first energy surface which touches the plane of discontinuity (111) at the point e. The energy at this point from the preceding consideration is equal to 6·5 eV. Further, we know that the energy contours near the square faces ($\pm$200), (0$\pm$20), (00$\pm$2) are the planes parallel to these bounding planes (see figures 1 and 2). As the energy at the point $(2\pi/a, 0, 0)$ is about 7·2 eV. greater than at $(\pi/a, \pi/a, \pi/a)$, we must conclude that the value of the energy contour in this place is approximately equal to 6·5 eV+7·2 eV=13·7 eV. Thus it is clear that in the *second* zone there exist electronic states with the energy of those near to the square faces in the first zone. Let us take, for example, the state at the point e in the second zone; its energy is equal to 6·5 eV$+\Delta$E=6·5 eV+3 eV=9·5 eV and this is smaller than the energy of the state at the point b in the first zone, which is equal to 13·7 eV. We can assume there is an overlap between the first and second zones in the (111)-direction.

To determine the position of the Fermi energy it is necessary to consider the distribution of three valence electrons per aluminium atom in these two zones. Let us assume that the first zone is *nearly* full; it can, therefore, comprise approximately two electrons. The third electron must overlap into the second zone. The energy contours near the point $(\pi/a, \pi/a, \pi/a)$ in the second zone are ellipsoids and their equation is given by the formula (35), with $\alpha=1+4\mathrm{E}_0'/\Delta\mathrm{E}=10$:

$$\mathrm{E}=9{\cdot}5\ \mathrm{eV.}+\frac{\hbar^2}{2m}(10(k_z-k_0)^2+k_x^2+k_y^2). \quad . \quad . \quad . \quad (41)$$

Now it is well known (10) that the number of electrons in the reciprocal space of the volume V per atom (atomic volume$=\tau$) is given by the formula

$$n=2\frac{\tau}{8\pi^3}\mathrm{V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

Assuming the validity of (41) for the energy of the electronic states in the second zone we have the following expression for the volume V :

$$\mathrm{V}=4\ .\ 4\pi\ .\ 10(k_z-k_0)^2_{\mathrm{max}}/3,$$

where $(k_z-k_0)_{\mathrm{max}}$ means the semi-axis of the ellipsoidal energy surface in the (111)-direction corresponding to the highest energy.

From the expression (42) we can deduce the value $(k_z-k_0)_{\mathrm{max}}$:

$$n=1=4\frac{\tau}{8\pi^3}\frac{8}{3}\pi(k_z-k_0)^2_{\mathrm{max}},$$

$$(k_z-k_0)_{\mathrm{max}}\doteq 0{\cdot}46\pi/a,$$

and the corresponding energy E_{max} in the second zone is approximately :

$$E_m \doteq 4{\cdot}4 \text{ eV}.$$

This result means that the Fermi energy would be about 4 eV. higher than the energy of the state at the point $(\pi/a, \pi/a, \pi/a)$ in the second zone. If we examine more carefully this consideration, we see at once that this energy lies in the region where the approximate formula (41) loses its validity. We see from our diagram 2 that in this region $k_z - k_0 \doteq 0{\cdot}46\pi/a$) the density of states is rather high and in any case higher than that given by our approximation (41). Thus we must conclude that the actual energy is a little smaller, say it is equal to 3·5 eV.

After these considerations we can describe the probable distribution of three valence electrons in the zone-picture as follows : the first zone is very nearly full, the Fermi surface lies near to the square faces in the $\{100\}$-directions and the corresponding energy is roughly 13 eV. There is no overlap in the $\{100\}$-directions. The third electron is situated in the second zone, where the Fermi surface has the shape of semi-ellipsoids surrounding the eight points $(\pm\pi/a, \pm\pi/a, \pm/\pi a)$ and the corresponding energy is naturally again : 6·5 eV.+3 eV.+3·5 eV.=13 eV., *i. e.*, nearly the same value as before.

§ 2. *The Soft X-ray Emission Spectra of Aluminium.*

The radiation emitted when the electrons from the first or second zone respectively make transitions to the ionized K and L levels forms a band. The breadth (in energy units) of the X-ray emission band gives directly the breadth of the Fermi distribution.

According to the measurements by Skinner (11) this breadth for aluminium is about 12 eV. This is in good agreement with our estimates, since we can expect from our preceding calculations that the width of Fermi distribution will be slightly over 13·0 eV.

In figures 4 *a* and *b* we show the intensity distribution in the emission spectra of aluminium. Let us first consider the emission at the Fermi edge in the spectrum AlL^3. This emission is caused by the transitions of electrons of Fermi energy from the first or second zone respectively to the ionized L-level. Thus the total intensity I(E) in this case is the sum of two intensities. The first one ($I^I(E)$) corresponds to the transitions from the electronic states lying in the first zone (near the square faces) to the L level, and the second one ($I^{II}(E)$) arises from the transitions of the electronic states of the Fermi energy lying in the second zone.

It is well known that the intensity of the emission is proportional to the product of the density of states for corresponding energy of the conducting product of the density of states for corresponding energy of the conducting electrons and the transition probability. For the intensity $I^I(E)$ the corresponding density of states in the first zone $N^I(E)$ can be calculated from equation (33), where the integration is to be taken over the Fermi surface in the first zone. Similarly we can determine the density $N^{II}(E)$ for the states lying in the second zone. Thus we can write :

$$\left.\begin{aligned} I(E)&=I^{I}(E)+I^{II}(E), \\ \text{and} \qquad I^{I}(E)&\propto N^{I}(E)p^{I}(E), \\ I^{II}(E)&\propto N^{II}(E)p^{II}(E), \end{aligned}\right\} \qquad (43)$$

where $p^{I}(E)$, $p^{II}(E)$ are the probabilities of transitions from the first or the second zone respectively to the L-level. $p(E)$ is proportional to the square of an integral of the type

$$\int \psi_f^* \operatorname{grad} \psi_i \, d\tau, \qquad (44)$$

where ψ_i, ψ_f are the initial and final wave functions. In our case the final wave function ψ_f, which vanishes except in the neighbourhood of the nucleus, has the symmetry of an atomic p-wave function ($\psi_f=\psi_p$). The simplest treatment of the wave function ψ_i of an electronic level of the lattice is to expand it in a series of functions ψ_s, ψ_p, ... (as we did in §1) having symmetry of s, p, d, ... atomic wave functions in the neighbourhood of the nuclei. Thus we write

Fig. 4 *a.*

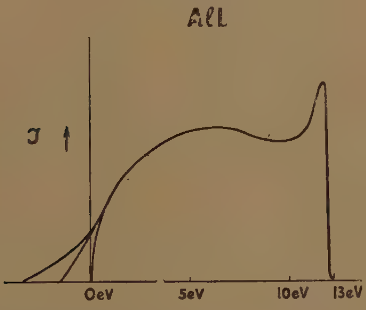
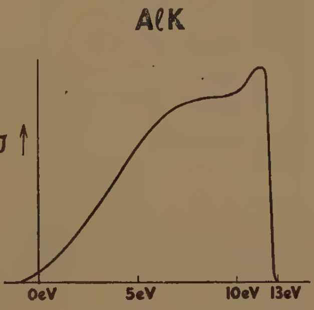

AlL

Fig. 4 *b.*


AlK

$$\psi_i=A_s\psi_s+A_p\psi_p+\ldots. \qquad (45)$$

From the equations (2), (3), (5), (6) and (35) it is easy to show that the wave function for the electronic states in the [100]-direction has the form :

$$\psi_{0k_x}=\phi_0 \cos \tfrac{1}{2}\theta+i\phi_1 \sin \tfrac{1}{2}\theta,$$

where

$$\cot \vartheta=\frac{M-N+8(A_0'+A_1')\cos \frac{1}{2}ak_x-4(A_0'+A_1')}{16 \sin \frac{1}{2}ak_x} \qquad (46)$$

and ϕ_0, ϕ_1 are the atomic wave functions from §1. In the neighbourhood of the Fermi energy, *i. e.* for the states with $k_x \doteq 2\pi/a$, we can see from the last formula that $\sin \frac{1}{2}\vartheta=0$ and the corresponding wave function is built up only from the atomic s-function. We can assume that the wave functions for all states of the Fermi energy in the first zone are composed practically only from the atomic s-wave function :

$$\psi_i^{I} \simeq \psi_s. \qquad (47)$$

Similarly it can be shown that the wave functions in the second zone in [111]-directions for the states corresponding to the Fermi energy have the form:

$$\psi_i^{\mathrm{II}} \simeq A_s\psi_s + A_p(\phi_1+\phi_2+\phi_3), \quad \ldots \ldots \quad (48)$$

where $|A_s| < |A_p| < 1$.

If we put these expressions in the formula for the transition probability and notice that the transition integrals between ψ_p and ϕ_1, ϕ_2, ϕ_3 vanish, we obtain

$$p^{\mathrm{I}}(E) \propto \left| \int \psi_p^* \operatorname{grad} \psi_s \, d\tau \right|^2,$$

$$p^{\mathrm{II}}(E) \propto \left| \int \psi_p^* \operatorname{grad} \psi_s \, d\tau \right|^2 . \ |A_s|^2.$$

For the total intensity I(E) we can thus write:

$$I(E) \propto \{N^{\mathrm{I}}(E) + |A_s|^2 N^{\mathrm{II}}(E)\} \left| \int \psi_p^* \operatorname{grad} \psi_s \, d\tau \right|^2. \quad \ldots \quad (49)$$

It is possible to say that the AlL spectrum in the energy range near the Fermi energy shows approximately the density of the state lying in the first zone.

Using our results derived at the end of paragraph (A), we must expect here a very sharp peak in the intensity of the emission spectrum, as the density of states $N^{\mathrm{I}}(E)$ in this energy interval is extremely high (see figure 4 *a*).

Further, we shall consider the intensity of the emission-spectrum AlL near the energy value at which the energy contour touches the planes {111}. It can be proved by our general method developed in paragraph (B) that there exists a peak in the density distribution at this point. Now we can show that the electronic wave functions for the states near the points $(\pm\pi/a, \pm\pi/a, \pm\pi/a)$ are composed only from the atomic *s*-wave function. Naturally, the wave functions corresponding to the states in the first zone, where the energy surface touching the plane of discontinuities {111} crosses the axis k_x, k_y and k_z, are composed from the *s*-wave function and slightly also from *p*-wave functions. But in the rough approximation it is possible to assume that the wave functions of these states on the energy contour going through the points $(\pm\pi/a, \pm\pi/a, \pm\pi/a)$ are built up from the *s*-wave function only. If we calculate the intensity of emission as before we can get the following formula:

$$I(E) \propto N(E) \left| \int \psi_p^* \operatorname{grad} \psi_s \, d\tau \right|^2.$$

Clearly we may expect the maximum of the intensity of radiation for the energy which corresponds to the energy of the electronic state at the points $(\pm\pi/a, \pm\pi/a, \pm\pi/a)$. From the diagram (4 *a*) we can see that this case occurs near the energy value 6·6 eV., which is again in good agreement with out expectations.

Let us now consider the intensity of emission in the AlK spectrum (fig. 4 *b*). This spectrum arises through the transitions of electrons from the states lying in the first or second zone to the ionized K-level. In this case the integral (44) determining the probability of transition is different from zero, when it has the following form :

$$A_p \int \psi_s{}^* \operatorname{grad} \psi_p \, d\tau,$$

i. e. when it corresponds to the transition from the *p*-states of the conduction band to the *s*-state of the atomic level K.

As we know already that the states with energy corresponding to the energy contour touching the planes {111} have only a small *p*-component, we must conclude that these states do not contribute very strongly to the K-spectra. But the states near the point *e* with (the energy 9·5 eV.) in the second zone (fig. 2) are composed almost completely from the *p*-wave functions. They contribute, therefore, strongly to the emissions in the K-spectra and we can expect the characteristic kink at this energy in the AlK spectrum. From the fig. 4 *b* we see that this kink is actually observed at the energy 9·6 eV. The energy difference between the peak of the AlL spectrum (6·6 eV.) and the corresponding kink on the emission curve in the AlK spectrum (9·6 eV.) gives the magnitude of the energy discontinuity across the planes {111} ; *i. e.* $\varDelta E_{111} \doteq 3$ eV., which is in accordance with our preceding expectations.

If we go still nearer to the Fermi surface in the second zone, the wave functions of these states begin to be a mixture of *s*- and *p*-wave functions. We may thus expect an increase in the emission intensity of the AlL spectrum for energies greater than 9·6 eV. This was observed, as is clear from the fig: 4 *a* ; we can see here a kink at the energy 10 eV. before the sharp peak ,which was described before.

In conclusion I wish to thank Prof. N. F. Mott (University of Bristol) for his advice and helpful interest in this work.

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LIV. *The Summation of a Particular Type of Power Series.*

By John Williams, M.Sc., University College, Exeter * †.

[Received April 24, 1947].

In a note to the 'Philosophical Magazine'⁽¹⁾, Dr. L. Silberstein deduced the sum of $\sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}$ (x real). This paper extends his analysis to sum the series

$$\sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \quad (z \text{ complex}), \qquad \ldots \ldots \quad (1)$$

where

$$\sum_{t=0}^{k} \lambda_t a_{t+n} = 0 \quad (k \geqslant 1), \text{ for all } n, \qquad \ldots \ldots \quad (2)$$

λ_t ($t=0, 1, \ldots, k$), $\lambda_0 \neq 0$, $\lambda_k \neq 0$ being given constants, complex if necessary.

Assume that $\sum_{t=0}^{k} \lambda_t \rho^t \equiv \lambda_k \prod_{\nu=1}^{m} (\rho-\rho_\nu)^{r_\nu+1}$, where r_ν ($\nu=1, 2, \ldots, m$) are positive integers or zero ($m<k$; $\sum_{\nu=1}^{m} r_\nu = k-m$), so that the general solution of (2) is ⁽²⁾ $a_n = \sum_{\nu=1}^{m} \sum_{s=0}^{r_\nu} A_{r_\nu, s} n^s \rho_\nu^n$, where $A_{r_\nu, s}$ are k-constants; $A_{r_\nu, s}=0$ when $s>r_\nu$. . . . (3)

Conversely, given a_n in the form (3), we can deduce a difference relation of the form (2).

Differentiating (1) term-by-term p times *w.r.t.*, z, we have the series

$$\sum_{n=0}^{\infty} a_{p+n} \frac{z^n}{n!}. \qquad \ldots \ldots \quad (4)$$

Let ρ be a real quantity greater than the greatest of $|\rho_\nu|$. Also, for each value of s, there exists a positive A_s greater than the greatest of $|A_{r_\nu, s}|$, ($\nu=1, 2, \ldots, m$). Consequently we write

$$\left| a_{p+n} \frac{z^n}{n!} \right| \leqslant \sum_{s=0}^{r} \frac{mA_s(n+p)^s \rho^{n+s} |z|^n}{n!},$$

where r denotes the greatest of the integers r_ν ($\nu=1, 2, \ldots, m$).

Now each $\sum_{n=0}^{\infty} (n+p)^s |z|^n/n!$ ($s=0, 1, \ldots r$) is convergent, so that series (4) is uniformly convergent in $0 \leqslant |z| \leqslant R$ (R finite), for each p.

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* Communicated by the Author.

† Formerly Lecturer in Mathematics, Sir John Cass Technical Institute, London.

Consequently, if $\sum\limits_{n=0}^{\infty} a_n \dfrac{z^n}{n!} = \mathrm{S}(z)$, then

$$\frac{d^p\mathrm{S}}{dz^p} = \sum_{n=0}^{\infty} a_{p+n} \frac{z^n}{n!} \text{ for all finite } z. \qquad (5)$$

Again, using (2),

$$\left.\begin{aligned} &\sum_{t=0}^{k} \lambda_t \frac{d^t\mathrm{S}}{dz^t} = \sum_{t=0}^{k} \sum_{n=0}^{\infty} \lambda_t a_{t+n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \sum_{t=0}^{k} \lambda_t a_{t+n} \frac{z^n}{n!} = 0, \\ \text{where}\quad &\frac{d^t\mathrm{S}}{dz^t} \equiv \mathrm{S}(z) \text{ when } t=0. \end{aligned}\right\} \qquad (6)$$

The solution of this differential equation for S(z) can, by (3), be written as [(3)]

$$\mathrm{S}(z) = \sum_{\nu=1}^{m} \sum_{s=0}^{r_\nu} \mathrm{B}_{r_\nu, s} z^s \exp(\rho_\nu z), \qquad (7)$$

where $\mathrm{B}_{r_\nu, s}$ $(s=0, 1, \ldots, r_\nu;\ \nu=1, 2, \ldots, m)$, $[\mathrm{B}_{r_\nu, s}=0$ if $s>r_\nu]$ are k-constants which can be expressed in terms $a_0, a_1, \ldots a_{k-1}$, as follows :—

$$\left.\begin{aligned} &\text{From (1), } a_t = \left[\frac{d^t\mathrm{S}}{dz^t}\right]_{z=0} = \sum_{\nu=1}^{m} \sum_{w=0}^{t} \frac{t!}{(t-w)!} \mathrm{B}_{r_\nu, w} \rho_\nu^{t-w} \text{ for } t=0, \\ &1, \ldots, (k-1), \text{ where } \mathrm{B}_{r_\nu, w}=0 \text{ when } w>r_\nu. \end{aligned}\right\} \qquad (8)$$

To extract expressions for the k-coefficients $\mathrm{B}_{r_\nu, w}$ in terms of the k-constants a_t, introduce functions $f_{\mu, q}(\rho)$ defined by

$$\left.\begin{aligned} &f_{\mu, q}(\rho) \equiv \frac{\lambda_k \prod\limits_{\nu=1}^{m} (\rho-\rho_\nu)^{r_\nu+1}}{(\rho-\rho_\mu)^q} = \sum_{r=0}^{k-q} \sigma_{\mu, q, r}\, \rho^r, \\ &\mu \text{ and } q \text{ being positive integers such that } 1 \leqslant \mu \leqslant m, \text{ and } \\ &1 \leqslant q \leqslant r_\mu+1. \end{aligned}\right\} \qquad (9)$$

Using (3), we have, provided $r \leqslant k-q$ $(\sigma_{\mu, q, r}=0$ when $r>k-q)$,

$$\left.\begin{aligned} \sigma_{\mu\ q, r} = r! \left[\frac{d^r}{d\rho^r}\left\{\frac{\sum\limits_{t=0}^{k} \lambda_t \rho^t}{(\rho-\rho_\mu)^q}\right\}\right]_{\varrho=0} &= \frac{(-)^q}{(q-1)!} \sum_{m=0}^{r} \frac{(q+m-1)!\,\lambda_{r-m}}{m!\,\rho_\mu^{q+m}} \\ = -\frac{1}{(q-1)!}\left[\frac{d^{q-1}}{d\rho^{q-1}}\left(\sum_{m=0}^{r} \lambda_m \rho^{m-r-1}\right)\right]_{\varrho=\varrho_\mu} &(\rho_\mu \neq 0 \text{ since } \lambda_0 \neq 0) \end{aligned}\right\}. \qquad (10)$$

Since $\left[\dfrac{d^r}{d\rho^r}\left(\sum\limits_{t=0}^{k} \lambda_t \rho^t\right)\right]_{\varrho=\varrho_\mu} = 0$ for $r=0, 1, \ldots, r_\mu$, $-\dfrac{1}{(q-1)!}$

$$\times \left[\frac{d^{q-1}}{d\rho^{q-1}}\left(\sum_{m=0}^{r} \lambda_m \rho^{m-r-1}\right)\right]_{\varrho=\varrho_\mu} = \frac{1}{(q-1)!}\left[\frac{d^{q-1}}{d\rho^{q-1}}\left(\sum_{m=r}^{k-1} \lambda_{m+1} \rho^{m-r}\right)\right]_{\varrho=\varrho_\mu}.$$

Denoting $$\frac{d^{q-1}}{d\rho^{q-1}}\left(\sum_{m=r}^{k-1} \lambda_{m+1}\rho^{m-r}\right) \text{ by } \mathrm{G}_r^{(q-1)}(\rho), \quad . \quad . \quad . \quad . \quad (11)$$

$$\left.\begin{aligned} \sigma_{\mu,\,q,\,r} &= \frac{1}{(q-1)\,!}\,[\mathrm{G}_r^{(q-1)}(\rho)]_{\varrho=\varrho_\mu} \text{ when } r \leqslant k-q, \\ &= 0 \qquad\qquad\qquad\qquad\quad\;\; \text{ when } r > k-q \end{aligned}\right\} \quad . \quad . \quad . \quad (12)$$

From (9), $$\left.\begin{aligned} f_{\mu,\,q} &= (\rho-\rho_\mu)^{r_\mu-q+1}\mathrm{F}_\mu(\rho), \\ \text{where} \quad \mathrm{F}_\mu(\rho) &= f_{\mu,\,r_\mu+1} = \frac{\lambda_k \prod\limits_{\nu=1}^{m} (\rho-\rho_\nu)^{r_\nu+1}}{(\rho-\rho_\mu)^{r_\mu+1}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Consequently,

$$\frac{d^s}{d\rho^s} f_{\mu,\,q} \equiv f_{\mu,\,q}^{(s)} = \sum_{r=0}^{s} \frac{s\,!\,(r_\mu-q+1)\,!\,(\rho-\rho_\mu)^{r_\mu-q+1-r}}{(s-r)\,!\,r\,!\,(r_\mu-q+1-r)\,!}\,\mathrm{F}_\mu^{\,(s-r)},$$

where $$\mathrm{F}_\mu^{\,(k)} \equiv \frac{d^k\mathrm{F}_\mu}{d\rho^k},$$

so that $$\left.\begin{aligned} [f_{\mu,\,q}^{(s)}]_{\varrho=\varrho_\nu} &= 0 \quad \text{when } \nu \neq \mu \text{ for all } s \text{ such that } s \leqslant r_\nu \\ &= 0 \quad \text{when } \nu = \mu \text{ for all } s \text{ such that } s \leqslant r_\mu - q \\ &= \frac{s\,!}{(s-r_\mu+q-1)\,!}\,[\mathrm{F}_\mu^{\,(s-r_\mu+q-1)}]_{\varrho=\varrho_\mu} \text{ when } \nu=\mu \\ &\qquad\qquad \text{and } r_\mu-q+1 \leqslant s \leqslant r_\mu. \end{aligned}\right\} \quad . \quad (14)$$

Now, from (8) we see

$$\sum_{r=0}^{k-1} a_r\sigma_{\mu,\,q,\,r} = \sum_{\nu=1}^{m}\sum_{s=0}^{r_\nu}\sum_{r=s}^{k-1} \sigma_{\mu,\,q,\,r}\,\frac{r\,!}{(r-s)\,!}\,\rho_\nu^{r-s}\mathrm{B}_{r_\nu,\,s},$$

whilst from (9) $$\sum_{r=s}^{k-q} \sigma_{\mu,\,q,\,r}\,\frac{r\,!}{(r-s)\,!}\,\rho^{r-s} = f_{\mu,\,q}^{(s)},$$

and since $\sigma_{\mu,\,q,\,r}=0$ when $r>k-q$, we write

$$\begin{aligned} \sum_{r=0}^{k-1} a_r\sigma_{\mu,\,q,\,r} &= \sum_{\nu=1}^{m}\sum_{s=0}^{r_\nu} \mathrm{B}_{r_\nu,\,s}[f_{\mu,\,q}^{(s)}]_{\varrho=\varrho_\nu} \\ &= \sum_{s=r_\mu-q+1}^{r_\mu} \frac{s\,!}{(s-r_\mu+q-0)\,!}\,[\mathrm{F}_\mu^{(s-r_\mu+q-1)}]_{\varrho=\varrho_\mu}\mathrm{B}_{r_\mu,\,s} \end{aligned}$$

(by virtue of (14)).

If now we put $$\mathrm{B}_{r_\mu,\,s} = \sum_{r=0}^{k-1} \omega_{\mu,\,s,\,r}\,a_r,$$

then from the preceding and (12)

$$\begin{aligned} \sigma_{\mu,\,q,\,r} &= \sum_{s=r_\mu-q+1}^{r_\mu} \frac{s\,!}{(s-r_\mu+q-1)\,!}\,[\mathrm{F}_\mu^{(s-r_\mu+q-1)}]_{\varrho=\varrho_\mu}\omega_{\mu,\,s,\,r} = \sum_{w=0}^{q-1} \frac{(r_\mu-w)\,!}{(q-w-1)\,!} \\ &\quad \times [\mathrm{F}_\mu^{(q-w-1)}]_{\varrho=\varrho_\mu}\omega_{\mu,\,r_\mu-w,\,r} = \frac{1}{(q-1)\,!}\,[\mathrm{G}_r^{(q-1)}]_{\varrho=\varrho_\mu} \quad . \quad . \quad . \quad (15) \end{aligned}$$

for all integers μ, q and r satisfying $1 \leqslant \mu \leqslant m$, $1 \leqslant q \leqslant r_\mu+1$, $0 \leqslant r \leqslant k-1$.

Putting $q=1$, $[\mathrm{G}_r]_{\varrho=\varrho_\mu} = (r_\mu)\,!\,[\mathrm{F}_\mu]_{\varrho=\varrho_\mu}\omega_{\mu,\,r_\mu,\,r}$ for $\mu=1, 2, \ldots, m$.

Since ρ_μ is n=zero for all μ $(1 \leqslant \mu \leqslant m)$, $[\mathrm{F}_\mu]_{\varrho=\varrho_\mu} \neq 0$, thus

$$\omega_{\mu,\, r_\mu,\, r} = \frac{1}{(r_\mu)!}\left[\frac{\mathrm{G}_r}{\mathrm{F}_\mu}\right]_{\varrho=\varrho_\mu}.$$

Again, putting $\mathrm{G}_r = \mathrm{F}_\mu . \Phi_r$, we have

$$\mathrm{G}_r^{(q-1)} = \sum_{w=0}^{q-1} \frac{(q-1)\,!}{w\,!\,(q-w-1)\,!} \mathrm{F}_\mu^{(q-w-1)} \Phi_r^{(w)}. \quad . \quad . \quad . \quad (16)$$

Since (15) and (16) hold for all integers q where $1 \leqslant q \leqslant r_\mu + 1$,

$$\omega_{\mu,\, r_\mu - w,\, r} = \frac{[\Phi_r^{(s)}]_{\varrho=\varrho_\mu}}{(r_\mu - w)\,!\ \ w\,!} = \frac{1}{(r_\mu - w)\,!\, w\,!}\left[\frac{d^w}{d\rho^w}\left\{\frac{\mathrm{G}_r}{\mathrm{F}_\mu}\right\}\right]_{\varrho=\varrho_\mu}, \quad . \quad (17)$$

when $0 \leqslant w \leqslant r_\mu$, the result being true for all integers μ where $1 \leqslant \mu \leqslant m$.

SUMMARY OF RESULTS :—

Given a series $\sum\limits_{n=0}^{\infty} a_n \dfrac{z^n}{n\,!}$,

where $\sum\limits_{t=0}^{k} \lambda_t a_{t+n} = 0$ for all n $(\lambda_k \neq 0,\ \ \lambda_0 \neq 0)$, and $\sum\limits_{t=0}^{k} \lambda_t \rho^t \equiv \lambda_k \prod\limits_{\nu=1}^{m} (\rho - \rho_\nu)^{r_\nu + 1}$, then the power series is uniformly convergent for all $0 \leqslant |z| \leqslant \mathrm{R}$ (finite) and . . (18)

$$\mathrm{S}(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n\,!} =$$

$$\sum_{\mu=1}^{m} \sum_{w=0}^{r_\mu} \sum_{r=0}^{k-1} \left[\frac{d^{r_\mu - w}}{d\rho^{r_\mu - w}}\left\{\frac{\mathrm{G}_r(\rho)(\rho-\rho_\mu)^{r_\mu+1}}{\lambda_k \prod\limits_{\nu=1}^{m} (\rho-\rho_\nu)^{r_\nu+1}}\right\}\right]_{\varrho=\varrho_\mu} \frac{a_r z^w \exp(\rho_\mu z)}{(r_\mu - w)\,!\, w\,!},$$

where $\mathrm{G}_r(\rho) = \sum\limits_{m=r}^{k-1} \lambda_{m+1} \rho^{m-r}$

Special Cases.

Consider $\mathrm{S}_k(z) = \sum\limits_{n=0}^{\infty} a_n \dfrac{z^n}{n\,!}$,

given a_n for $n=0, 1, \ldots, (k-1)$, and $a_{k+n} = a_n$ for all n.

In this case $r_\mu = 0$ for all μ $(=1, 2, \ldots, m)$, $m=k$ and $\rho_\mu = \exp\left(\dfrac{2\pi\mu i}{k}\right)$, so that

$$\mathrm{S}_k(z) = \sum_{\mu=1}^{k} \sum_{r=0}^{k-1} a_r \left(\frac{\rho_\mu{}^{-r}}{k}\right) \exp(\rho_\mu z)$$

$$= \frac{1}{k} \sum_{\mu=1}^{k} \sum_{r=0}^{k-1} a_r (\cos q_\mu + i \sin q_\mu) e^{p_\mu},$$

where $$p_\mu = x\cos\left(\frac{2\pi\mu}{k}\right) - y\sin\left(\frac{2\pi\mu}{k}\right);$$

$$q_\mu = x\sin\left(\frac{2\pi\mu}{k}\right) + y\cos\left(\frac{2\pi\mu}{k}\right) - \frac{2\pi\mu r}{k};\quad z = x+iy.$$

If z is real,

$$S_{2m}(x) = \frac{1}{m}\sum_{r=0}^{2m-1} a_r\left[\frac{e^x+(-)^r e^{-x}}{2} + \sum_{\mu=1}^{m} e^{x\cos\frac{\pi\mu}{m}}\cos\left\{x\sin\left(\frac{\pi\mu}{m}\right) - \frac{\pi\mu r}{m}\right\}\right],$$

where m is an integer >1;

$$S_{2m+1}(x) =$$
$$\frac{2}{2m+1}\sum_{r=0}^{2m} a_r\left[\frac{1}{2}e^x + \sum_{\mu=1}^{m} e^{x\cos\frac{2\pi\mu}{2m+1}}\cos\left\{x\sin\left(\frac{2\pi\mu}{2m+1}\right) - \frac{2\pi\mu r}{2m+1}\right\}\right],$$

where m is an integer $\geqslant 1$,

$$S_1(x) = a_0 e^x, \qquad S_2(x) = a_0\cosh x + a_1\sinh x.$$

Furthermore, if

$$u_k(x) = \sum_{n=0}^{\infty}\frac{x^{nk}}{(nk)!}, \qquad v_k(x) = \sum_{n=0}^{\infty}(-)^n\frac{x^{nk}}{(nk)!},$$

$$u_{2m}(x) = \frac{1}{m}\left[\cosh x + \sum_{\mu=1}^{m} e^{x\cos\frac{\pi\mu}{m}}\cos\left(x\sin\frac{\pi\mu}{m}\right)\right],$$

when m is an integer >1;

$$u_{2m+1}(x) = \frac{2}{2m+1}\left[\frac{1}{2}e^x + \sum_{\mu=1}^{m} e^{x\cos\frac{2\pi\mu}{2m+1}}\cos\left(x\sin\left(\frac{2\pi\mu}{2m+1}\right)\right)\right],$$

when m is an integer $\geqslant 1$;

$$v_{2m}(x) = \frac{1}{m}\sum_{\mu=0}^{m-1} e^{x\cos\left(\frac{2\mu+1}{2m}\right)\pi}\cos\left[x\sin\left(\frac{2\mu+1}{2m}\right)\pi\right],$$

when m is an integer $\geqslant 1$;

$$v_{2m+1}(x) =$$
$$\frac{2}{2m+1}\left\{\frac{1}{2}e^{-x} + \sum_{\mu=0}^{m-1} e^{x\cos\left(\frac{2\mu+1}{2m+1}\right)\pi}\cos\left[x\sin\left(\frac{2\mu+1}{2m+1}\right)\pi\right]\right\},$$

when m is an integer $\geqslant 1$;

$$u_1(x) = e^x, \qquad u_2(x) = \cosh x, \qquad v_1(x) = e^{-x}.$$

In conclusion, I wish to thank Dr. A. E. R. Church, M.A., and Mr. T. V. Davies, M.Sc., for their kindness in reading this note and offering their instructive criticisms.

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LV. *A Formula for Expanding an Integral as a Series.*

By H. F. Willis, M.Sc., Ph.D. (Admiralty Research Laboratory)[*].

[Received April 11, 1946.]

A number of problems in mathematical physics lead to integrals of the type

$$I=\int_a^b f(x)\,.\,F(x)\,dx,$$

where $F(x)$ is an oscillatory function, such as $\sin mx$ or $J_0(mx)$. These integrals, in principle, can be evaluated by expanding the integrand as a power series in x and integrating term by term. This, however, is not usually practicable if the term $F(x)$ contains many oscillations in the region (a, b) because the power series would need many terms to represent adequately the integrand. The following expansion is suitable for many such cases, though it is not confined only to cases in which $F(x)$ is oscillatory.

We consider the more general integral

$$I(\alpha)=\int_a^b f(x)\,.\,F(x)e^{-\alpha(x-c)}\,dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $b>a$, and where c has any value. The range of integration can be of infinite extent, but in this case it is of course assumed that the original integral is convergent.

Suppose now that $f(x)$ admits of a Taylor series expansion around the point $x=c$, the radius of convergence of this expansion tentatively being assumed to cover the entire region of integration. Thus

$$f(x)=\sum_{r=0}^{\infty}\frac{(x-c)^r}{r\,!}f^r(c). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We then have

$$I(\alpha)=\int_a^b F(x)e^{-\alpha(x-c)}\left[\sum_{r=0}^{\infty}\frac{(x-c)^r}{r\,!}f^r(c)\right]dx$$

$$=\sum_{r=0}^{\infty}\frac{f^r(c)}{r\,!}\int_a^b F(x)\,.\,(x-c)^r\,.\,e^{-\alpha(x-c)}\,dx. \quad . \quad . \quad . \quad (3)$$

We now impose on $F(x)$ the limitation that the integral

$$\int_a^b F(x)\,.\,e^{-\alpha(x-c)}\,dx$$

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admits of expansion in positive integral powers of α, so that we can write

$$\int_a^b F(x)\,.\,e^{-\alpha(x-c)}\,dx=\phi(\alpha)\quad\text{(say)}\quad\ldots\ldots\quad(4)$$

$$=\sum_{r=0}^{\infty}\frac{\phi^r(0)}{r\,!}\alpha^r$$

$$=\sum_{r=0}^{\infty}A_r\alpha^r,\qquad\text{where } A_r=\frac{\phi^r(0)}{r\,!}.$$

Equation (4) is now differentiated r times with respect to α and we get

$$\int_a^b F(x)(x-c)^r e^{-\alpha(x-c)}\,dx=(-1)^r\phi^r(\alpha).\quad\ldots\quad(5)$$

Introducing this result in equation (3) we obtain

$$I(\alpha)=\sum_{r=0}^{\infty}(-1)^r\frac{f^r(c)}{r\,!}\phi^r(\alpha).\quad\ldots\ldots\quad(6)$$

If we now suppose α to approach zero we obtain the following expression for the original integral :—

$$I=\underset{\alpha=0}{\mathrm{L}t}\,I(\alpha)=\sum_{r=0}^{\infty}(-1)^r\frac{f^r(c)}{r\,!}\phi^r(0)$$

$$=A_0f(c)-A_1f'(c)+A_2f''(c)-\ldots.$$

No attempt has been made to justify the various steps by which this expansion has been obtained, so that we cannot always expect the expansion to be a valid one. It would be an intricate problem to study the conditions under which the expansion represents the integral ; it is probably better to consider each particular case on its own merits. The expansion sometimes yields a convergent series representing the integral, sometimes it only gives an asymptotic series, whilst at other times the procedure breaks down altogether. These possibilities will be illustrated when we apply the formula to particular cases.

Special Formulæ.

By giving special forms to $F(x)$ and to the limits of integration we can derive a large number of special formulæ. The following example illustrates the procedure.

In equation (1) we take $a=0$, $b=\infty$, $c=0$, and $F(x)=J_0(mx)$. An expansion for the integral

$$\int_0^{\infty} f(x)J_0(mx)dx$$

is then obtained from the known integral

$$\int_0^{\infty} e^{-\alpha x}J_0(mx)dx=\frac{1}{\sqrt{\alpha^2+m^2}}.$$

Expanding this latter expression in the form

$$\frac{1}{\sqrt{\alpha^2+m^2}}=\frac{1}{m}-\frac{1}{2}\cdot\frac{\alpha^2}{m^3}+\frac{1\,.\,3}{2^2\,.\,2!}\cdot\frac{\alpha^4}{m^5}-\frac{1\,.\,3\,.\,5}{2^3\,.\,3\,!}\cdot\frac{\alpha^6}{m^7}+\ldots.$$

we obtain the appropriate coefficients A_r, thus

$$A_1=A_3=A_5=\ldots=0\,;\ A_0=\frac{1}{m};\ A_2=-\frac{1}{2m^3};\ A_4=\frac{1\,.\,3}{2^2\,.\,2!}\cdot\frac{1}{m^5},\ \text{etc.}$$

We therefore have the expansion

$$\int_0^\infty f(x)J_0(mx)dx=\frac{f(0)}{m}-\frac{1}{2}\frac{f''(0)}{m^3}+\frac{1\,.\,3}{2^2\,.\,2!}\frac{f^{\text{iv}}(0)}{m^5}-\ldots$$

$$\ldots(-1)^r\frac{1\,.\,3\,.\,5\ldots(2r-1)}{2^r\,.\,r\,!\,m^{2r+1}}f^{2r}(0).$$

The following list of expansions has been derived by a similar procedure :—

List of Special Formulæ.

$$(a)\ \int_0^\infty f(x)e^{-mx}\,dx=\frac{1}{m}\left\{f(0)+\frac{f'(0)}{m}+\frac{f''(0)}{m^2}+\frac{f'''(0)}{m^3}+\ldots\right\}$$

$$(b)\ \int_0^\infty f(x)\sin mx\,dx=\frac{1}{m}\left\{f(0)-\frac{f''(0)}{m^2}+\frac{f^{\text{iv}}(0)}{m^4}-\ldots\right\}$$

$$(c)\ \int_0^\infty f(x)\cos mx\,dx=\frac{1}{m}\left\{\frac{f'(0)}{m}-\frac{f'''(0)}{m^3}+\frac{f^{\text{v}}(0)}{m^5}-\ldots\right\}$$

$$(d)\ \int_0^\infty f(x)J_0(mx)\,dx=\frac{1}{m}\left\{f(0)-\frac{1}{2}\cdot\frac{f''(0)}{m^2}+\frac{1\,.\,3}{2^2\,.\,2\,!}\cdot\frac{f^{\text{iv}}(0)}{m^4}-\frac{1\,.\,3\,.\,5}{2^3\,.\,3\,!}\cdot\frac{f^{\text{vi}}(0)}{m^6}\ldots\right\}$$

$$(e)\ \int_0^\infty f(x)J_1(mx)\,dx=\frac{f(0)}{m}+\frac{1}{m}\left\{\frac{f'(0)}{m}-\frac{1}{2}\cdot\frac{f'''(0)}{m^3}+\frac{1\,.\,3}{2^2\,.\,2\,!}\frac{f^{\text{v}}(0)}{m^5}-\frac{1\,.\,3\,.\,5}{2^3\,.\,3\,!}\cdot\frac{f^{\text{vii}}(0)}{m^7}+\ldots\right\}$$

$$(f)\ \int_0^\infty f(x)\frac{\sin mx}{x}\,dx=\frac{\pi}{2}f(0)+\left\{\frac{f'(0)}{m}-\frac{f'''(0)}{3m^3}+\frac{f^{\text{v}}(0)}{5m^5}-\ldots\right\}$$

$$(g)\ \int_0^\infty f(x)e^{-m^2x^2}\,dx=\frac{\sqrt{\pi}}{2m}\left\{f(0)+\frac{1}{2\,!\,2}\cdot\frac{f''(0)}{m^2}+\frac{1\,.\,3}{4\,!\,2^2}\cdot\frac{f^{\text{iv}}(0)}{m^4}+\ldots\right\}$$
$$+\frac{1}{2m}\left\{\frac{f'(0)}{m}+\frac{1\,!}{3\,!}\cdot\frac{f'''(0)}{m^3}+\frac{2\,!}{5\,!}\cdot\frac{f^{\text{v}}(0)}{m^5}+\frac{3\,!}{7\,!}\cdot\frac{f^{\text{vii}}(0)}{m^7}\ldots\right\}$$

Examples of the Foregoing Formulæ.

In order to show the working of the expansion and to illustrate some points of interest we now apply the procedure to several integrals which can be evaluated by alternative means.

(1) Applying formula (*d*) to the integral

$$\int_0^\infty e^{-x^2} J_0(mx)\, dx$$

we obtain the asymptotic series

$$\frac{1}{m}+\frac{1^2}{1\,!}\cdot\frac{1}{m^3}+\frac{1^2\,.\,3^2}{2\,!}\cdot\frac{1}{m^5}+\frac{1^2\,.\,3^2\,.\,5^2}{3\,!}\cdot\frac{1}{m^7}+\ldots.$$

The integral is evaluated in Watson's 'Bessel Functions' (p 394) in the form

$$\sqrt{\pi}/2\,.\,e^{-m^2/8} I_0(m^2/8).$$

An asymptotic expansion of this expression has been obtained (*ibid.* p. 205) which agrees with the one given above.

(2) Applying formula (*d*) to the integral

$$\int_0^\infty \frac{J_0(mx)}{1+x}\, dx$$

we obtain the asymptotic series

$$\frac{1}{m}-\frac{1^2}{m^3}+\frac{1^2\,.\,3^2}{m^5}-\frac{1^2\,.\,3^2\,.\,5^2}{m^7}+\ldots.$$

The integral can be otherwise evaluated in the form

$$\pi/2\,\{H_0(m)-Y_0(m)\} \quad (\textit{ibid. } \text{p. } 436),$$

and an asymptotic expansion of this function, identical with the one given above, has been obtained (*ibid.* p. 333).

This case has the following point of interest. In the derivation of the general formula it was assumed tentatively that the domain of convergence of the expansion of $f(x)$ should cover the whole range of the original integral. In the present case the function $f(x)$ does not conform to this stipulation, for the function $1/(1+x)$ admits of expansion in powers of x only in the region $(1, -1)$, whilst the range of integration is $(0, \infty)$. The validity of the asymptotic expansion, in spite of this fact, shows that our restriction on $f(x)$ is not always a necessary one.

(3) The integral

$$\int_0^\infty e^{-m^2x^2} \cos x\, dx$$

brings out another point of interest. If we apply formula (*c*) to this, taking $f(x)$ to be $e^{-m^2x^2}$, the procedure breaks down, all terms of the expansion being zero. In this respect the expansion is similar to the Taylor expansion of the function e^{-1/x^2} round the point $x=0$, where all the terms of the expansion are zero though the function itself is non-zero. If, in the expansion of the integral, we invert the rôle of the functions and apply formula (*g*), taking $f(x)$ to be $\cos x$, the method is successful and we obtain the expansion

$$\frac{\sqrt{\pi}}{2m}\left\{1-\frac{1}{2\,!\,2}\cdot\frac{1}{m^2}+\frac{1\,.\,3}{4\,!\,2^2}\cdot\frac{1}{m^4}-\ldots\right\}$$

This series is convergent, and is the expansion of

$$\frac{\sqrt{\pi}}{2m} \cdot e^{-1/4m^2},$$

which can be otherwise shown to be the value of the integral.

(4) A further case in which it is profitable to manipulate the rôles of the functions $f(x)$ and $\mathrm{F}(x)$ is provided by the integral

$$\int_0^\infty \frac{\sin x \,.\, \sin mx}{x} dx.$$

Using formula (f) and taking $f(x)=\sin x$, we get the series

$$\frac{1}{m}+\frac{1}{3m^3}+\frac{1}{5m^5}+\ldots\ldots,$$

which is convergent if $m>1$, but fails to tell us anything of the value of the integral if $m \leqslant 1$. If, however, we use formula (b) and take

$$f(x)=\frac{\sin mx}{x},$$

we obtain an expansion in the form

$$m+\frac{m^3}{3}+\frac{m^5}{5}+\ldots\ldots,$$

which is convergent for $m<1$. The two series obtained are expansions of

$$\log\sqrt{\left|\frac{1+m}{1-m}\right|}$$

for $m>1$ and for $m<1$.

Summary.

If the integral

$$\int_a^b \mathrm{F}(x) \,.\, e^{-\alpha(x-c)}\, dx$$

can be evaluated in the form

$$\sum_{r=0}^{\infty} \mathrm{A}_r \alpha^r$$

for any suitable value of c, then the integral

$$\int_a^b f(x) \,.\, \mathrm{F}(x)\, dx$$

will normally possess an expansion in the form

$$\sum_{r=0}^{\infty} (-1)^r \mathrm{A}_r f^r(c).$$

LVI. *A Geometrical Illustration of a Principle of Experimental Directives.*

By J. M. Hammersley *.

[Received October 8, 1947.]

Summary.

In designing experiments, particularly when they are to be performed by other people, it is sometimes valuable to make use of geometrical models representing the fundamental issues of the investigations. This paper gives an example of such an approach as applied to a very simple " experiment," and shows how it can lead to a concise statement of the appropriate practical procedure.

Introduction.

In large-scale experimental research, such as is coming into prominence nowadays in industrial, civil and military fields, the shortage of adequately trained scientific staff and the extensiveness of practical experiments necessitate that a small number of trained scientists and mathematicians direct the work of a much larger number of ancillary research workers. Thus in the author's war-time experience a specialist staff of fifteen experimental officers (including only three mathematicians) assisted by an ancillary staff of over a hundred had to deal in four years with nearly 500 military experiments each ranging in duration from about two weeks to two years. In such cases the policy, planning, and statistical design of the experiments together with the interpretation of their results resides with the specialist nucleus ; but the main burden of the practical work, the recording and collation of data, preparation of equipment, photography, evaluation, analysis, and computing must as a rule be delegated to the ancillary staff. There are very obvious dangers and drawbacks to this arrangement ; and indeed it is unlikely to be successful even in favourable cases unless the trained directing staff issue experimental directives, laying down in considerable detail the experimental procedure to be adopted and guarding against vitiating contingencies. It is a cardinal principle, to be observed when writing such directives, that the actual experimental procedure shall as far as possible be given in a cut-and-dried form, free from qualifying exceptions and associated alternatives, and leading to an experimental result that can be stated in unequivocal terms. The nature of the experiment will usually determine when and by what means it is possible to achieve this object. Rather than attempt to generalize, the following very simple example will illustrate the principle. It is, however, interesting to note that here, as in many other cases, the use of geometrical analogues can throw considerable light on the essential issues of an experiment and lead to a concise combinatorial formulation of the appropriate experimental procedure.

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* Communicated by the Author.

Statement of a Simple Problem and its Popular Solution.

There is a well-known " parlour problem " which runs as follows :—

> " Of 12 given pennies it is known that 11 are genuine and 1 is counterfeit. The genuine pennies are each of the same (correct) weight, but the counterfeit is of incorrect weight. This difference in weight is the only available means of detecting the counterfeit. You are required to detect the counterfeit and to state whether it is too heavy or too light, using a balance but no weights for the scale-pans except the pennies themselves. How may this be done in three weighings ? "

This problem is, of course, very easy to solve ; but it is remarkable how many people give a solution of the following type :—

> " I should place pennies Nos. 1, 2, 3 and 4 in the left-hand pan and pennies Nos. 5, 6, 7, 8 in the right-hand pan for the first weighing. If these pennies balanced, I should then for the second weighing place pennies 6, 7, 8 in the left-hand pan and pennies 9, 10, 11 in the right-hand pan. If at the second weighing the pennies balanced, I should finally weigh penny 12 (the counterfeit according to the above results) against penny 11 to see whether penny 12 was too heavy or too light. But, if at the second weighing the left-hand pan descended, I should know that the counterfeit was too light and that it was one of pennies 9, 10, or 11 : and I should then decide which of these three pennies was the counterfeit by weighing penny 9 against penny 10. Again, if at the second weighing the right-hand pan descended, I should know that the counterfeit was too heavy ; and once more for the third weighing I should weigh penny 9 against penny 10 to decide which of pennies 9, 10, or 11 was too heavy. On the other hand, if at the original first weighing the left-hand pan descended, I should then for the second weighing place, etc., etc. . . ."

The reader who cares to complete the foregoing method of solution will find that it contains quite a fair number of " if's " and " and's " and " but's " ; and it is clear that, were we required to find in seven weighings the single counterfeit from amongst a thousand pennies, this method would be quite impracticable.

Combinatorial Solution.

For the case of 12 pennies consider instead the following cut-and-dried experimental directive :—

" Place the pennies :—

1st weighing : left-hand pan, Nos. 1, 5, 8, 10 ;
right-hand pan, Nos. 2, 6, 9, 12.
2nd weighing : left-hand pan, Nos. 2, 4, 8, 11 ;
right-hand pan, Nos. 3, 5, 9, 10.
3rd weighing : left-hand pan, Nos. 2, 5, 7, 12 ;
right-hand pan, Nos. 3, 6, 8, 11.

Record the results of the three weighings by entering one cross on each line of the proforma :—

Weighing	Left-hand pan descends	Right-hand pan descends	Balance
1st			
2nd			
3rd			

and submit the proforma for interpretation."

To see how this directive was constructed and how the entry on the proforma will immediately indicate the required solution of the problem, consider the matrix

$$\begin{bmatrix} +1 & -1 & 0 & 0 & +1 & -1 & 0 & +1 & -1 & +1 & 0 & -1 \\ 0 & +1 & -1 & +1 & -1 & 0 & 0 & +1 & -1 & -1 & +1 & 0 \\ 0 & +1 & -1 & 0 & +1 & -1 & +1 & -1 & 0 & 0 & -1 & +1 \end{bmatrix} \quad \ldots\ldots \quad (i)$$

This matrix possesses the following properties :—

(*a*) each element is either +1, or 0, or −1 ;
(*b*) no two columns are identical, nor is any column with the signs of its elements reversed identical with any other column ;
(*c*) no column consists entirely of zeros ;
(*d*) the sum of the elements in each row is zero ;
(*e*) the above four properties are true of the sub-matrices formed by omitting the last $3k$ columns of the complete matrix, where k is any integer such that any such sub-matrix has at least three columns *.

Comparing the directive with the matrix we see that at the ith weighing the jth penny has been placed in the left-hand, the right-hand, or neither pan according as the (i, j) element of the matrix is +1, −1, or 0. Remembering that we are working from the matrix to the directive in writing the latter, we see that property (*a*) ensures that every element of the matrix can be interpreted by the above rule. Property (c) ensures that no penny goes unweighed; for if the counterfeit were to go unweighed, it would be impossible to know whether it were too heavy or too light. Property (d) ensures that an equal number of pennies are placed upon each side of the balance at each weighing; so that any unbalance can only be due to the incorrect weight of the counterfeit. Let the entry on the proforma be translated as a column matrix whose ith element is $+1$, -1, or 0 according as the cross in the ith row of the proforma falls in the first, second, or third column. Then it is obvious that if this column matrix is identical with the jth column of (i), the jth penny is too heavy; while if this column matrix is identical to the jth column of (i) with its sign reversed, the jth penny is too light. Moreover, a correspondence of this kind must exist, because there is one and only one counterfeit coin present; and also this correspondence (and hence the resulting solution) is unique because of property (b).

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* The reason for stating (*e*) in these general terms will become apparent when considering the generalized problem.

Generalization of the Problem and its Geometrical Solution.

Before discussing the origin of the matrix (i), let us consider the more general problem where there is one counterfeit coin amongst a total of n coins, and it is required to solve the problem as before in the minimum number of weighings.

Firstly, we note that the problem is insoluble for $n=2$. For $n=3$, two weighings suffice corresponding to the matrix

$$\begin{pmatrix} +1 & 0 & -1 \\ -1 & +1 & 0 \end{pmatrix}.$$

The case $n=12$ has already been solved; and by property (e) the cases $n=6$ and $n=9$ are solved by deleting from (i) the last six and the last three columns respectively. The cases $n=5$, 8, 11 follow in similar fashion from the matrix

$$\begin{bmatrix} 0 & +1 & -1 & +1 & -1 & +1 & 0 & -1 & 0 & +1 & -1 \\ -1 & 0 & +1 & -1 & +1 & 0 & +1 & -1 & 0 & +1 & -1 \\ -1 & -1 & 0 & +1 & +1 & 0 & -1 & +1 & +1 & 0 & -1 \end{bmatrix}, \qquad \ldots\ldots \text{(ii)}$$

and the cases $n=4$, 7, 10 from the matrix

$$\begin{bmatrix} 0 & -1 & 0 & +1 & -1 & 0 & +1 & 0 & +1 & -1 \\ +1 & -1 & +1 & -1 & 0 & 0 & 0 & +1 & 0 & -1 \\ +1 & 0 & -1 & 0 & 0 & +1 & -1 & 0 & +1 & -1 \end{bmatrix}, \qquad \ldots\ldots \text{(iii)}$$

since both matrices (ii) and (iii) possess properties (a) to (e) inclusive.

Having disposed of the special cases $3 \leqslant n \leqslant 12$, consider the aggregate of points in the $(m-1)$-dimensional space W of a finite geometry with modulus 3. The modulus is chosen to be 3 because the problem concerns three alternatives in two ways: firstly, the result of a weighing may be unbalance to the left, or unbalance to the right, or balance; and, secondly, any coin may be *a priori* too heavy, or too light, or of correct weight.

The homogeneous coordinates of a point P in W may be represented by a column matrix **P**, each of whose elements is either $+1$, or 0, or -1. There are altogether $\frac{1}{2}(3^m-1)$ distinct points in W; and if the corresponding column matrices are written side by side to form an $m\times\frac{1}{2}(3^m-1)$ matrix **A**, then it is clear that **A** is the largest possible matrix of m rows which possesses properties (a), (b), and (c). Moreover, all the m-rowed matrices possessing these three properties are sub-matrices of **A** or the matrices obtained from **A** by reversing the signs of some columns. However, each row of **A** contains 3^{m-1} non-zero elements, and hence **A** cannot possess property (d). To obtain a matrix satisfying properties (d) and (e), we note that the points of W lie in sets of four on the lines of W. In particular, apart from the unit point U, all the points of W lie in sets of three on the $\frac{1}{2}(3^{m-1}-1)$ lines which pass through U. If P is any point other than U, then the two other points on the line PU have coordinates $(\mathbf{P}+\mathbf{U})$ and $(\mathbf{P}-\mathbf{U})$, summation here being with modulus 3 and the sums being expressed as either $+1$, or 0, or -1. To see that $\mathbf{P}+(\mathbf{P}+\mathbf{U})+(\mathbf{P}-\mathbf{U})=\mathbf{0}$, it is not sufficient to remark that $3\mathbf{P}\equiv\mathbf{0}$ (mod 3), since there is a (1, 2) correspondence between points and the results of weighings. However, since the elements of **P** are either $+1$, or 0, or -1, a check of the validity of the equation $\mathbf{P}+(\mathbf{P}+\mathbf{U})+(\mathbf{P}-\mathbf{U})=\mathbf{0}$ for the particular case $\mathbf{P}=\{+1, 0, -1\}$, $\mathbf{U}=\{+1, +1, +1\}$ is sufficient to prove the general case. Now generate a sequence of points $\{P_r\}$, of which P_1 is any point distinct from U, and $\mathbf{P}_2=(\mathbf{P}_1+\mathbf{U})$, and $\mathbf{P}_3=(\mathbf{P}_1-\mathbf{U})$; and in general for $k=1, 2, \ldots, \frac{1}{2}(3^{m-1}-3)$ P_{3k+1} is chosen to be any point distinct from U and P_r $(r=1, 2, \ldots, 3k)$ and $\mathbf{P}_{3k+2}=(\mathbf{P}_{3k+1}+\mathbf{U})$ and $\mathbf{P}_{3k+3}=(\mathbf{P}_{3k+1}-\mathbf{U})$. Then it follows that the matrix

$$\mathbf{B}=(\mathbf{P}_1,\ \mathbf{P}_2,\ \ldots,\ \mathbf{P}_{(3^m-3)/2}),$$

possesses properties (a) to (e) inclusive. It is clear that no m-rowed matrix larger than **B** can possess the five stated properties, though in this respect the class of **B**-matrices which may be generated from the class of sequences $\{P_r\}$ is not unique. **B**, however, comprises one possible enumeration of the points of W excluding U, arranged in triads on the aggregate of lines through U and so signed that they satisfy property (e). The generation of $\{P_r\}$ is one of the " automatic " methods of achieving this enumeration.

It is evident that **B**, together with its sub-matrices (formed according to the rule of property (e)), will solve the cases $\frac{1}{2}(3^{m-1}-3)<n\equiv0$ (mod 3) $\leqslant\frac{1}{2}(3^m-3)$. The cases $\frac{1}{2}(3^{m-1}-3)<n\equiv-1$(mod 3)$<\frac{1}{2}(3^m-3)$ can be solved by forming (in the same way as before) sequences $\{Q_r\}$ from the aggregate of points included neither in the first eleven columns of the matrix

$$\mathfrak{C}=\left[\begin{array}{ccccccccccc|l} 0 & +1 & -1 & +1 & -1 & +1 & 0 & -1 & 0 & +1 & -1 & \\ -1 & 0 & +1 & -1 & +1 & 0 & +1 & -1 & 0 & +1 & -1 & \\ -1 & -1 & 0 & +1 & +1 & 0 & -1 & +1 & +1 & 0 & -1 & \mathbf{Q}_1, \mathbf{Q}_2 \ldots, \mathbf{Q}_{(3^m-3^2)/2} \\ \mathbf{o} & +\mathbf{u} & -\mathbf{u} & +\mathbf{u} & -\mathbf{u} & +\mathbf{u} & \mathbf{o} & -\mathbf{u} & \mathbf{o} & +\mathbf{u} & -\mathbf{u} & \end{array}\right]$$

nor the two points $\mathbf{V}_1=\{0, +1, 0, \mathbf{o}'\}$, $\mathbf{V}_2=\{+1, 0, +1, +\mathbf{u}'\}$, where $\mathbf{o}$ and $\mathbf{u}$ are column matrices of $(m-3)$ rows, the former consisting entirely of zeros and the latter entirely of elements equal to $+1$. That it is possible to form such a sequence $\{Q_r\}$ follows from the fact that if $\mathbf{C}_j$ is the jth column of $\mathbf{C}$, the triads of points (C_1, C_5, C_6), (C_2, C_3, C_7), (C_8, C_9, C_{10}), (C_4, V_1, V_2) are collinear with U. Although $\mathbf{C}$ is not unique, it suffices together with its sub-matrices to solve the cases stated. Finally, we may form sequences $\{R_r\}$ from the aggregate of points included in the first ten columns of the matrix

$$\mathbf{D}=\left[\begin{array}{cccccccccc|c} 0 & -1 & 0 & +1 & -1 & 0 & +1 & 0 & +1 & -1 & \\ +1 & -1 & +1 & -1 & 0 & 0 & 0 & +1 & 0 & -1 & \\ +1 & 0 & -1 & 0 & 0 & +1 & -1 & 0 & +1 & -1 & \mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_{(3^m-3^3)/2} \\ +\mathbf{u} & \mathbf{o} & -\mathbf{u} & \mathbf{o} & \mathbf{o} & +\mathbf{u} & -\mathbf{u} & \mathbf{o} & +\mathbf{u} & -\mathbf{u} & \end{array}\right]$$

nor the points $\mathbf{V}_3=\{-1, +1, +1, +\mathbf{u}'\}$, $\mathbf{V}_4=\{+1, +1, -1, -\mathbf{u}'\}$, $\mathbf{V}_5=\{+1, -1, +1, +\mathbf{u}'\}$; since the triads of points (D_1, D_5, V_3), (D_2, D_6, V_4), (D_8, D_9, V_5), (D_3, D_4, D_7) are collinear with U. $\mathbf{D}$ is not unique, but suffices with its sub-matrices to solve the cases

$$\tfrac{1}{2}(3^{m-1}-3)<n\equiv+1 \pmod 3)<\tfrac{1}{2}(3^m-3).$$

To sum up these results we have shown how to solve all the cases $\frac{1}{2}(3^{m-1}-3)<n\leqslant\frac{1}{2}(3^m-3)$. When n satisfies this inequality, m weighings are also necessary. For each weighing can only give one of three possible results; and hence m weighings provides a population of at most 3^m-1 distinct results, since balance at every weighing will not solve the problem. Also for n coins, there are *a priori* $2n$ possible conclusions, because any one coin may be a heavy or a light counterfeit. Hence we must have $3^m-1\geqslant 2n$. Moreover, the case $3^m-1=2n$ must be excluded; since this would imply solution by means of an $\mathbf{A}$-matrix, which does not possess property (d) *. Since 3^m-2 is odd, $\frac{1}{2}(3^m-3)\geqslant n$: so that, given n coins (where $n>2$), it is necessary and sufficient to make m weighings, where m is the smallest integer not less than $\log_3(2n+3)$.

Conclusion.

This paper has been confined to the discussion of an almost trivial example of a problem in experimental design; because to employ an example of actual practical occurrence would call for too lengthy a description of experimental conditions, and also because a mass of special features germane to a real experiment might tend to obscure the underlying principle discussed. The example given here in the text lacks an element of reality in that it would be much simpler in practice to place any one penny in the left-hand pan, and then weigh the remaining pennies

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* It is, however, interesting to note that if we had, in addition to $\frac{1}{2}(3^m-1)$ untested coins, one specified coin known to be genuine, the solution by an $\mathbf{A}$-matrix would be possible. For instance given 14 coins, of which the 14th is known to be genuine, three weighings will detect the counterfeit and determine whether it is heavy or light. The method of doing this by adding an extra column $\{+1, +1, +1, +\mathbf{u}'\}$ to $\mathbf{A}$ is obvious.

(in the right-hand pan) individually against this arbitrary standard. At least two and at most eleven weighings (with an expectation of $5\frac{3}{4}$ weighings) would be necessary in this straightforward approach, on which no time has to be spent in considering combinatorial possibilities. The combinatorial solution would, however, be worth while if the weighings were difficult and lengthy practical jobs. In many research projects the labour of practical work is indeed a militant factor ; and then a few hours study of the theoretical structure of an experimental procedure and the fundamentals of the issues at stake may save weeks or months of practical work. Not only this, but also consideration of a geometrical analogue sometimes enables one to enumerate the totality of possible experimental procedures *, and thus select a treatment for use either according to further desiderata or perhaps by randomization.

Methods of the foregoing kind may also prove useful in preparing programme coding instructions for electronic computors.

LVII. *On an Identity involving E-functions.*

By T. M. MacRobert,

Professor of Mathematics, University of Glasgow †.

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§ 1. *Introductory.*

THE E-functions were defined, in the 'Proceedings of the Royal Society of Edinburgh,' lviii. p. 3 (1937) and in the 'Philosophical Magazine,' ser. 7, xxxi. p. 254 (1941), by the formulæ

$$E(p\,;\ \alpha\ :\ q\,;\ \rho_s : z) \equiv \Gamma(\alpha_p)\left\{\prod_{n=1}^{q}(\rho_n-\alpha_n)\right\}^{-1}$$

$$\times\prod_{n=1}^{q}\int_0^1 \lambda_n^{\alpha_n-1}(1-\lambda_n)^{\rho_n-\alpha_n-1}d\lambda_n \prod_{n=q+1}^{p-2}\int_0^\infty e^{-\lambda_n}\lambda_n^{\alpha_n-1}d\lambda_n$$

$$\times\int_0^\infty e^{-\lambda_{p-1}}\lambda_{p-1}^{\alpha_{p-1}-1}(1+\lambda_1\lambda_2\ldots\lambda_{p-1}/z)^{-\alpha_p}d\lambda_{p-1}, \quad \ldots\ldots \quad (1)$$

where $p \geqq q+1$, $|\mathrm{amp}\, z|<\pi$, $R(\alpha_n)>0$, $n=1, 2, \ldots, p-1$, and $(R\rho_n-\alpha_n)>0$, $n=1, 2, \ldots, q$; and ‡

$$E(p\,;\ \alpha_r : q\,;\ \rho_s : z) \equiv \frac{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\ldots\Gamma(\alpha_p)}{\Gamma(\rho_1)\,\Gamma(\rho_2)\ldots\Gamma(\rho_q)} F(p\,;\ \alpha_r : q\,;\ \rho_s : -1/z), \quad \ldots \quad (2)$$

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* Thus in the example cited for 12 pennies there are 152 distinct **B**-matrices quite apart from permutations in the order of rows and columns ; from amongst which matrices it might be desirable to select, for instance, a solution in which there was a minimum shifting of pennies from pan to pan between weighings.

† Communicated by the Author.

‡ $F(p\,;\ \alpha\ :\ q\,;\ \rho_s : z)$ is an alternative notation for the function ${}_pF_q(\alpha_r\,;\ \rho_s\,;\ z)$.

where $p \leqq q$, $|z| \neq 0$. If $p=q+1$, the latter formula holds provided that $|z|>1$.

In § 2 an identical relation between E-functions will be established; and in § 3 multiple integrals involving modified Bessel Functions will be evaluated by means of this relation. Some further identities for simple E-functions and Kummer Functions and also for modified Bessel Functions are deduced in § 4.

§ 2. *An Identity involving E-functions.*

The formula to be established is

$$\sum_{r=0}^{n} {}^{n}C_{r}(\alpha_1-n+r\,;\, n-r)(\alpha_1-\alpha_2-n\,;\, r)z^{-r}$$

$$\times \mathrm{E}(\alpha_1-n+r, \alpha_2+n, \alpha_3+r, \alpha_4+r, \ldots, \alpha_p+r : q\,;\, \rho_s+r : z)$$

$$=(\alpha_2\,;\, n)\mathrm{E}(p\,;\, \alpha_r : q\,;\, \rho_s : z), \quad . \quad . \quad . \quad (3)$$

where n, p, q are any positive integers (or zero) and

$$(k\,;\, r) \equiv k(k+1)(k+2)\ldots(k+r-1), \quad r=1, 2, 3, \ldots, \quad . \quad . \quad . \quad (4)$$

$$(k\,;\, 0) \equiv 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4')$$

To prove it, start with the formulæ (Phil. Mag. ser. 7, xxxi. p. 260 1941))

$$\mathrm{E}(p\,;\, \alpha_r : q\,;\, \rho_s : z) = \frac{1}{\alpha_1}\mathrm{E}(\alpha_1+1, \alpha_2, \ldots, \alpha_p : q\,;\, \rho_s : z)$$

$$+\frac{1}{\alpha_1 z}\mathrm{E}(p\,;\, \alpha_r+1 : q\,;\, \rho_s+1 : z), \quad . \quad . \quad . \quad . \quad (5)$$

$$(\alpha_1-\alpha_2)\,\mathrm{E}(p\,;\, \alpha_r : q\,;\, \rho_s : z) = \mathrm{E}(\alpha_1+1, \alpha_2, \ldots, \alpha_p : q\,;\, \rho_s : z)$$

$$-\mathrm{E}(\alpha_1, \alpha_2+1, \alpha_3, \ldots, \alpha_p : q\,;\, \rho_s : z), \quad . \quad . \quad . \quad (6)$$

where p and q are any positive integers (or zero). Multiply (5) by $\alpha_1-\alpha_2$, subtract (6) and multiply by α_1; thus

$$\alpha_2\mathrm{E}(\alpha_1+1, \alpha_2, \ldots, \alpha_p : q\,;\, \rho_s : z) - \alpha_1\mathrm{E}(\alpha_1, \alpha_2+1, \alpha_3, \ldots, \alpha_p : q\,;\, \rho_s : z)$$

$$=\frac{\alpha_1-\alpha_2}{z}\mathrm{E}(p\,;\, \alpha_r+1 : q\,;\, \rho_s+1 : z). \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Here replace α_1 by α_1-1, and get

$$(\alpha_1-1)\mathrm{E}(\alpha_1-1, \alpha_2+1, \alpha_3, \ldots, \alpha_p : q\,;\, \rho_s : z)$$

$$+\frac{\alpha_1-\alpha_2-1}{z}\mathrm{E}(\alpha_1, \alpha_2+1, \ldots, \alpha_p+1 : q\,;\, \rho_s+1 : z)$$

$$=\alpha_2\mathrm{E}(p\,;\, \alpha_r : q\,;\, \rho_s : z), \quad . \quad . \quad . \quad (8)$$

which is formula (3) when $n=1$.

Now assume (3) for a particular value of n, multiply by α_2+n and apply (8) to each term on the left-hand side. This then becomes

$$\sum_{r=0}^{n} {}^{n}C_r(\alpha_1-n+r\,;\,n-r)(\alpha_1-\alpha_2-n\,;\,r)z^{-r}$$
$$\times\left\{\begin{array}{l}(\alpha_1-n+r-1)E(\alpha_1-n+r-1,\ \alpha_2+n+1,\ \alpha_3+r,\ \ldots,\\ \qquad\qquad\qquad\qquad\alpha_p+r:q\,;\,\rho_s+r:z)\\ +(\alpha_1-\alpha_2-2n+r-1)z^{-1}E(\alpha_1-n+r,\ \alpha_2+n+1,\ \alpha_3+r+1,\ \ldots,\\ \qquad\qquad\qquad\qquad\alpha_p+r+1:q\,;\,\rho_s+r+1:z).\end{array}\right\}.$$

But

$${}^{n}C_r(\alpha_1-\alpha_2-2n+r-1)+{}^{n}C_{r+1}(\alpha_1-\alpha_2-n+r)={}^{n+1}C_{r+1}(\alpha_1-\alpha_2-n-1).$$

Therefore (3) holds with $n+1$ in place of n. It holds, however, when $n=1$; hence it holds for all positive integral values of n.

Note.—As (3) holds for all positive integral values of p and q, it follows, by applying (2) and replacing z by $-1/z$, that

$$\sum_{r=0}^{n}(-1)^r\,{}^{n}C_r(\alpha_1-\alpha_2-n\,;\,r)\,\frac{(\alpha_3\,;\,r)\ldots(\alpha_p\,;\,r)}{(\rho_1\,;\,r)\ldots(\rho_q\,;\,r)}\,z^r$$
$$\times F(\alpha_1-n+r,\ \alpha_2+n,\ \alpha_3+r,\ \ldots,\ \alpha_p+r:q\,;\,\rho_s+r:z)$$
$$=F(p\,;\,\alpha_r:q\,;\,\rho_s:z),\qquad\ldots\ldots\ldots\quad(9)$$

where $p<q+1$.

§ 3. *Multiple Integrals involving Modified Bessel Functions.*

The integral formula

$$\prod_{n=1}^{q}\int_0^1\lambda_n^{\alpha_n-1}(1-\lambda_n)^{\rho_n-\alpha_n-1}d\lambda_n\prod_{n=q+1}^{p-2}\int_0^\infty e^{-\lambda_n}\lambda_n^{\alpha_n-1}\,d\lambda_n$$
$$\times\int_0^\infty\left[\begin{array}{r}e^{-(a\lambda_1\ldots\lambda_{p-2}+b)\lambda}\,\lambda^{\alpha-1}(1+\lambda\lambda_1\ldots\lambda_{p-2})^{-\alpha}\\ \times I_{2m}[2\surd\{ab\lambda(1+\lambda\lambda_1\ldots\lambda_{p-2})\}]\end{array}\right]d\lambda$$
$$=\frac{a^m\prod\limits_{n=1}^{q}\Gamma(\rho_n-\alpha_n)}{b^\alpha\Gamma(\alpha-m)\Gamma(2m+1)}E(\alpha+m,\ \alpha-m,\ \alpha_1,\ \ldots,\ \alpha_{p-2}:q\,;\,\rho_s:b)\,F\left({\alpha+m\,;\,a \atop 2m+1}\right),$$
$$\ldots\quad(10)$$

where $R(a)>0$, $R(b)>0$, $p\geqq q+2$, $R(\alpha_n)>0$ for $n=1,\ 2,\ \ldots,\ p-2$, $R(\rho_n-\alpha_n)>0$ for $n=1,\ 2,\ \ldots,\ q$, $R(\alpha+m)>0$, will now be established.

First assume that $p=q+2$, $m>-\frac{1}{2}$, $R(b)>|a|$. Expand* $\exp(-a\lambda\lambda_1\ldots\lambda_q)$ and I_{2m} in series; then the integrand in the final integral becomes

$$(ab)^m e^{-b\lambda}\lambda^{\alpha+m-1}(1+\lambda\lambda_1\ldots\lambda_q)^{-\alpha+m}$$
$$\times\sum_{r=0}^{\infty}\sum_{s=0}^{r}(-1)^s\frac{(ab)^{r-s}\{\lambda(1+\lambda\lambda_1\ldots\lambda_q)\}^{r-s}(a\lambda\lambda_1\ldots\lambda_q)^s}{\Gamma(2m+r+1-s)\,.\,(r-s)\,!\,.\,s\,!},$$

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* *Cf.* Bromwich, 'Infinite Series,' Second Edition, p. 500.

and therefore the left-hand side of (10) is equal to

$$(ab)^m \prod_{n=1}^{q} \Gamma(\rho_n-\alpha_n) \sum_{r=0}^{\infty} \frac{(ab)^r}{r\,!}$$

$$\times \sum_{s=0}^{r} (-1)^s\, {}^r\mathrm{C}_s \frac{\mathrm{E}(\alpha-m-r+s,\ \alpha+m+r,\ \alpha_1+s,\ \ldots,\ \alpha_q+s : q\,;\ \rho_t+s : b)}{\Gamma(2m+r+1-s)\Gamma(\alpha-m-r+s)b^{\alpha+m+r+s}}$$

$$=\frac{a^m}{b^\alpha} \prod_{n=1}^{q} \Gamma(\rho_n-\alpha_n) \times \sum_{r=0}^{\infty} \frac{a^r(\alpha+m\,;\,r)\ \mathrm{E}(\alpha-m,\ \alpha+m,\ \alpha_1,\ \ldots,\ \alpha_q : q\,;\ \rho_t : b)}{r\,!\ \Gamma(2m+r+1)\ \Gamma(\alpha-m)},$$

by (3). From this (10), with $p=q+2$, follows, the restrictions $m>-\frac{1}{2}$, $\mathrm{R}(b)>|a|$ being now removed.

To extend the formula to the cases in which $p>q+2$, multiply (10), with $p=q+2$, by b^α, make the substitution

$$(a\lambda_1\lambda_2\ldots\lambda_q+b)\lambda=\mu,$$

and then apply the formula (Phil. Mag. ser. 7, xxxi. p. 255 (1941))

$$\int_0^\infty e^{-\lambda}\lambda^{\alpha_{p+1}-1}\mathrm{E}(p\,;\ \alpha_r : q\,;\ \rho_s : b/\lambda)d\lambda=\mathrm{E}(p+1\,;\ \alpha_r : q\,;\ \rho_s : b). \quad . \quad (11)$$

This process may be repeated as often as required.

Note.—When $p=2$, $q=0$, it is clear, from the asymptotic expansion of I_{2m}, that the integral does not converge if a and b are equal. The condition $\mathrm{R}(b)>|a|$ is therefore retained.

Again, in (10) replace m by $-m$, subtract (10) from the result and apply the formulæ

$$\frac{\pi}{2\sin m\pi}\left\{\mathrm{I}_{-m}(z)-\mathrm{I}_m(z)\right\}=\mathrm{K}_m(z), \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$\sum_{m,-m} \Gamma(2m)\Gamma(\alpha-m)z^{\alpha-m}\mathrm{F}(\alpha-m\,;\ 1-2m\,;\ z)=\mathrm{E}(\alpha-m,\ \alpha+m : : z). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Then

$$\prod_{n=1}^{q}\int_0^1 \lambda_n^{\alpha_n-1}(1-\lambda_n)^{\rho_n-\alpha_n-1}\,d\lambda_n \prod_{n=q+1}^{p-2}\int_0^\infty e^{-\lambda_n}\lambda_n^{\alpha_n-1}\,d\lambda_n$$

$$\times\int_0^\infty\left[\begin{matrix} e^{-(a\lambda_1\ldots\lambda_{p-2}+b)\lambda}\ \lambda^{\alpha-1}(1+\lambda\lambda_1\ldots\lambda_{p-2})^{-\alpha} \\ \times\mathrm{K}_{2m}[2\surd\{ab\lambda(1+\lambda\lambda_1\ldots\lambda_{p-2})\}]\end{matrix}\right]d\lambda$$

$$=\frac{\prod_{n=1}^{q}\Gamma(\rho_n-\alpha_n)\ \mathrm{E}(\alpha-m,\ \alpha+m : : a)}{2(ab)^\alpha\Gamma(\alpha-m)\ \Gamma(\alpha+m)}\mathrm{E}(\alpha-m,\alpha+m,\alpha_1,\ldots,\alpha_{p-2} : q;\ \rho_s : b), \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where $\mathrm{R}(a)>0$, $\mathrm{R}(b)>0$, $p\geqq q+2$, $\mathrm{R}(\alpha_n)>0$ for $n=1,\ 2,\ \ldots,\ p-2$, $\mathrm{R}(\rho_n-\alpha_n)>0$ for $n=1,\ 2,\ \ldots,\ q$, and $\mathrm{R}(\alpha\pm m)>0$.

Special cases of (10) and (14) were given by W. T. Howell (Phil. Mag. ser. 7, xxviii. p. 495 (1939)); for example, see (15) below.

§ 4. *Further Identities.*

When $p=2$, $q=0$, formula (10) becomes

$$\int_0^\infty e^{-(a+b)\lambda}\lambda^{\alpha-1}\mathrm{I}_{2m}[2\sqrt{\{ab\lambda(1+\lambda)\}}](1+\lambda)^{-\alpha}\,d\lambda$$
$$=\frac{a^m b^{-\alpha}}{\Gamma(\alpha-m)\Gamma(2m+1)}\,\mathrm{E}(\alpha+m,\,\alpha-m::b)\,\mathrm{F}\binom{\alpha+m;\ a}{2m+1}, \qquad (15)$$

where $\mathrm{R}(b)>|a|$, $\mathrm{R}(\alpha+m)>0$.

Assume that $m>-\frac{1}{2}$, expand I_{2m} and apply formula (1); the left-hand side of the equation then becomes

$$\sum_{r=0}^{\infty}\frac{(ab)^{m+r}\mathrm{E}(\alpha+m+r,\,\alpha-m-r::a+b)}{\Gamma(2m+1+r)\,.\,r!\,\Gamma(\alpha-m-r)(a+b)^{\alpha+m+r}}$$
$$=\frac{(ab)^m(a+b)^{-\alpha-m}}{\Gamma(2m+1)\,\Gamma(\alpha-m)}\sum_{r=0}^{\infty}\left(\frac{-ab}{a+b}\right)^r\frac{(1-\alpha+m\,;r)}{(2m+1\,;r)\,.\,r!}$$
$$\times\,\mathrm{E}(\alpha+m+r,\,\alpha-m-r::a+b).$$

Thus, if $\mathrm{R}(b)>|a|$,

$$\mathrm{F}(\alpha+m\,;2m+1\,;a)\,\mathrm{E}(\alpha+m,\,\alpha-m::b)=\left(\frac{b}{a+b}\right)^{\alpha+m}$$
$$\times\sum_{r=0}^{\infty}\frac{(1-\alpha+m\,;r)}{(2m+1\,;r)\,.\,r!}\left(\frac{-ab}{a+b}\right)^r\mathrm{E}(\alpha+m+r,\,\alpha-m-r::a+b). \qquad (16)$$

On applying the substitution $(a+b)\lambda=\mu$ to the integral on the left of (15) it becomes possible, by revolving the μ-axis through angles between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, to give amp b any value between $-\frac{3}{2}\pi$ and $\frac{3}{2}\pi$. If, now, the formula

$$e^{i\pi\alpha}\,\mathrm{E}(1-\alpha,\,\rho-\alpha::ze^{i\pi})-e^{-i\pi\alpha}\,\mathrm{E}(1-\alpha,\,\rho-\alpha::ze^{-i\pi})$$
$$=2i\pi\Gamma(\rho-\alpha)[\Gamma(\rho)]^{-1}e^{-z}z^{\rho-\alpha}\,\mathrm{F}(\alpha\,;\rho\,;z), \qquad (17)$$

with b in place of z, $1-\alpha+m$ in place of α and $1+2m$ in place of ρ is applied to the left-hand side of (16), and with $a+b$ in place of z, $1-\alpha+m+r$ in place of α and $1+2m+2r$ in place of ρ to the right-hand side, the formula

$$\mathrm{F}(\alpha+m\,;1+2m\,;-a)\,\mathrm{F}(1-\alpha+m\,;1+2m\,;b)$$
$$=\sum_{r=0}^{\infty}\frac{(1-\alpha+m\,;r)\,(\alpha+m\,;r)}{(2m+1\,;r)\,.\,r!\,(2m+1\,;2r)}\,e^{-a}(-ab)^r\,\mathrm{F}\binom{1-\alpha+m+r;\ a+b}{1+2m+2r},$$

is obtained. From this, on applying the identity

$$\mathrm{F}(\alpha\,;\rho\,;z)=e^z\mathrm{F}(\rho-\alpha\,;\rho\,;-z), \qquad (18)$$

and then replacing α by $1-\alpha$, it follows that

$$\mathrm{F}(\alpha+m\,;1+2m\,;a)\,\mathrm{F}(\alpha+m\,;1+2m\,;b)$$
$$=\sum_{r=0}^{\infty}\frac{(\alpha+m\,;r)\,(1-\alpha+m\,;r)}{(1+2m\,;r)\,.\,r!\,(1+2m\,;2r)}(-ab)^r\,\mathrm{F}\binom{\alpha+m+r;\ a+b}{1+2m+2r}, \qquad (19)$$

where m is not a negative integer or half a negative odd integer. [*Cf.* Burchnall and Chaundy, Quart. Jour. xii. p. 121 (1941)].

On again applying (17) in the same way to (16), but with $-m$ in place of m and $-m-r$ in place of $m+r$, it is found that

$$F(\alpha+m\,;\,1+2m\,;\,a)\,F(\alpha-m\,;\,1-2m\,;\,b)$$

$$=\left(\frac{b}{a+b}\right)^{2m}\sum_{r=0}^{\infty}\frac{(m\,;\,r)\,(m+\frac{1}{2}\,;\,r)}{(2m+1\,;\,r)\,.\,r\,!}\left\{\frac{2\sqrt{(ab)}}{a+b}\right\}^{2r}F\left(\begin{matrix}\alpha-m-r\,;\;a+b\\ 1-2m-2r\end{matrix}\right), \quad \ldots \quad (20)$$

where $R(b)>|a|$ and m is not an integer or half an odd integer.

Finally, in (19), (20) and (16) replace a and b by $2a$ and $2b$, put $\alpha=\frac{1}{2}$ and apply the formulæ

$$F(\tfrac{1}{2}+n\,;\,1+2n\,;\,2z)=\Gamma(n+1)(\tfrac{1}{2}z)^{-n}e^{z}I_n(z), \quad \ldots \quad (21)$$

$$E(\tfrac{1}{2}+n,\,\tfrac{1}{2}-n\,::\,2z)=\sec\,(n\pi)\,.\,\sqrt{(2\pi z)}e^{z}K_n(z). \quad \ldots \quad (22)$$

Then it is found that

$$I_m(a)I_m(b)=\frac{1}{\Gamma(m+1)}\left(\frac{\frac{1}{2}ab}{a+b}\right)^{m}$$

$$\times\sum_{r=0}^{\infty}\frac{(\frac{1}{2}+m\,;\,r)}{(1+2m\,;\,r)\,.\,r\,!}\left(\frac{-2ab}{a+b}\right)^{r}I_{m+r}(a+b), \quad \ldots \quad (23)$$

$$I_m(a)I_{-m}(b)=\frac{1}{\Gamma(m+1)}\left(\frac{\frac{1}{2}ab}{a+b}\right)^{m}$$

$$\times\sum_{r=0}^{\infty}\frac{(\frac{1}{2}+m\,;\,r)}{(1+2m\,;\,r)\,.\,r\,!}\left(\frac{-2ab}{a+b}\right)^{r}I_{-m-r}(a+b), \quad \ldots \quad (24)$$

where $R(b)>|a|$, and

$$I_m(a)K_m(b)=\frac{1}{\Gamma(m+1)}\left(\frac{\frac{1}{2}ab}{a+b}\right)^{m}$$

$$\times\sum_{r=0}^{\infty}\frac{(\frac{1}{2}+m\,;\,r)}{(1+2m\,;\,r)\,.\,r\,!}\left(\frac{2ab}{a+b}\right)^{r}K_{m+r}(a+b), \quad \ldots \quad (25)$$

where $R(b)>|a|$.

LVIII. *On the Record Registered on a Cylinder with Uniform Screw Motion.*

By C. G. Pendse, M.A., Ph.D.*

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Introduction.

A standard method of obtaining a continuous record of a measurable physical quantity varying with time, like a component of the ground motion, amounts ultimately to making a marking point remain in contact with paper wrapped round a right circular cylinder which has a screw motion along its axis. In the case of $\left\{\begin{matrix}\text{mechanical}\\ \text{optical}\end{matrix}\right\}$ registration, the marking point is $\left\{\begin{matrix}\text{the end of the stylus}\\ \text{a light speck}\end{matrix}\right\}$ and the paper wrapped round the cylinder is $\left\{\begin{matrix}\text{often smoked}\\ \text{photographic}\end{matrix}\right\}$. If the physical quantity in question remains constant, the recording point remains at rest relative to the frame of the recording instrument with respect to which the drum has its screw motion.

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* Communicated by the Author.

In course of work with seismographs it has been found necessary to study the effects on the appearance of the record when the speed of translation and the speed of rotation of the recording cylinder are changed so as to vary the " time scale " of seismograms. The present note, which consists of four sections, shows the connection between the radius and the translatory and rotational speeds of the recording cylinder and the appearance of the record when the chart (*i. e.*, the record-bearing paper) is flattened out.

§ 1 deals with the kinematics of the recording point and the recording cylinder. § 2 gives the analytical transformation applicable to the flattening of the chart on a plane. § 3 gives the details about the lines on the flattened-out paper and the relations between them. The effects of sudden changes in the screw motion on the appearance of the record are considered in § 4.

A special feature of the transformation used in § 2 is that it involves discontinuities ; these discontinuities are responsible for the breaking up of a continuous curve on the cylinder into a series of straight lines and/or curves when the record-bearing paper is flattened out. Without this transformation it is not possible to obtain the analytical representation of the flattening on a plane. The formulæ in §§ 3 and 4 would be useful for selecting the rotational and translatory speeds so as to suit requirements (including the spacing between the lines).

§ 1. *Kinematics of the Recording Point and the Recording Cylinder.*

Let F be the frame determined by the instrument.

Let P be the recording point in contact with the cylinder.

Let P_0 be the point, fixed relative to F and coincident with P when P is at rest relative to F.

Let $\mathbf{s}$ be the unit vector in one of the two directions determined by the axis of the cylinder.

Let $-V\mathbf{s}$ and $-\Omega\mathbf{s}$ be the translatory velocity and angular velocity respectively of the cylinder relative to F, at any instant.

Let a be the radius of the cylinder.

Let O be a point fixed relative to the cylinder and situated on its axis, and

let $Oxys$ be a right-handed Cartesian frame fixed relative to the cylinder and such that the positive direction of Os is parallel to $\mathbf{s}$, Ox and Oy being perpendicular to $\mathbf{s}$ and to each other.

Let x, y, s be the Cartesian coordinates and r, θ, s the cylindrical polar coordinates of a point relative to $Oxys$, so that $x=r\cos\theta$, $y=r\sin\theta$.

When the point P or P_0 describes any continuous path on the cylinder, θ will change continuously. In order to avoid a discontinuity in the coordinates, it is assumed that θ can increase or decrease without limit.

We consider the motion of P_0 relative to $Oxys$ and of P relative to F and $Oxys$.

P will be shifted from P_0 in F when the physical quantity has departed from its reference value, which corresponds to P_0. The nature of the trajectory of P in F, viz., rectilinear or curved, and the total permissible range for the motion of P in F, will depend on the construction and the constants (including the geometrical dimensions) of the instrument. The simplest and the most desirable path for P in F would be in a straight line parallel to Os. If, during any interval of time, the physical quantity either oscillates about the reference value or departs from it in one sense only, P will describe one path either repeatedly or without repetition. In the case of a repeated description of the path, P may describe either the whole of the total path or varying portions thereof on each occasion. Provided each of V and Ω never vanishes or changes its sign, the effect of the screw motion is that we obtain a record in the frame $Oxys$ of the motion of P in F such that only one point of the record corresponds to the position of P at an instant, though P may be in the same position relative to F at some other instant.

We now consider the paths of P_0 and P relative to $Oxys$.

Let the suffix "$_0$" be used to distinguish the θ and s coordinates of P_0 from those of P.

The velocity of P_0 relative to $Oxys$ is compounded of $V\mathbf{S}$ and $\Omega\overrightarrow{OP_0}\times\mathbf{S}$, so that for the motion of P_0 we have* $r=a$, $\dot{\theta}_0=\Omega$ and $\dot{s}_0=V$. The equations of the path of P_0 in $Oxys$ are

$$r=a,\quad \frac{\theta_0-\theta_0'}{\Omega}=\frac{s_0-s_0'}{V}=t,$$

where θ_0' and s_0' are constants, when V and Ω remain constant. $\theta-\theta_0$ and $s-s_0$ at any instant measure the deviation of P from P_0 in $Oxys$. The functional relationship between $\theta-\theta_0$ and $s-s_0$, if any, will be determined by the constants of the instrument. As indicated above, the simplest case is one in which θ always equals θ_0.

Let C and C_0 denote respectively the paths of P and P_0 in $Oxys$.

§ 2. *Transformation for the Flattening out of the Surface of the Cylinder in a Plane without Stretching or Contraction or Tearing* †.

When the chart is spread out into a plane sheet, the process is similar to the development of the surface of the cylinder on a tangent plane. We have to find a transformation from the r, θ, s system of coordinates to another system ξ, η, σ in one-one correspondence with the former one, such that one of the new coordinates, say ξ, will be constant over a plane and such that the length of any line element in the ξ, η, σ system equals the length of the corresponding line element in the, r θ, s system when $r=a$.

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* The overhead dot denotes differentiation with respect to t (time).

† This section is geometrical and the statements in it are independent of the motion of the recording cylinder in the physical problem.

Let $z=x+iy$ and $\zeta=\xi+i\eta$, where $i=\sqrt{-1}$. We consider the transformation

$$\zeta=a \log \frac{z}{a}=a \log \left|\frac{z}{a}\right|+ia \arg z, \text{ where } 0\leqslant \arg z<2\pi$$

and $\sigma=s$. The transformation from z to ζ is conformal at all points in the finite portion of the z-plane except at $z=0$. We can write

$$\xi=a \log \left|\frac{z}{a}\right|, \quad \eta=a \arg z, \quad \sigma=s.$$

In the ξ, η, σ system the cylinder $r=a$ is transformed into the rectangular strip $\xi=0$, $0\leqslant\eta<2\pi a$.

It should be noticed that the line $\xi=0$, $\eta=2\pi a$ is *not* a part of the representation of the cylinder $r=a$ in the ξ, η, σ system. The cylinder therefore corresponds to an infinite rectangular strip *minus* one edge in the ξ, η, σ system.

It is easy to verify that the length of the line element on the cylinder $r=a$ determined by two neighbouring points on it, *not* on the opposite sides of the barrier $\arg z=0$, is equal to the length of the corresponding line element in the ξ, η, σ system. Hence the length of any curve between two points on the cylinder $r=a$, not crossing the barrier $\arg z=0$, is equal to the length of the corresponding curve in the ξ, η, σ system.

If, in the case of the whole or a part of a continuous curve on the cylinder $r=a$, θ passes through 2π or a multiple of 2π, $\arg z$ will have a discontinuity of 2π and the whole or the part of the corresponding curve in the ξ, η, σ system will have a corresponding discontinuity in the η-co-ordinate ; as the point of discontinuity of $\arg z$ is approached, η will tend to $2\pi a$ and at the point of discontinuity η will vanish.

§ 3. *Paths of* P_0 *and* P *in the* ξ, η, σ *System when* V *and* Ω *remain constant.*

The equations of C_0 are

$$r=a, \quad \frac{\theta_0-\theta_0'}{\Omega}=\frac{s_0-s_0'}{V}=t,$$

where θ_0' and s_0' are constants, as mentioned in § 1. From the above equations it is easy to verify that C_0 is a circular helix on a cylinder of radius a, the tangent to the curve making an angle $\tan^{-1}\left(\frac{V}{a\Omega}\right)$ with the generators of the cylinder.

If $2n\pi\leqslant\theta<2(n+1)\pi$, where n is an integer,

$$\arg z=\theta-2n\pi,$$

i. e.

$$\eta=a(\theta-2n\pi).$$

Hence, taking into account § 2 above, the portion of C_0 for which $2n\pi \leqslant \theta < 2(n+1)\pi$ is transformed into the finite straight line $L_0^{(n)}$ defined by the relations

$$\xi=0, \quad 0 \leqslant \eta_0 < 2\pi a, \quad \sigma_0-\sigma_0'=\frac{V}{a\Omega}\{\eta_0+a(2n\pi-\theta_0')\}. \quad . \quad . \quad (1)$$

From (1) we find that the lines $L_0^{(n)}$, for different values of n, are parallel to each other, that any two successive lines of the set $L_0^{(n)}$ intercept the same length $\frac{2\pi V}{\Omega}$ on the σ-axis, for any value of n, and that, consequently, the distance between any two successive lines $L_0^{(n)}$ is independent of n.

Let α be the angle made by $L_0^{(n)}$ with the η-axis, let p be the perpendicular distance between any two lines $L_0^{(n)}$ and $L_0^{(n+1)}$, and let L_0 be the length of the line $L_0^{(n)}$. Also let $A_0^{(n)}$ and $B_0^{(n)}$ be points at which $L_0^{(n)}$ meets the lines $\eta=0$ and $\eta=2\pi a$, while $\xi=0$, respectively.

Then $$A_{n+1}A_n=\frac{2\pi V}{\Omega}=B_{n-1}B_n \quad \text{and}$$

$$p=\frac{2\pi V}{\Omega}\cos\alpha, \quad L_0=2\pi a \sec\alpha, \quad \text{where } \tan\alpha=\frac{V}{a\Omega}.$$

Consequently the straight line $A_{n+1}B_n$ is parallel to the σ-axis. On simplification we have

$$\tan\alpha=\frac{V}{a\Omega}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$p=\frac{2\pi}{\left(\frac{1}{a^2}+\frac{\Omega^2}{V^2}\right)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$L_0=2\pi\left(a^2+\frac{V^2}{\Omega^2}\right)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$pL_0=\frac{4\pi^2a^2V^2}{\Omega^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In the ξ, η, σ system, C will be split up, exactly as in the case of C_0, into a series of separate curves. Here also, as in the case of C_0, we denote by $L^{(n)}$ a typical member of the series for C.

During the interval of time for which the physical quantity does not depart from the reference value, P will coincide with P_0 and the relevant portions of C in the x, y, s and also the ξ, η, σ systems will coincide with the corresponding portions of C_0.

We now consider the case in which the physical quantity departs from the reference value for an interval of time.

If the construction of the instrument and its constants are such that $\theta=\theta_0$ throughout the motion, there will be a one-one correspondence between points of $L^{(n)}$ and $L_0^{(n)}$, the straight line joining any two corresponding points being parallel to the η-axis. When, however, θ does not equal θ_0, the end points of $L^{(n)}$ will not correspond to the end points of $L_0^{(n)}$, and judgment has to be used in measuring deviation of the physical quantity from the reference value.

In practice, when P coincides with P_0 for a sufficiently long interval of time, it is possible to know the orientation of the $L_0^{(n)}$ lines and the spacing between them, and even when the actual record does not contain all the $L_0^{(n)}$ lines, they can be drawn on the chart for purposes of comparison with the record obtained.

§ 4. *Effect of Changes in the Screw Motion.*

In this section we consider some subsidiary questions, such as changes in the paths of P and P_0 in the ξ, η, σ system when the translatory and rotational speeds of the drum are changed.

It is seen from (2), (3) and (4) that no change whatever will be discerned in the lines $L_0^{(n)}$ when the speeds are altered in such a way that $\frac{V}{\Omega}$ remains unchanged. In this case, the "time scale" will be altered without any change in the spacing of the record, so that, in any assigned interval of time, the amount of paper used up will increase or decrease according as V increases or decreases.

If, however, the ratio $\frac{V}{\Omega}$ is changed, α, p and L_0 will all change, the changed values being given by (2), (3) and (4).

Suppose, now, that at a stage in the interval $2n\pi < \theta < 2(n+1)\pi$, the speeds of translation and of rotation change from V' to V'' and from Ω' to Ω'' respectively. Let α' and α'' be the angles made by the paths of P_0 with the η-axis before and after the discontinuity.

Then $\tan\alpha' = \frac{V'}{a\Omega'}$, $\tan\alpha'' = \frac{V''}{a\Omega''}$, and the angle $(\alpha'' - \alpha')$ through which the path has turned is given by

$$\tan(\alpha'' - \alpha') = \frac{1}{a} \frac{\frac{V''}{\Omega''} - \frac{V'}{\Omega'}}{1 + \frac{V'V''}{a^2\Omega'\Omega''}},$$

i. e.,

$$\alpha'' - \alpha' = \tan^{-1}\left\{ \frac{1}{a} \frac{\frac{V''}{\Omega''} - \frac{V'}{\Omega'}}{1 + \frac{V'V''}{a^2\Omega'\Omega''}} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

(6) enables us to determine quantitatively the change in $\frac{V}{\Omega}$ from the record when the physical quantity does not depart from the reference value at the time of the discontinuity in $\frac{V}{\Omega}$.

The path of P in the ξ, η, σ system will also undergo corresponding changes when V and Ω change. But in the absence of the $L_0^{(n)}$ lines, either actual or calculated, it will be difficult to determine the changes accurately.

LIX. *Note on some Theoretical Discussions of Palladium-hydrogen and Cognate Systems.*

By Donald P. Smith*.

[Received February 24, 1947.]

Palladium-hydrogen has in recent years formed the subject of various discussions which seek to account for the hystereses and other characteristics of this system, and presumably of others of similar type, upon thermodynamical or statistical-mechanical grounds. Because of the interest which such questions possess in relation to many important properties of metallic bodies, it may not be amiss to point out that in certain instances the conclusions which are reached do not rest directly upon the basic principles from which they purport to be derived, but upon ancillary assumptions, explicit or tacit, which detract in no small measure from their significance.

As an example of the type of discussion concerned we may take that of Lacher (1), who endeavoured to explain the hysteresis of the pressure-concentration isotherms of palladium-hydrogen on a statistical-mechanical basis, but seems to have found it necessary to support his argument with at least two explicit assumptions.

The first is that, of the two solid phases α and β which occur together in the concentration range within which alone the hysteresis is found, the one phase exists finely dispersed through the other. This supposition is directly at variance with experimental evidence afforded by the diffusion study of Duhm (2), (3), and confirmed by the X-ray results of C. S. Barrett and the writer (4), to the effect that the two phases are completely separated by a single-phase boundary which migrates inward from the periphery of the metallic mass as the concentration of hydrogen either increases or diminishes.

Throughout the $(\alpha+\beta)$-range of concentrations, or realm of hysteresis, the solid body of palladium-hydrogen consists of a core composed wholly of one type of lattice, and giving the diffractions of this lattice alone; while the core is surrounded by a shell made up entirely of the other lattice. During absorption of hydrogen the α-lattice is core and the β-lattice constitutes the mantle, whilst during evolution after full saturation the positions of the two are interchanged. From the evidence just mentioned, seemingly supported by the recent results of Chaudron and his fellow-workers (5), (6), (7), there can be no inter-dispersion of phases.

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* Communicated by the Author.

The second hypothesis of Lacher is that of the persistence of metastable states described as supersaturated α- and an unsaturated β-phase. As has been pointed out by Owen (8), such persistence is difficult to reconcile with the circumstance that both the width and the area of the hysteresis loops increase at higher temperatures. Moreover, the relief of a supersaturated state by inoculation, which is further assumed, implies that the formation of new solid phases in palladium-hydrogen occurs by nucleation and grain growth. While this supposition is perhaps not so clearly contradicted by existing evidence as is the one just above discussed, it is certainly sufficiently questionable to require justification. As has been amply demonstrated by more than a score of X-ray studies, both the α- and β-phases of this system are of face-centred cubic symmetry and differ crystallographically only in respect to the magnitudes of their parameters; whereas the growth of new grains in metals or alloys is in general accompanied by change of symmetry. Transformations in metallic solids are, on the other hand, already familiar which do not involve grain-growth from nuclei, but which consist solely in dilatations (positive or negative), and such translations as these may entail. As an instance, we may recall the polymorphic transition of zirconium, near 870° C., of which Burgers (9) appears clearly to have demonstrated the dilatational character. Particularly in view of the absence of any change of symmetry, and of the uniform advance of the phase boundary, it is more natural to regard the formation of the α- from the β-phase of palladium-hydrogen as a simple dilatation, occurring when a certain concentration of hydrogen is reached, rather than as an inoculable process of grain-growth. It is, indeed, not improbable that inoculability belongs only to changes by which lattice of a new symmetry is produced. As lending support to the simple dilatational nature of lattice formation in the case under consideration, it may be noted that, in a detailed microscopic study of the effects of absorption of hydrogen by palladium (10), no indications were observed of the production of new grain boundaries or of growth from centres of crystallization.

Not only in the paper to which we have thus far devoted attention, but in various other theoretical discussions of palladium-hydrogen and kindred systems, the formation is assumed of simple solid-solution phases, evidently conceived as realms in which hydrogen diffuses freely in all directions through the normal interstices of the metallic lattice, and within which there are therefore no discontinuities of the concentration gradients. There already exists, however, a very considerable body of experimental evidence which makes it probable that in solid metal-hydrogen systems the hydrogen is not thus distributed, but is at least in large part occluded in rifts or widened spacings of the lattice.

While a summary of such evidence is not here in place, we may indicate some of its varieties and bearings and in particular point out a relation which supports the other evidences and affords a simple conception of the hystereses of palladium-hydrogen and of certain changes in other systems.

Openings of structure have by several different investigators, evidently unaware of one another's conclusions, been found to afford the simple explanation of such varied matters as the effects of hydrogen on the strength and mode of fracture of steel ((11), (12), on the magnetic properties of iron (13) and on the micro-structure of palladium (10); while they have yielded equally satisfactory interpretations of the effects of plastic deformation and annealing upon the permeability and capacity of palladium for hydrogen, and the inconstant stability of palladium-hydrogen (14); as well as of the comportment of this system with alternating currents and the variations of its electrostatic capacity with hydrogen content (15).

Apart from papers already cited, there exists an extensive literature showing the influence, upon permeability and capacity for hydrogen, of the state of strain of the absorbing metal, its cold-worked or annealed condition; while the converse of this, the plastic deformation of the metal by hydrogen, has long been recognized and has been much emphasized in the publications of Guichard and of Chaudron, together with many collaborators during the 1930's. It is hence of interest to note that the conception of lattice rifting as a condition precedent to absorption of hydrogen by metal, and of occluded hydrogen as largely concentrated in the rifts rather than generally distributed through the lattice interstices in "solid solution," appears to be in accord with the latest findings as to the effects of plastic deformation and the structural state of a metal after cold-working. From numerous studies of the broadening of X-ray spectral lines by deformation, it seems evident that the worked metallic lattice is either subdivided into crystallites (18), or is markedly distorted (17), (18), (19). Either effect must apparently be accompanied by enlargement of some of the lattice spacings, and it would be not inconsistent with the progressive character of cold-working brought about by absorption of hydrogen, and the disintegration which this at length produces in certain metals, that a sufficient degree of distortion and consequent rifting may finally amount to full subdivision. Metal as it is commonly examined regarding absorption of hydrogen, in the form of foil or wire, is somewhat cold-worked and hence doubtless rifted; while metal in the almost strain-free state secured by severe annealing *in vacuo* is practically impervious to hydrogen, as seems to have been sufficiently demonstrated both for iron (12), (20) and for palladium (21), (14), the principal representatives of the two major classes of occluding metals.

The relation directly connected with hysteresis, to which reference has been made, is the following: In solid metal-hydrogen systems, a change accompanied by *increase* of lattice volume brings about a *decrease* of permeability and capacity for hydrogen; while decrease of volume as a rule produces increase of occlusive openness.

For palladium-hydrogen the effect of this may be seen in the results of many investigators (22), (23), (8) and is very clearly evident in fig. 1 of the study of Lambert and Gates (24). Here the increase of lattice volume of nearly 10 per cent. (from $a_{\alpha}=3{\cdot}894$ to $a_{\beta}=4{\cdot}018$ Å.), which accompanies absorption and the conversion of the α- to the β-lattice, lessens permeability so that on the absorption isotherm a pressure greater than that of the equilibrium (evolution) curve corresponds to any given concentration of hydrogen ; while it correspondingly diminishes occlusive capacity, causing a smaller concentration of absorbed hydrogen to obtain at given pressure on the absorption isotherm, as compared with the evolution curve for the same temperature. It is to be noted that the alterations of occlusive capacity and permeability thus brought about are opposite in sense to those which might be expected if the normal interstices of the lattice were the paths of ready diffusion and places of storage of the hydrogen, and agree rather with the view that the effective voids are scattered rifts or occasional widened spacings which are throttled by expansion of the surrounding unrifted lattice.

Other changes of lattice volume occur in many metal-hydrogen systems. Data regarding lattice volumes are sometimes lacking, but all such changes appear to produce hysteresis of the solubility curves, and it is not unlikely that any such disturbance causes a temporary opening of structure which persists at least through the period of transition and is responsible for the displacements observed. For the changes which accompany polymorphic transitions in the system iron-hydrogen present information is relatively full and illuminating. Taking hydrogen concentrations as ordinates and centigrade temperatures as abscissæ, the solubility isobar for a pressure of 1 atmosphere, for this endothermic system, rises gradually as an almost straight line, with perhaps slight upward curvature, from a temperature near 300°, where solubility first becomes measurable. At the A_3-point near 910°, where body-centred α-iron gives place to the face-centred γ-lattice, solubility suddenly increases by some 55 per cent. ; while in the neighbourhood of 1400° the γ-form in turn goes over to the body-centred δ-lattice, with a diminution of approximately 3·8 per cent. in solubility (25). Since solubility, both at A_3 and at A_4, is greater in the denser γ-lattice, occlusive capacity appears to be determined not by the volume of the lattice, but possibly by its susceptibility to deformation. In addition, hysteresis occurs in both regions of transition, where the opening or closing of structure attendant upon alteration of lattice volume appears to extend over a range of some degrees. Quantitatively, the positions of the curves are still somewhat uncertain, especially at the high-temperature transformation A_4

Although the process involving change of lattice volume is in the case of palladium-hydrogen the isothermal formation of a secondary alloy phase (β), taking place without alteration of symmetry, in an exothermic system, and in the case of iron-hydrogen consists in a polymorphic transformation of the lattice, occurring with change of temperature in an endothermic system, it is then possible to find a qualitative explanation of both as due merely to throttling of rifts or widened spacings by the expansion of the surrounding lattice. This conception of the mode of containment of absorbed hydrogen in endothermic systems, and of a part of it in exothermic instances like palladium-hydrogen where another part is probably held in quite a different way, does not appear to be inconsistent with the results of such recent studies as those of Owen and his co-workers (26), (27), (8) upon the hysteresis phenomena in the system last named.

The brief considerations just given, although they mention only a small part of the relevant literature, are perhaps enough to make evident that theoretical treatments of metal-hydrogen systems, which without further discussion assume the existence in such systems of simple solid soluotins, in which hydrogen is distributed uniformly or with continuous concentration gradients, disregard a diverse body of experimental evidence sufficient to warrant a question as to the significance of conclusions thus derived ; while similar doubts are not improbably in order with regard to the assumption, for such systems, of phase-aggregates or inter-dispersed phases, or of phases produced in ready-formed solid metal by processes of nucleation and grain-growth.

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LX. *Rotatory Flow on the Surface of the Earth.—Part I. Cyclostrophic Motion.*

By T. V. Davies, M.Sc. †

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1. *Introduction.*

In 1880 Lord Kelvin, in a paper "On the Vibrations of a Columnar Vortex" ‡, made a detailed study of the behaviour of small vibratory motions superposed on a basic circular-vortex type of flow. The density in his paper is treated as a constant. A similar type of problem is investigated in the present paper; but we propose to treat the case of a variable-density field of a special type, and we introduce in addition the Coriolis terms into the equations of motion with a view to possible meteorological applications of the results.

It is assumed that a rotatory disturbance is composed of

(*a*) a flow which remains symmetrical and invariable with respect to a central vertical axis O*z*, this being known as the basic flow;

(*b*) a flow which is asymmetrical about O*z* and variable with respect to the time, this being known as the perturbation flow.

The further assumption is made that the basic flow is everywhere and always greater in magnitude than the perturbation flow. The results will apply, therefore, to disturbances in nature which have a predominantly circular-vortex type of flow. Much of the evidence pertaining to well-developed tropical cyclones (or typhoons) suggests that the circular flow remains the predominant feature with increasing height and the present results will apply, but the important temperate depressions certainly cannot be included since their basic flow rapidly changes from a rotatory to a rectilinear type with increasing height.

The pressure density relation is assumed in the present paper to be isothermal.

It is found that the perturbation pressure and flow can be analysed into an infinite set of harmonic components, and the important features of these components have been described. Due to the asymmetry, the variable density and the angular velocity of the earth (introduced via the Coriolis Terms), it is found that there is a resultant force acting upon and tending to give translational motion to the disturbance. A brief mention is made of the stability of small perturbations. It is found that a perturbation will remain small provided its angular motion about the centre of the cortex lies between determinate limits.

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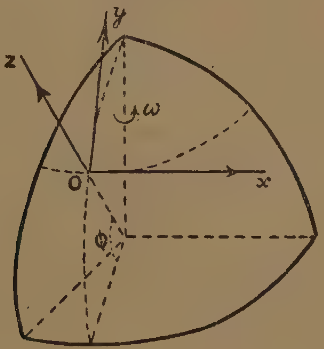
† Communicated by Prof. G. Temple.

‡ W. Thomson, Proc. Roy. Soc. Edinburgh (1880).

2. *The adaptation of Kelvin's method to the variable-density state, and the derivation of a partial differential equation for the perturbation pressure. The fundamental boundary condition at the surface of the earth.*

When the equations of motion of a fluid are expressed in Cartesian coordinates on the surface of the earth, Ox is measured due East, Oy due North and Oz vertically upwards at a point O, latitude ϕ. In cylindrical coordinates (r, θ, z) we choose the azimuthal angle θ to be measured from the East coordinate axis Ox. The angular velocity of the earth is denoted by ω.

Fig. 1.



If (u_2, v_2, w_2) are the velocity components in the directions (r, θ, z) increasing, p_2 the pressure and ρ_2 the density, the complete equations of motion in cylindrical coordinates may then be written in the form

$$\frac{\mathrm{D}u_2}{\mathrm{D}t}-\frac{v_2^2}{r}-2\omega\sin\phi v_2+2\omega\cos\phi w_2\cos\theta=-\frac{1}{\rho_2}\frac{\partial p_2}{\partial r}, \quad \ldots \quad (2.01)$$

$$\frac{\mathrm{D}v_2}{\mathrm{D}t}+\frac{u_2v_2}{r}+2\omega\sin\phi u_2-2\omega\cos\phi w_2\sin\theta=-\frac{1}{\rho_2 r}\frac{\partial p_2}{\partial\theta}, \quad . \quad (2.02)$$

$$\frac{\mathrm{D}w_2}{\mathrm{D}t}-2\omega\cos\phi(u_2\cos\theta-v_2\sin\theta)=-\frac{1}{\rho_2}\frac{\partial p_2}{\partial z}-g, \quad (2.03)$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t}\equiv\frac{\partial}{\partial t}+u_2\frac{\partial}{\partial r}+\frac{v_2}{r}\frac{\partial}{\partial\theta}+w_2\frac{\partial}{\partial z} \quad . \quad . \quad . \quad . \quad (2.04)$$

and we assume $p_2=\kappa^2\rho_2$ (κ=constant).

The equation of continuity is

$$\frac{\partial\rho_2}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho_2 r u_2)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho_2 v_2)+\frac{\partial}{\partial z}(\rho_2 w_2)=0. \quad . \quad . \quad (2.05)$$

We assume that the general motion is composed of a basic cyclostrophic flow together with a perturbation flow which is unsteady and asymmetrical, and we express this analytically by writing

$$\left.\begin{aligned} u_2&=u_1, \\ v_2&=\mathrm{V}(r)+v_1, \\ w_2&=w_1, \\ p_2&=\mathrm{P}(r, z)+p\rho, \\ \rho_2&=\rho+\rho_1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.06)$$

In the set (2.06) (u_1, v_1, w_1, p) are functions of r, θ, z and t, while V and P are functions of r and (r, z) respectively. The velocity components (u_1, v_1, w_1) are assumed to be small in absolute magnitude compared with V, likewise p is small in absolute magnitude compared with P.

Substituting from (2.06) in the set of equations (2.01) . . . (2.04), and bearing in mind the assumption with regard to absolute magnitudes, two courses are now possible according as V and $2r\omega \sin \phi$ are of equal or unequal magnitudes.

When V is of equal magnitude to $2r\omega \sin \phi$ the equation governing the basic flow derived from (2·01) is the well-known gradient-wind equation

$$-\frac{\mathrm{V}^2}{r}-2\omega\mathrm{V}\sin\phi=-\frac{1}{\rho}\frac{\partial \mathrm{P}}{\partial r}.$$

When $2r\omega \sin \phi$ is small compared with V we may alternatively use the cyclostrophic wind equation

$$-\frac{\mathrm{V}^2}{r}=-\frac{1}{\rho}\frac{\partial \mathrm{P}}{\partial r}.$$

Following is a table indicating the values of $2r\omega \sin \phi$ for varying r and ϕ :—

TABLE I.

$2r\omega \sin \phi$, in miles per hour.

r mls. ϕ° degrees	5	25	50	100	200
0	0	0	0	0	0
30°	1·3	6·6	13·1	26·2	52·5
60°	2·3	11·4	22·7	45·4	90·9
90°	2·6	13·1	26·2	52·5	105·0

It is evident from Table I. that in order to make an adequate attack upon the motion of the air in the large-scale temperate depressions it is certainly necessary to choose the gradient-wind equation to govern the basic flow. The cyclostrophic-wind equation will be valid for small-scale rotatory disturbances such as tropical cyclones, tornadoes, waterspouts, whirlwinds and so on. It must be noted, however, that near the equator cyclostrophic motion can be dominant for wide ranges of r.

In the development of the problem it has been found that the analysis becomes extremely involved if the gradient-wind equation is taken for the basic flow, and in the present paper we restrict ourselves to the cyclostrophic case which admits of relatively simpler treatment. Accordingly we take as the equations governing the basic flow the cyclostrophic set

$$\left.\begin{aligned} -\frac{V^2}{r}&=-\frac{1}{\rho}\frac{\partial P}{\partial r},\\ 0&=-\frac{1}{\rho r}\frac{\partial P}{\partial \theta},\\ 0&=-\frac{1}{\rho}\frac{\partial P}{\partial z}-g,\\ \frac{\partial V}{\partial \theta}&=0,\\ P&=\kappa^2\rho. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (2.07)$$

In deriving the third of these equations the coefficients neglected on the left-hand side of (2.03) are shown in standard treatises † to be small compared with the right-hand side.

Taking the terms of next-highest order we obtain the equations which govern the perturbation flow, bearing in mind throughout that $2r\omega\sin\phi$ is neglected in comparison with V. These become :

$$\left.\begin{aligned} \frac{\partial u_1}{\partial t}+\frac{V}{r}\frac{\partial u_1}{\partial \theta}-\frac{2Vv_1}{r}-2\omega V\sin\phi&=-\frac{\partial p}{\partial r},\\ \frac{\partial v_1}{\partial t}+u_1\frac{\partial V}{\partial r}+\frac{V}{r}\frac{\partial v_1}{\partial \theta}+\frac{u_1V}{r}&=-\frac{1}{r}\frac{\partial p}{\partial \theta},\\ \frac{\partial w_1}{\partial t}+\frac{V}{r}\frac{\partial w_1}{\partial \theta}+2\omega V\sin\theta\cos\phi&=-\frac{\partial p}{\partial z}, \end{aligned}\right\} \quad . \; . \quad (2.08)$$

and

$$r\lambda\,.\,\rho_1+\frac{\partial}{\partial r}(\rho u_1 r)+\rho\frac{\partial v_1}{\partial \theta}+\frac{\partial}{\partial z}(\rho w_1 r)=0. \quad . \; . \; . \quad (2.09)$$

The perturbation equations (2·08) may also be written in the symbolic form

$$\left.\begin{aligned} \lambda\,.\,u_1-\frac{2V}{r}v_1&=-\frac{\partial p}{\partial r}+2\omega V\sin\phi,\\ \left(\frac{dV}{dr}+\frac{V}{r}\right)u_1+\lambda\,.\,v_1&=-\frac{1}{r}\frac{\partial p}{\partial \theta},\\ \lambda\,.\,w_1&=-\frac{\partial p}{\partial z}-2\omega V\sin\theta\cos\phi, \end{aligned}\right\} \quad . \; . \quad (2.10)$$

---

† D. Brunt, ‘Physical and Dynamical Meteorology,’ ch. ix. p. 187.

where

$$\lambda \equiv \frac{\partial}{\partial t} + \frac{V}{r}\frac{\partial}{\partial \theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.11)$$

Using a determinantal process, it is now easily shown that

$$\left.\begin{aligned}
\left\{\lambda^2 + \frac{2V}{r}\left(\frac{dV}{dr} + \frac{V}{r}\right)\right\} . u_1\rho &= -\lambda . \left(\rho\frac{\partial p}{\partial r}\right) - \frac{2V}{r^2}\left(\rho\frac{\partial p}{\partial \theta}\right) &&= f_1, \\
\left\{\lambda^2 + \frac{2V}{r}\left(\frac{dV}{dr} + \frac{V}{r}\right)\right\} . v_1\rho &= -\lambda . \left(\frac{\rho}{r}\frac{\partial p}{\partial \theta}\right) - \left(\frac{dV}{dr} + \frac{V}{r}\right) && \\
&\qquad \times\left(-\rho\frac{\partial p}{\partial r} + 2\omega\rho V \sin\phi\right) &&= f_2, \\
\lambda . w_1\rho &= -\rho\frac{\partial p}{\partial z} - 2\omega\rho V \sin\theta\cos\phi &&= f_3.
\end{aligned}\right\} \quad . \quad . \quad . \quad (2.12)$$

Written in symmetrical form the set (2.12) becomes

$$\left.\begin{aligned}
\lambda(\lambda^2+f) . u &= \lambda . f_1, \\
\lambda(\lambda^2+f) . v &= \lambda . f_2, \\
\lambda(\lambda^2+f) . w &= (\lambda^2+f) . f_3,
\end{aligned}\right\}, \quad . \quad . \quad (2.13)$$

while (2.09) becomes

$$r\lambda . \rho_1 + \frac{\partial}{\partial r}(ur) + \frac{\partial v}{\partial \theta} + \frac{\partial (wr)}{\partial z} = 0, \quad . \quad . \quad . \quad . \quad (2.09)'$$

where we have written

$$\left.\begin{aligned}
f &= \frac{2V}{r}\left(\frac{dV}{dr} + \frac{V}{r}\right) \\
\text{and} \qquad (u, v, w) &\equiv (u_1\rho, v_1\rho, w_1\rho).
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.14)$$

In order to derive a partial differential equation for p, we now eliminate (u, v, w) between the four equations included in (2.13) and (2.09)′. It is evident that

$$\left.\begin{aligned}
\lambda(\lambda^2+f) . (ur) &= \lambda . (rf_1), \\
\lambda(\lambda^2+f) . v &= \lambda . f_2, \\
\lambda(\lambda^2+f) . (wr) &= (\lambda^2+f) . (rf_3),
\end{aligned}\right\}, \quad . \quad . \quad (2.15)$$

so that

$$\begin{aligned}
&\frac{\partial}{\partial r}\{\lambda . (rf_1)\} + \frac{\partial}{\partial \theta}\{\lambda . f_2\} + \frac{\partial}{\partial z}\{(\lambda^2+f) . (rf_3)\} \\
&= \frac{\partial}{\partial r}\{\lambda(\lambda^2+f) . (ur)\} + \frac{\partial}{\partial \theta}\{\lambda(\lambda^2+f) . v\} + \frac{\partial}{\partial z}\{\lambda(\lambda^2+f) . (wr)\}. \quad (2.16)
\end{aligned}$$

Since

$$\frac{\partial}{\partial r}\{\lambda(\lambda^2+f)\}\equiv\lambda(\lambda^2+f)\cdot\frac{\partial}{\partial r}+3\frac{d}{dr}\left(\frac{V}{r}\right)\frac{\partial^3}{\partial t^2\partial\theta}+3\frac{d}{dr}\left(\frac{V^2}{r^2}\right)\frac{\partial^3}{\partial t\partial\theta^2}$$
$$+\frac{d}{dr}\left(\frac{V^3}{r^3}\right)\frac{\partial^3}{\partial\theta^3}+\frac{df}{dr}\left(\frac{\partial}{\partial t}+\frac{V}{r}\frac{\partial}{\partial\theta}\right)+f\frac{d}{dr}\left(\frac{V}{r}\right)\frac{\partial}{\partial\theta}, \qquad (2.17)$$

$$\frac{\partial}{\partial\theta}\{\lambda(\lambda^2+f)\}\equiv\lambda(\lambda^2+f)\cdot\frac{\partial}{\partial\theta}, \quad \ldots\ldots\ldots\ldots \quad (2.18)$$

$$\frac{\partial}{\partial z}\{\lambda(\lambda^2+f)\}\equiv\lambda(\lambda^2+f)\cdot\frac{\partial}{\partial z}, \quad \ldots\ldots\ldots\ldots \quad (2.19)$$

it follows that

$$\frac{\partial}{\partial r}\{\lambda\cdot(rf_1)\}+\frac{\partial}{\partial\theta}\{\lambda\cdot f_2\}+\frac{\partial}{\partial z}\{(\lambda^2+f)\cdot(rf_3)\}$$
$$=\lambda(\lambda^2+f)\cdot\left\{\frac{\partial}{\partial r}(ur)+\frac{\partial v}{\partial\theta}+\frac{\partial}{\partial z}(rw)\right\}$$
$$+\left\{3\frac{d}{dr}\left(\frac{V}{r}\right)\frac{\partial^3}{\partial t^2\partial\theta}+3\frac{d}{dr}\left(\frac{V^2}{r^2}\right)\frac{\partial^3}{\partial t\partial\theta^2}+\frac{d}{dr}\left(\frac{V^3}{r^3}\right)\frac{\partial^3}{\partial\theta^3}\right.$$
$$\left.+\frac{df}{dr}\left(\frac{\partial}{\partial t}+\frac{V}{r}\frac{\partial}{\partial\theta}\right)+f\frac{d}{dr}\left(\frac{V}{r}\right)\frac{\partial}{\partial\theta}\right\}\cdot(ur). \quad \ldots \quad (2.20)$$

The first term on the right-hand side of (2.20) appears in the equation (2.09)′, and if we operate on the remainder of (2.20) by the operator (λ^2+f) and note that the differentials on the right-hand side are commutative with this operator, we obtain

$$(\lambda^2+f)\left\{\frac{\partial}{\partial r}[\lambda\cdot(rf_1)]+\frac{\partial}{\partial\theta}[\lambda\cdot f_2]+\frac{\partial}{\partial z}[(\lambda^2+f)\cdot(rf_3)]\right\}$$
$$=-r\lambda^2(\lambda^2+f)^2\cdot\rho_1+\left\{3\frac{d}{dr}\left(\frac{V}{r}\right)\frac{\partial^3}{dt^2\partial\theta}+3\frac{d}{dr}\left(\frac{V^2}{r^2}\right)\frac{\partial^3}{\partial tc\theta^2}+\frac{d}{dr}\left(\frac{V^3}{r^3}\right)\frac{\partial\theta^3}{\partial^3}\right.$$
$$\left.+\frac{df}{dr}\left(\frac{\partial}{\partial t}+\frac{V}{r}\frac{\partial}{\partial\theta}\right)+f\frac{d}{dr}\left(\frac{V}{r}\right)\frac{\partial}{\partial\theta}\right\}(rf_1). \quad \ldots \quad (2.21)$$

In the equation (2.21) f_1, f_2, f_3 are expressible in terms of the partial differentials of p, and equation (2.21) thus gives the general partial differential equation satisfied by the perturbation pressure p. In order to derive the explicit partial differential equation for p we note that

$$\frac{\partial}{\partial r}\cdot\lambda \equiv\lambda\cdot\frac{\partial}{\partial r}+\frac{d}{dr}\left(\frac{V}{r}\right)\frac{\partial}{\partial\theta},$$
$$\frac{\partial}{\partial\theta}\cdot\lambda \equiv\lambda\cdot\frac{\partial}{\partial\theta},$$
$$\frac{\partial}{\partial z}\cdot(\lambda^2+f)\equiv(\lambda^2+f)\cdot\frac{\partial}{\partial z}.$$

Thus the left-hand side of equation (2.21) becomes

$$\lambda(\lambda^2+f)\frac{\partial}{\partial r}(rf_1)+(\lambda^2+f)\frac{d}{dr}\left(\frac{V}{r}\right)\frac{\partial}{\partial\theta}(rf_1)+\lambda(\lambda^2+f)\frac{\partial f_2}{\partial\theta}+(\lambda^2+f)^2\frac{\partial}{\partial z}(rf_3),$$

and the complete equation (2.21) becomes

$$\begin{aligned}\lambda(\lambda^2+f)\,.\,\frac{\partial}{\partial r}(rf_1)+\frac{d}{dr}\left(\frac{V}{r}\right)&\left\{\frac{\partial^3}{\partial t^2\partial\theta}+\frac{2V}{r}\frac{\partial^3}{\partial t\partial\theta^2}+\frac{V^2}{r^2}\frac{\partial^3}{\partial\theta^3}+f\frac{\partial}{\partial\theta}\right\}(rf_1)\\&+r\lambda^2(\lambda^2+f)^2\rho_1+\lambda(\lambda^2+f)\,.\,\frac{\partial f_2}{\partial\theta}+(\lambda^2+f)^2\,.\,\frac{\partial}{\partial z}(rf_3)\\=\left\{3\frac{d}{dr}\left(\frac{V}{r}\right)\frac{\partial^3}{\partial t^2\partial\theta}\right.&+3\frac{d}{dr}\left(\frac{V^2}{r^2}\right)\frac{\partial^3}{\partial t\partial\theta^2}+\frac{d}{dr}\left(\frac{V^3}{r^3}\right)\frac{\partial^3}{\partial\theta^3}\\&\left.+\frac{df}{dr}\left(\frac{\partial}{dt}+\frac{V}{r}\frac{\partial}{\partial\theta}\right)+f\frac{d}{dr}\left(\frac{V}{r}\right)\frac{\partial}{\partial\theta}\right\}(rf_1),\end{aligned}$$

which reduces to

$$\begin{aligned}\lambda(\lambda^2+f)\,.\left\{\frac{\partial}{\partial r}(rf_1)+\frac{\partial f_2}{\partial\theta}\right\}&+(\lambda^2+f)^2\,.\,\frac{\partial}{\partial z}(rf_3)+r\lambda^2(\lambda^2+f)^2\rho_1\\=\left\{2\frac{d}{dr}\left(\frac{V}{r}\right)\frac{\partial^3}{\partial t^2\partial\theta}\right.&+2\frac{d}{dr}\left(\frac{V^2}{r^2}\right)\frac{\partial^3}{\partial t\partial\theta^2}+\frac{2}{3}\frac{d}{dr}\left(\frac{V^3}{r^3}\right)\frac{\partial^3}{\partial\theta^3}\\&\left.+\frac{df}{dr}\left(\frac{\partial}{\partial t}+\frac{V}{r}\frac{\partial}{\partial\theta}\right)\right\}.\,(rf_1).\qquad(2.22)\end{aligned}$$

Using the expressions (2.12) for f_1, f_2, f_3 in terms of p, we thus obtain

$$\begin{aligned}\lambda(\lambda^2+f)\,.&\left\{\frac{\partial}{\partial r}\left[-r\lambda\,.\left(\rho\frac{\partial p}{\partial r}\right)-\frac{2V}{r}\left(\rho\frac{\partial p}{\partial\theta}\right)\right]-\frac{\rho}{r}\lambda\,.\,\frac{\partial}{\partial\theta}\left(\frac{\partial p}{\partial\theta}\right)\right.\\&\left.+\rho\left(\frac{dV}{dr}+\frac{V}{r}\right)\frac{\partial}{\partial\theta}\left(\frac{\partial p}{\partial r}\right)\right\}-r(\lambda^2+f)^2\frac{\partial}{\partial z}\left(\rho\frac{\partial p}{\partial z}\right)\\&-r(\lambda^2+f)^2\,.\,2\omega V\sin\theta\cos\phi\frac{\partial\rho}{\partial z}+r\left\{2\frac{d}{dr}\left(\frac{V}{r}\right)\frac{\partial^3}{\partial t^2\partial\theta}\right.\\&\left.+2\frac{d}{dr}\left(\frac{V^2}{r^2}\right)\frac{\partial^3}{\partial t\partial\theta^2}+\frac{2}{3}\frac{d}{dr}\left(\frac{V^3}{r^3}\right)\frac{\partial^3}{\partial\theta^3}+\frac{df}{dr}\left(\frac{\partial}{\partial t}+\frac{V}{r}\frac{\partial}{\partial\theta}\right)\right\}\\&\times\left\{\lambda\,.\left(\rho\frac{\partial p}{\partial r}\right)+\frac{2V}{r^2}\left(\rho\frac{\partial p}{\partial\theta}\right)\right\}+r\lambda^2(\lambda^2+f)^2\rho_1=0.\qquad(2.23)\end{aligned}$$

The final form for the general partial differential equation for p is as follows :—

$$\begin{aligned}\lambda(\lambda^2+f)\,.&\left\{-\frac{\partial}{\partial r}\left[r\lambda\,.\left(\frac{\partial p}{\partial r}\right)+\frac{2V}{r}\frac{\partial p}{\partial\theta}\right]-\frac{1}{r}\lambda\,.\,\frac{\partial^2 p}{\partial\theta^2}+\left(\frac{dV}{dr}+\frac{V}{r}\right)\frac{\partial^2 p}{\partial\theta\partial r}\right\}\\&-r(\lambda^2+f)^2\,.\left\{\frac{\partial^2 p}{\partial z^2}-\frac{g}{\kappa^2}\frac{\partial p}{\partial z}\right\}+r\left\{2\frac{d}{dr}\left(\frac{V}{r}\right)\frac{\partial^3}{\partial t^2\partial\theta}+2\frac{d}{dr}\left(\frac{V^2}{r^2}\right)\frac{\partial^3}{\partial t\partial\theta^2}\right.\\&\left.+\frac{2}{3}\frac{d}{dr}\left(\frac{V^3}{r^3}\right)\frac{\partial^3}{\partial\theta^3}+\frac{df}{dr}\left(\frac{\partial}{\partial t}+\frac{V}{r}\frac{\partial}{\partial\theta}\right)\right\}\left\{\lambda\,.\left(\frac{\partial p}{\partial r}\right)+\frac{2V}{r^2}\frac{\partial p}{\partial\theta}\right\}\\&+2\omega Vr\sin\theta\cos\phi\frac{g}{\kappa^2}\left(\frac{V^2}{r^2}-f\right)^2-\frac{V^2}{r\kappa^2}\lambda(\lambda^2+f)\left\{\lambda+\frac{\partial p}{\partial r}+\frac{2V}{r}\frac{\partial p}{\partial\theta}\right\}\\&+\frac{r\lambda^2}{\kappa^2}(\lambda^2+f)^2p=0.\qquad(2.24)\end{aligned}$$

To conclude this section we shall introduce a boundary condition for the motion. For all flow over the surface of the earth the vertical velocity w must vanish at the surface of the earth. With the choice of co-ordinate axes in the present paper making $w=0$ at $z=0$ means that the boundary condition will be satisfied only at one point $r=0$, $z=0$, since curvature considerations will invalidate it elsewhere. For the small-scale rotatory disturbances which we are considering the discrepancy will be small if we continue to adopt $w=0$ at $z=0$ as the principal boundary condition. Expressed in terms of p, this boundary condition implies that

$$\left.\frac{\partial p}{\partial z}\right|_{z=0}+2\omega\cos\phi\,.\,V\sin\theta=0 \quad . \quad . \quad . \quad . \quad . \quad (2.25)$$

for all values of r, θ and t.

The boundary condition (2.25) must be true for all values of θ and for all time, and it follows that a special solution for p must be sought which obeys these conditions. This special solution which enables us to satisfy the boundary condition at $z=0$ is referred to as the "boundary perturbation pressure." In addition to this special solution it will be shown that there exists an infinite number of solutions which can be utilized to satisfy a boundary condition for p at any height; this will be called the "auxiliary perturbation pressure."

3. *The Boundary and Auxiliary Perturbation Pressures when $V=\beta r$ $(0\leqslant r\leqslant a)$.*

In the tropical cyclone one of the principal features is a relatively calm core with the wind velocity increasing rapidly away from the central axis, wind speeds of 100 m.p.h. or higher being encountered about 5–10 miles from the axis. The simplest representation of such a scheme is an assumed basic velocity distribution increasing linearly from the central axis, and we assume henceforth that

$$V=\beta r \qquad (0\leqslant r\leqslant a). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.01)$$

This velocity distribution will not be valid for the tornado or waterspout type of phenomenon.

When we substitute $V=\beta r$ in (2.24) and assume that the solution for p is of the form $\sin\theta\chi(r)$, the function $\chi(r)$ must satisfy

$$\frac{d^2\chi}{dr^2}+\frac{1}{r}\left(1+\frac{\beta^2r^2}{\kappa^2}\right)\frac{d\chi}{dr}-\chi\left(\frac{1}{r^2}+\frac{\beta^2}{\kappa^2}\right)=-\frac{6\omega\beta g\cos\phi}{\kappa^2}r. \qquad (3.02)$$

κ is the velocity of sound and $\beta r/\kappa\ll 1$, and we may express the particular solution of (3.02) in ascending powers of $\beta r/\kappa$. We find that the particular solution is

$$p_1=\frac{\omega\beta g\cos\phi}{\kappa^2}r^3\sin\theta\left\{-\frac{3}{4}+\frac{1}{16}\frac{r^2\beta^2}{\kappa^2}\dots\right\}. \quad . \quad . \quad (3.03)$$

To satisfy (2.25) we investigate a solution for p of the form

$$\sin\theta e^{gz/2\kappa^2}\{A_{m1}\cos(\alpha_m z)+B_{m1}\sin(\alpha_m z)\}\chi_m(r),$$

and we find that $\chi_m(r)$ must satisfy the equation

$$\frac{d^2\chi_m}{dr^2}+\frac{1}{r}\left(1+\frac{\beta^2r^2}{\kappa^2}\right)\frac{d\chi_m}{dr}+\left(\alpha_m^{*2}-\frac{1}{r^2}\right)\chi_m=0, \quad . \quad . \quad . \quad (3.04)$$

where

$$\alpha_m^{*2}=3\alpha_m^2+\frac{3}{4}\frac{g^2}{\kappa^4}-\frac{\beta^2}{\kappa^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.05)$$

In solving (3.04) we may assume that β^2r^2/κ^2 is small compared with unity, and in this case the solution which is zero at $r=0$ is

$$\chi_m=J_1(\alpha_m^* r).$$

Thus the general solution for the boundary perturbation pressure is

$$p=p_1+\sin\theta^{gz/2\kappa^2}\sum_m\{A_{m1}\cos(\alpha_m z)+B_{m1}\sin(\alpha_m z)\}J_1(\alpha_m^* r). \qquad (3.06)$$

The boundary condition (2.25) now leads to the identity

$$\sum_m\left(A_{m1}\frac{g}{2\kappa^2}+B_{m1}\alpha_m\right)J_1(\alpha_m^* r)\equiv-2\omega\beta r\cos\phi \qquad (0\leqslant r\leqslant a), \quad (3.07)$$

hence, using the results of the Fourier-Bessel Expansion, it follows that

$$J_1(\alpha_m^* a)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.08)$$

and

$$A_{m1}\frac{g}{2\kappa^2}+B_{m1}\alpha_m=\frac{4\omega\beta\cos\phi J_0(\alpha_m^* a)}{\alpha_m^* J_2{}^2(\alpha_m^* a)}. \quad . \quad . \quad . \quad . \quad . \quad (3.09)$$

A second relation between A_{m1} and B_{m1} may be obtained from the $\sin\theta$ term in the surface pressure distribution. Equation (3.08) shows that there are an infinity of eigen-values for α_m, and thus the boundary perturbation pressure will consist of an infinite set of characteristic pressure terms each with a distinctive pressure variation in the vertical.

To derive the general solution of (2.24) we investigate further a solution for p of the form

$$\cos(l\theta-\sigma t)e^{gz/2\kappa^2}\{A_{ml}\cos(\beta_m z)+B_{ml}\sin(\beta_m z)\}\chi_{ml}(r), \quad . \quad (3.10)$$

and substituting in (2.24) we find that χ_{ml} must satisfy the equation

$$\frac{d^2\chi_{ml}}{dr^2}+\frac{1}{r}\left(1+\frac{\beta^2r^2}{\kappa^2}\frac{d\chi_{ml}}{dr}+\left(\beta_m^{*2}-\frac{l^2}{r^2}\right)\chi_{ml}=0,\right. \quad . \quad . \quad . \quad (3.11)$$

where

$$\beta_m^2=\left[\{4\beta^2-(\sigma-\beta l^2\}\left\{\beta_m^2+\frac{g^2}{4\kappa^4}-\frac{(\sigma-\beta l)^2}{\kappa^2}\right\}-\frac{2\beta^3 l}{\kappa^2}(\sigma-\beta l)\right]\Big/(\sigma-\beta l)^2.$$

$$. \quad . \quad . \quad . \quad . \quad (3.12)$$

Hence with $\beta^2r^2/\kappa^2 \ll 1$ the solution of (3.11) will be $J_l(\beta^*_m r)$ and the general solution of (2.24) will contain an infinite number of terms of the type

$$\cos(l\theta-\sigma t)e^{gz/2\kappa^2}\{A_{ml}\cos(\beta_m z)+B_{ml}\sin(\beta_m z)\}J_l(\beta^*_m r), \quad . \quad (3.13)$$

which represents a wave of pressure travelling around the circular vortex with angular velocity σ/l. These solutions may be ultilized to satisfy a boundary condition for p at any height by making use of the Fourier-Bessel Expansion, and for this to be possible we must have

$$J_l(\beta^*_m a)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.14)$$

as in (3.08). The assumption is made above that β^{*2}_m is a positive quantity, and this is evidently the criterion for the stability of the perturbations. It follows from (3.12) that for stability $(\sigma-\beta l)$ must be between certain limits, but, since these are more complicated than those in Lord Kelvin's paper, though of the same nature, this aspect will not be enlarged upon.

4. *The Force Components acting upon the Vortex* $V=\beta r$ $(0 \leqslant r \leqslant a)$.

The formulæ for the velocity components of the perturbation flow can now be discussed using (2.12) and the results of Section 3, but the most interesting result arising from Section 3 relates to the resultant force exerted by the vortex $V=\beta r$ on its surroundings. If we denote by (F_E, F_N) the easterly and northerly components of this force, then

$$F_E=\int_{z=0}^{z}\int_{\theta=0}^{2\pi}(p\rho)\Big|_{r=a} a\cos\theta\, d\theta, \quad . \quad . \quad . \quad (4.01)$$

$$F_N=\int_{z=0}^{z}\int_{\theta=0}^{2\pi}(p\rho)\Big|_{r=a} a\sin\theta\, d\theta. \quad . \quad . \quad . \quad . \quad (4.02)$$

Using (3.08) and (3.14) it follows that

$$p\Big|_{r=a}=\frac{\omega\beta g\cos\phi}{\kappa^2}a^3\sin\theta\left\{-\frac{3}{4}+\frac{1}{16}\frac{a^2\beta^2}{\kappa^2}\dots\right\},$$

and thus

$$(p\rho)\Big|_{r=a}=\frac{a^3\rho_0\omega\beta g\cos\phi}{\kappa^2}e^{-\frac{gz}{\kappa^2}-\frac{1}{2}\frac{\beta^2a^2}{\kappa^2}}\left\{-\frac{3}{4}+\frac{1}{16}\frac{a^2\beta^2}{\kappa^2}\dots\right\}\sin\theta,$$

$$. \quad . \quad . \quad . \quad (4.03)$$

where ρ_0 is the density at $r=0$, $z=0$.

It follows that

$$F_E=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.04)$$

$$F_N=\rho_0a^4\omega\beta\cos\phi e^{-\frac{1}{2}\frac{\beta^2a^2}{\kappa^2}}(1-e^{\;gz/\kappa^2})\left\{-\frac{3}{4}+\frac{1}{16}\frac{a^2\beta^2}{\kappa^2}\dots\right\}, \quad (4.05)$$

so that the resultant force exerted by the vortex on its surroundings is a southerly force, and the absolute magnitude is approximately equal to $\frac{3}{4}\rho_0a^4\omega\beta\cos\phi$. The direction of this force is reversed in the southern hemisphere.

LXI. *Notice respecting New Book.*

The Naming of the Telescope. By Edward Rosen. [Pp. xvi+110.] (New York: Henry Schuman, 1947.) $2.50.

In this elegantly produced little volume the author attempts to find the answer to the question, " Who originated the name telescope ? " From early in 1611, *telescopio* began to replace *occhiale* and *perspicillum* in Galileo's letters, and the word first appears in a printed book, Lagalla's *Lunar Phenomena*, in 1612. The " public unveiling " of the term telescope is here traced to a banquet given for Galileo on April 14, 1611, by Frederick Cesi, founder and president of the Academy of the Lynxes. It is to Cesi that the new word is usually attributed. The evidence for this and other attributions, collected from contemporary records, books and letters, is marshalled and critically discussed. In the course of the inquiry many interesting glimpses are afforded of scientific life in Italy at the beginning of the seventeenth century, in particular of the activities of the Lynxes. It is finally concluded that the originator of the now widespread term was John Demisiani, one of the Academy group, but a poet and theologian rather than a scientist, who thus fostered, though he did not initiate, the use of Greek words in coining names for new scientific instruments. Among other matters discussed are the limitation in modern Italian of *telescopio* to the reflecting instrument, and the use, in a more general sense, of the term *cannocchiale*, a combination of the words for tube and lens.

Concentration on limited, deliberately isolated, problems is characteristic of the scientist and of the modern historian, though writers on the history of science commonly spread their wings in wider flights. The fascination of an intensive search for a particular piece of information is admirably conveyed in this book. It can hardly be denied, however, that the problem is a very minor one, or that the solution throws little light on the development of science. The author is one of the few who combine with enthusiasm for the history of science the temperament and the technical qualifications which are necessary for effective detailed research in the field. It is to be hoped that this brilliant treatment of a minor question is but a prelude to future work by the author on some of the abounding major problems. E. C. S.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

LXII. *Technique of the Step-by-Step Integration of Ordinary Differential Equations* *.

By W. J. Duncan, D.Sc., F.R.S.,
Professor of Aerodynamics at the College of Aeronautics, Cranfield.†

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Summary.

In Part I, step-by-step methods are examined critically and emphasis is placed on the dependence of the error on the number n of steps used for a given range of the independent variable. The index of a process is defined and it is shown that the errors can be assessed and partially corrected when the index is known and results obtained for two or more values of n. Attention is drawn to the advantages in certain cases of a part-analytical process.

In Part II, methods of numerical integration in general are classified and briefly reviewed. The chart, Table 2.2.1, summarizes the classification.

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* The paper is substantially the same as Report No. 4 of the College of Aeronautics, dated February, 1947.

† Communicated by the Author.

0. INTRODUCTION.

Processes for the step-by-step integration of ordinary differential equations are amongst the most valuable of the methods for the numerical solution of such equations with one-point boundary conditions. Some of the advantages are :—

(*a*) They are universally applicable to ordinary differential equations of any order and to simultaneous sets of such equations, provided only that there are no singularities in the range of integration.

(*b*) Results of any required degree of accuracy can be obtained.

(*c*) Some of the most generally useful of the processes are intuitively obvious. They are therefore easy to learn and to remember.

In the present paper emphasis is placed on the simpler techniques of step-by step integration and only brief reference is made to those depending on the use of finite differences, such as the Adams-Bashforth process. It is stressed that the error in the approximate solution for a given range of the independent variable is a function of that variable and of the number n of equal steps used. Provided that exactly the same process is used throughout, and the same numerical accuracy maintained at all times, it becomes possible to estimate the error when the calculation is made with two or more values of n ; it even becomes possible to improve the solution obtained with the greatest number of steps. Accordingly it is suggested that it should be a normal procedure in serious work to make the computation with two different numbers of intervals ; if it is anticipated that $2m$ steps will give the required accuracy a preliminary calculation with, say, m steps should be made.

In Part I, the technique of step-by-step integration of ordinary differential equations or sets of such equations is considered in a general way and the *index* of the process is defined. Part II contains a brief review of some other methods for the numerical solution of ordinary differential equations. It is believed that with the development of digital calculating machines of great power and rapidity the numerical methods of integration will oust all others, including notably those depending on physical or kinematical analogies (differential analysers, etc.).

PART I.

STEP-BY-STEP INTEGRATION.

1.1. *Reduction of any Equation or Set to a First Order Set.*

It is well-known that an ordinary differential equation of any order or a simultaneous set of such equations can always be reduced to a set of first order equations by the use of auxiliary variables. The auxiliary variables can be chosen in an indefinitely great number of ways, but the most obvious choice consists of all the differential coefficients of the dependent variables, excepting only the one of highest order for each variable. For example, the pair of second order equations.

$$\left.\begin{aligned}\frac{d^2x_1}{dt^2}&=f_1\left(t,\ x_1,\ x_2,\frac{dx_1}{dt},\frac{dx_2}{dt}\right),\\ \frac{d^2x_2}{dt^2}&=f_2\left(t,\ x_1,\ x_2,\frac{dx_1}{dt},\frac{dx_2}{dt}\right)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1.1:1)$$

can be replaced by the following set of four first order equations :—

$$\left.\begin{aligned}\frac{dx_1}{dt}&=x_3,\\ \frac{dx_2}{dt}&=x_4,\\ \frac{dx_3}{dt}&=f_1(t,\ x_1,\ x_2,\ x_3,\ x_4),\\ \frac{dx_4}{dt}&=f_2(t,\ x_1,\ x_2,\ x_3,\ x_4).\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1.1:2)$$

In the remainder of this paper, it will be supposed that the equation or equations under consideration have been reduced to a first order set.

1.2. *Formulation of the Problem and Outline of Method.*

At present we confine attention to one-point boundary problems. The independent variable is t and the dependent variables are $x_1, x_2, \ldots x_m$. The data are the differential equations themselves and the values for t equal to a of each dependent variable or information from which these initial values can be found. It is required to construct an approximate numerical solution for the range a to b of t.

We shall suppose that the general march of the solution within the range is known independently, for example, from physical intuition, or from a rough preliminary exploration, that singularities are absent and all the differential coefficients finite. Valuable hints about the preliminary exploration and suggestions for overcoming difficulties are given in the treatise by Runge (2). As an example, an infinite differential coefficient can be avoided by taking a new independent variable, which is a linear combination of the original independent and dependent variables, throughout the whole or a part of the range of integration. This is equivalent to rotation of the coordinate axes.

We divide the range a to b of t into n equal parts h, so

$$h=\frac{b-a}{n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.2:1)$$

and write

$$\left.\begin{aligned}t_0&=a\\ t_r&=a+rh=\frac{(n-r)a+rb}{n}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1.2:2)$$

The approximate value of a dependent variable x_j corresponding to t_r will be written $x_j(r)$.

The first step in the solution consists in finding $x_j(1)(j=1, 2, \ldots . m)$. To do this we calculate, by a process characteristic of the particular method adopted, a mean slope $s(j, 1)$ for the interval; in this symbol the first number corresponds to the dependent variable and the second to the interval. Then

$$x_j(1)=x_j(0)+hs(j, 1). \qquad \ldots \ldots \ldots \quad (1.2:3)$$

The second and further steps are carried out in the same manner. Thus

$$x_j(r+1)=x_j(r)+hs(j, r+1). \qquad \ldots \ldots \quad (1.2:4)$$

We regard it as part of the definition of a step-by-step method that $s(j, r+1)$ is derived from the values $x_j(r)$ by *exactly* the same process as $s(j, 1)$ is derived from the values $x_j(0)$. Without this proviso the process of solution would lack uniformity and the difficulty of the discussion would be increased.

In the original method, given by Euler in the 18th century, $s(j, r)$ is simply dx_j/dt at the beginning of the rth interval as given by the differential equation in terms of t_{r-1} and the already calculated corresponding values of the dependent variables. This method, although still much used on account of its simplicity, is relatively crude and various more refined methods have been devised.

1.3. *Definition of the Index of a Step-by-Step Process.*

For the present we shall regard the following as fixed :—

(*a*) The differential equation or equations.

(*b*) The initial conditions.

(*c*) The range of the independent variable.

(*d*) The uniform process of step-by-step integration.

The only variable left is the number n of equal steps used. We propose to study the manner in which the errors in the approximate solution at the end of the range of integration $(t=b)$ depend on n.

In the first place, we shall suppose that the actual computations have been carried out with perfect accuracy, so that the only errors are those inherent in the process adopted. Consequently, the errors are, in the conditions postulated, functions of the number n only. In the simple process of Euler (*see* § 1.2) only the first two terms in the Taylor expansion are retained. Hence the error in each interval will be an infinite power series in h beginning with a term in h^2 or, what amounts to the same thing, an infinite power series in n^{-1} beginning with a term in n^{-2}. Evidently here the error when $t=b$ at the end of n intervals can be expanded as an infinite power series in n^{-1} beginning with the *first* power, for the leading term will be the sum of n terms each proportional to n^{-2}. *We shall accordingly say that the Euler process has the index* 1. For other processes the lowest power may be higher, and we define the index of a process to be the index of the lowest power of n^{-1} which appears in the expressions for the errors at the end of the total interval of integration when these are expanded as infinite power series in the reciprocal of n.

The process of Runge and Kutta (7), (8), (2), (4) has the index 4, but is not on that account to be assumed to be necessarily superior to processes having lower indices; the relative merits of processes are briefly considered in § 1.7.

A very simple and useful process of index 2 is as follows. Use the initial slopes for the $(r+1)$th interval as given by the differential equations and the already computed values of the dependent variables at t_r to compute approximate values of the dependent variables for

$$t_{r+\frac{1}{2}}=t_r+\frac{h}{2},$$

i. e., at the middle of the $(r+1)$th interval. Then take the slope $s(j, r+1)$ to be the value of dx_j/dt as computed from the differential equation for the foregoing mean values of t and of the dependent variables. Examples illustrating this process and that of Euler are given in § 1.5.

1.4. *Use of the Index in Assessment of Errors and Partial Correction.*

In accordance with § 1.3 we shall assume that the error $\epsilon_r(t)$ in the value of $x_r(t)$ can be expanded as

$$\epsilon_r(t)=n^{-k}(e_0+e_1n^{-1}+e_2n^{-2}+\text{etc.}), \quad . \quad . \quad . \quad (1.4:1)$$

where k is the index of the process, and will be supposed known. Then, provided the interval h is sufficiently small, the first term in (1.4 : 1) will be dominant. Hence we shall have approximately

$$\epsilon_r(t)=e_0n^{-k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.4:2)$$

Now let the process be carried out with two different numbers n_1 and n_2 of intervals and let the corresponding approximations to $x_r(t)$ be $x_{r1}(t)$ and $x_{r2}(t)$. Then we shall have with close approximation

$$\left.\begin{aligned} x_r(t)&=x_{r1}(t)+e_0n_1^{-k} \\ &=x_{r2}(t)+e_0n_2^{-k} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.4:3)$$

Hence $$e_0=\frac{x_{r2}(t)-x_{r1}(t)}{n_1^{-k}-n_2^{-k}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.4:4)$$

from which e_0 can be computed since all the terms on the right hand side of the equation are known. We can then calculate $x_r(t)$ from (1.4 : 3) and so obtain an improved approximation to $x_r(t)$. The index of the corrected value will be raised by 1.

If three different numbers n_1, n_2 and n_3 of intervals be used, it becomes possible to calculate both e_0 and e_1 if it be assumed that the higher terms in the expansion (1.4 : 1) can be neglected. When this is done, a very helpful comparison can be made between the approximations to $x_r(t)$ obtained by taking all possible pairs and finally all three together.

The following procedure is suggested in the practical use of this method. First, estimate (or guess) the number n_1 of intervals required to give an approximate solution of the required accuracy. Begin by taking a smaller number n_2 of intervals, say half the number, and carry through the process. Repeat the process using n_1 intervals and compute the corrections as explained above ; quite apart from the use of the corrections, the repetition of the process with a different number of intervals is a most valuable check. If it appears that the errors in the corrected results are likely to be too large, the process must be carried out a third time with a number n_3 of intervals greater than n_1 and final corrections applied.

The regularity in the computed results required for the application of the process of correction will be somewhat impaired by "rounding-off." This will not give rise to trouble provided that a suitable degree of numerical accuracy is maintained throughout the work, as is sufficiently demonstrated in the examples given in § 1.5.

The process of extrapolation towards the true solution described above was devised by the writer in ignorance that substantially the same method had been given by L. F. Richardson (9), (10) as an example of what he calls ' the deferred approach to the limit.' The writer's attention was drawn to this by R. A. Fairthorne.

1.5. *Illustrative Examples.*

Examples of great simplicity have been selected, but they suffice to illustrate the methods already described.

Example (*a*).

The differential equation is

$$\frac{dx}{dt}=1+0\cdot2\,t-0\cdot5\,x, \quad . \quad . \quad . \quad . \quad . \quad (1.5:1)$$

and the initial condition is

$$x=1 \quad \text{when} \quad t=0.$$

It is required to compute x when $t=1$.

We shall use Euler's process, (*see* § 1.2) for which the index is 1, and we shall begin by using three intervals, which is obviously too small to yield anything but a rough approximation. The calculation proceeds as shown in Table 1.5.1

The computation has been repeated with 5 and 10 intervals in use and the results are given in Table 1.5.2.

Now let us derive a corrected value of x for $t=1$ from the results obtained with 3 and 5 intervals. Since the index is 1, we get

$$\left.\begin{aligned}1\cdot4843&=x(1)+\frac{e_0}{3},\\ 1\cdot4819&=x(1)+\frac{e_0}{5}\end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad . \quad (1.5:2)$$

TABLE 1.5.1.

Step-by-Step Integration of a Differential Equation by Euler's Process. Index 1.

Number of Intervals 3. $h=1/3$.

Column Number					
1	2	3	4	5	6
r	t_r	x_r	$\left(\frac{dx}{dt}\right)_r$	$\triangle x_r$ $=h\times$ col. 4	x_{r+1} =col. 3 +col. 5
0	0	1	$\frac{1}{2}$	1/6	7/6
1	1/3	1 1/6	29/60	29/180	239/180
2	2/3	1 59/180	169/360	169/1080	$\frac{1603}{1080}=1{\cdot}4843$

TABLE 1.5.2.

Step-by-Step Integration of a Differential Equation by Euler's Process. Index 1.

Number of intervals : 5 and 10.

t	x	
	5 intervals used	10 intervals used
0	1·0	1·0
0·1		1·05
0·2	1·1	1·0995
0·3		1·1485
0·4	1·198	1·1971
0·5		1·2452
0·6	1·2942	1·2929
0·7		1·3403
0·8	1·3888	1·3873
0·9		1·4339
1·0	1·4819	1·4802

Hence, by difference,

$$0{\cdot}0024=\frac{2e_0}{15},$$

$$e_0=0{\cdot}0180,$$

and

$$\frac{e_0}{5}=0{\cdot}0036.$$

Hence
$$x(1)=1\cdot4819-0\cdot0036,$$
$$=1\cdot4783.$$

From the results for $n=5$ and 10 we obtain

$$\left.\begin{aligned}1\cdot4819&=x(1)+\frac{e_0}{5},\\ 1\cdot4802&=x(1)+\frac{e_0}{10}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.5:3)$$

Hence
$$e_0=0\cdot0170,$$
and
$$x(1)=1\cdot4785.$$

Similarly from the results for $n=3$ and 10 we derive
$$e_0=0\cdot0176,$$
and
$$x(1)=1\cdot4784.$$

Lastly, if we take all three results and retain the first two terms in the expansion for the error we have

$$\left.\begin{aligned}1\cdot4843&=x(1)+\frac{e_0}{3}+\frac{e_1}{9},\\ 1\cdot4819&=x(1)+\frac{e_0}{5}+\frac{e_1}{25},\\ 1\cdot4802&=x(1)+\frac{e_0}{10}+\frac{e_1}{100}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1.5:4)$$

These equations yield
$$e_0=0\cdot0157,$$
$$e_1=0\cdot0043,$$
$$x(1)=1\cdot4786.$$

An inspection of these results would lead us to assert with considerable confidence that $x(1)$ is $1\cdot4786$ with an error probably not exceeding 1×10^{-4} and almost certainly not exceeding 2×10^{-4}. The true solution is in fact

$$x(t)=0\cdot4t+1\cdot2-0\cdot2e^{-0\cdot5t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.5:5)$$

which yields
$$x(1)=1\cdot4787.$$

Since the whole computation was made with only four places of decimals * in use, a more accurate result could only have been obtained by a favourable accident. The assessment of the error is confirmed.

From each pair of corresponding results contained in Table 1.5.2, we can compute a corrected value of x, and the corrections to be applied to the intermediate results can be obtained by interpolation. In this way a corrected table can be constructed giving x for every $0\cdot1$ increment of t.

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* When the computations are carried to 6 places of decimals the final approximation to $x(1)$ is $1\cdot478706$, which is correct to 6 places.

In the practical working of this example, three different numbers of intervals would not be used, but one of the pairs 3, 5 or 5, 10 adopted according to the accuracy desired.

Example (*b*).

The differential equation is

$$\frac{dx}{dt}=\sqrt{(1-x^2)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.5:6)$$

and the boundary condition $x=0$ when $t=0$. Find x when $t=1$. The true solution is

$$x=\sin t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.5:7)$$

and $x=0{\cdot}8415$ when $t=1$. Here the solution can be obtained by interchange of the dependent and independent variables, quadrature for a range of values of x and interpolation. However, the example provides a good illustration of the methods at present being applied.

We begin by adopting Euler's method (index 1) and shall use 3, 5 and 10 intervals. Only the final results for $x(1)$ will be quoted, namely

$$x(1)_3=0{\cdot}9016,$$
$$x(1)_5=0{\cdot}8766,$$
$$x(1)_{10}=0{\cdot}8586.$$

The corrected results obtained in the manner given under Example (*a*) are :—

from 3 and 5 intervals	$x(1)=0{\cdot}8391$,
from 3 and 10 intervals	$x(1)=0{\cdot}8402$,
from 5 and 10 intervals	$x(1)=0{\cdot}8406$,
from 3, 5 and 10 intervals	$x(1)=0{\cdot}8413$.

True value $x(1)=0{\cdot}8415$ (to 4 places).

Example (*c*).

The problem is the same as in Example (*b*), but we shall now adopt the process of index 2 briefly described in § 1.3. The tubular computation with 5 intervals is given in Table 1.5.3 (p. 502).

In the Table the number under the radical in column 4 is 1 minus the square of the entry in column 3, and the final entry in column 4 is the initial slope for the interval. This is used to find an approximate value of x for the middle of the interval (column 5); the increment of x is $0{\cdot}1\times$entry in column 4. Column 6 is derived from column 5 in the same manner as 4 from 3; the entry is a mean slope for the interval. The entries in column 5 are much less accurate than those in column 7, and are to be discarded when the calculation is complete. Column 3 is not really necessary since all the entries, except the first, already appear in column 7.

The great superiority of this process over the Euler process in accuracy is shown by comparison of the error in the last entry of Table 1.5.3 with that obtained before correction in Example (*b*) with 5 intervals ;

TABLE 1.5.3.

Step-by-Step Integration of a Differential Equation by a Process of Index 2.

Number of intervals 5. $h=0{\cdot}2$.

Column Number						
1	2	3	4	5	6	7
r	t_r	x_r	$\left(\frac{dx}{dt}\right)_r$	$x_{r+\frac{1}{2}}$	$\left(\frac{dx}{dt}\right)_{r+\frac{1}{2}}$	x_{r+1}
0	0	0	1·0	0·1	$\sqrt{0{\cdot}99}=0{\cdot}9950$	0·1990
1	0·2	0·1990	$\sqrt{0{\cdot}9604}=0{\cdot}9800$	0·2970	$\sqrt{0{\cdot}9118}=0{\cdot}9549$	0·3900
2	0·4	0·3900	$\sqrt{0{\cdot}8479}=0{\cdot}9208$	0·4821	$\sqrt{0{\cdot}7676}=0{\cdot}8761$	0·5652
3	0·6	0·5652	$\sqrt{0{\cdot}6805}=0{\cdot}8249$	0·6477	$\sqrt{0{\cdot}5805}=0{\cdot}7619$	0·7176
4	0·8	0·7176	$\sqrt{0{\cdot}4851}=0{\cdot}6965$	0·7872	$\sqrt{0{\cdot}3803}=0{\cdot}6167$	0·8409

the errors are 0·0006 and 0·0351 respectively. Since the additional labour in the present process is small, it must be regarded as much preferable to that of Euler.

A comparison of the results obtained by the method of index 2 with 5 and 10 intervals is provided in Table 1.5.4.

The process of correction will now be applied to the two results obtained for $x(1)$. Since the index is 2 and the numbers of intervals 5 and 10, we have

$$\left.\begin{aligned} 0{\cdot}8409&=x(1)+\frac{e_0}{25}, \\ 0{\cdot}8413&=x(1)+\frac{e_0}{100} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.5:8)$$

Hence $\qquad e_0=-0{\cdot}0133,$

and $\qquad x(1)=\ 0{\cdot}8414.$

Since the correction to the value of $x(1)$ obtained with 10 intervals is only 0·0001, we should conclude that the error probably does not exceed this. The error (in the four-figure answer) is just 0·0001. A more thorough test would have involved making the computations to five or six places of decimals throughout.

Note on the Correction Process.

In all the examples we have taken equal intervals throughout the range of integration, but the same process of assessment of error and partial correction can be used with unequal intervals, provided that each of the original intervals is evenly subdivided into the same number of smaller intervals.

1.6. *Use of an Analytical Solution in Each Interval of a Step-by-Step Integration.*

It sometimes happens that over a limited range of the independent variable the equation or equations to be solved approximate closely to an equation or set having a closed analytical solution, and that the soluble equations can be adjusted to the set to be integrated in each interval by suitable changes of parameters. Such a process of substitution is likely to be advantageous when the interval throughout which the parameters can be kept constant is much larger than the interval

TABLE 1.5.4.

Step-by-Step Integration of a Differential Equation by a Process of Index 2.

Numbers of Intervals : 5 and 10.

t	Uncorrected approximations to $x(t)$		Correction to value for 10 intervals deduced from corresponding values	Interpolated correction	Final corrected approximation	True value of $x(t)$. (To 4 figures)	Error in final approximation
	5 intervals	10 intervals					
0	0	0	0	—	0	0	0
0·1	—	0·0999	—	0	0·0999	0·0998	—0·0001
0·2	0·1990	0·1988	—0·0001	—	0·1987	0·1987	0
0·3	—	0·2957	—	—0·0001	0·2956	0·2955	—0·0001
0·4	0·3900	0·3896	—0·0001	—	0·3895	0·3894	—0·0001
0·5	—	0·4796	—	—0·0001	0·4795	0·4794	—0·0001
0·6	0·5652	0·5648	—0·0001	—	0·5647	0·5646	—0·0001
0·7	—	0·6443	—	—0·0001	0·6442	0·6442	0
0·8	0·7176	0·7174	—0·0001	—	0·7173	0·7174	+0·0001
0·9	—	0·7833	—	0	0·7833	0·7833	0
1·0	0·8409	0·8413	+0·0001	—	0·8414	0·8415	+0·0001

which must be used in the ordinary step-by-step integration. This can perhaps be made clearer by considering dynamical applications. In the ordinary step-by-step solution of dynamical equations, the time intervals must be so small that the velocities and displacements have only small percentage changes in each interval; for oscillatory motions the time intervals must be much smaller than the periods. However, a single analytical approximation can be used so long as the percentage changes of the parameters are small, and this may be true for intervals covering many periods.

An example illustrating what has been said is provided by R. A. Frazer's method of mean coefficients for the solution of sets of linear ordinary differential equations with variable coefficients. Here throughout each interval the coefficients are treated as constants and given values equal to their arithmetical mean values for the interval. The initial values of the dependent variables for the rth interval are obtained from the initial values for the first interval by continued matrix multiplication in Frazer's process; the introduction of matrices is convenient, but not essential. It should be emphasized that the method described in the present section applies to non-linear as well as to linear equations.

An aeronautical application of the analytical step-by-step solution is contained in an investigation of the free motion of a stable glider in an atmosphere of variable density (11). Here the rate of change of each dependent variable is expressed as the product of a linear function of these variables and of the square root of the relative air density. Since the relative density is a function of the dependent variables, the equations are non-linear. However, as it varies only relatively slowly it can be treated as constant during a time interval of many seconds. In the numerical example worked out in detail it was shown that satisfactory accuracy was obtained with an interval of 50 seconds whereas, the periodic time of the phugoid oscillation was 36 seconds.

A final and obvious remark is that the method described here is of no value unless the closed analytical solution is suitable for reasonably easy computation.

1.7. *Choice of a Step-by-Step Process.*

The circumstances in which the analytical step-by-step method can be used with advantage have been indicated in § 1.6. Further discussion relates to the numerical step-by-step processes and is restricted to one-point boundary conditions.

Provided that all the methods under consideration are reliable and of reasonable simplicity, the criterion for selection should be the attainment of given accuracy with minimum labour. In accordance with § 1.4 we shall assume that the dominant term in the error is $e_0 n^{-k}$, and that this represents the error with enough accuracy for the purpose of comparison of processes. Now we can easily ascertain the index k for any process, but to set bounds to e_0 would require detailed investigation in each instance. Manifestly, we have no right to assume that e_0 has like values for processes of differing indices. Nevertheless, we may fairly assume that a method of high index has been so designed that a given accuracy is attained with considerably wider intervals than would be required with processes of smaller indices. This advantage is, however, bought at the cost of greater labour per interval. For instance, the labour per interval in the Kutta process of index 4 is very much greater than in the Euler process of index 1.

An important consideration is that the interval used in the integration should not be larger than the interval of the tabulation required in the result unless this is very small, for it would almost certainly be wasteful to use elaborate methods to integrate accurately with wide intervals and then to use interpolation to fill out the tabulation.

The present discussion is inevitably inconclusive, but the writer's belief is that processes of low index are usually to be preferred. On the whole, he would favour a process of index 2, such as that described in § 1.3. This is much more accurate than the Euler process when the interval is the same while the extra labour is small.

1.8. *Two-Point Boundary Conditions.*

Step-by-step methods are not well adapted to the solution of differential equations with two-point boundary conditions, but they can be used. To take first the simplest case, suppose that x_1 is given for $t=a$, and that x_2 is given for $t=b$, these being the only dependent variables (*see* § 1.1). Then we may start a step-by-step integration at $t=a$ with the known value of x_1 and some trial value λ of x_2, which should be guessed or estimated on the best evidence available. The integration is carried through, and the value $x_2(b)$ found. Unless by a fortunate accident, this will not be the assigned value and the calculation must be repeated with another value for λ. The process must be continued until the true value of λ can be found by interpolation with the required accuracy. Clearly the process is relatively laborious.

The same general method could be used for two-point boundary problems with more dependent variables and for n-point boundary problems.

1.9. *Aeronautical Applications.*

Step-by-step integration of differential equations or sets of them has often been used in aeronautical investigations and some representative examples are included in the list of references (12), (13), (14), (15), (16). The authors of these papers have not always described the process of integration used, but it appears that these were usually of the simplest. Jones and Trevelyan (13) state, "the initial forces acting upon the aeroplane are supposed constant for one-tenth of a second; the change of the motion under the action of these forces during this time is then calculated, and taken to be the initial motion for the next tenth of a second, and so on."

Step-by-step integration has been most frequently applied to problems of looping, pull-outs from dives and stalled flight. In general, the accuracy of the data has been so low that much refinement in the integration would not have been justified.

Very extensive applications of step-by-step integration have been made to exterior ballistics (motion of projectiles, bombs and guided missiles), but further consideration of these is beyond the scope of this paper.

PART II.

CLASSIFICATION AND REVIEW OF METHODS FOR THE NUMERICAL INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS.

2.1. *Introduction.*

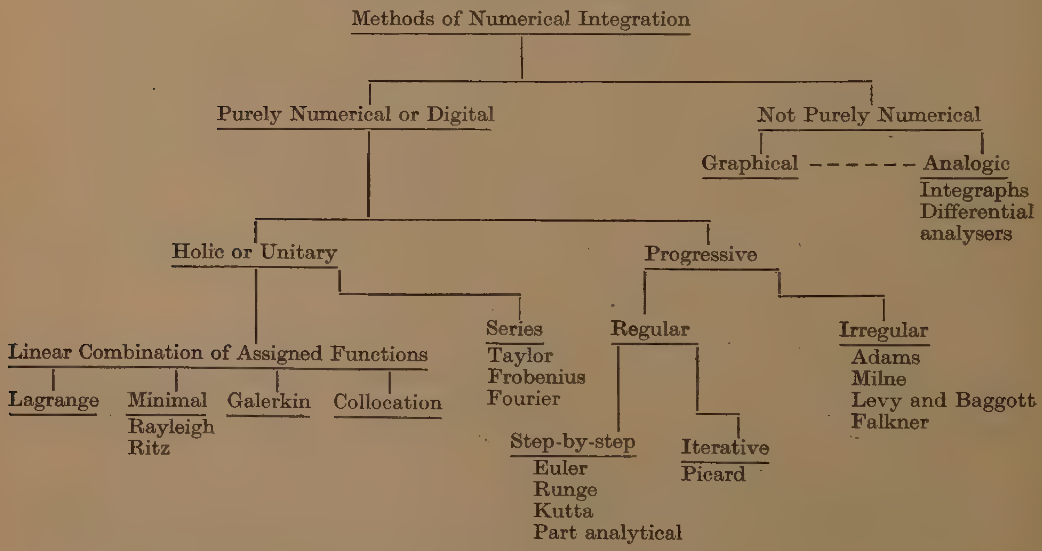
The aim is to classify methods of numerical solution of differential equations and sets, and to review existing methods very briefly.

2.2. *Classification of Methods.*

The first and broadest division is into purely numerical or digital methods and those which are not entirely numerical, though capable of giving numerical results (*see* the classification chart.—Table 2.2.1).

TABLE 2.2.1.

Classification Chart for Methods of Numerical Integration of Ordinary Differential Equations.


Methods of Numerical Integration
Purely Numerical or Digital
Not Purely Numerical
Graphical
Analogic
Integraphs
Differential analysers
Holic or Unitary
Progressive
Linear Combination of Assigned Functions
Lagrange
Minimal
Rayleigh
Ritz
Galerkin
Collocation
Series
Taylor
Frobenius
Fourier
Regular
Irregular
Adams
Milne
Levy and Baggott
Falkner
Step-by-step
Euler
Runge
Kutta
Part analytical
Iterative
Picard

Notes.—Series solutions could be logically included under " Linear Combinations of Known Functions," but are separated on account of the distinctive methods used. The methods called minimal depend on the use of a stationary property. The method of Picard is called iterative, since indefinitely repeated integration is used, and progressive, since each operation of integration is essentially progressive.

The methods which are not purely digital can be divided into analogic and graphical. Analogic methods depend on the use of a physical or kinematic analogy, and include the use of such instruments as the differential analyser of Bush. Graphical methods are, strictly, analogic and this is indicated by the dotted line in the chart.

The digital methods can be divided broadly into progressive and holic or unitary. In the progressive methods, the process of solution begins at one point and passes to other points in regular sequence; in some the whole process may be repeated until the required accuracy is obtained. The characteristic of the holic methods is that the range of integration is treated as a whole throughout the process of integration.

Progressive methods may be classified as regular, when the process of integration is the same throughout the range, and irregular. The irregular methods usually start with a very accurate computation of the solution over a short initial part of the range and then proceed step-by-step. The ordinary step-by-step and iterative methods are regular.

The holic methods, so far as they are known to the writer, depend either on approximate analytical solutions (series, etc.) or representation of the solution as a linear combination of known functions which satisfies the boundary conditions for all values of the parameters. The latter are so determined that an approximate solution is obtained.

2.3. *Brief Review of Methods.*

2.3.1. *Graphical Methods.*—These may be of value in rough work or in preliminary explorations (2), (5). The attainable accuracy is low.

2.3.2. *Analogic Methods.*—These are probably of most value for rather complicated equations or systems and when the range of integration is great. The attainable accuracy is not high.

2.3.3. *Regular Progressive Methods.*—These include the step-by-step methods, numerical or part analytical, discussed in Part I of this paper, and the iterative method of Picard. The latter is discussed at length by Scarborough (4).

2.3.4. *Irregular Progressive Methods.*—The prototype is the method of Adams (1) where the highly accurate solution in the first part of the range is obtained from Taylor's series. The solution proceeds by extrapolation and the notation of finite differences is adopted.

Adams' method is the only one described by Whittaker and Robinson (3) who declare roundly, "the best method of integrating differential equations numerically is one devised by J. C. Adams———." On the other hand Scarborough (4) makes the following critical comments, "the success of the method in starting a solution depends upon (i) the ease with which successive derivatives of the unknown function can be calculated, and (ii) the rapidity with which the Taylor series converges. If the successive derivatives are easily calculated and the Taylor series converges rapidly, the method furnishes the best means of starting a solution and should be used in preference to any other. But if, on the other hand, the successive derivatives are not easily calculated, or if the Taylor series is such that the interval of convergence is not easily determined, the method should not be used." "Because of the difficulty of calculating the successive derivatives of the air-resistance function the method of Adams is not suitable for starting the computation of trajectories." A refinement of Adams' method is described by Falkner (18).

The method of W. E. Milne (17) starts with a relatively accurate solution in an initial part of the range, obtained by the method of Adams, Runge-Kutta or other suitable method. The solution is extended by using two integrated forms of Newton's interpolation formula. An interesting feature of the method is that each value of a dependent variable is obtained by two processes and, if the results do not agree sufficiently closely, the interval must be reduced.

2.3.5. *Unitary or Holic Methods.*—These can be exemplified by minimal methods in which use is made of some stationary property of the solution. The approximate solution may be expressed as a linear combination of a finite number of known linearly independent functions so selected that the expression satisfies the boundary conditions for all values of the coefficients of some (or all) of the functions; these coefficients are then determined by the stationary property. The methods of Rayleigh and Ritz can be included here.

In the method of Galerkin (19), (20) the solution is expressed in the same way, but the coefficients are determined by making n linearly independent weighted means of the residual error in the differential equation zero, where n is the number of functions with unknown coefficients.

The method here called Lagrangian depends on finding a dynamical analogy of the differential equation or equations; when the equations are derived from a mechanical problem, the analogy is already provided. The solution is expressed in the same way as before and the unknown coefficients of the selected functions are regarded as generalised coordinates and are found from the Lagrangian dynamical equations. The minimal method may be regarded as a special case of the Lagrangian method and the Galerkin method becomes identical with the Lagrangian when the "weight functions" used in forming the mean errors are chosen in a particular manner.

In the method of collocation (21) the representation of the solution is the same as before but the unknown coefficients are found by making the residual error exactly zero at n selected points within the range of integration.

2.3.6. *Relaxation.*—The relaxation technique can be applied to aid in the solution of equations obtained by various methods of approximation, *e. g.*, the minimal. It is described by Southwell in two books (22), (23) and in numerous papers.

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LXIII. *The Electric Field of a Condenser of which One Plate is an Arc and the other a Radius of a Circular Cylinder.*

By N. Davy,
University College, Nottingham *.

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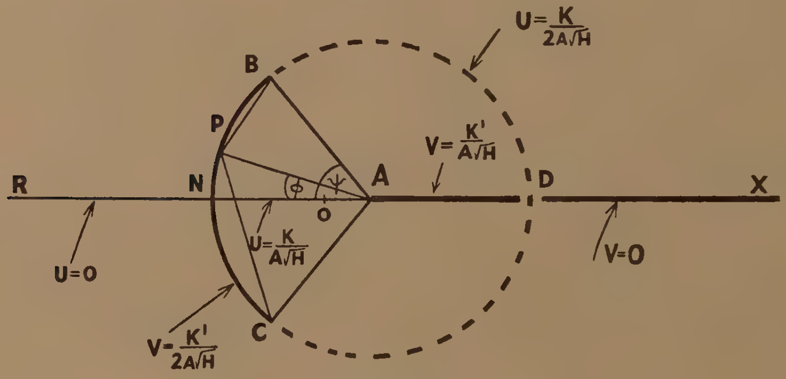
§ 1. *Introduction.*

It is proposed to investigate the electrical properties of a two-dimensional condenser, of which one plate is a radius AD of a circular cylinder and the other an arc BC of the same cylinder, facing the end of the first plate, symmetrically (see fig. 1).

Fig. 1.



The z-plane.

The electrodes are (a) the radius AD, (b) the circular arc BNC, and (c) the line DX extending to infinity. AD is insulated from DX at D and there is a cut along the arc BDC, impassable to lines of force. The origin of coordinates is not A but O.

The field, with certain limitations, applies to both the internal and external regions. The problem appears to be solved by the conformal transformation

$$z=\wp(AW), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\wp(AW)$ denotes the Weierstrassian elliptic function of (AW). $W=U+iV$, where U is the stream function and V the electrostatic potential. A is a constant to which a physical meaning is given later and $z=x+iy$, as usual.

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* Communicated by the Author.

If we adopt Halphen's definition of $\wp(\mathrm{AW})$, equation (1) becomes

$$\mathrm{AW}=\int_z^\infty \frac{dz}{\{4(z-e_1)(z-e_2)(z-e_3)\}^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The present problem is solved by taking the case when one of the quantities e_2 is real and the other two e_1 and e_3 are complex and conjugate. Thus let

$$e_2=+2\alpha, \quad e_1=-\alpha+i\beta \quad \text{and} \quad e_3=-\alpha-i\beta, \quad . \quad . \quad . \quad (3)$$

where α and β are real. e_1 and e_3 are the values of z at the ends of the arc BC, in fig. 1, and e_2 is the value of z at the centre A of the circle, the radius AD of which forms the other electrode.

For our present purpose it appears better to transform equation (1) to Jacobian elliptic functions by writing

$$z=e_2+\mathrm{H}\left\{\frac{1+\mathrm{cn}\, 2\mathrm{A}\sqrt{\mathrm{H}}\mathrm{W}}{1-\mathrm{cn}\, 2\mathrm{A}\sqrt{\mathrm{H}}\mathrm{W}}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where H is a constant, which proves to be the radius of the circle BCD. The modulus of these elliptic functions is k, where

$$k^2=\frac{1}{2}-\frac{3e_2}{4\mathrm{H}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Fig. 1 represents the z-plane. The origin of coordinates O is not at the centre of the circle. $\mathrm{OA}=2\alpha=e_2$. Since at B $z=-\alpha+i\beta$, the length of the radius of the circle is equal to $(9\alpha^2+\beta^2)^{\frac{1}{2}}$.

§ 2. *The Electrodes, Equipotentials, and Lines of Force.*

(a) The equipotential $\mathrm{V}=0$. In equation (4) put $\mathrm{W}=\mathrm{U}+i\mathrm{V}$ and then $\mathrm{V}=0$. (4) becomes

$$z=e_2+\mathrm{H}\left\{\frac{1+\mathrm{cn}\, 2\mathrm{A}\sqrt{\mathrm{H}}\mathrm{U}}{1-\mathrm{cn}\, 2\mathrm{A}\sqrt{\mathrm{H}}\mathrm{U}}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

When $\mathrm{U}=0$, $z=+\infty$ and when

$$2\mathrm{A}\sqrt{\mathrm{H}}\mathrm{U}=\mathrm{K}, \; z=e_2+\mathrm{H}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

so that at all points along the real axis in fig. 1, from the point D to plus infinity, the potential V is zero.

(b) The line of force $\mathrm{U}=0$.

If, in equation (4), we put $\mathrm{U}=0$ it becomes

$$z=e_2+\mathrm{H}\left\{\frac{1+\mathrm{cn}\, i2\mathrm{A}\sqrt{\mathrm{H}}\mathrm{V}}{1-\mathrm{cn}\, i2\mathrm{A}\sqrt{\mathrm{H}}\mathrm{V}}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Since $$\mathrm{cn}\,(iu, k)=\frac{1}{\mathrm{cn}\,(u, k')}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

equation (8) becomes

$$z=e_2+\mathrm{H}\left\{\frac{\mathrm{cn}\,(2\mathrm{A}\sqrt{\mathrm{H}}\mathrm{V}, k')+1}{\mathrm{cn}\,(2\mathrm{A}\sqrt{\mathrm{H}}\mathrm{V}, k')-1}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

When $\mathrm{V}=0$, $z=-\infty$, and when

$$2\mathrm{A}\sqrt{\mathrm{H}}\mathrm{V}=\mathrm{K}', \; z=e_2-\mathrm{H}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Hence the real axis from the point N to minus infinity is the line of force $U=0$.

(*c*) The equipotential $V=\dfrac{K'}{A\sqrt{H}}$ e.s.u.

If, in (4), we put $V=\dfrac{K'}{A\sqrt{H}}$ we get, since $\operatorname{cn}(u+2iK')=\operatorname{cn} u$,

$$z=e_2+H\left\{\frac{1-\operatorname{cn} 2A\sqrt{H}U}{1+\operatorname{cn} 2A\sqrt{H}U}\right\}. \qquad (12)$$

When $U=0$, $z=e_2$ and when

$$2A\sqrt{H}U=K,\ z=e_2+H. \qquad (13)$$

Hence the radius AD in fig. 1 is the equipotential $V=\dfrac{K'}{A\sqrt{H}}$

(*d*) In a similar manner the radius AN in fig. 1 can be shown to be the line of force

$$U=\frac{K}{A\sqrt{H}}, \qquad (14)$$

and the arc BNC to be the equipotential

$$V=\frac{K'}{2A\sqrt{H}}. \qquad (15)$$

§ 3. *Survey of the Problems Solved by the Transformation.*

The results obtained in § 2 show that the transformation represented by equations (1) and (4) solves several problems, even when we consider its application to electrostatics only. First it solves the case of a circular cylinder of dielectric with centre A carrying as electrodes (*a*) a radius AD, and (*b*) the symmetrical arc BNC of fig. 1. These two form a condenser restricted to the region inside the cylinder. Then, by taking half of the system, it is seen that (4) also solves the case of a semicircular cylindrical condenser such as NBDAN in fig. 1, with the radius AD forming one electrode and the arc BN the other. A third case solved is that of a condenser formed by a finite plane plate AD as one electrode and a collinear semi-infinite plane plate DX as the other. The field of force is not unlimited in extent. There is a cut, *i. e.*, an impassable gap formed by the circular arc BDC, across which no lines of force can pass. They must all proceed across the arc BNC, which they intersect normally. A fourth case solved is that of a condenser of which one electrode is the semi-infinite plate DX and the other the circular arc BC. The region inside the circle BNCD is excluded from the field. A fifth case is that in which one side of AD is one plate and and one side of DX the other. In a sixth case one half of the arc BC, *i. e.* the arc BN, forms one electrode and one side of DX forms the other (see figs. 1 and 5).

§ 4. *Electric Intensity and Surface Density on the Electrodes.*

The electric intensity or field strength

$$F=\left|\frac{dW}{dz}\right|. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

If we use equation (2), this gives

$$F=\left|\frac{1}{\wp'(AW)}\right|=\left|\frac{-1}{A\{4(z-e_1)(z-e_2)(z-e_3)\}^{\frac{1}{2}}}\right|. \quad . \quad . \quad . \quad (17)$$

The real part is therefore

$$F=\left|\frac{dW}{dz}\right|=\frac{1}{2A\{AP\,.\,BP\,.\,CP\}^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

i.e., it is inversely proportional to the square-root of the product of the distances of any point P from the points A, B and C, respectively.

If, instead of equation (2), we use (4)

$$z=e_2+H\left\{\frac{1+\text{cn}\,2A\sqrt{H}W}{1-\text{cn}\,2A\sqrt{H}W}\right\}$$

and

$$\frac{dz}{dW}=\frac{-4H^{\frac{3}{2}}A\,\text{sn}\,2A\sqrt{H}W\,.\,\text{dn}\,2A\sqrt{H}W}{\{1-\text{cn}^2\,2A\sqrt{H}W\}^2}. \quad . \quad . \quad . \quad (19)$$

Inverting gives

$$\frac{dW}{dz}=\frac{\{1-\text{cn}^2\,2A\sqrt{H}W\}^2}{4AH^{\frac{3}{2}}\,\text{sn}\,2A\sqrt{H}W\,.\,\text{dn}\,2A\sqrt{H}W}. \quad . \quad . \quad . \quad . \quad (20)$$

Either (17) or (20) shows that at the points A, B and C, the field is infinite as expected.

The surface density at any point is $\sigma=\frac{F}{4\pi}$.

(*a*) Surface density at P, on the electrode BC (see fig. 1). By equations (17) and (18) the surface density at P

$$=\sigma=\frac{F}{4\pi}=\frac{1}{8\pi A\{AP\,.\,BP\,.\,CP\}^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad (21)$$

Now the distance AP=H is constant in length, hence

$$\sigma=\frac{1}{8\pi A\sqrt{H}\{BP\,.\,CP\}^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Let the angle BAN$=\psi$ and PAN$=\phi$. Then by elementary geometry we can prove that

$$\{BP\,.\,CP\}^{\frac{1}{2}}=\sqrt{2}H\{\cos\phi-\cos\psi\}^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad (23)$$

so that

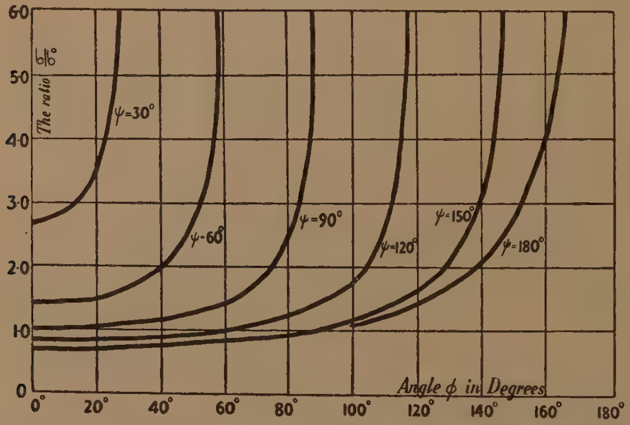
$$\sigma=\frac{1}{8\sqrt{2}\pi AH^{\frac{3}{2}}\{\cos\phi-\cos\psi\}^{\frac{1}{2}}}=\frac{\sigma_0}{\{\cos\phi-\cos\psi\}^{\frac{1}{2}}}, \quad . \quad . \quad (24)$$

where
$$\sigma_0=\frac{1}{8\sqrt{2}\pi A H^{\frac{3}{2}}}, \quad \ldots \ldots \ldots \quad (25)$$
which is a constant. Also $\cos\psi$ is a constant in any given problem and $\cos\phi$ is a variable.

Suppose we are considering the problem in which the arc BNC is one electrode and the radius AD the other, then, by equation (15), the potential difference between them may be called V_0 and
$$V_0=\frac{K'}{2A\sqrt{H}}. \quad \ldots \ldots \ldots \quad (26)$$

In this equation K' is a function of H and ψ, so that we cannot express σ_0 as a function of H and V_0 alone. To show the variation of σ over the face of the electrode BNC, it will perhaps suffice to keep σ_0 constant and exhibit graphs in which ϕ is the abscissa and σ/σ_0 the ordinate (see fig. 2) in which a few special values of σ are given, for various lengths of arc, *i. e.* of ψ and angular coordinates ϕ of the point P.

Fig. 2.



Showing the variation of surface density at points along the arc NB in fig. 1, assuming constant values of A and H.

By making the substitution $y=\cos\phi$ and $x=\frac{1}{\sigma^2}$ we can re-write equation (24) in the form
$$y=\sigma_0^2x+\cos\psi, \quad \ldots \ldots \ldots \quad (27)$$
giving a straight-line graph.

§ 5. *Total Charges on Electrodes.*

The total charge Q per cm. depth on any length of any electrode is equal to $\int_{z_1}^{z_2}\sigma\,.\,dz$.

Hence
$$Q=\int\sigma\,.\,dz=\frac{1}{4\pi}\left|\int_{z_1}^{z_2}\frac{dW}{dz}dz\right|=\frac{W_2-W_1}{4\pi}. \quad \ldots \quad (28)$$

To find W, use equation (4) giving

$$\operatorname{cn} 2A\sqrt{H}W=\frac{z-H-e_2}{z+H-e_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and

$$W=\frac{1}{2A\sqrt{H}}\operatorname{cn}^{-1}\frac{z-H-e_2}{z+H-e_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and the charge on one side of any electrode, per cm. depth, is

$$Q=\frac{1}{8\pi A\sqrt{H}}\left\{\operatorname{cn}^{-1}\frac{z_2-H-e_2}{z_2+H-e_2}-\operatorname{cn}^{-1}\frac{z_1-H-e_2}{z_1+H-e_2}\right\}. \quad . \quad . \quad (31)$$

Thus, to find the charge on one side only of BC, write $e_2=+2\alpha$, $z_2=-\alpha+i\beta$ and $z_1=-\alpha-i\beta$. This gives

$$Q=\frac{1}{8\pi A\sqrt{H}}\left\{\operatorname{cn}^{-1}\frac{i\beta}{H-2\alpha}-\operatorname{cn}^{-1}\frac{-i\beta}{H-2\alpha}\right\}. \quad . \quad . \quad . \quad (32)$$

Then, since $\operatorname{cn}(u+2K)=-\operatorname{cn} u$, where u has any value

$$Q=\frac{K}{4\pi A\sqrt{H}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

We reject the solution $Q=\frac{K'}{4\pi A\sqrt{H}}$ because it decreases when the angle ϕ increases. It is really the charge on the complementary arc BCD in the case when that arc becomes an electrode. The charge on one side of AD is found to be one-half of the charge on one side of BNC, as expected.

§ 6. *Capacities and Resistances per Unit Length.*

The capacity per cm. depth between the inner side of BC and the whole of AD is found by dividing the appropriate charge by the potential difference. By equation (15) the potential difference is $\frac{K'}{2A\sqrt{H}}$. The charge, by (33), on one side of BNC is $\frac{K}{4\pi A\sqrt{H}}$.

Dividing, we find the capacity to be

$$C=\frac{K}{2\pi K'} \text{ e.s.u.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

In the corresponding electrodynamic case, the electric resistance between the same two electrodes is

$$R=\frac{\rho}{4\pi C}=\frac{\rho K'}{2K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

in the same units as ρ, the resistivity. In each case the medium is restricted to that inside the circle BNCD of fig. 1. The external region is "out of bounds." The capacity between half of one side of BC and the whole of one side of AD is $\frac{K}{4\pi K'}$, and the corresponding resistance is

$$\frac{\rho K'}{K}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Now, by equation (28),

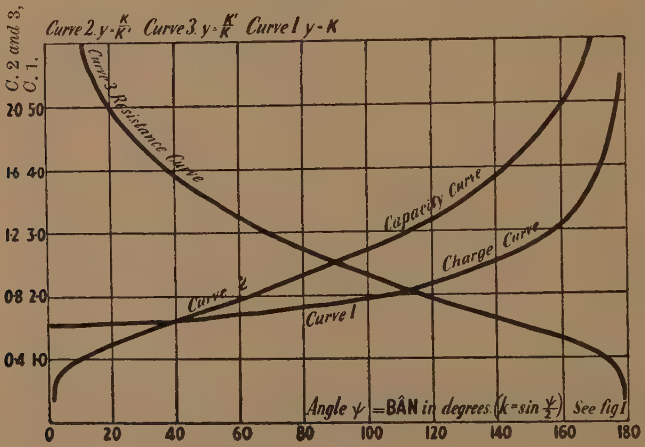
$$k^2=\frac{1}{2}-\frac{3e_2}{4\mathrm{H}}=\frac{1}{2}-\frac{3\alpha}{2\mathrm{H}}.$$

Also, in fig. 1,
$$\cos\psi=\frac{3\alpha}{\mathrm{H}}=1-2\sin^2\frac{\psi}{2}.$$

Hence
$$k=\sin\frac{\psi}{2}. \qquad \qquad (37)$$

Graphs showing how Q, R and C vary with ψ are shown in fig. 3. Capacities and resistances per cm. between (a) AD and DX via BNC, and between (b) BNC and DX, in both cases involving the region external to the circle BNCD, are calculated in a similar manner to the above.

Fig. 3.



Curve 1.—Relation of the charge on BC to half the angle it subtends at A. Multiply ordinates by $\frac{1}{4\pi\mathrm{A}\sqrt{\mathrm{H}}}$ to get the charge per cm. in e.s.u.

Curve 2.—Relation of the capacity between the inner side of BC and the whole of AD. Multiply ordinates by $\frac{1}{2\pi}$ to get the capacity per cm. in e.s.u.

Curve 3.—Relation of the resistance between the inner side of BC and the whole of AD. Multiply ordinates by $\frac{\rho}{2}$ to get the resistance per cm.

§ 7. *Mechanical Forces on Small Bodies in the Field.*

The mechanical force on a small electric dipole of constant electric moment is proportional to $\left|\frac{d\mathrm{F}}{dz}\right|$. Also $\frac{d\mathrm{F}}{dz}=\frac{\mathrm{F}d(\log\mathrm{F})}{dz}$. Then, by the aid of (17), we soon find that

$$\left|\frac{d\mathrm{F}}{dz}\right|=\left|\frac{\mathrm{F}d(\log\mathrm{F})}{dz}\right|=\left|\frac{\Sigma(z-e_2)(z-e_3)}{4\mathrm{A}\{(z-e_1)(z-e_2)(z-e_3)\}^{\frac{3}{2}}}\right|, \qquad (38)$$

where $e_1=-\alpha+i\beta$, $e_2=+2\alpha$, and $e_3=-\alpha-i\beta$.

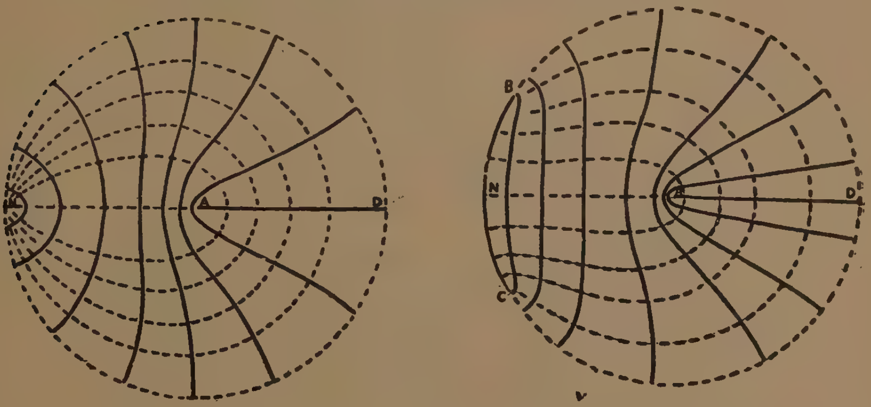
The mechanical force on a small dipole whose electric moment is wholly induced is proportional to $\left|\frac{dF^2}{dz}\right|$.

Again, using (37), we find that

$$\left|\frac{dF^2}{dz}\right|=\left|\frac{2F^2d(\log F)}{dz}\right|=\left|\frac{-(3z^2+\Sigma e_1e_2)}{4A^2(z-e_1)^2(z-e_2)^2(z-e_3)^2}\right|. \quad . \quad (39)$$

Fig. 4.

Internal fields for various values of $\widehat{BAN}=\psi$.
(*a*) $\psi=0$, (*b*) $\psi=30°$, (*c*) $\psi=60°$, (*d*) $\psi=90°$.

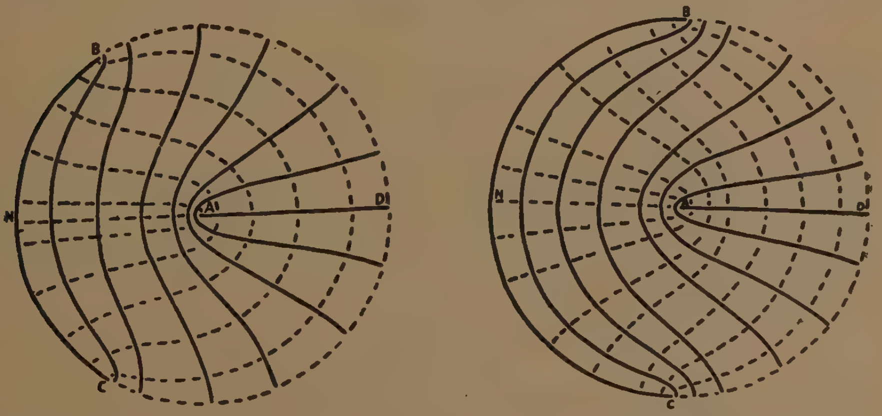


(*a*)

Internal field when $\widehat{BAN}=\psi=$zero.

(*b*)

Internal field when $\widehat{BAN}=\psi=30°$.



(*c*)

Internal field when $\widehat{BAN}=\psi=60°$.

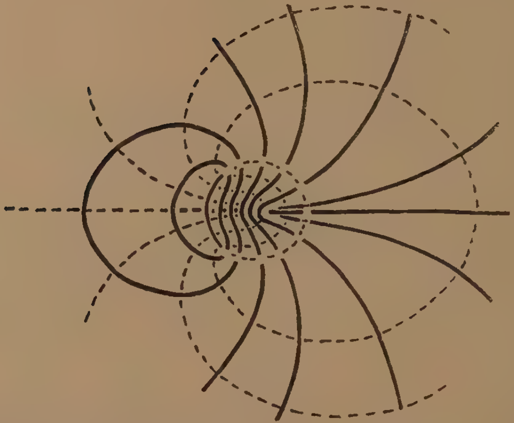
(*d*)

Internal field when $\widehat{BAN}=\psi=90°$.

Both $\left|\frac{dF}{dz}\right|$ and $\left|\frac{dF^2}{dz}\right|$ become infinite at the edges $z=e_1$, e_2 and e_3.

Experimentally obtained equipotentials for various cases are shown in figs. 4 *a*, *b*, *c*, *d*, and 5.

Fig. 5.



Field between the electrodes AD and DX when there is a circular cut BDC, impassable to lines of force.
(For the lettering see fig. 1. $\psi=45°$ approx.)

LXIV. *Synchronization of Controlled Relaxation Oscillators.*

By O. I. Butler, M.Sc.
(Senior Lecturer in Electrical Engineering, University of Liverpool)* .

[Received April 6, 1947.]

(1) *Summary.*

In a previous paper † Harker has shown that synchronization of the linear time-base type of controlled relaxation oscillator is practically realizable. The present paper constitutes a quantitative investigation of the phenomena of synchronization for controlled relaxation oscillators in general, and demonstrates that a mode of synchronization materially different from that envisaged by Harker and other writers ‡ is practically realizable. A measure of the rigidity of the "synchronous lock" is obtained, which allows the effect of changing conditions of operation to be more definitely assessed.

(2) *Introduction.*

An oscillator may be classed as being of the relaxation type when its oscillations are characterized by a sudden change, or relaxation, from one state of unstable equilibrium to another. Thus, the output voltage waveform of a relaxation oscillator is essentially non-sinusoidal. A controlled relaxation (c.r.) oscillator is one in which an injected oscillation is used to control the relaxation oscillation frequency to an integral sub-multiple of the injected oscillation frequency.

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* Communicated by the Author.

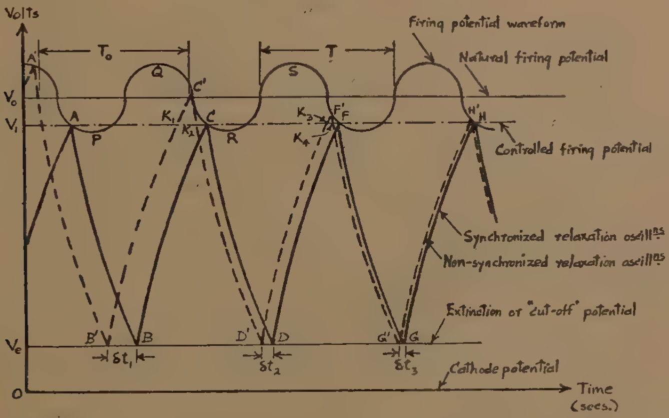
† G. F. H. Harker, Phil. Mag. xxvi. p. 193 (1938).

‡ *E.g.* O. S. Puckle, 'Time Bases,' p. 131 (Chapman & Hall, Ltd., 1938).

Harker has described a simple type of c.r. oscillator and discussed in some detail the manner in which it functions. It is not proposed to repeat this discussion, or to extend it to cover the more complex types of c.r. oscillator such as that due to Puckle*. It is sufficient to note that the phenomena of synchronization of many c.r. oscillators, of different physical form, may be investigated by a consideration of the anode potential waveforms of the discharge, or " trigger ", valve with respect to the potential of the cathode.

Thus, in fig. 1, V_0 represents the natural firing potential and V_e represents the potential of the anode at which the valve returns to its non-conducting state. The controlled relaxation oscillations of the anode potential are represented by ABCDF for synchronized operation between the voltage levels V_e and V_1, where the latter is evidently the controlled firing voltage level.

Fig. 1.


Volts
T_0
T
A'
Q
S
C'
V_0
V_1
A
P
K_1
K_2
C
R
K_3
K_4
F'
F
H'
H
Firing potential waveform
Natural firing potential
Controlled firing potential
Synchronized relaxation oscill^{ns}.
Non-synchronized relaxation oscill^{ns}.
Extinction or "cut-off" potential
V_e
B'
B
D'
D
G'
G
δt_1
δt_2
δt_3
Cathode potential
0
Time (secs.)

In practice, ABCDF may be composed of falling voltage portions which follow an exponential law and rising voltage portions which are linear with respect to time. Alternatively, both the falling and rising voltage portions may follow exponential laws, depending upon the particular application of the oscillator.

The special attribute of the oscillator is that when the natural periodic time T_0 (see fig. 1) of the relaxation oscillations approximates to some integral multiple of the periodic time T of the firing potential oscillations PQRS, the relaxation oscillations assume automatically a periodic time T_r which coincides exactly with this particular multiple of T. Thus,

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* *Loc. cit.*, p. 30.

$T_r = mT$, where m is any integer, and the oscillator provides a stable method of frequency demultiplication irrespective of limited changes in T or T_0; the particular value of $m=1$ being used in fig. 1 for the purpose of illustration.

In general, an interval of transition from non-synchronized to synchronized operation occurs after switching on the oscillator, due to an initial departure from the phase relationship between PQRS and the relaxation oscillations appropriate to synchronized operation. Such an interval of transition is illustrated by the relaxation oscillations A′B′C′D′E′ as the appropriate phase relationship is approached automatically with each successive oscillation.

(3) *Basic Relationships.*

The basis of the present method of investigation is that instead of studying the deviation in voltage level of the successive firing points A′, C′, F′, H′, from the ultimate voltage level V_1, the deviation in time of the discharge voltage portions A′B′, C′D′, etc., from the ultimate portions AB, CD, etc., respectively, is examined. Such deviations in time may be represented by $\delta t_1, \delta t_2, \ldots \delta t_n$, respectively.

The fundamental property utilized in the present investigation is that the deviation in time of the rising voltage portions B′C′, D′F′, etc. from the ultimate portions BC, DF, etc., respectively, is also $\delta t_1, \delta t_2, \ldots \delta t_n$. This may be readily proved, irrespective of whether the relaxation oscillations are composed entirely of portions which follow exponential laws or partly of portions which are linear with respect to time.

It follows that the relationship between δt_1 and δt_2 may be derived from the terminal conditions of the first cycle of the relaxation oscillations; *i. e.*, by considering the double triangular portion $C'CK_2K_1$ shown in fig. 1, where K_1 and K_2 are points on B′C′ and C′D′, respectively, at the common voltage level V_1.

An enlarged view of $C'CK_2K_1$ is shown for convenience of illustration in fig. 2, from which it will be seen that β represents the inclination of the firing potential waveform at C, whilst ρ and ϕ represent respectively the inclinations of the rising and falling portions of the synchronized relaxation oscillation at C. It may be readily proved that ρ and ϕ represent also the inclinations of the rising and falling portions of the non-synchronized relaxation oscillation at K_1 and K_2 respectively.

Let C'_1 be the point of intersection of the tangents $C_1'C$ and $C_1'K_2$ to the curves $C'C$ and $C'K_2$ at C and K_2 respectively; *i. e.* let $C_1'C$ and $C_1'K_2$ be straight lines of inclination β and ϕ respectively. Also, let $C_1'K_1'$ be a tangent to the curve $C'K_1$ when displaced through the horizontal distance K_1K_1'; *i. e.* let $C_1'K_1'$ be a straight line through C_1' of inclination ρ and intersecting K_1C at K_1' such that $K_1K_1' = \alpha_1$ and $K_1'C = \delta t_1'$, say. It is assumed that the firing potential does not attain a stationary (*i. e.*, maximum or minimum) value in the interval corresponding to $C'C$.

Evidently, in the triangle $C_1'K_1'C$,

$$\frac{\delta t_1'}{\sin(\beta-\rho)}=\frac{C_1'C}{\sin\rho},$$

and in the triangle $C_1'K_2C$,

$$\frac{\delta t_2}{\sin(\beta-\phi)}=\frac{C_1'C}{\sin\phi}.$$

Eliminating $C_1'C$ from these equations gives

$$\delta t_2=\frac{\sin\rho\sin(\beta-\phi)}{\sin\phi\sin(\beta-\rho)}\delta t_1',$$

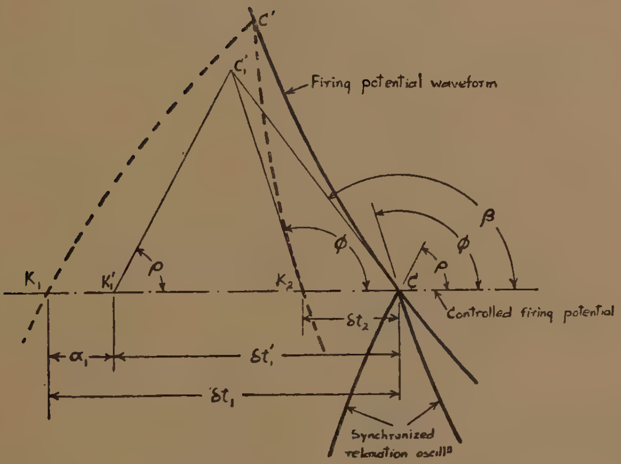
which, by expansion and simplification, becomes

$$\delta t_2=k\,.\,\delta t_1'=k(\delta t_1-\alpha_1), \qquad \text{(1)}$$

where

$$k=\frac{\cot\beta-\cot\phi}{\cot\beta-\cot\rho}. \qquad \text{(2)}$$

Fig. 2.


C′
C₁′
Firing potential waveform
β
φ
φ
ρ
ρ
K₁
K₁′
K₂
C
Controlled firing potential
δt₂
α₁
δt₁′
δt₁
Synchronized relaxation oscill^n

Similarly, the relationship between δt_2 and δt_3 may be derived from the terminal conditions of the second cycle of the relaxation oscillations; *i. e.*, by considering the double triangular portion $F'FK_4K_3$ shown in fig. 1, where K_3 and K_4 are points on $D'F'$ and $F'G'$ respectively at the common voltage level V_1.

An enlarged view of $F'FK_4K_3$ would be identical with that of $C'CK_2K_1$ shown in fig. 2, except for C', C, K_2, K_1, δt_1, α_1, $\delta t_1'$ and δt_2 being replaced by F', F, K_4, K_3, δt_2, α_2, $\delta t_2'$ and δt_3 respectively. It follows that by writing δt_2, α_2, $\delta t_2'$ and δt_3 in place of δt_1, α_1, $\delta t_1'$ and δt_2 in equation (1), the relationship between δt_2 and δt_3 is correctly given by

$$\delta t_3=k\,.\,\delta t_2'=k(\delta t_2-\alpha_2), \qquad \text{(3)}$$

the coefficient k, as given by equation (2), being unaltered.

Substituting for δt_2 from equation (1),

$$\delta t_3 = k^2 \delta t_1 - k\alpha_2 - k^2 . \alpha_1. \quad \ldots\ldots \quad (4)$$

It follows in a similar manner that the relationship between δt_3 and δt_4 may be derived from the terminal conditions of the third cycle of the relaxation oscillations, and is

$$\delta t_4 = k^3 . \delta t_1 - k\alpha_3 - k^2\alpha_2 - k^3\alpha_1,$$

whence, by the method of mathematical induction, the terminal conditions of the nth cycle of the relaxation oscillations permits the general equation

$$\delta t_{n+1} = k^n . \delta t_1 - k\alpha_n - k^2 . \alpha_{n-1} \ldots\ldots - k^n\alpha_1 \quad \ldots\ldots \quad (5)$$

to be derived.

Also, in general, the smaller the value of δt_p, where p is any integer from 1 to n, the smaller is the value of α_p; *i. e.*, $\alpha_n < \alpha_{n-1} \ldots\ldots < \alpha_1$. Consequently, α_p/α_{p-1} is a fractional quantity, and if K represents the smallest value it attains then $\alpha_2 \nless K\alpha_1$, $\alpha_3 \nless K\alpha_2 \nless K^2\alpha_1$, etc., and in general $\alpha_p \nless K^{p-1} . \alpha_1$. Alternatively, if K represents the largest value attained by α_p/α_{p-1}, then $\alpha_2 \ngtr K\alpha_1$, $\alpha_3 \ngtr K\alpha_2 \ngtr K^2 . \alpha_1$, etc., and in general $\alpha_p \nless K^{p-1} . \alpha_1$. It follows, by substitution for α_p in equation (5), that numerically

$$\delta t_{n+1} \ngtr k^n . \delta t_1 - [kK^{n-1} + k^2K^{n-2} + \ldots + k^{n-1}K + k^n]\alpha_1,$$

i. e.,

$$\delta t_{n+1} \ngtr k^n . \delta t_1 - k\frac{k^n - K^n}{k - K}\alpha_1 \quad \ldots\ldots \quad (6)$$

and, in particular, when $k = K$,

$$\delta t_{n+1} \ngtr k^n . \delta t_1 - nk^n\alpha_1. \quad \ldots\ldots \quad (6\,a)$$

It may be shown that equations (6) and (6 *a*) are valid irrespective of whether α_1 is a positive or a negative quantity.

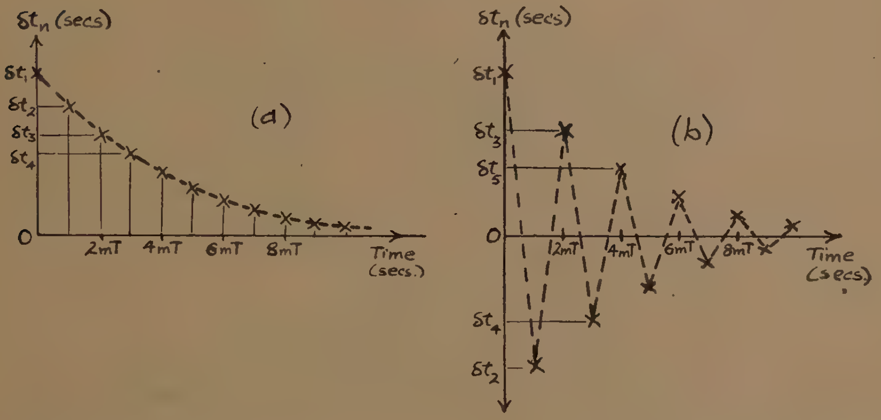
Thus, as n increases indefinitely, it will be evident that δt_{n+1}, and therefore δt_n, approaches zero value when k is a positive or negative fraction, bearing in mind that K is always a fractional quantity. It will be further evident that the same argument and conclusions are applicable when δt_1 has a negative value; *i. e.*, when the oscillation A′B′C′ occurs later than the synchronized oscillation ABC. Hence, in order that synchronization of the oscillator may be practically realizable, the terminal conditions of the relaxation oscillations must be such as to provide a value of k which lies within the limits of ± 1.

Obviously, provided $-1 < k < 1$, any small disturbance from the synchronized state is subjected to conditions which result in a reversion to the synchronized state. Thus, synchronization will persist, once it has been established, and the condition of $-1 < k < 1$ defines the limits for which synchronization constitutes a regime of stable equilibrium.

In particular, when k is a positive fraction it will be seen from equations (6) and (6 a) that δt_n approaches asymptotically to zero as n increases, to give the mode of synchronization depicted diagrammatically in fig. 3 a. On the other hand, when k is a negative fraction, the mode of synchronization depicted diagrammatically in fig. 3 b is obtained. They may be referred to as the "non-oscillatory" and "oscillatory" modes of synchronization, respectively. As far as the author is aware, the latter mode of synchronization has previously received little, if any, attention.

It is of interest to note that the foregoing condition of $-1<k<1$, which must be satisfied to obtain an inherent synchronizing action, is independent of the amplitude V_1-V_e ($=V_a$ say) and natural periodic time T_0, of the relaxation oscillation, and also of the amplitude v_a and periodic time T of the firing potential wave-form. Nevertheless, it must be borne in mind that the values of V_a, v_a and T may be such as to prevent synchronization at particular values of T_0, as demonstrated in some detail by Harker * in his concluding paragraphs.

Fig. 3.



(4) *General Conditions for Synchronization.*

It has been shown that the angles ϕ, ρ and β must be limited to those values which satisfy the condition of $-1<k<1$, in order that an inherent synchronizing action may be obtained. For some applications of the oscillator the duration of the falling voltage portion of the relaxation oscillation is very small (*e.g.* in the time base of a cathode-ray oscillograph), and it may reasonably be assumed that cot $\phi=0$, in which case equation (2) simplifies to

$$k=\cot\beta/(\cot\beta-\cot\rho). \qquad \qquad (2\,a)$$

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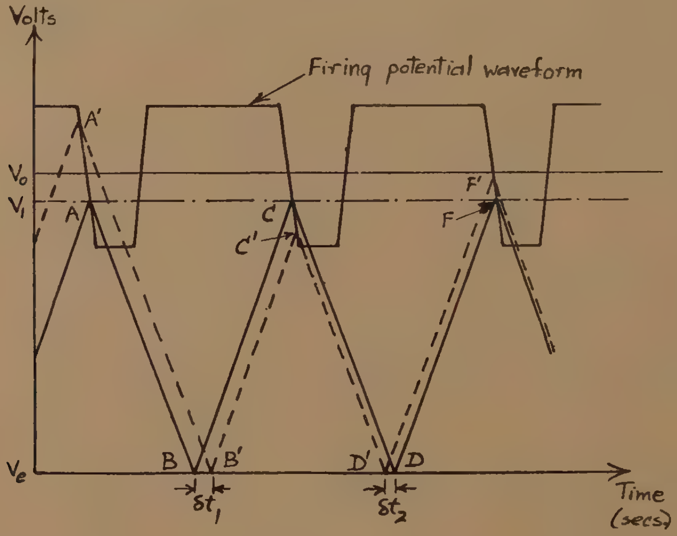
* *Loc. cit.*

However, when the oscillator is applied to frequency measuring equipment the duration of the falling voltage portion may be increased to extend the permissible frequency limits of operation of the oscillator. It is proposed, therefore, to investigate the operating conditions which give inherent synchronization when $\cot\phi$ has an appreciable finite value.

It may be noted at the outset that $\cot\rho$ is always a positive quantity and $\cot\phi$ is always a negative quantity. Also, it appears obvious that $\cot\beta$ in general is a negative quantity, in which case it may be noted that the denominator of the expression for k is a negative quantity.

First consider the case of $\cot\beta$ being a finite negative quantity; that is, the peaks of the relaxation oscillation wave-form occur on the negative-slope portions of the firing potential wave-form, as shown in fig. 1. It follows from equation (2) that $0<k<1$ when $\beta>\phi$. Also, $-1<k<0$ when $\beta<\phi$ and

Fig. 4.



$$\cot\beta<\tfrac{1}{2}(\cot\phi+\cot\rho). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Thus, the non-oscillatory mode of synchronization is practically realizable when $\beta>\phi$, whilst the oscillatory mode of synchronization is practically realizable when $\beta<\phi$ and condition (7) is satisfied.

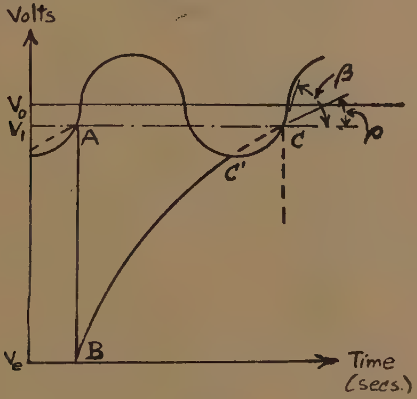
An example of $\beta>\phi$ is afforded by fig. 1, whilst fig. 4 affords an example of $\beta<\phi$ with condition (7) satisfied. For simplicity of illustration both the rising and falling voltage portions of the relaxation oscillations shown in fig. 4 are assumed to vary linearly with respect to time, and it clearly demonstrates how δt_n alternates in algebraical sign; *i. e.*, the first non-synchronized cycle A′B′C′ *lags* the synchronized cycle ABC by δt_1, whilst the second non-synchronized cycle C′D′F′ *leads* the synchronized cycle by δt_2.

Secondly, it will be seen from equation (2) that k=unity when $\beta=180°$. Thus, a stable state of synchronization is impracticable when the peaks of the relaxation oscillation wave-form occur on a zero slope portion of the firing potential wave-form.

Thirdly, when $\beta=90°$, $k=\cot\phi/\cot\rho$. Therefore, since $\cot\phi$ is always a negative quantity whilst $\cot\rho$ is always a positive quantity, only the oscillatory mode of synchronization can be obtained. Evidently this latter mode of synchronization is possible only when $\cot\phi$ is numerically less than $\cot\rho$.

Finally, consider the case of $\cot\beta$ being a possible positive quantity; that is, when the peak of the relaxation oscillation wave-form occurs on the positive-slope portion PQ of the firing potential wave-form shown in fig. 1. It follows from equation (2) that it is now impossible to obtain a value of k in the range $0<k<1$, but a value of k in the range $-1<k<0$ is obtained when condition (7) is satisfied.

Fig. 5.

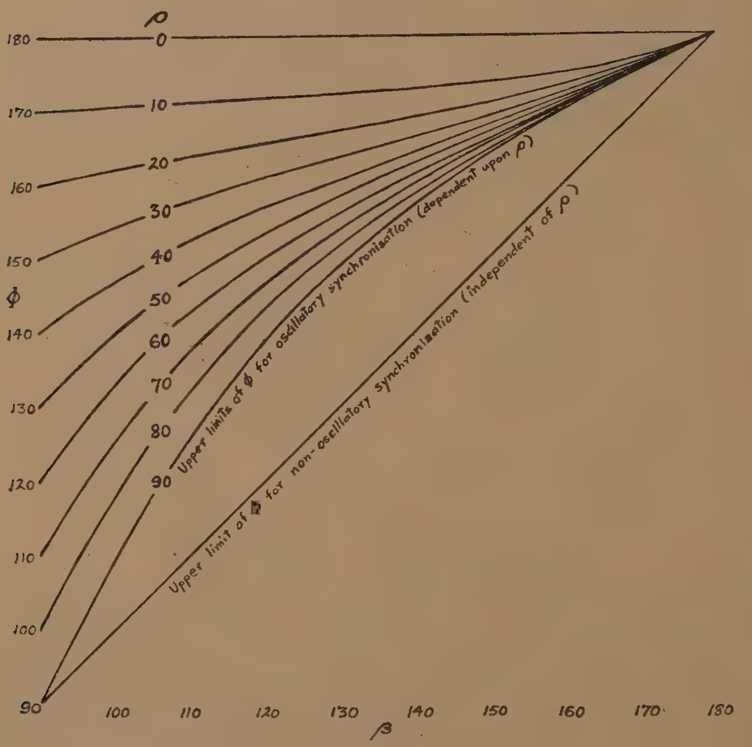


However, referring to fig. 5, it will be evident that it is a practical impossibility to satisfy even the condition of $\cot\beta<\cot\rho$ (*i.e.* $\beta>\rho$) because, instead of the peak of the relaxation oscillation wave-form coinciding with C, at the correct voltage level V_1, it will occur at the intervening point C′, at which point $\cot\beta$ has a negative or zero value depending upon the particular wave-form of the firing potential. Clearly, then, it is a practical impossibility to satisfy condition (7).

Thus, for the case of β having a positive value, stable synchronization of the oscillator is impracticable; *i.e.*, the peaks of the relaxation oscillation wave-form cannot occur successively on the positive slope portions of the firing potential wave-form and be a stable condition, no matter what may be the respective wave-forms of the relaxation oscillation and firing potential.

The foregoing results may be conveniently summarized by the graphs of fig. 6, which give the limiting values of ϕ, β and ρ for which synchronization is practically realizable. It will be seen that for any value of ρ the non-oscillatory mode of synchronization is practically realizable when the values of β and ϕ are such that the point (β, ϕ) lies below the straight-line graph of $\phi=\beta$. Also, depending upon the value of ρ, *e. g.* $\rho=60°$, the oscillatory mode of synchronization is practically realizable when the values of β and ρ are such that the point (β, ϕ) lies below the curve applicable to $\rho=60°$ and above the graph of $\phi=\beta$. The upper limits of the oscillatory mode of synchronization have, of course, been determined from condition (7).

Fig. 6.



(5) *Rigidity of Synchronous Lock.*

Obviously, the value of k given by equation (2) is a measure of the rigidity with which synchronization is inherently maintained, *i. e.* a measure of the rigidity of the "synchronous lock." The smaller the value of k, the more rapidly will any disturbance from the synchronized state be eliminated and synchronization recovered, *i. e.*, the more rigid is the synchronous lock.

In particular, the most powerful synchronous lock exists when $\phi=\beta$, since k is then zero. That is, any disturbance which may occur without upsetting the equality of ϕ and β fails to break the synchronous lock for any finite interval of time. Hence, the condition of $\phi=\beta$ represents the ideal to which operation of a relaxation oscillator should tend.

It is probably more convenient to refer to $1/k=\mathrm{S}$, say, as a measure of the rigidity of the synchronous lock. Thus, an increase in the value of S above unity corresponds to an increase in the rigidity of the synchronous lock. It follows that the relationship

$$\mathrm{S}=(\cot\beta-\cot\rho)/(\cot\beta-\cot\phi) \quad . \quad . \quad . \quad . \quad (8)$$

allows a quantitative investigation to be made of the efficacy of the synchronous lock as the peak of the synchronized relaxation oscillation wave-form changes its position on the firing potential wave-form, *i.e.*, as the natural relaxation frequency varies.

For example, consider the case of a relaxation oscillator used as the linear time-base of a cathode-ray oscillograph, the " fly-back " being assumed to be negligible and the firing potential wave-form being given by $v=\mathrm{V}_a+v_a\cos 2\pi ft$, where $f=1/\mathrm{T}$. The anode potential wave-forms are therefore as shown in fig. 7. It follows that

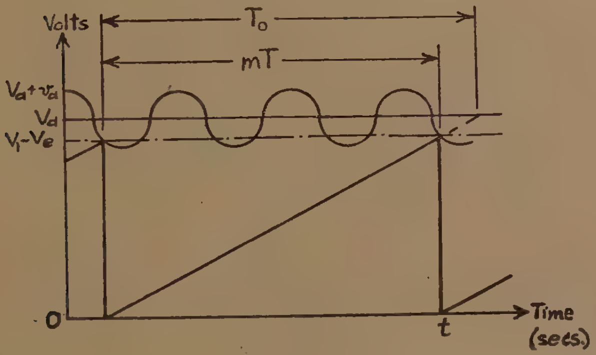
$$\tan\beta=dv/dt=-2\pi f v_a\sin 2\pi ft,$$

$$\tan\rho=(\mathrm{V}_a+v_a\cos 2\pi ft)/m\mathrm{T}$$

and

$$\cot\phi=0.$$

Fig. 7.



Hence, by substitution in equation (8),

$$\mathrm{S}=1-\frac{\tan\beta}{\tan\rho}$$

$$=1+\frac{2\pi mr\sin 2\pi ft}{1+r\cos 2\pi ft}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where $r=v_a/\mathrm{V}_a$.

The variation of S when $m=1$ and t varies from T/2 to T, also when $m=2$ and t varies from 3T/2 to 2T, and again when $m=3$ and t varies from slightly more than 5T/2 to 3T, is shown graphically in fig. 8; the value of r being arbitrarily fixed at 0·1. The reason why synchronization is not practically realizable for certain values of T_0 when $m=3$ and $r=0{\cdot}1$ has been discussed in some detail by Harker * in his concluding paragraphs.

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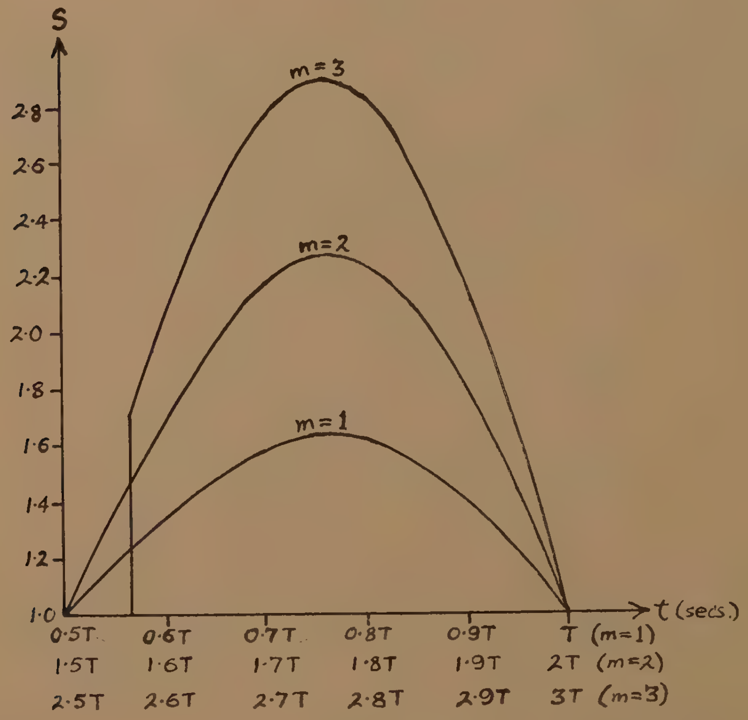
* *Loc. cit.*

The improvement in the rigidity of the synchronous lock with increase in m is clearly demonstrated by fig. 8. It will be further apparent from equation (9) that S is improved by increasing the ratio $r=v_a/V_a$ although the range of T_0 for which synchronization is practically realizable is thereby decreased.

It may be noted, further, that equation (8) allows a quantitative comparison of the efficacy of the synchronous lock obtained when different firing potential wave-forms are utilized. In particular, it may be noted that the presence of harmonics in the firing potential wave-form may produce values of S greater than unity outside the ranges of t shown in fig. 8 and also values of S less than unity inside these ranges.

Fig. 8.



I wish to take this opportunity of thanking Prof. L. Rosenhead, D.Sc., F.R.S., and Mr. J. Durnford, B.Sc., for a most helpful discussion of the subject.

LXV. *Theory of Plasticity—Elements of Simple Theory.*

By J. W. Roderick, M.A., M.Sc., Ph.D., A.M.I.C.E., A.F.R.Ae.S.*

[Received October 17, 1947.]

1. *Introduction.*

The behaviour of mild steel when strained beyond the yield point has long been studied by scientists and engineers. The subject has always received certain attention in connection with industrial processes such as rolling and cold bending, but it is only comparatively recently that structural engineers have become interested. This interest has arisen since they realized that the failure of a highly redundant steel structure, such as a welded rigid frame, is not imminent when yield of the material is first developed (1), but simply that a redistribution of bending moment is about to take place so that the structure may absorb energy in other parts not so highly stressed. In order to study this range it is necessary to consider the effect of the plastic properties of steel on the ability of components of the frame to resist straining actions when the yield point of the material has been exceeded at certain points. The purpose of this short paper is therefore to set out the elements of the simple theory of flexure, as established by Robertson and Cook (2), for mild steel members in the elasto-plastic condition, and to extend it to cover the case of bending combined with axial loading.

2. *The Transition of Mild Steel from the Elastic to the Plastic State.*

In the initial stages of straining the behaviour of the majority of materials used in engineering structures is very similar in that they are elastic and obey Hooke's Law, but as soon as any plastic deformation occurs the reaction to further straining depends upon the plastic properties which may vary widely from one material to another, and for a given material, according to the mechanical and heat treatment it has undergone. Hence in formulating any theory of plasticity for mild steel it is necessary to define the state of the material to which it refers.

A specimen of mild steel containing approximately 0·2 per cent carbon, chosen at random and tested in tension in an autographic spring loaded machine, would give a stress-strain diagram lying somewhere between those shown at (*a*) and (*b*) in fig. 1. If the specimen were one of coarse grain previously overstrained, the diagram would tend to be of the form (*a*) and plastic deformation would therefore be continuous over a considerable range of loading. On the other hand, a steel of fine grain undistorted by any cold working would show no indication of plasticity until complete breakdown of the crystal aggregate occurred, as instanced by the appearance of Lüders' lines. The stress-strain diagram would be as shown at (*b*). In this case the material is elastic up to the point A, known as the upper yield point, when the whole crystal aggregate behaves very much as a single crystal of iron and breakdown occurs in the form of slip along planes coinciding approximately with the direction of maximum shear stress, the intersection of these slip planes with the surface being the Lüders' lines referred to above. At the commencement of slip there is a drop in stress to the point B, the lower yield point, and further straining continues at approximately this value from B to C. Beyond the point C considerable strain hardening takes place, with the result that plastic deformation now requires the application of additional load.

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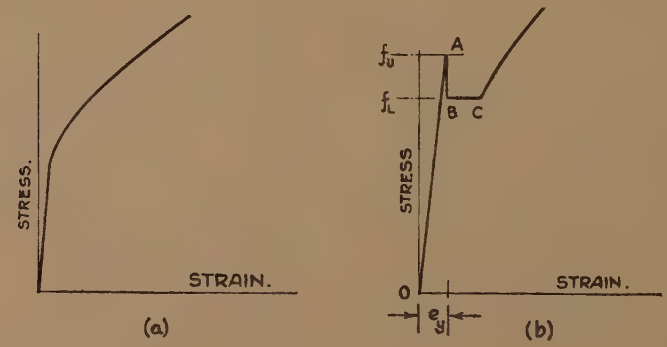
* Communicated by Dr. J. F. Baker.

For the present purpose it is proposed to confine attention to the annealed or normalized condition for which mild steel has a stress-strain

Fig. 1.



(a) (b)

relationship of the form shown at (*b*) in fig. 1. Let the stresses at the upper and lower yield points be f_U and f_L respectively, then assuming the strain to be the same at A and B, let its value be e_y. Hence, when the material is elastic as from 0 to A, $e<e_y$ where e is the actual strain; according to Hooke's law the stress in this range will always have the value

$$f=\mathrm{E}e. \qquad (1)$$

When yield has occurred and the material is behaving plastically, the condition of straining will be $e>e_y$ from B to C, so that in this part of the diagram

$$f=f_L. \qquad (2)$$

A series of assumptions can now be set down which provide the basis for the derivation of expressions for slope and deflection of a member bent to a sufficient extent to cause some of the material to be strained beyond the yield point.

3. *Theory of Simple Bending.*

The initial assumptions are :—

(1) the material obeys Hooke's law until the stress reaches the upper yield value ; on further straining the stress drops to the lower yield value and thereafter remains constant ;

(2) the upper and lower yield-stresses and the modulus of elasticity have the same values in compression as in tension ;

(3) the material is homogeneous and isotropic in both the elastic and plastic states ;

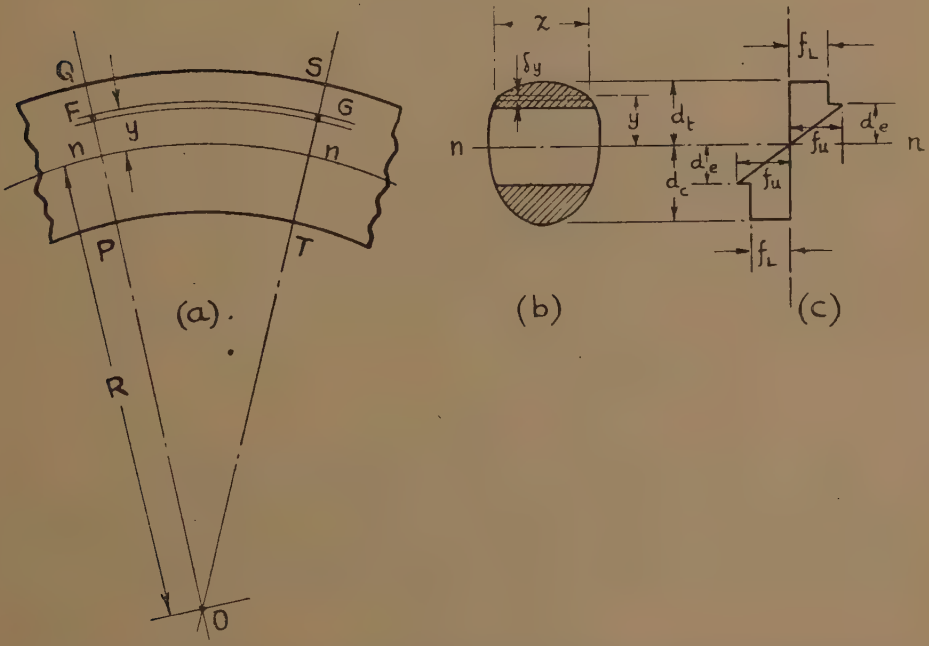
(4) plane transverse sections remain plane and normal to the longitudinal axis after bending ;

(5) when considering bending action only, there is no resultant axial force on the beam ;

(6) the cross-section of the beam is symmetrical about an axis through its centroid parallel to the plane of bending ;

(7) every layer of the material is free to expand and contract longitudinally and laterally under stress as if separated from the other layers.

Fig. 2.



In fig. 2 (*a*), let PQ and ST be the final positions of two adjacent cross-sections of a beam after bending by pure couples applied to its ends and so causing an increase in length of those fibres above the neutral axis and a decrease in those below. For any fibre FG distance *y* from the neutral axis it can be shown from the geometry of the deformation that it has been strained by an amount

$$e=\frac{y}{R}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where R is the radius of curvature of the member. This expression is unaffected by the condition of the material, and it is therefore in order to assume that when the end couples are of sufficient magnitude to cause part of the section to become plastic, FG may represent a layer of either elastic or plastic material. In the former case it is evident from equations (1) and (3) that the stress can be expressed as,

$$f=\mathrm{E}\frac{y}{\mathrm{R}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or in other words, the stress in the elastic region is proportional to the distance from the neutral axis. If, as assumed, the applied moment is such as to make f reach the maximum value f_{U} within the section, then for fibres more distant from the neutral axis the stress in them, having attained the value f_{U} will drop back to the lower yield stress f_{L}. It is further assumed that the stress in these fibres never exceeds the lower yield value f_{L}, even when yield has taken place across the whole cross-section. At first sight this might not appear to be justified, but it can be demonstrated that strains in excess of the value at C (fig. 1 (*b*)) do not occur until plasticity has almost reached the neutral axis, and it is evident from tests that the error involved in assuming that the stress in the yielded fibre never exceeds f_{L} is of little consequence.

On these assumptions it is possible to build up the complete diagram of stress distribution (fig. 2 (*c*)) for an elasto-plastic section (fig. 2 (*b*)). Consider an elementary strip of the cross-section, width z, thickness δ_y and distance y from the neutral axis, if the stress on it is some value f the corresponding force will be $fzdy$ and the moment about the neutral axis $fzydy$. Hence for the whole section the moment of resistance can be expressed as

$$\begin{aligned}\mathrm{M_R}&=\int_{-d_c}^{+d_t} fzydy\\ &=\int_{-d_e}^{+d_e} f_1zydy+\int_{d_e}^{d_c} f_2zydy+\int_{d_e}^{d_t} f_3zydy,\end{aligned}$$

the three integrals being necessary since there are one elastic and two plastic regions. In accordance with equations (4) and (2) respectively, the above stresses can be written

$$f_1=\frac{\mathrm{E}}{\mathrm{R}}y \text{ and } f_2=f_3=f_{\mathrm{L}}$$

and the expression becomes

$$\mathrm{M_R}=\frac{\mathrm{E}}{\mathrm{R}}\int_{-d_e}^{+d_e} zy^2dy+f_{\mathrm{L}}\int_{d_e}^{d_c} zydy+f_{\mathrm{L}}\int_{d_e}^{d_t} zydy.$$

It will be seen that

$\int_{-d_e}^{+d_e} zy^2dy = \mathrm{I}_e$, the second moment of area of the elastic region;

$\int_{d_c}^{d_c} zydy = \mathrm{A}_c\bar{y}_c$, the first moment of area of the plastic region on the compression side;

$\int_{d_e}^{d_t} zydy = \mathrm{A}_t\bar{y}_t$, the first moment of area of the plastic region on the tension side;

therefore,

$$\mathrm{M_R} = \frac{\mathrm{EI}_e}{\mathrm{R}} + f_\mathrm{L}(\mathrm{A}_t\bar{y}_t + \mathrm{A}_c\bar{y}_c).$$

From fig. 2 (*c*), it is evident that when $y = d_e$, $f = f_\mathrm{U}$ and from equation (4),

$$\frac{\mathrm{E}}{\mathrm{R}} = \frac{f_u}{d_e},$$

that is,

$$\mathrm{M_R} = f_u\frac{\mathrm{I}_e}{d_e} + f_\mathrm{L}(\mathrm{A}_t\bar{y}_t + \mathrm{A}_c\bar{y}_c).$$

Further, writing

$$\mathrm{Z}_e = \frac{\mathrm{I}_e}{d_e},$$

$$\mathrm{Z}_p = (\mathrm{A}_t\bar{y}_t + \mathrm{A}_c\bar{y}_c),$$

and finally,

$$\mathrm{M_R} = f_\mathrm{U}\mathrm{Z}_e + f_\mathrm{L}\mathrm{Z}_p \quad \ldots\ldots\ldots \quad (5)$$

The position of the neutral axis can be determined from the assumption that there is no resultant force on the beam; the total tensile force is

$$\mathrm{F}_t = f_\mathrm{L}\mathrm{A}_t + \int_0^{d_e} f_t z\,dy,$$

and, for the compression side,

$$\mathrm{F}_c = f_\mathrm{L}\mathrm{A}_c + \int_0^{d_e} f_c z\,dy,$$

where f_t and f_c are the stresses in any fibre on the tension and compression sides respectively in the elastic region. The condition for no resultant force is therefore

$$f_\mathrm{L}\mathrm{A}_t + \int_0^{d_e} f_t z\,dy = f_\mathrm{L}\mathrm{A}_c + \int_0^{d_e} f_c z\,dy,$$

but

$$f_t = \frac{\mathrm{E}}{\mathrm{R}}y_t = \frac{f_\mathrm{U}}{d_e}y_t \text{ and } f_c = \frac{f_\mathrm{U}}{d_e}y_c,$$

hence

$$f_\mathrm{L}(\mathrm{A}_t - \mathrm{A}_c) = \frac{f_\mathrm{U}}{d_e}\left\{\int_0^{d_e} zy_c\,dy - \int_0^{d_e} zy_t\,dy\right\},$$

and it is evident that

$\int_0^{d_e} zy_c\,dy=\mathrm{A}_\gamma\bar{y}_\gamma$, the first moment of area of the elastic region on the compression side ;

$\int_0^{d_e} zy_t\,dy=\mathrm{A}_\tau\bar{y}_\tau$, the first moment of area on the elastic region on the tension side.

The above equation can therefore be written as

$$f_{\mathrm{L}}(\mathrm{A}_t-\mathrm{A}_c)=f_{\mathrm{U}}(\mathrm{A}_\gamma\bar{y}_\gamma-\mathrm{A}_\tau\bar{y}_\tau), \quad . \ . \ . \ . \ . \ . \quad (6)$$

which together with equation (5) determines the position of the neutral axis.

As in the simple theory of elastic bending the deflection of an elasto-plastic beam can be obtained from the fact that its curvature at any section, irrespective of the condition of the material, is given with sufficient accuracy by the relation

$$\frac{1}{\mathrm{R}}=\frac{d^2y}{dx^2}, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (7)$$

and, as already demonstrated from equation (4),

$$\frac{\mathrm{E}}{\mathrm{R}}=\frac{f_{\mathrm{U}}}{d_e},$$

therefore

$$\frac{d^2y}{dx^2}=\frac{f_{\mathrm{U}}}{\mathrm{E}d_e}.$$

With the aid of equation (5) it will usually be possible to express d_e as a function of x, the longitudinal distance of the section from a given origin, namely

$$d_e=\phi(x),$$

so that

$$\frac{d^2y}{dx^2}=\frac{f_{\mathrm{U}}}{\mathrm{E}\phi(x)}. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (8)$$

Integrating

$$\frac{dy}{dx}=\frac{f_{\mathrm{U}}}{\mathrm{E}}\int\frac{dx}{\phi(x)}+\mathrm{A}, \quad . \ . \ . \ . \ . \ . \ . \ . \quad (9)$$

and repeating the process, the expression for deflection is obtained,

$$y=\frac{f_{\mathrm{U}}}{\mathrm{E}}\int\int\frac{dx}{\phi(x)}dx+\mathrm{A}x+\mathrm{B}, \quad . \ . \ . \ . \ . \ . \quad (10)$$

the constants being evaluated from known boundary conditions.

4. *Bending combined with Axial Loading.*

In this section it is proposed to examine the case of a compressive stress of uniform intensity superimposed on a bending stress distribution where the combined effect is sufficient to produce plastic zones. Referring to fig. 3, consider the section of a member subjected to an axial compressive force P and bending moment M; then, for elastic conditions,

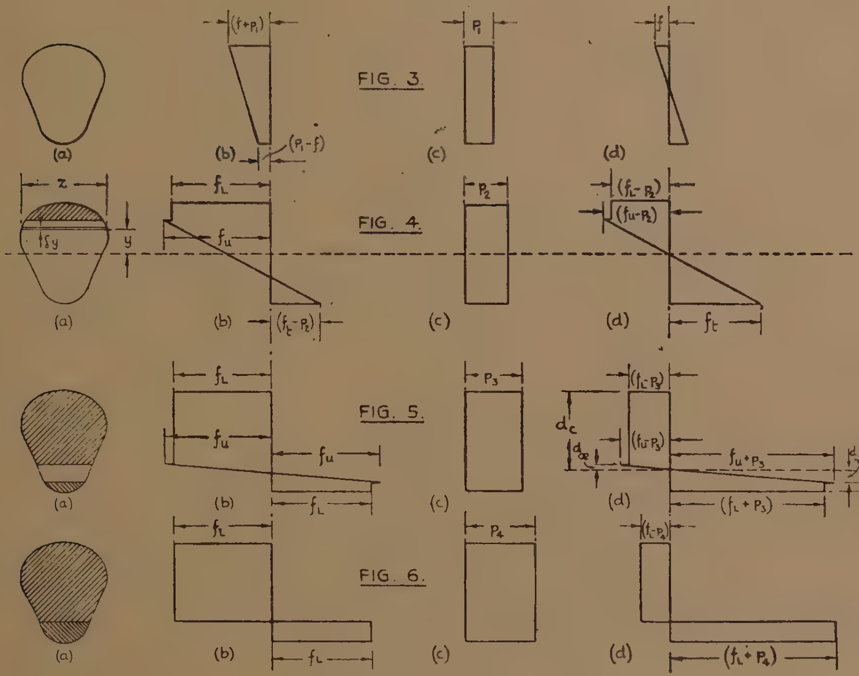
$$P = Ap_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and

$$M = fZ, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where A is the area, and Z the modulus of section. The resultant stress distribution (*b*) is therefore made up of a uniform compressive distribution (*c*) and one of varying bending stress (*d*) having an extreme fibre value *f*. It is evident from the resultant distribution that yield will occur first at the extreme fibre, where the stress is $(f+p_1)$. It is

Figs. 3–6.



assumed that, on increasing the straining action beyond the point where yield occurs, each fibre as it reaches the upper yield stress (f_U) drops back to the lower yield value (f_L), and thereafter remains constant, the contenton again being that the fibres are never sufficiently extended to cause strain-hardening. Accordingly, as in fig. 4, after yield a resultant stress distribution (*b*) is obtained where the material at the one extreme fibre is plastic, while at the other the stress, having decreased, eventually changes sign. If now the axial stress p_2 is deducted from the resultant distribution, the bending stress distribution (*d*) is obtained. The condition represented by the diagrams of fig. 4 terminates when the resultant stress $(p_2 - f_t)$, having become tensile, eventually reaches the upper yield value for the material, so giving rise to the distributions shown in fig. 5. This is the last process in the development of the moment of resistance, and in the course of it more and more fibres on both the tension and compression side attain the upper yield stress and revert to the lower yield value, until finally the whole section is in a state of plasticity. This results in the distributions of fig. 6.

It is now proposed to derive a set of equations similar to those given in the previous section but to consider only the condition where yielding has taken place on both the tension and compression sides, since corresponding equations can be obtained for the case of one plastic zone by omission of appropriate terms. Referring to fig. 5 (*d*), the moment of resistance of the section can be expressed as

$$\begin{aligned} M_R &= \int_{-d_t}^{+d_c} fzy\,dy \\ &= \frac{E}{R}\int_{-d_{te}}^{+d_{ce}} zy^2\,dy + (f_L+p)\int_{d_{te}}^{d_t} zy\,dy + (f_L-p)\int_{d_{ce}}^{d_c} zy\,dy, \end{aligned}$$

where

$\int_{-d_{te}}^{+d_{ce}} zy^2\,dy = Ie,$ the second moment of area of the elastic region ;

$\int_{d_{ce}}^{d_c} zy\,dy = A_c\bar{y}_c,$ the first moment of area of the plastic region on the compression side ;

$\int_{d_{te}}^{d_t} zy\,dy = A_t\bar{y}_t,$ the first moment of area of the plastic moment on the tension side ;

and, from equation (4),

$$\frac{E}{R} = \frac{f}{y} = \left(\frac{f_U-p}{d_{ce}}\right).$$

Making these substitutions in the above equation,

$$M_R = (f_U-p)\frac{I_e}{d_{ce}} + (f_L-p)A_c\bar{y}_c + (f_L+p)A_t\bar{y}_t,$$

which can be further simplified by writing

$$\frac{I_e}{d_{ce}} = Z_e,$$

$$A_c\bar{y}_c = Z_{pc},$$

$$A_t\bar{y}_t = Z_{pt},$$

so that

$$M_R = (f_U-p)Z_e + (f_L-p)Z_{pc} + (f_L+p)Z_{pt} \quad . \quad . \quad . \quad . \quad (13)$$

It is also known that the bending stress distribution must produce no axial load, a condition which determines the position of the neutral axis. Equating the total force arising from the compressive stresses to that on the tension side,

$$(f_L-p)A_c + \int_0^{d_{ce}} f_c z\,dy = (f_L+p)A_t + \int_0^{d_{ce}} f_t z\,dy,$$

where f_t and f_c are the stress in any fibre on the tension and compression sides respectively in the elastic region. By making the following substitutions,

$$f_c=\frac{\mathrm{E}}{\mathrm{R}}y_c=\frac{(f_u-p)}{d_{ce}}y_c \text{ and } f_t=\frac{(f_u-p)}{d_{ce}}y_t,$$

the expression becomes

$$d_{ce}\{(f_{\mathrm{L}}+p)\mathrm{A}_t-(f_{\mathrm{L}}-p)\mathrm{A}_c\}=(f_{\mathrm{U}}-p)\left\{\int_0^{d_{ce}}zy_c\,dy-\int_0^{d_{ce}}zy_t\,dy\right\},$$

in which

$\int_0^{d_{ce}}zy_c\,dy=\mathrm{A}_\gamma\bar{y}_\gamma,$	the first moment of area of the elastic region on the compression side ;
$\int_0^{d_{te}}2y_t\,dy=\mathrm{A}_\tau\bar{y}_\tau,$	the first moment of area of the elastic region on the tension side.

Finally, the relationship can be written as

$$d_{ce}\{(f_{\mathrm{L}}+p)\mathrm{A}_t-(f_{\mathrm{L}}-p)\mathrm{A}_c\}=(f_{\mathrm{U}}-p)\{\mathrm{A}_\gamma\bar{y}_\gamma-\mathrm{A}_\tau\bar{y}_\tau\}. \quad . \quad . \quad (14)$$

Expressions for slope and deflection can be derived in a similar manner as for the case of simple bending by expressing the curvature for this elasto-plastic condition in the same approximate form, namely

$$\frac{1}{\mathrm{R}}=\frac{d^2y}{dx^2};$$

hence

$$\frac{d^2y}{dx^2}=\frac{(f_{\mathrm{U}}-p)}{\mathrm{E}d_{ce}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15\,a)$$

In many problems of bending combined with axial loading it will be possible to express d_{ce} in terms of either the deflection y or x the distance of the particular section from a given origin, that is

$$d_{ce}=\phi(x,y)$$

and

$$\frac{d^2y}{dx^2}=\frac{(f_{\mathrm{U}}-p)}{\mathrm{E}\phi(x,y)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15\,b)$$

Integrating,

$$\frac{dy}{dx}=\frac{(f_{\mathrm{U}}-p)}{\mathrm{E}}\int\frac{dx}{\phi(x,y)}+\mathrm{A}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and repeating the process,

$$y=\frac{(f_{\mathrm{U}}-p)}{\mathrm{E}}\iint\frac{dx}{\phi(x,y)}dx+\mathrm{A}x+\mathrm{B}, \quad . \quad . \quad . \quad . \quad . \quad (17)$$

the constants being evaluated according to known conditions.

5. *Conclusion.*

The simple plastic theory given here is still the subject of controversy, though much of the criticism springs from the inability to explain tests on beams of steel having indefinite properties. However, a certain amount of work has been done (3), (4), & (5), and that by Morrison (5) on carefully controlled steel has shown beyond doubt that this simple theory will interpret the behaviour of a member subjected to pure bending. There is, unfortunately, only a little experimental evidence of the influence of shearing force on the plastic condition, but the indication (1, *b*) is that it is not perceptible until the shearing stresses attain values considerably in excess of those produced by the normal location of loads on structural members. When these high shearing forces do occur, it would appear that while the load to cause yield of the material is lowered the effect upon the collapse load is negligible. More experimental work is necessary before this point can be dismissed altogether, but in most practical cases little error will result in applying the simple theory of plasticity to members subjected to transverse loading.

In examining the behaviour of a structural member subjected to straining actions sufficient to produce plastic zones, the first task is to derive, in general terms for the known loads, the appropriate equations for slope and deflection for both the elastic and plastic lengths. Constants of integration can then be determined from the particular forms of the equations at common boundaries and from known conditions at various points along the member. Where the cross-sections are simple geometrical forms complete solutions can be obtained without much difficulty, but for standard rolled sections the process is more laborious and necessitates arithmetical integration.

In conclusion, the author would like to take this opportunity to express his indebtedness to Professor J. F. Baker for valuable comments, and to the British Welding Research Association for permission to publish the paper.

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LXVI. *Problems Involving Flexural Vibrations treated by the Wave-train Method.*

By G. Green, D.Sc., Lecturer in Physics in the University of Glasgow *.

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The wave-train method has hitherto been applied to problems in wave-motion in which only a single condition has to be fulfilled at any boundary of the medium through which the waves are being transmitted. In the present paper the purpose in view is to record a preliminary attempt to apply the wave-train method to problems which involve the fulfilment of more than one condition at each boundary, or at any source or place of discontinuity within the medium itself. In the case of flexural wave problems, owing to the double conditions imposed at each boundary, the methods of solution hitherto employed are generally indirect involving the use of infinite series, and recorded results are mainly approximations. It is therefore of some interest to investigate the possibilities of an alternative method of solution which may present advantages in certain cases.

Wave-trains.

The differential equation governing flexural vibrations in a straight rod or beam of length l is

$$\frac{d^2y}{dt^2}+\frac{\mathrm{EI}}{\rho\mathrm{A}}\frac{d^4y}{dx^4}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

terms depending on the angular motion of the sections of the rod being neglected: where E represents the Young's modulus and ρ the density of the material of the rod, I the moment of inertia of a section and A the area of a section of the rod. In the notation used in Rayleigh's 'Sound,' $\frac{E}{\rho} \cdot \frac{A}{I}$ is replaced by $\kappa^2 b^2$. Assuming that

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* Communicated by the Author.

$$y = e^{ikt - i\lambda x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

represents a wave-train solution of the above equation, the relation between k and λ is expressed by

$$k^2 = \kappa^2 b^2 \lambda^4, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which indicates that, corresponding to a specified value of k, four wave-train solutions are obtainable. In these four solutions the $e^{-i\lambda x}$ term may appear in any one of the four forms—$e^{i\lambda x}$, $e^{-i\lambda x}$, $e^{i\lambda' x}$, $e^{-i\lambda' x}$, in which λ' is written to represent $i\lambda$. The first two of the above forms, inserted in (2), give respectively the usual negative and positive wave-train solutions of equation (1). The third and fourth forms do not represent wave solutions with respect to the x coordinate: nevertheless the solutions obtained by inserting them in (2) above can be treated exactly in the same way as the usual wave-train solutions in the process of summation of wave-trains employed in the application of the wave-train method.

Sources.

The first problem which arises in relation to wave-train solutions of type (2) above is to determine in what manner they are to be combined in order to represent a source or discontinuity of any kind situated at any section $x = x_1$. The simplest form of source is represented by the two wave-trains

$$y = e^{ikt - i\lambda(x_1 - x)} \ldots x < x_1 \quad : \quad y' = e^{ikt - i\lambda(x - x_1)} \ldots x > x_1 \quad . \quad . \quad (4)$$

At each point of the medium these equations determine, in addition to displacements, the gradient $\frac{dy}{dx}$, the bending moment $\mathrm{EI}\frac{d^2y}{dx^2}$, and the force acting on any cross-section $-\mathrm{EI}\frac{d^3y}{dx^3}$. The above source equations involve equal displacements on the two sides of section $x = x_1$, equal and opposite gradients, equal bending moments, and equal and opposite forces on a section. Forces transmitted across a section being of the nature of stresses, $-\mathrm{EI}\frac{d^3y}{dx^3}$ represents a force acting in the positive direction of y applied by the material on the right of section x to the material on the left of section x, the positive direction of x being to the right.

In relation to the main problems to be considered later we require source equations, similar to (4), to represent an externally applied periodic force, $\mathrm{F}e^{ikt}$, acting at the point $x = x_1$ in the direction of y positive. This external force must give rise to a discontinuity in the force transmitted across a section on either side of $x = x_1$, since the stress arising from it must operate equally but in opposite senses on the two sides of the section $x=x_1$. This discontinuity in the force acting on a section is represented by the equation

$$-\mathrm{EI}\left\{\frac{d^3y'}{dx^3}-\frac{d^3y}{dx^3}\right\}_{x=x_1}=\mathrm{F}e^{ikt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This equation therefore expresses the condition for a periodic force of strength F acting externally at section $x=x_1$.

As e^{ikt} is a constituent of all periodic solutions, it may for convenience be omitted from this point. We require a solution of (1) which satisfies (5) at the same time: and as a trial solution fulfilling the source condition (5), at $x=x_1$, we assume

$$\left.\begin{aligned} y&=\mathrm{A}\left\{\frac{1}{(i\lambda)^3}e^{-i\lambda(x_1-x)}+\frac{1}{(i\lambda')^3}e^{-i\lambda'(x_1-x)}\right\}\ldots x<x_1, \\ y'&=\text{same function of } (x-x_1). \quad . \quad . \quad \ldots x>x_1. \end{aligned}\right\} \quad . \quad . \quad . \quad (6)$$

This solution satisfies conditions of symmetry at $x=x_1$ in respect to the gradient which is zero on either side of x_1, and to the bending moment which is continuous at x_1, and it satisfies (5) in addition provided that $\mathrm{A}=\mathrm{F}/(4\mathrm{EI})$. With $\mathrm{F}=1$, the solution therefore represents the effect of a periodic force of unit amplitude applied at point $x=x_1$ in a medium extending to infinite distance on either side of x_1; and it may be referred to as the periodic force source solution of unit strength.

If in (6) above we replace the plus by the minus sign, we obtain a solution representing a periodic displacement without the application of external force ($\mathrm{F}=0$), at $x=x_1$: that is, a displacement source solution.

Boundary Conditions.

The solution (6) requires modification to apply to a finite medium under prescribed boundary conditions. Of the large number of possible boundary conditions we take for illustration the case of a medium both whose boundaries, $x=0$ and $x=l$, are clamped, the medium being subjected to an external periodic force at point x_1. These boundary conditions are represented by the equations

$$y=0, \quad \text{and} \quad \frac{dy}{dx}=0, \quad \text{at} \quad x=0 \quad \text{and at} \quad x=l. \quad . \quad . \quad . \quad (7)$$

In the present case corresponding to each wave-train incident on a boundary there would in general be two reflected wave-trains, thus rendering the methods formerly used to determine the continuing wave-system after several reflections very difficult to apply. The difficulty of determining the reflected system for the above boundary conditions does not arise however when the boundary is at the origin, $x=0$. A simple relation holds between each term in the incident wave-system and the corresponding terms in the reflected wave-system. In short, where sin λx, cos λx, sinh λx, cosh λx appear in the expression for the incident wave-system these expressions are to be replaced by $(\sin \lambda x - \sinh \lambda x)$, $(\cos \lambda x - \cosh \lambda x)$, $-(\sin \lambda x - \sinh \lambda x)$, $-(\cos \lambda x - \cosh \lambda x)$, respectively in order to form the incident and reflected wave-systems combined at boundary $x=0$. The same process can be applied at boundary $x=l$, by change of variable from x to x' defined by $x+x'=l$, thus transforming all expressions to an origin at $x=l$, $x'=0$. The continuation terms arising from reflection at each boundary in turn form by summation the complete wave-system representing the total effect of the original source at any point within the medium.

Summation of Terms.

For convenience we introduce an abbreviated notation indicated by

$$\left.\begin{aligned}
\mathrm{C}(\lambda x)&=(\cos \lambda x-\cosh \lambda x) : \\
\mathrm{S}(\lambda x)&=(\sin \lambda x-\sinh \lambda x) : \\
C(\lambda x)&=(\cos \lambda x+\cosh \lambda x) : \\
S(\lambda x)&=(\sin \lambda x+\sinh \lambda x) : \\
\alpha&=\cos \lambda l \cosh \lambda l+\sin \lambda l \sinh \lambda l : \\
\beta&=\sin \lambda l \cosh \lambda l+\cos \lambda l \sinh \lambda l : \\
\gamma&=\sin \lambda l \cosh \lambda l-\cos \lambda l \sinh \lambda l : \\
\delta&=\cos \lambda l \cosh \lambda l-\sin \lambda l \sinh \lambda l :
\end{aligned}\right\}, \quad . \quad . \quad . \quad (8)$$

as these expressions are regularly formed at each reflection. The functions $\mathrm{C}(\lambda x)$ and $\mathrm{S}(\lambda x)$ may, where (λx) is the variable, be written simply as C and S respectively. We shall require also the functions :

$$\left.\begin{aligned}
\mathrm{P}(\lambda x)&=\mathrm{S}(\lambda l)\mathrm{C}-\mathrm{C}(\lambda l)\mathrm{S} \ : \quad \mathrm{Q}(\lambda x)=\mathrm{C}.(\lambda l)\mathrm{C}+S(\lambda l)\mathrm{S}, \\
P(\lambda x)&=\mathrm{S}(\lambda l)C-\mathrm{C}(\lambda l)S \ : \quad Q(\lambda x)=\mathrm{C}(\lambda l)C+S(\lambda l)S.
\end{aligned}\right\}. \quad (9)$$

It is sufficient to indicate the process of summation of terms arising from a source by reflections at boundaries in its application to the source represented by (4) above, as successive steps are repeated in its application to (6) and other forms of source. The initial negative train from the source (4) being

$$\begin{aligned}
y=\cos kt&[\cos \lambda x_1 \cos \lambda x+\sin \lambda x_1 \sin \lambda x] \\
&+\sin kt[\sin \lambda x_1 \cos \lambda x-\cos \lambda x_1 \sin \lambda x], \quad . \quad . \quad . \quad . \quad (10)
\end{aligned}$$

this train, along with its first reflection, at $x=0$ is given by

$$y_1=\cos kt[\cos \lambda x_1\mathrm{C}+\sin \lambda x_1\mathrm{S}]+\sin kt[\sin \lambda x_1\mathrm{C}-\cos \lambda x_1\mathrm{S}]. \quad (11)$$

The second reflection takes place at $x=l$, and the third at $x=0$. Putting y_2 to represent the second and third reflections combined we find

$$\begin{aligned}
y_2=\cos kt&[\cos \lambda x_1\{\delta\mathrm{C}+\beta\mathrm{S}\}+\sin \lambda x_1\{-\gamma\mathrm{C}+\alpha\mathrm{S}\}] \\
&+\sin kt[\sin \lambda x_1\{\delta\mathrm{C}+\beta\mathrm{S}\}-\cos \lambda x_1\{-\gamma\mathrm{C}+\alpha\mathrm{S}\}]. \quad . \quad . \quad (12)
\end{aligned}$$

The next term y_3 represents the fourth and fifth reflections combined, and so on. Each term in this succession is formed from its predecessor by replacing each C by $(\delta C+\beta S)$ and each S by $(-\gamma C+\alpha S)$. The initial positive wave-train from the source represented by

$$y=\cos kt[\cos\lambda x_1\cos\lambda x+\sin\lambda x_1\sin\lambda x]$$
$$-\sin kt[\sin\lambda x_1\cos\lambda x-\cos\lambda x_1\sin\lambda x] \quad . \quad . \quad . \quad . \quad (13)$$

provides a similar succession of terms of which the first term y_1 is given by

$$y_1=\cos kt[\cos\lambda x_1\{\alpha C-\beta S\}+\sin\lambda x_1\{\gamma C+\delta S\}]$$
$$+\sin kt[\cos\lambda x_1\{\gamma C+\delta S\}-\sin\lambda x_1\{\alpha C-\beta S\}]. \quad . \quad . \quad (14)$$

Here y_1 represents the first and second reflections combined, y_2 the third and fourth reflections combined, and so on: each term of this succession being formed from its predecessor by replacing each C by $(\alpha C-\beta S)$, and each S by $(\gamma C+\delta S)$.

These two successions have now to be summed for the two regions, $x<x_1$, and $x>x_1$, separately. The initial negative train (10) belongs only to region $x<x_1$; the initial positive train belongs to region $x>x_1$: all other terms belong to both regions. The details of summation are therefore given below for the $x<x_1$ region only, the first term of this summation being (11) above. The two successions are readily reduced to one by combining like terms, *e. g.* (12) and (14). Thereafter the terms can be simplified by means of the relation $\alpha\delta+\beta\gamma=1$, and the summations of terms containing cos kt and sin kt performed separately, as indicated below. By writing

$$T_1=\cos kt[\cos\lambda x_1\,C+\sin\lambda x_1\,S]: \text{ and } p=(\delta+\alpha), \quad . \quad (15)$$

the sum, S_1, of the cos kt group may be expressed in the form

$$S_1=T_1\{1+p+(p^2-2)+(p^3-3p)+\text{etc.}\}, \quad . \quad . \quad . \quad (16)$$

where $T_n=pT_{n-1}-T_{n-2}$ for $n>2$. From this we find that $S_1=0$. Again, to obtain S_2, the sum of all the sin kt group except the first term, we write

$$T_1=\sin kt[\sin\lambda x_1\{(\delta-\alpha)C+2\beta S\}+\cos\lambda x_1\{2\gamma C+(\delta-\alpha)S\}], \quad . \quad (17)$$

and the required summation may be expressed in the form

$$S_2=T_1\{1+p+(p^2-1)+(p^3-2p)+\text{etc.}\}, \quad . \quad . \quad . \quad (18)$$

where $T_n=pT_{n-1}-T_{n-2}$, and $p=(\delta+\alpha)$ as before. The value obtained for S_2 is accordingly given by

$$S_2=\frac{T_1}{2-p}=\frac{T_1}{2D}, \text{ where } D=(1-\cos\lambda l\cosh\lambda l). \quad . \quad . \quad (19)$$

To S_2 there falls to be added the sin kt term in (11) above in order to obtain the complete effect of the original source in the region $x<x_1$, which is therefore represented by

$$y=\sin kt[\sin\lambda x_1\{(1-\alpha)C+\beta S\}+\cos\lambda x_1\{\gamma C+(\delta-1)S\}]/D \;.\;.\;.\; x<x_1. \quad (20)$$

The corresponding result for the region $x>x_1$ is given by

$$y'=\text{same expression with } x \text{ and } x_1 \text{ replaced by } (l-x) \text{ and } (l-x_1) \text{ respectively.} \quad \ldots x>x_1. \quad . \quad (21)$$

These results apply to the source represented by (4) above. If in (4) we replace λ by λ' we obtain, to represent the complete effect of the source so defined, the expressions

$$\left.\begin{aligned} y&=-ie^{ikt}[\sinh \lambda x_1\{(1-\delta)\mathrm{C}-\beta\mathrm{S}\}-\cosh \lambda x_1\{\gamma\mathrm{C}+(1-\alpha)\mathrm{S}\}]/\mathrm{D}, \\ y'&=\text{same expression, with } x \text{ and } x_1 \text{ replaced by } (l-x) \text{ and} \\ &\qquad (l-x_1) \text{ respectively} \end{aligned}\right\}. \quad (22)$$

by a repetition of the steps outlined above in the case of the original source (4). The results obtained above for the two types of source considered may be summarized in the following statements :—
All even terms in λ appearing in the initial negative and positive wave-trains contribute a zero total effect in the summations.

Odd terms in λ appear with different signs in the initial negative and positive trains: According as the terms appear in the initial negative wave-train

$\cos \lambda x$	yields	$\{(1-\alpha)\mathrm{C}+\beta\mathrm{S}\}/\mathrm{D}$	in the final summations.
$\sin \lambda x$	,,	$\{\gamma\mathrm{C}-(1-\delta)\mathrm{S}\}/\mathrm{D}$	,, ,, ,, ,, .
$\cosh \lambda x$	,,	$-\{(1-\delta)\mathrm{C}-\beta\mathrm{S}\}/\mathrm{D}$	,, ,, ,, ,, .
$\sinh \lambda x$	,,	$-\{\gamma\mathrm{C}+(1-\alpha)\mathrm{S}\}/\mathrm{D}$	,, ,, ,, ,, .

These statements apply to all summations subject to the clamped-clamped boundary conditions.

Periodic Force Solution for Clamped-clamped Bar.

A combination of the above results yields the expressions required to represent the complete effect of a source defined by equations (6) above, that is, the complete effect of a periodic force $\mathrm{F}e^{ikt}$ acting at the point $x=x_1$. The result, arranged in a symmetrical form, is given by

$$\left.\begin{aligned} y&=\frac{\mathrm{F}e^{ikt}}{4\mathrm{EI}\lambda^3\mathrm{D}}\Big[\mathrm{P}\{\lambda(l-x_1)\}\mathrm{C}-\mathrm{Q}\{\lambda(l-x_1)\}\mathrm{S}\Big] \ldots x<x_1 \\ y'&=\text{same expression, with } x \text{ and } x_1 \text{ replaced by } (l-x) \\ &\qquad \text{and } (l-x_1) \text{ respectively.} \quad \ldots x>x_1. \end{aligned}\right\}. \quad . \quad (23)$$

Periodic Force Solution for "Supported" Bar.

Solutions similar to (23) are obtainable for each of the numerous possible boundary conditions by the methods indicated above. In the case of a "supported" bar, the ordinary method of summation requires in effect only one boundary coefficient, and leads to the periodic force solution

$$y=-\frac{\mathrm{F}e^{ikt}}{2\mathrm{EI}\lambda^3}\left[\frac{\sin\lambda(l-x_1)\sin\lambda x}{\sin\lambda l}-\frac{\sinh\lambda(l-x_1)\sinh\lambda x}{\sinh\lambda l}\right]\ldots x>x_1, \quad \ldots\ldots (24)$$

with the corresponding expression for $y'(x>x_1)$. Other cases may be discussed later. Meantime a short statement only is given of the main results derivable from the periodic force solution recorded in (23) above in order to facilitate its application to the solution of practical problems.

Instantaneous and Continuous Force Solutions.

The solution representing the effect of an instantaneous or impulsive application of force of strength F, at the point $x=x_1$ is $\frac{1}{\pi}\int_0^\infty dk\, y$, where y is the periodic force solution given in (23) above for $x<x_1$ and $x>x_1$. The evaluation of this integral yields an infinite series of terms, each term being determined in value by a particular root of the equation, D=0. In view of this equation certain simplifications of the process of integration are obtained

$$\frac{d\mathrm{D}}{d\lambda}=-l\,\mathrm{S}(\lambda l):\ \mathrm{P}(\lambda x_1)=-\mathrm{P}\{\lambda(l-x_1)\}\ :\ \mathrm{Q}\{\lambda(l-x_1)\}=\frac{\mathrm{C}(\lambda l)}{\mathrm{S}(\lambda l)}\,\mathrm{P}\{\lambda(l-x_1)\}. \quad \ldots\ldots (25)$$

The distinction hitherto maintained between the forms of the solutions for the regions $x<x_1$ and $x>x_1$ now becomes unnecessary: and the evaluation of the instantaneous integral for any value of x may be written in the form

$$y=-\frac{\mathrm{F}}{\mathrm{EI}l}\sqrt{\frac{\mathrm{EI}}{\rho\mathrm{A}}}\sum_1^\infty\frac{e^{ikt}\mathrm{P}(\lambda x_1)\mathrm{P}(\lambda x)}{\{\mathrm{S}(\lambda l)\}^2\lambda^2}, \quad \ldots\ldots (26)$$

where the summation extends to all the positive roots of equation D=0. It is worthy of note that the above expression for the effect of an instantaneous force F is an expansion of the instantaneous integral in terms of the normal functions, defined by P(λx), for the case of a bar clamped at both ends. The normal functions are in fact formed in the process of summation of trains arising from a source.

The instantaneous force solution, (26), may be employed to give the effects of forces distributed in space or in time. In the former case replace F in (26) by $\mathrm{F}(x_1)dx_1$ and integrate with respect to x_1 throughout the range $x_1=0$ to $x_1=l$. If the force acting at x_1 varies with the time, replace F in (26) by $\mathrm{F}(t')dt'$ and t by $(t-t')$, and integrate with respect to t' from $t'=0$ to $t'=t$, in accordance with a process familiarly employed in wave-motion problems. In the case of a force moving at speed V along a bar, replace F by $\mathrm{F}(t')dt'$, x_1 by $\mathrm{V}t'$, and t by $(t-t')$ and integrate as in the preceding case. The process indicated yields the continuous force source solution, representing the effect of the application of a steady force F at x_1, for a time interval indefinitely great, in the form

$$y = \frac{F}{EIl} \sum_{1}^{\infty} \frac{(1 - e^{ikt}) P(\lambda x_1) P(\lambda x)}{\{S(\lambda l)\}^2 \lambda^4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

the summation extending to all positive roots of D=0 as before. In this result the periodic term, with $t \rightarrow \infty$, presents an apparent anomaly in the representation of a steady effect: but in accordance with the rule that a periodic term of this kind should be assessed at its mean value, the value of the term in this case would be zero.

Verification.

The expression arrived at in (27) for the displacement effect of a force F steadily applied to a bar at any specified point x_1 (e^{ikt} omitted), can readily be compared with the well-known solution of this problem by the usual method employed in statics. Taking for comparison the case in which the force F is applied at the mid-point of the bar, $x_1 = \frac{l}{2}$, the displacement curve of the neutral axis is given by

$$y = \frac{F}{16EI}\left(lx^2 - \frac{4}{3}x^3\right) \ldots x < \frac{l}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

with y' the same function of $(l-x)$, the curve being symmetrical about the mid-point. By expanding the expressions y and y' in terms of the normal functions we recover (27), omitting the periodic term, as required for verification.

Summary.

The above discussion illustrates, by means of a particular case, the general process of forming periodic, instantaneous and continuous solutions, by summation of wave-trains or their equivalents. It illustrates also the formation of the appropriate "normal functions" in any particular case, and shows their relation to fundamental Fourier wave-trains. Similar results are obtainable from any periodic source solution. From (24), the fundamental periodic force solution for a "supported" beam a group of results corresponding to those given for the "clamped-clamped" beam have also been worked out in detail, including the corresponding verification in the case of the problem of the steadily applied force: but it is considered unnecessary to record these in the present introductory paper.

References.

Rayleigh, 'Sound, Vol. i.'
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LXVII. *The Susceptibility of Nitroglycerine to Explosion by Impact in the Presence of Various Gases.*

By M. F. R. Mulcahy, M.Sc.*

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1. Introduction.

The impact sensitivity of liquid nitroglycerine and other explosives in the liquid state is increased very considerably when a small amount of air or other suitable gas is entrapped in the explosive during the impact (Bowden, Eirich, Mulcahy, Vines and Yoffe, 1943 *b*) (Bowden, Mulcahy, Vines and Yoffe, 1946, 1947). For the study of the effect with nitroglycerine and explosives liquid at room temperature, the entrapping of the gas may be effected conveniently in three ways, viz. :—

(1) by allowing a hammer with a flat surface to fall on a pool of explosive containing a small gas bubble,

(2) by arranging the explosive in small droplets or narrow strips on a flat anvil and striking it with a flat hammer, and

(3) by striking a continuous layer of explosive with a hammer with a small cavity in its surface, provided that the orifice of the cavity is so levelled relative to the anvil that a small pocket of air is entrapped in the cavity as the hammer enters the liquid. This last device, particularly if the hammer is hemispherical, is the most effective in increasing the sensitivity, though the effect is large with all three.

The relative effects of several different gases have been compared (*loc. cit.*) by carrying out impact experiments of types 2 and 3 in a chamber which could be filled with any required gas. An important result of these experiments was that if the gas entrapped is one of high specific heat, viz., ether or carbon tetrachloride vapour, the explosion efficiency, *i. e.*

$$\frac{\text{number of explosions}}{\text{number of impacts}}\times 100$$

is lower than if air or nitrogen or hydrogen is entrapped under the same conditions. It was also found that, with cavity impact (3) in air at different pressures, the explosion efficiency falls off with decreasing initial pressure, though this general result was not obtained with impacts on droplets or parallel strips.

On the whole the results are compatible with the view that the increase in sensitivity arising from the inclusion of gas is due to the high temperature flash resulting from sudden adiabatic compression of the entrapped gas spaces during impact. In addition, there is some evidence that the chemical nature of the gas can contribute to its sensitizing action since nitrogen has been found to be less effective than oxygen (impacts of types 1 and 2) and nitrous oxide (impacts of type 1).

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* Communicated by the Author.

The work reported in this paper continues the experiments described above. Using the method of cavity impact, the effect of variation of the initial gas pressure on the sensitivity has been investigated for a number of gases and a fairly exhaustive qualitative comparison of the relative sensitizing actions of several different gases has been made.

The absolute temperature T reached in the adiabatic compression of an ideal gas from initial pressure P_0 and volume V_0 to the final pressure P and volume V is given by

$$T=T_0\left(\frac{V_0}{V}\right)^{\gamma-1}=T_0\left(\frac{P}{P_0}\right)^{\frac{\gamma-1}{\gamma}},$$

where T_0 is the initial temperature and γ the specific heat ratio of the gas. It follows that, under a given set of conditions of compression, a gas of a given γ value will generate a higher temperature than a gas of lower γ, and would therefore be expected at least at first sight to initiate explosion more readily in the explosive in contact with it.

The influence of variation of the initial pressure on the final temperature reached with any one gas will differ for the two cases in which a fixed initial volume of the gas is compressed (*a*) to a constant final volume or (*b*) to a constant final pressure. In the former case the final temperature is independent of the initial pressure, in the latter it increases with increasing pressure ratio, *i. e.*, with decreasing initial pressure. The total quantity of heat generated during the compression may easily be shown to be proportional to P_0 in the first case and approximately to $P_0^{1/\gamma}$ in the second for a fixed initial volume of the gas *. In the experiments with cavity impact, the initial volume of gas entrapped is evidently determined by the dimensions of the cavity and is therefore independent of the initial pressure. The compression of this initial volume may conceivably result simply from reduction in the volume of the cavity as it is driven into the anvil or, as suggested by Bowden and collaborators (1947 *a*) it may be caused by the setting up of a pressure within the liquid during impact approaching in magnitude the flow pressure of the metal anvil which is being deformed †. With different initial pressures and the same hammer and anvil, the first of these explanations corresponds to the conditions of (*a*) above and the second approximates to the case of (*b*), viz. compression to a constant final pressure. It will be shown below, however, that there is evidence against the first explanation and that it is probable that the flow pressure of the material of the anvil † rather than the reduction in the cavity volume is the determining factor in cavity initiation.

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* Provided, in the latter case, that $P \gg P_0$.

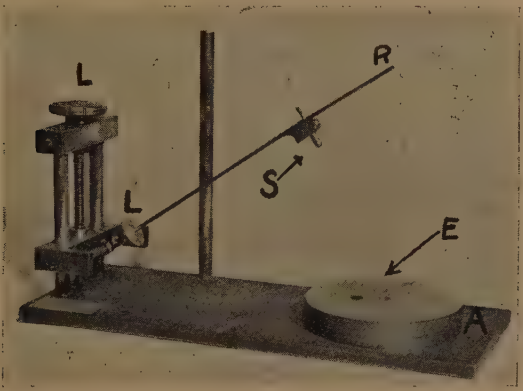
† In the case where the deformation is confined to the anvil, *i. e.*, where the hammer is harder than the anvil.

2. EXPERIMENTAL.

The impact apparatus used (*cf.* Bowden and collaborators, 1943 *a* and *b*, 1947 *a*) is shown in fig. 1. It consists of a light pendulum fall hammer swinging on ball bearings. The impacting surfaces are the cavity striker S and the lapped brass anvil A. S is a piece of brass rod 2 cm. long and 0·6 cm. in diameter in the hemispherical end of which a cavity 1 mm. in diameter and 1 mm. deep has been turned so as to leave a thin projecting rim of metal about the circumference. This rod is fitted into a socket in the pendulum arm R and secured by the screw shown. The arm is weighted so that the impact occurs close to its centre of percussion. It is released from an electromagnet (not shown) and the striker falls into a small pool of explosive E on the anvil. The striker S may be replaced whenever necessary.

Before each series of impacts the striker was *carefully "levelled"* by means of the screws LL; it was judged "level" when a light tap gave a faint but complete circular imprint of the cavity rim on the anvil. The whole apparatus was then placed in a vacuum chamber, the nitroglycerine put on the anvil, and the arm secured at the required height by the electromagnet. The chamber was exhausted by a Megavac pump to the required residual pressure as measured by a McLeod gauge; the appropriate gas (or air) was admitted to the desired pressure (mercury manometer), the arm released from the electromagnet, and the result—explosion or failure—recorded. Unless damaged and therefore requiring replacement, the striker after cleaning with alcohol was ready for another impact without further relevelling. The striker was always arranged to strike the anvil surface clear of the imprints of previous impacts. Immediately before each experiment the nitroglycerine was de-gassed in a separate apparatus under a pressure of 1–3 × 10^{-2} cm. Occasionally one or two bubbles were observed to form in the nitroglycerine pool during evacuation in the vacuum chamber. In this case, care was taken to delay the impact until these disappeared.

Fig. 1.



The gases, other than air, were unpurified, direct from commerical cylinders. For most cylinders used rough determinations of the moisture content of the gases were made ; it was found to be negligible in all cases. The helium was analyzed for hydrocarbon content, which was found to be not greater than 0·2 per cent. (as methane) *. The pressure of residual air in the chamber varied somewhat in different experiments but was always in the region of 3×10^{-2} cm. For nearly all experiments this represents a smaller proportion of the total pressure than the oxygen and nitrogen impurity contents of the gases. The small variations which occurred in the residual air pressure produced no noticeable effect on the results.

The height of fall was 22 cm. and the effective weight of the hammer 35 gm. in each impact. In the experiments of series I and II the strikers became damaged after a few explosions and were replaced, so that the explosion efficiencies given represent the aggregate results of several strikers. In series III–VI the same striker, which withstood several hundred impacts, was used throughout. However, it was re-levelled at the beginning of each series. The explosion efficiency is sensitive to the degree of levelling and, as it was not found possible to level the striker in an absolutely reproducible manner, the results of one series, while self-consistent, are not *quantitatively* comparable with those of another.

Series I is a comparison of the effect of air at three different pressures. As far as possible, the impacts were made under each pressure in rotation. Series II is a similar experiment with nitrogen. Here the impacts were made alternately at the two low pressures with an occasional impact at atmospheric pressure. Each of series III–VI is a comparison of the effects of two gases at various pressures. Since, during a long series of impacts, the striker shows some variation of effectiveness in initiating explosion (due, no doubt, to small variations in the degree of levelling), the gases were taken alternately for a small number of impacts at each of the pressures in rotation. In this way the comparison of the effects under the different pressures was spread as far as possible over the whole series, so that a valid inference could be drawn concerning the effect of change of pressure as well as change of character of the gas.

3. Results.

A. *The Effect of Variation of the Initial Pressure on the Explosion Efficiency.*

Series I : *In Air.*

The results are given in Table I.

In this series five strikers were used. It is clear from the results that the explosion efficiency decreased with decreasing initial pressure of air. This result is in agreement with the result of similar experiments reported previously for rather heavier impacts (Bowden and collaborators, 1947 *a*).

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* The author's thanks are due to Mr. R. Williamson for this determination.

Series II : *In Nitrogen.*

The results are given in Table II.

It will be seen that when the impacts are carried out in an atmosphere of nitrogen, the effectiveness of the gas in initiating explosion falls off with decreasing initial pressure, as in the case of impacts in air.

TABLE I.

Pressure of Air	$\frac{\text{No. of explosions.}}{\text{No. of impacts}}$	Explosion Efficiency
76 cm. Hg.	$\frac{15}{15}$	100%
6·4 cm.	$\frac{9}{15}$	65%
0·2 cm.	$\frac{0}{6}$	~0%

TABLE II.

Pressure of N_2	$\frac{\text{No. of explosions}}{\text{No. of impacts}}$	Explosion Efficiency
76 cm. Hg.	$\frac{5}{5}$	100%
15 cm.	$\frac{22}{26}$	85%
2 cm.	$\frac{11}{26}$	40%

Series III–VI : *In Helium, Hydrogen, Oxygen and Nitrogen.*

These results are given in Tables III.–VI. below, and will be referred to in more detail in the next section of the paper. Here attention is drawn to the fact that the experiments show that with each of the gases the explosion efficiency decreases with decreasing initial pressure of gas *. This behaviour therefore appears to be general under the conditions of impact used.

B. *The Relative Effects of Helium, Hydrogen, Nitrogen and Oxygen in Initiating Explosion.*

In these experiments the gases were taken in pairs and their relative effects on the explosion efficiency compared by performing alternate impacts in each gas. In order to determine whether the effectiveness of one gas relative to the other is altered by variation in the initial pressure, comparisons were made at two or more pressures in the manner described in Section 2. As mentioned previously, some variation in the effectiveness of the striker in initiating explosion at any one pressure of the same gas was discernible during a long series of impacts. The effect of this on the results was minimized by taking each pressure in rotation, but, in view of the considerable tedium attaching to the work, the number of impacts was not adequate for quantitative statistical analysis of the results. It is emphasized therefore, that the result obtained from the comparison is little better than qualitative in each case.

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* When the explosion efficiencies are plotted against pressure for each of the gases in Tables III. and VI., the points fall very roughly on the expected S-shaped curve.

Series III : *Comparison of Helium* with Hydrogen.*

The values of γ for helium and hydrogen are respectively 1·66 and 1·41 †. Consequently it would be expected that helium would be more effective than hydrogen in initiating explosion. The results obtained are given in Table III.

TABLE III.

Pressure	Helium		Hydrogen	
	$\frac{\text{Explosions}}{\text{Impacts}}$	Explosion Efficiency	$\frac{\text{Explosions}}{\text{Impacts}}$	Explosion Efficiency
60 cm.	$\frac{5}{5}$	~100%	$\frac{5}{5}$	~100%
35 cm.	$\frac{14}{16}$	85%	$\frac{12}{15}$	80%
30 cm.	$\frac{9}{16}$	55%	$\frac{4}{16}$	25%
25 cm.	$\frac{8}{20}$	40%	$\frac{2}{20}$	10%
Total impacts at 35, 30 and 25 cm.	$\frac{31}{52}$	60%	$\frac{18}{51}$	35%

It will be seen that with both gases the explosion efficiency is ca. 100 per cent. at 60 cm. but falls at lower pressures. However, it is clear from Table III. that at each pressure below 60 cm. the explosion efficiency with hydrogen is somewhat less than with helium. This is in accordance with expectation though it may be remarked that the difference is not very great. Interpolating from the figures to obtain the pressure at which 50 per cent. explosions would be expected in each case (at which point the maximum difference should be shown) gives ~29 cm. for helium and ~34 cm. for hydrogen. This shows that the extra sensitizing effect produced by substituting helium for hydrogen, though definite, is quite small.

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* The helium (Medical Quality, from Ohio Chemical Co., U.S.A.) was stated to contain 1·8 per cent. nitrogen.

† The values quoted for physical properties in this paper are taken from 'International Critical Tables.'

Series IV : *Comparison of Nitrogen with Hydrogen.*

Nitrogen and hydrogen have very clearly the same value for γ, 1·404 and 1·410 respectively, and it would therefore be expected at first sight that they would have about equal sensitizing effects. However, the results given in Table IV. show that although with both gases the explosion efficiency falls from ca. 100 per cent. at 60 cm. to ca. 0 per cent. at 2 cm., the explosion efficiency in nitrogen is still 85 per cent. at 12 cm., whereas in hydrogen it has fallen to 35 per cent. This shows that the sensitizing effect of nitrogen is rather greater than that of hydrogen. This conclusion is also supported by the results of series V. given below. A similar difference between nitrogen and hydrogen is noticeable in the result of impact experiments of type 2 described by Bowden and collaborators (1947 *a*). The explanation of this effect is not certain, but it is suggested that the lower sensitizing effect of hydrogen may be connected with its higher thermal conductivity, which is seven times that of nitrogen at ordinary temperatures (*cf.* discussion).

TABLE IV.

Pressure	Nitrogen		Hydrogen	
	$\frac{\text{Explosions}}{\text{Impacts}}$	Explosion Efficiency	$\frac{\text{Explosions}}{\text{Impacts}}$	Explosion Efficiency
60 cm.	$\frac{6}{6}$	~100%	$\frac{6}{6}$	~100%
25 cm.	—	—	$\frac{1}{2}$	—
15 cm.	—		$\frac{0}{2}$	—
12 cm.	$\frac{10}{12}$	85%	$\frac{4}{12}$	35%
2 cm.	$\frac{0}{6}$	~0%	$\frac{0}{6}$	~0%

Series V : *Comparison of Nitrogen with Helium.*

These experiments were done as a check on the results of series III and IV. Series III indicates that the sensitizing effect of helium is greater than that of hydrogen, and series IV that the effect of nitrogen is greater than that of hydrogen. Consequently a comparison of nitrogen and helium should reveal a smaller difference in behaviour than either of the two previous comparisions. The results given in Table V. show that this expectation is fulfilled.

The results indicate that the sensitizing effect of helium and nitrogen are about equal. Possibly the effect of the low specific heat of the helium is compensated by its relatively high thermal conductivity.

TABLE V.

Pressure	Helium		Nitrogen	
	Explosions/Impacts	Explosion Efficiency	Explosions/Impacts	Explosion Efficiency
18 cm. Hg.	$\frac{9}{16}$	55%	$\frac{8}{16}$	50%
15 cm. Hg.	$\frac{6}{13}$	45%	$\frac{4}{13}$	35%
Total Impacts	$\frac{15}{29}$	50%	$\frac{12}{29}$	40%

Series VI: *Comparison of Nitrogen with Oxygen.*

The γ values for nitrogen and oxygen under normal conditions of temperature and pressure are respectively 1·404 and 1·401 and the thermal conductivities are in the ratio 1 : 1·02. In the absence of any effect other than compressional heating it would be expected that these two gases would behave practically identically. On the other hand, it has been found with impacts of types 1 and 2 that oxygen is more effective than nitrogen, the difference being attributed to the chemical nature of the oxygen (Bowden and collaborators, 1943 *b*, 1947). However, as the method of entrapping the gas by cavity impact seems to lead to a certain divergence in the results from those obtained from other methods (*cf.* Bowden et al., 1947 *a* and below), and as preliminary experiments at a single pressure (not reported) indicated that with cavity impact the difference between the sensitizing actions of oxygen and nitrogen was small or non-existent, it was decided to carry out a comparison at various pressures with cavity impacts. Table VI. gives the results obtained at 5, 10, 12, 15 and 14 cm. The experiments at 14 cm. are not strictly comparable with those at 5, 10, 12 and 15 since they were not done in rotation with them, but all together at the end of the series (without, however, re-levelling the hammer). Actually, as may be seen from Table VI., the explosion efficiencies at 14 cm. are at about the values expected from consideration of those obtained at the other pressures.

The results show that oxygen is a little more effective than nitrogen. The pressure required to give 50 per cent. explosions under the conditions of the experiments of Table VI. would be $\sim$13 cm. for oxygen and $\sim$15 cm. for nitrogen. It is clear, therefore, that with cavity impact the extra sensitizing effect of oxygen above that of nitrogen is quite small. In the absence of any well-marked difference in the physical properties of the two gases, the greater effectiveness of oxygen may be ascribed, as previously, to the oxidizing properties of the gas.

TABLE VI.

Pressure	Nitrogen		Oxygen	
	$\frac{\text{Explosions}}{\text{Impacts}}$	Explosion Efficiency	$\frac{\text{Explosions}}{\text{Impacts}}$	Explosion Efficiency
15 cm.	$\frac{13}{25}$	50%	$\frac{18}{25}$	70%
12 cm.	$\frac{2}{20}$	10%	$\frac{8}{20}$	40%
10 cm.	$\frac{1}{12}$	10%	$\frac{2}{12}$	15%
5 cm.	$\frac{1}{7}$	$\sim$15%	$\frac{1}{7}$	$\sim$15%
14 cm.	$\frac{3}{13}$	25%	$\frac{7}{13}$	55%
Total impacts at 10, 12, 14 and 15 cm.	$\frac{19}{70}$	25%	$\frac{35}{70}$	50%

C. *An Experiment on the Mechanism of Action of the Cavity.*

The sensitizing effect of the cavity is clearly associated with compression of the air or other gas trapped in it, but it is not clear by what mechanism the pressure is applied to the gas on impact. Under the conditions of the experiments described above a circular imprint of the cavity rim about 0·05 mm. deep is made on the anvil, and, as mentioned in the introduction, the simplest view would be that the compression results from the cavity being partly filled by the metal of the anvil as the periphery sinks into the anvil surface. On the other hand, it has been suggested (Bowden and collaborators, 1947 *a*) that, on impact, a pressure of the order of magnitude of the flow pressure of the metal surfaces is communicated to the liquid and the enclosed gas space, and that it is to this sudden pressure increase that the temperature rise is due. In support of this explanation they observed that cavity impacts on a very soft (indium) anvil failed to cause explosion, whereas impacts on harder anvils gave explosions readily. A further experiment of essentially similar type carried out under more precise conditions of comparison is described below.

The experiment was carried out by Mr. J. J. Batten.

The apparatus was exactly the same as in the experiments described in Section 2, except that the brass anvil was hollowed out at the centre and antimonial (2 per cent.) lead alloy cast into the space. Precautions were taken to ensure that the lead fitted tightly into the space so that lead and surrounding brass were held rigidly together. The surface of the anvil was then turned and lapped carefully on a cast iron lap. In this way an anvil was prepared, the surface of which was completely flat but consisted of antimonial lead at the centre surrounded by a *coplanar* annulus of brass. Thus a cavity striker " levelled " to the brass surface was also " level " with respect to the lead. The hardness of the brass and lead were respectively 104 (Vickers) and 22 (Vickers). The strikers used were made from somewhat harder brass than that of the anvil, viz. : 159 (Vickers). The hardness of the cavity rim was probably somewhat increased by work-hardening during the levelling operation and by subsequent impacts.

With this apparatus and the vacuum chamber it was thought that it might be possible to decide between the two explanations given above for the mechanism of action of the cavity. If the compression is merely due to the penetration of the anvil surface by the cavity rim, impacts on the softer lead surface should be more effective in causing explosion than (otherwise identical) impacts on the brass surface. If, on the other hand, the degree of compression in the gas space is directly dependent on the flow pressure of the anvil, the impacts on brass should be the more effective. Since, as shown above, the effect of the entrapped gas decreases with diminishing initial pressure, it is clear that impacts on the more effective anvil, lead or brass, should give explosions at a lower initial pressure of air than those on the less effective one.

Table VII.

Initial Pressure of Air	Brass	Lead	Remarks
	$\frac{\text{Explosions}}{\text{Impacts}}$	$\frac{\text{Explosions}}{\text{Impacts}}$	
20 cm.	$\frac{5}{5}$	$\frac{5}{5}$	Explosions on the lead surface were generally considerably weaker than those on the brass surface (see text).
15 cm.	$\frac{10}{10}$	$\frac{4}{10}$	
10 cm.	$\frac{2}{10}$	$\frac{0}{10}$	These impacts were interspersed with impacts on the brass surface at ∼76 cm. which gave 100% efficiency, viz. $\frac{22}{22}$.
5 cm.	$\frac{0}{5}$	$\frac{0}{5}$	

In each of four series with air at various pressures, impacts were carried out on nitroglycerine alternately on the lead and brass surfaces. The results are given in Table VII.

Table VII. shows clearly that the brass surface is the more effective, for, although with both surfaces the explosion efficiency rises from ∼0 per cent at 5 cm. to ∼100 per cent at 20 cm., at the intermediate pressures the efficiency for impacts on brass is greater. It will be seen that it has reached 100 per cent at 15 cm., at which pressure the efficiency for impacts on lead is only 40 per cent. The other observation of interest is that the explosions on the lead surface were for the most part weaker than those on the brass. About half of the lead explosions were accompanied by only a feeble sound and in most the nitroglycerine was only partly consumed. The impacts on brass all gave very loud explosions accompanied, with a few exceptions, by complete consumption (or dispersal) of the surrounding explosive. The depth of the imprint of the cavity rim from an impact which did not cause explosion was ∼0·05 mm. on the brass and ∼0·25 mm. on the lead surface.

It is apparent from this experiment that the effectiveness of the cavity is not increased by allowing it to impinge further into the surface of the anvil. Furthermore, since the metal of higher flow pressure is the more effective it may be inferred that the heating effect produced in the air space is the greater. And this would seem to indicate that the *final pressure* reached is greater than when the impact is made on the metal of lower flow pressure. The evidence is therefore against the simple volume contraction mechanism and, as far as it goes, is in favour of the view that the flow pressure of the deforming metal surface is the important factor in the action of the cavity.

4. DISCUSSION.

A. The results show that the sensitizing effect of entrapping a gas pocket in the cavity decreases with decreasing initial pressure of the gas. As shown in the previous section, it is probable that the final pressure reached in compression of the gas is directly dependent on the flow pressure of the metal deformed by the impact. In this case, the final pressure will be independent of the initial pressure of the gas and consequently the adiabatic temperature rise will increase with decreasing initial pressure*. It might, therefore, be expected that the effectiveness of the gas in initiating explosion would increase rather than diminish with decreasing initial pressure. On the other hand, the quantity of heat generated in the gas during the compression decreases with decreasing initial pressure. Previous experiments have shown that, when the cavity is filled with nitroglycerine and a bubble of air at atmospheric pressure introduced into it before impact, there is a critical volume of air below which the bubble does not initiate explosion (Bowden and collaborators, 1947 *a* ; Richards, 1946). Since the adiabatic temperature rise would not be expected to be less in a small bubble than in a large one, this is further evidence that the total quantity of heat generated in the compression is an important factor.

The absence of any substantial decrease in sensitivity with impacts on parallel strips or droplets (type 2) in low pressures of air down to 10^{-2} cm. (Bowden et al., 1947 *a*) is in sharp contrast to the behaviour found with cavity impact. In the light of the available evidence (Bowden et al.,

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* *Cf.* Introduction.

1947 *a* and *b*; and this paper), it does not seem possible to find a satisfactory explanation for this unexpected difference. In view of this difficulty it would not be profitable, at this stage, to attempt an explanation of the pressure dependance of sensitizing action found when the gas is trapped by the cavity*.

B. The relative effectiveness of the gases tried may be expressed

$$He \approx N_2 > H_2$$

and

$$O_2 > N_2,$$

but it is clear from the experiments that, with cavity impact at least, the relative magnitude of the sensitizing effect of any one of these gases is not very different from that of any other†.

The greater effect of helium relative to hydrogen may be correlated with its greater γ value and the consequent greater temperature rise on compression. The difference between nitrogen and hydrogen must be explained differently. As mentioned previously, it is possible that because of the much greater thermal conductivity of hydrogen (seven times that of nitrogen) a greater part of the heat generated is conducted to the walls of the cavity during the compression and consequently the temperature rise is less than when nitrogen is compressed under the same conditions. On the other hand, the initiation of the explosion nucleus is (presumably) brought about by the activation of nitroglycerine molecules by impinging gas molecules, and activation (and de-activation) are frequently found to be remarkably specific processes.

It is not possible to explain the difference between oxygen and nitrogen as being due to differences in either γ values or thermal conductivities. The assumption of a specific effect with oxygen is not unreasonable, since it seems likely that reaction should be initiated in the explosive rather more easily by contact with heated oxygen than by an inert gas under similar conditions. In fact, it seems surprising that the difference between the effects of oxygen and nitrogen is not greater than is the case.

The results of this and previous work (Bowden and collaborators, 1943 *b*, 1947) have been discussed with the assumption that the accepted values of γ for the various gases are applicable to the conditions of rapid compression which obtain in the impact experiments. It is possible that in fact this assumption may not be valid. Lewis and von Elbe (1939) have described experiments in which a gas was subjected to rapid compression and the γ value which obtained calculated from simultaneous readings of the pressure and volume during the compression. According to these authors, when compression of nitrogen, oxygen and carbon monoxide is brought about within 12×10^{-3} second, the apparent value of γ is 1·43, which is 0·03 above the normal value. When the compression is accomplished in $7 \cdot 5 \times 10^{-3}$ sec., the apparent value of γ is still higher, viz., 1·46. The rise of γ above the normal value in these experiments is ascribed to the partial failure of the rotational degrees of freedom to be excited in the short time of the compression. (The γ value of argon was found to be normal and unaffected by the speed of compression.) In the cavity impact experiments (in air), it has been shown that initiation of the explosion takes place within 50×10^{-6} sec. after the first instant of contact between striker and anvil (Bowden and collaborators, 1947 *b*). As the major part of the compression of the entrapped gas space evidently takes place when the striker and anvil are in contact, it follows that the time taken in compressing the gas to the temperature required to effect initiation is much less than the time of compression in Lewis and von Elbe's experiments. It has also been shown (*ibid.*) that, with impacts on parallel strips (type 2), initiation again takes place within 50×10^{-6} sec. after the first instant at which the air spaces are entrapped in the liquid. Consequently in these experiments, as in those with the cavity, the time of compression is short compared with that found by Lewis and von Elbe to involve anomalous γ values for oxygen and nitrogen. It seems possible, therefore, that polyatomic gases may behave more or less as monatomic under the conditions of the impact experiments. This could account for the small difference in sensitizing action between helium and hydrogen. On the other hand, since the low γ values of carbon tetrachloride and ether depend not only on rotational but also on vibrational degrees of freedom, which are well known to be slowly excited, it is difficult at the same time to explain the low sensitivity in these vapours unless it is assumed that they are condensed to liquid in the short time of compression. However, in the absence of more data on the anomalous effect and also on the mechanism by which the activation energy necessary to initiate reaction is conveyed from the hot gas to the explosive, it is not possible at present to gauge the importance of the effect in the impact experiments.

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* It may be observed that the effect of heat loss by conduction on the final temperature reached in the compression will be the greater the lower the initial pressure. It is possible that at sufficiently low pressures the temperature necessary to initiate explosion may not be reached (or maintained for a sufficient length of time). Consequently, although the ideally expected adiabatic temperature rise may actually increase at low pressures, the decrease in sensitizing action could be explained. The relatively low effect of hydrogen compared with nitrogen lends some plausibility to this explanation. However, it is difficult to see why the effect of heat loss during the compression should not be equally operative in the impacts of type 2.

† It will be recalled that the absolute sensitizing effect of these gases, *i. e.* when comparison is made with impacts in which no gas is entrapped, is very large.

It is clear that, provided allowance is made for the high thermal conductivities of hydrogen and helium and for a specific effect of oxygen, the results given in this and previous papers are in a general way compatible with an explanation based on the normal γ values for the various gases.

SUMMARY.

The effect of decreasing initial pressure on the sensitizing action of gas entrapped in nitroglycerine under conditions of cavity impact (*cf.* Bowden and collaborators, 1943 *a* and *b*, 1947 *a*) has been investigated for several gases. It was found that for all the gases examined, viz., air, nitrogen,

oxygen, hydrogen and helium, the effectiveness of the gas in causing initiation of explosion decreases with decreasing initial pressure.

The relative sensitizing effects of the several gases were examined. For nitrogen, oxygen, helium and hydrogen the relative effects, though not greatly different in magnitude, were found to be in the order $He \approx N_2 > H_2$ and $O_2 > N_2 > H_2$.

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LXVIII. *On Terminology and Symbols in Physical Chemistry.*

By J. E. VERSCHAFFELT, D.Sc. (University of Ghent, Belgium) *.

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1. *Introduction.*

Two years ago this journal published in no. 241 (February, 1944) two interesting papers on physical terminology ⁽¹⁾ written by Prof. W. E. Duncanson. As nomenclature, specially in thermomechanics, is a matter which has also my full interest (as secretary of the Association Internationale du Froid I published a report on that subject in 1932 ⁽²⁾), I should like to make some remarks concerning Mr. Duncanson's papers and also some suggestions, justifying at the same time my own use of terms and symbols.

2. *Molecular Weight.*

Among the first notions which are acquired by students in natural philosophy in the lectures on chemistry we find those of *atomic weight* and *molecular weight*. Since several years objection has been made against these terms, because they are inaccurate in a literal sense. Indeed, they do not refer to the physical weight of an atom or a molecule, *i. e.* to the force which acts on those particles in the field of gravitation, a force which varies from one point to another in that field. The mass and not the weight of a particle being invariable, many prefer the use of the terms *atomic mass* and *molecular mass.* But even these expressions are incorrect, as what is hinted at is not the *absolute* mass of the particles (as names for the absolute masses the terms would be quite right), but the *relative* (or better the *reduced*) mass, the ratio of the absolute mass to that of a well-defined particle (such as an atom of hydrogen); for that reason we should speak of *relative* (or *reduced*) *atomic* and *molecular mass.* The ratio of two masses being, however, the same as the ratio of the weights, there is no reason for replacing weight by mass, the more so since our first knowledge of those quantities resulted from weighing.

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* Communicated by the Author.

Mr. Duncanson suggests the name *relative molar mass* (or *weight*), that is, the relative mass or weight of a mole. That includes, however, that already the definition of a mole is given, and that is, it seems to me, not logical: we usually speak of molecular weight before we have to give the definition of a mole; therefore I prefer the qualification *molecular.* Besides, I prefer to use the word *reduced* instead of *relative*: in relativity (Mr. Duncanson points also to the fact) a relative magnitude is not the *ratio* but the *difference* of two absolute magnitudes, whereas we speak of a reduced quantity when we mean its ratio to another quantity. I think, however, that we may continue to use simply the original terms of atomic and molecular weight. For, next to its physical meaning, the word weight has also an abstract mathematical meaning, that of a numerical coefficient of importance attributed to an object or, as in the probability calculus, to the result of a measurement. It is in this abstract sense that I use the expressions of atomic and molecular weight, for in using them we do not mind the way in which those numbers are found.

As for the symbols by which atomic and molecular weight should be represented, therefore, are logically indicated the usual notations A and M. I do not agree with Duncanson's suggestion to use A also as symbol for molecular weight; I rather do the contrary, as I use A as symbol for affinity, and after all, a single atom is a monoatomic molecule, that is to say, that an atomic weight is also a molecular weight.

Mr. Duncanson would keep the symbol M for the *molar mass*, the mass of a *mole* (or gram molecule), that is, the mass of a quantity of substance the mass of which is expressed in grams by the molecular weight. But I do not see the necessity of using a special symbol for 1 mol; one mol is a special unit of mass, and that does not want a special symbol. We do not, for instance, use a special symbol for a calorie, and so we may write

$$1 \text{ mol} = \text{M gm.},$$

so as we write

$$1 \text{ cal. (calorie)} = 4{,}185 \text{ j. (joules)}.$$

Besides, even if we use another unit of mass than the gram, the measure of the mass of a mole does not change, for it is still M units of mass, M being the molecular weight; if we take, for instance, the kilogram as unit of mass instead of the gram, a mole will logically be no more a molecule gram but a molecule kilogram or M kgm.

What is now the right definition of the molecular weight? It is the mass of a molecule measured with a suitable unit, which is the mass of a conveniently chosen particle. Originally was chosen the atom of hydrogen, as being at that time the smallest particle known, but when was it discovered that nearly all elements in the natural state are mixtures of isotopes, all atomic weights were reduced (*i. e.* divided by a fixed number) so as to make that for ordinary oxygen, that is, for the most abundant isotope in natural, atmospheric oxygen, the atomic weight hould be exactly 16.

That may be expressed in this manner: the unit of mass used for the determination of atomic and molecular weights is the mass of the atom of an imaginary element, the atomic mass of which is exactly the sixteenth part of the atomic mass of ordinary oxygen. To that fictive element I wish to give the name of *Dalton*, in memory of John Dalton, the English chemist who in 1808 founded the atomic theory. To that unit of mass I give also the name of *dalton*, so that a molecule with molecular weight M has a mass of M daltons. If δ is the mass in grams of the imaginary fundamental atom, then

$$1 \text{ dalt.} = \delta \text{ gm.}$$

and the molecular mass of a substance with molecular weight M is Mδ gm.

It has been found that

$$\delta = 1{,}650 \,.\, 10^{-24}.$$

3. *Avogadro's Number.*

Inversely we may write

$$1 \text{ gr.} = N \text{ dalt.} \quad \text{with} \quad N = \frac{1}{\delta} = 6{,}062 \,.\, 10^{23}.$$

The number N, known under the name of *number of Avogadro*, is the number of molecules in a gram molecule; indeed, that number of molecules is $M/M\delta = 1/\delta = N$.

Mr. Duncanson has suggested to use the word *avogadron* for denoting that quantity; that suggestion I readily admit. So we may say that an avogadron of daltons makes a gram, and that an avogadron of molecules makes a mole or gram molecule, for $N \times M\delta$ gm. $=$ M gm., as $N \times \delta = 1$.

I do not agree with Mr. Duncanson when he says that in his equation (1) M_0 is a mass (16 gm.) and not a number (16); m is the mass of an oxygen atom in gm. and is consequently a number, so that M_0, the mass in gm. of an avogadron of oxygen atoms, is also a number, the number 16, the atomic weight of oxygen. Certainly, if in equation (2)

m represents the mass of a molecule (measured with an arbitrary unit), then M_0, the mass of an avogadron of molecules, is also a mass; but then M_0 is no more the molecular weight, unless we admit that Avogadro's number depends upon the unity of mass used.

Even when the substance is built up with several kinds of molecules, we may define a mole as a quantity of homogeneous matter containing an avogadron of particles. Is M gram the mass of that quantity, then M is the mean molecular weight. This may be expressed in the molecular weights M_1, M_2, etc., up to M_n of the different pure constituents of the mixture, the number of which is n (first letter of number), and the mixing ratios of those constituents. An arbitrary constituent I indicate by its number in the series ν (the Greek letter corresponding to the Latin letter n); so its molecular weight is M_ν.

Let, now, m_ν be the mass in grams of the ν-th constituent in the mixture, then

$$m = m_1 + m_2 + \ldots + m_n = \Sigma m_\nu$$

is the total mass and

$$x_\nu = \frac{m_\nu}{m}$$

(the letter x has not been chosen for a special reason, but it seemed the most convenient to me) is the *ponderal mixing ratio* or *mass fraction* of the νth constituent. Dividing m_ν by M_ν, we get the *molar quantity*

$$\mu_\nu = \frac{m_\nu}{M_\nu}$$

(μ being again the Greek letter corresponding to m) or the *number of moles* of the νth constituent in the mixture, while

$$\mu = \Sigma \mu_\nu$$

is the total molar quantity of the mixture. So the *molar mixing ratio* or *molar fraction* of the νth constituent is

$$\xi = \mu_\nu / \mu$$

(once more a Greek letter has been taken going with x). Evidently

$$\Sigma x_\nu = 1 \qquad \text{and} \qquad \Sigma \xi_\nu = 1.$$

The mean molecular weight of the mixture being determined by $M\mu = m$, we have

$$M = \frac{m}{\mu} = \Sigma \frac{m_\nu}{\mu} = \Sigma M_\nu \frac{\mu_\nu}{\mu} = \Sigma \xi_\nu M_\nu,$$

and also

$$\frac{1}{M} = \frac{\mu}{m} = \Sigma \frac{\mu_\nu}{\mu} = \Sigma \frac{1}{M_\nu} \frac{m_\nu}{m} = \Sigma \frac{x_\nu}{M_\nu}.$$

Such are the rules for the calculation of the mean molecular weight of a mixture.

4. *Density and Specific Volume.*

If v is the volume of an arbitrary quantity of homogeneous matter the mass of which is m, then

$$c=\frac{m}{v}$$

is the mass per unit of volume, a magnitude which is commonly called the *density* of the matter. I prefer to call it the *concentration* (hence the symbol c) and in particular the *ponderal concentration*, because it is determined by weight. Obviously is

$$\gamma=\frac{\mu}{v}$$

(the letter γ corresponding again to c) the *molar concentration*, the number of moles per unit of volume.

If we have to do with a mixture, then

$$c_\nu=\frac{m_\nu}{v} \qquad \text{and} \qquad \gamma_\nu=\frac{\mu_\nu}{v}$$

are the *partial* ponderal and molar concentrations of the νth constituent, and

$$c=\Sigma c_\nu \qquad \text{and} \qquad \gamma=\Sigma\gamma_\nu$$

are the *total* concentrations of the mixture.

The reciprocal of the concentration

$$v'=\frac{1}{c}=\frac{v}{m}$$

(the notation v' will be forthwith explained), the volume per unit of mass is usually called the *specific volume*. I wish to preserve that name; the term *rarity*, proposed by Mr. Duncanson, as expressing the reciprocal of density, does not please me.

Mr. Duncanson objects to the word *specific* that it has no precise meaning when it is used. I do not quite agree with that: a specific property of matter is a property characteristic for the kind, the species of matter in the state in which it is observed; specific properties are functions of state, *i. e.* functions of the factors (temperature, pressure, etc.) which determine the state of the substance. So refractive index and dielectric constant are specific properties. I propose, however, to use the word *specific* in a quite definite sense.

We know several magnitudes which, belonging to the bulk of an arbitrary quantity of a homogeneous part of a body (phase of a substance), are proportional to the mass; of such a kind are the heat capacity, the magnetization under the influence of a uniform magnetic field, etc. Such magnitudes I call *spatial magnitudes*, because they depend upon the space occupied by the bulk of the phase of the matter; other magnitudes which belong to the interface between two bodies or phases, such as the mass of adsorbed matter or electrical charge, I call *superficial magnitudes* (the spatial and superficial magnitudes constitute together the *extensive* magnitudes of a *thermomechanical system*, this being ordinarily composed of homogeneous phases and inhomogeneous interfacial layers).

If G (the first letter of great) is the measure of such a spatial magnitude we may write

$$\mathrm{G}=m\mathrm{G}',$$

where $\mathrm{G}'=\mathrm{G}/m$ is the magnitude per unit of mass, and that is what I wish to be called the *specific magnitude*; so we may speak of specific heat capacity, specific magnetization and also specific volume, as the volume also is a spatial magnitude, after the mass itself the simplest of all.

As for a given state of the matter G′ is the partial derivative of

$$\mathrm{G}=\mathrm{G}(\mathrm{T}, p, m)=G(\mathrm{T}, p, \mu);$$

with regard to m (the factors of state remaining constant), and mathematicians use to indicate the derivative of a function by an accent, I write

$$\frac{\partial \mathrm{G}}{\partial m}=\mathrm{G}',$$

and therefore I represent the specific volume by v'.

Of course, there are many other characteristic properties of matter, such as the refractive index, which are not related to a spatial magnitude, and some of them have received also the qualification of specific (specific resistance, specific rotatory power), but most of them bear already shorter names (refractivity, resistivity, viscosity) and, for those who have not yet got one, appropriate names will certainly be found easily, so that the qualification of *specific magnitudes* may be reserved for the G′.

We may also write

$$\mathrm{G}=\mu G,$$

wherein

$$G=\frac{\mathrm{G}}{\mu}=\frac{\partial \mathrm{G}}{\partial \mu}=\mathrm{M}\mathrm{G}'$$

is the *molar magnitude*, the magnitude per mole; that is, the specific magnitude multiplied by the molecular weight, or, more accurately, by the molar mass.

Surely G′ and G have not the same dimensions, for μ is a number and m is a mass; so G has the same dimensions as G. Mr. Duncanson also has called attention to the dimensional difference between molar and specific magnitudes.

Let N be the number of molecules contained in the mass m of a phase; that is a spatial magnitude, and so $\mathrm{N}'=\mathrm{N}/m$ is the specific number of molecules, whereas

$$N=\mathrm{M}\mathrm{N}'=\frac{\mathrm{MN}}{m}=\frac{\mathrm{MN}}{\mathrm{NM}\partial}=\frac{1}{\partial}$$

(being now the number indicating in the chosen unit of mass the mass which I named dalton) is the number of molecules per mole, *i. e.* Avogadro's number in case the unit of mass is the gram.

It may be remarked that a mole is, as the calorie, a secondary and not a fundamental unit of mass. So there is no need for using it consistently throughout and building up upon that base a system of derived units that would be characteristic for the considered kind of matter.

It does not often occur that we have to reduce a spatial magnitude to the unit of volume and therefore it is not very necessary to give a special name to the magnitude G/v nor to introduce a new symbol for it. The magnitude $c=m/v$, which is commonly known under the name of density and which I denote by the name of concentration is, evidently, of such a kind; it is, for that reason, rather indicated that we should call G/v the *density of* G (a terminology which is sometimes used; we speak, *e. g.*, of the density of an electrical charge), and so the common density would be more explicitly the *density of mass*. But never should we denote those densities as specific magnitudes, although they are, as well as the real specific magnitudes just defined, characteristic for the kind of matter.

For that reason also the name of *specific weight* given to the weight of the unit of volume must, in my opinion, be rejected. In my terminology the specific weight would be $mg/m=g$, that is the intensity of the field of gravitation, and so it would be the same for all matter. But I think that many will agree with me that the notion of specific weight, even in the first meaning, may be left out of the text-books.

If the spatial magnitude G belongs to a mixture, then

$$\mathrm{G}=\mathrm{G}(\mathrm{T}, p, m_1, \ldots, m_n)=G(\mathrm{T}, p, \mu_1, \ldots, \mu_n),$$

and we know that in that case

$$\mathrm{G}=\Sigma m_\nu \mathrm{G}_\nu'=\Sigma \mu_\nu G_\nu,$$

wherein $$\mathrm{G}'_\nu=\frac{\partial \mathrm{G}}{\partial m_\nu} \quad \text{and} \quad \mathrm{G}_\nu=\frac{\partial \mathrm{G}}{\partial \mu_\nu}=\mathrm{M}_\nu \mathrm{G}_\nu'$$

are magnitudes which may be denoted by the names of *partial* specific and molar magnitudes.

5. *Specific Heat.*

The name of *specific heat* is also one of those against which objections have been raised; it is an inheritance from the time where one thought that matter contained something that was called heat and so in my terminology specific heat would mean the heat per unit mass. Evidently it has not that signification; it means properly the *specific heat capacity*, if we call *heat capacity* of a body the amount of heat necessary for raising through one degree the temperature of the body in a given state. Even that expression is not quite logical, but it seems to me that it is still the best we dispose of; and so I continue to use for the present the shorter term of specific heat.

The heat capacity is a spatial magnitude; I represent it by C, and so its derivative with regard to the mass, the specific heat capacity or shortly the specific heat, I represent by C′, with the indices p or v for denoting that it is taken at constant pressure or volume. Of course, the *molar heat capacity* (*molar heat*) is $C=MC'$.

Mr. Duncanson suggests to use instead of specific heat the term *thermity*. I cannot accept this denomination. In my opinion the prefix *therm* must only be used in words which concern the temperature; for notions related to heat we should by preference use the prefix *calor* (I point to the difference between thermometer and calorimeter; accordingly I do not accept the expression *thermal capacity*), and so I should prefer the term *calority* for specific heat. Unhappily, however, no more than the first, the second word has a meaning by itself.

The phenomenon of the raising of temperature of a body by supply of heat is what I call a *thermic effect*, and in particular a *calorithermic effect*; so the rise of the temperature of the unit of mass of the body by supply of one calorie might be called its *specific calorithermic power* or its *calorithermity*. Inversely, the heat emitted by a gram of the body when its temperature is lowered through one degree might be named its *specific thermocaloric power* or its *thermocalority*; that would certainly be better, plainer than calority without more, but I fear that such a term will be considered as too pedantic and will never be admitted in elementary physics. So, I think, the expressions heat capacity and even specific heat will further be used in the future.

6. *Latent Heat.*

The term *latent heat*, which means concealed heat, is another relic of the time when one thought that matter contained heat. I think it would be advisable that we use the word *latent* in this connection no longer; I speak always of *heat of transition*, when I do not say precisely what transition (melting, vaporization, etc.) is hinted at.

A heat of transition, that is heat of isothermic transformation, I represent by l (first letter of latent) and so the heat per unit mass, the specific heat of transition, is l'; the heat per mole Ml' I represent by λ (in a chemical reaction λ is the heat of reaction per equivalent). Again λ and l' have not the same dimensions: λ is an energy, l' an energy divided by a mass.

Mr. Duncanson writes L instead of λ; I use the letter L for work (labour) and write the equation formulating mathematically the first law of *thermomechanics* (which I divide into *thermostatics* and *thermodynamics*)

$$dQ=dU+dL,$$

wherein dQ is the caloric effect in an elementary transformation of the considered body, dL the elementary external work done by it and dU its elementary change of internal energy U (I do not use the letter E for energy, because I prefer to reserve that letter for electromotive force). Ordinarily, $dL = pdv$, and so

$$dQ = dU + pdv = C_v dT + l_v dv,$$

l_v being the *heat of dilatation.* We may also write

$$dQ = dW - vdp = C_p dT + l_p dp,$$

W (first letter of warmth or Wärme) $= U + pv$ being the enthalpy, and l_p the *heat of compression.*

7. *Temperature and Gas Constant.*

Mr. Duncanson considers temperature as a derived magnitude with the dimensions of energy. On the thermomechanical scale the (absolute) temperature is defined by

$$\tfrac{3}{2} kT = \theta,$$

θ being the mean kinetic energy of a molecule of a perfect gas and k Boltzmann's constant. When temperature is measured on the Kelvin-scale, $k = 1{,}372 \, . \, 10^{-16}$.

I prefer considering the unit of temperature as a fundamental unit, and so I give to the universal constant k the dimensions energy/temperature. We know that if we make undimensionate a universal constant we lessen by one the number of fundamental magnitudes; considering, for instance, the velocity of light in vacuo as dimensionless, we make the unit of time derive from the unit of length. Inversely, rendering dimensionate a constant that was formerly considered as dimensionless, we introduce a new fundamental unit.

The value of the temperature

$$T = \frac{2}{3} \frac{\theta}{k}$$

on a thermomechanical scale depends upon the value we assume for k. We could put $k = 1$ (and so we could do with every universal constant) and obtain in this manner a special scale of temperature which would, of course, be very unpractical. For obtaining a practical unit of temperature we should take for k a suitable power of 10; 10^{16} would give a scale not very different from the Kelvin scale and so a practical temperature scale based on logical fundaments would be obtained.

According to the dynamical theory of gases the equation of state of a perfect gas may be written

$$pv = \tfrac{2}{3} N\theta = RT,$$

when $R = Nk$; R is a spatial magnitude with the same dimensions as k. Per unit mass the equation is

$$pv' = R'T, \qquad \text{with} \qquad R' = R/m = N'k,$$

and per mole

$$Mpv'=RT, \qquad \text{where} \qquad R=Nk$$

is a universal constant, the *gas constant*; its value is 8,316 joules/degree when the unit of mass is the gram.

It is not logical to speak of R′ (the R of Mr. Duncanson) as the gas constant per gram, for R and R′ are not constants; only R is.

8. *Some more Definitions and Symbols.*

In general we may write the elementary external work in the form

$$dL=Xdx+Ydy+\ldots,$$

where X, Y, etc., are *generalized forces*, and x, y, etc., *generalized displacements.*

The second law may be formulated

$$TdS=dQ+Ad\omega,$$

wherein the always positive term $Ad\omega$ (the *dissipated energy*) refers to a spontaneous, *irreversible* transformation (a chemical reaction for instance); A is the *affinity* in that transformation, ω the *degree of progress* of it; S is the entropy, T the temperature on the thermomechanical scale.

Combining the two laws we find

$$dU=TdS-Xdx\ldots-Ad\omega,$$

where S, x, y, etc., and ω are *extensive magnitudes* (see § 4) determining the state of the thermomechanical system, and are therefore to be considered as the *state factors*, whereas the *conjugated factors* T, X, Y, etc., and A, which are called *intensive magnitudes*, are functions of the extensive ones and may therefore be considered as *state functions.* The equations expressing how the different intensive factors depend upon the extensive ones are the *equations of state* of the system. Of course, we may as well consider the intensive factors, at least some of them (T and p, for instance) as state factors and their conjugates (S and v) as state functions.

From dU may be derived expressions for the differentials dF and dG of the *free energy* F=U−TS and the *thermal potential* (function of Gibbs) G=F+pv.

Such are, in a nutshell, the fundaments of physico-chemical thermomechanics; they give an idea of the terminology and symbols I use.

9. *Chemical Reactions.*

A phase or system is said to be *open* when it may exchange matter with bodies with which it is in contact, as a liquid that is evaporating in the atmosphere or a hygroscopic substance which absorbs vapour from the ambient air. Let us consider such a phase or system and suppose that at constant temperature and pressure the masses of its constituents vary by dm_ν; then (supposing that the state does not depend on other factors) the thermal potential varies by

$$dG=\Sigma G_\nu'\,dm_\nu=\Sigma M_\nu G_\nu'\,d\mu_\nu=\Sigma G_\nu\,d\mu_\nu,$$

G_ν' being the *specific potential* of the νth constituent when it exists as pure phase, or the *partial potential* when it belongs to a mixed phase; G_ν is the *molar potential*. We suppose moreover that there is equilibrium of contact between the different phases and so G_ν' has, as we know, the same value for each constituent in all phases.

But even when the system is *closed* the masses of the constituents may vary; this occurs, namely, when a chemical reaction is going on in the system. The closed system behaves then as an open one, from which some quantities of matter are taken away and replaced by equivalent quantities of other substances. We may then write

$$d\mu_\nu=-q_\nu d\omega,$$

wherein ω represents what we have already denoted as the *degree of progress* of the reaction, whilst the q_ν are (positive or negative) integral numbers called *stoichiometric coefficients*; these coefficients are taken as small as possible so that they have no common divisor other than 1. I take them positive for the constituents which vanish in the reaction, negative for those which are generated by it.

The total mass being invariable, we have evidently

$$0=\Sigma dm_\nu=\Sigma M_\nu d\mu_\nu=-\Sigma q_\nu M_\nu d\omega \quad \text{or} \quad \Sigma q_\nu M_\nu=0.$$

This last equation is a numerical relation between the molecular weights of the substances which participate in the reaction; if, however, we consider M_ν as a symbol for the molecular constitution, we may interpret it as the *equation of reaction*, that is, the formula for the reaction, such as

$$3H_2+N_2\rightarrow 2NH_3$$

in the case of the synthesis of ammonia, written in the mathematical form of an equation

$$3H_2+N_2-2NH_3=0.$$

According to the definition of affinity given in § 8, we have

$$A=-\left(\frac{\partial G}{\partial\omega}\right)_{Tp}=-\Sigma MG_\nu'\frac{d\mu_\nu}{d\omega}=\Sigma q_\nu M_\nu G_\nu'=\Sigma q_\nu G_\nu.$$

We know that $A=0$ is the condition for chemical equilibrium.

In the case if isomerism, that is, when the reaction is a molecular transformation by which the structure but not the constitution of the molecule is changed, $q_1=1$ and $q_2=-1$ and the equation of reaction is

$$M_1-M_2=0,$$

the reaction formula being

$$M_1 \rightarrow M_2.$$

The chemical equilibrium between the two isomeric substances is realized when

$$G_1' = G_2',$$

whether the isomers are pure in separate phases or mixed in one.

Such is also the condition of physical equilibrium of two phases of a same substance, the transition of which from one phase to another may be represented by the same formula as above, provided there is no change in the molecule, which would mean a chemical transformation superposed on the physical one. On account of this analogy I call the transition of matter between two *homomolecular* phases a *physical isomerism*.

10. *Physically and Chemically Pure Substances.*

We say that a substance is *physically pure* when there is no possibility of changing its specific properties by the usual methods of purification, as distillation and crystallization. A physically pure substance has under every pressure one fixed boiling point and one freezing point. Water, for instance, is a physically pure substance.

But a physically pure substance is not necessarily *chemically pure*, when we require that a chemically pure substance should be homomolecular, *i. e.* built up by one kind of molecule only. I consider the molecule as characterizing the kind of matter: each kind of molecule is characteristic for a *chemical species*. From this point of view water is not chemically pure, if we admit that it is built up by molecules which are partially aggregates of the simplest one H_2O; so water would be a mixture of different chemical species, to which W. Sutherland gave the names of hydrols (monohydrol, dihydrol, etc.). In the same way the nitrogen-peroxide (or azotyl), constituted by the two kinds of molecule NO_2 and N_2O_4, may be considered as a mixture of two substances (the monoazotyl and the diazotyl), each of which would have its own specific properties and its own crystalline phase, if one succeeded in isolating them.

It is known that several substances which are considered as pure, such as benzene, lose their characters of physical purity when they are intensely dried. That is probably due to the fact that in absence of water molecules the molecules of the substance undergo association, or, in other words, that water molecules act as a catalysator against this association. So water itself would be its own catalysator, with the effect that for every system of values of T and p an equilibrium is obtained between the different hydrols; practically, moreover, this state of chemical equilibrium is established instantaneously and so water behaves as if it were a homomolecular substance. After all, I should say that really pure substances only exist theoretically, except perhaps in the crystalline state, for a crystal is not a chaotic heap of molecules.

As I suppose that chemical reactions only can take place between molecules, in my mind a crystalline phase can only then participate in a reaction when it has previously dissolved in molecular form in the adsorbed layer of a fluid which is in contact with it. That recalls the old assertion that a substance only then can act when it is fluid (*corpora non agunt nisi fluida*).

References.

(1) W. E. Duncanson, "Molecular Weight, Gram Molecule and Avogadro's Number," Phil. Mag. xxxv. pp. 73–80 (1944); 'Terminology and Symbols in Physics,' *ibid.* pp. 81–90.

(2) J. E. Verschaffelt, "Rapports sur la Terminologie et les Notations en Thermodynamique," *Bull. int. des rens. frig.* xiii. annexe spéciale au nº 2, pp. 2–20, 22–27; ann. spéc. au nº 5, pp. 31–33 (1932).

XLIX. *The Restricted Problem of the Random Walk.*

By A. N. Gordon, M.Sc.*

[Received April 8, 1947.]

In the Phil. Mag. of August 1944, Dr. Silberstein solves the following problem :—

A particle, moving in a plane, starts from a fixed point O, taking n unit steps at random, these steps, however, being restricted to directions parallel to two fixed straight lines. It is required to find the frequency distribution of possible end points to such paths.

The two given lines can be taken as coordinate axes with origin O, there being no loss of generality in assuming that the axes are rectangular. Thus at any stage the particle is equally likely to move to the left, right, up or down.

The solution is obtained by forming a difference equation in four variables, a simple change of variable enabling this to be split into two simpler, (and similar), difference equations each in two variables. These equations are each of the form

$$\phi_{n+1}(s)=\phi_n(s+1)+\phi_n(s-1)$$

and Dr. Silberstein states that, as this is the difference equation derived from the simpler problem of n steps taken at random along a line, which problem has a known solution

$$\phi_n(s)={}^nC_{\frac{n+s}{2}},$$

then this is the solution required.

---

* Communicated by the Author.

It can, however, be easily shown by standard methods that a solution of the above equation is

$$\phi_n(s)=\psi(n+s)+{}^nC_1\psi(n+s-2)+{}^nC_2\psi(n+s-4)+\ldots+\psi(-n+s),$$

where ψ is an arbitrary function.

This becomes the solution given when ψ is chosen so that

$$\psi(m)=0,\ m\neq0;\quad \psi(m)=1,\ m=0.$$

Such a function is for example

$$\psi(m)=\frac{\sin m\pi}{m\pi}.$$

Hence, although his choice of solution is reasonable, it requires further justification, (other than that it gives the right answer), if it is to be adopted.

Further, although it is stated that the only way of solving the problem is by the formation of a difference equation, the following very simple solution shows that this is completely unnecessary.

Simple Solution of the Problem.

Consider the point P whose coordinates with respect to the given straight lines are the integers r, s. Then the number of different ways of arriving at P in n steps is given by the coefficient of x^ry^s in the expansion of

$$(x+x^{-1}+y+y^{-1})^n.$$

This has a total of 4^n terms each of which can be associated with a distinct path (contributions from x^{-1}, for example, corresponding to steps to the left). Hence, if $f_n(r, s)$ is the required distribution function,

$$\Sigma f_n(r, s)x^ry^s=(x+x^{-1}+y+y^{-1})^n$$
$$=\left\{(x+y)\left(1+\frac{1}{xy}\right)\right\}^n.$$

Make the substitution $xy=u$, $x/y=v$, and this becomes

$$\Sigma f_n(r, s)u^{\frac{r+s}{2}}v^{\frac{r-s}{2}}=\left\{\frac{(u+1)(v+1)}{\sqrt{uv}}\right\}^n=\left(\sqrt{u}+\frac{1}{\sqrt{u}}\right)^n\left(\sqrt{v}+\frac{1}{\sqrt{v}}\right)^n,$$

giving

$$f_n(r, s)={}^nC_{\frac{n+r+s}{2}}\,.\,{}^nC_{\frac{n-r+s}{2}},$$

which is Silberstein's solution.

Extension to Three Dimensons.

By suitable choice of generating function this method can be extended to any number of dimensions. Thus for three dimensions the required distribution function is given by

$$\sum_{r,s,t} f_n(r, s, t)x^r y^s z^t = (x+x^{-1}+y+y^{-1}+z+z^{-1})^n. \quad . \quad . \quad . \quad (1)$$

I have not succeeded in deriving as simple a solution in this case, but the substitution used above does lead to some simplification, giving

$$\sum_{r,s,t} f_n(r, s, t)u^{\frac{r+s}{2}} v^{\frac{r-s}{2}} z^t \equiv \left\{\frac{(1+u)(1+v)}{\sqrt{uv}} + (z+1/z)\right\}^n$$

$$= \sum_{\alpha=-t}^{n-t} {}^n\mathrm{C}_{t+\alpha} \left\{\frac{(1+u)(1+v)}{\sqrt{uv}}\right\}^{n-t-\alpha} (z+1/z)^{t+\alpha}$$

$$\therefore \; f_n(r, s, t) = \sum_{\alpha=0}^{n-(r+s+t)} {}^n\mathrm{C}_{t+\alpha} {}^{n-t-\alpha}\mathrm{C}_{\frac{n+r+s+t-\alpha}{2}} {}^{n-t-\alpha}\mathrm{C}_{\frac{n+r-s-t-\alpha}{2}} {}^{t+\alpha}\mathrm{C}_{t+\alpha/2},$$

where even values of α are to be taken in the summation. It also follows that $f_n(r, s, t)=0$ when $n-(r+s+t)$ is odd, which is otherwise obvious from the problem itself.

Putting $n-(r+s+t)=2k$ and $\alpha=2\lambda$, we have finally, after making some simplifications,

$$f_n(r, s, t) = {}^n\mathrm{C}_k \sum_{\lambda=0}^{k} {}^{n-k}\mathrm{C}_{t+\lambda} {}^k\mathrm{C}_\lambda {}^{n-t-2\lambda}\mathrm{C}_{k+r-\lambda}.$$

Calculation of the frequencies by this formula is very laborious except when k is small. In this case it is almost as easy to pick out the required coefficient in the generating function itself.

Deductions from Equation (1).

(*a*) By differentiating both sides of equation (1) with respect to x, it is easy to deduce the following recurrence relation

$$rf_n(r, s, t) = n\{f_{n-1}(r-1, s, t) - f_{n-1}(r+1, s, t)\}$$

with similar formulæ for s and t.

It may also be noted that similar formulæ hold for any number of dimensions, including the one and two dimensional cases.

(*b*) From the symmetry of the problem it is obvious that $f_n(r, s, t)$ is an even function of r, s and t. Therefore equation (1) can be written

$$\sum_{r,s,t\geq 0} f_n(r, s, t)\left(x^r+\frac{1}{x^r}\right)\left(y^s+\frac{1}{y^s}\right)\left(z^t+\frac{1}{z^t}\right) = \left(x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}\right)^n.$$

Putting $x=e^{j\theta}$, $y=e^{j\phi}$, $z=e^{j\psi}$, this becomes

$$8\sum_{r,s,t\geq 0} f_n(r, s, t) \cos r\theta \cos s\phi \cos t\psi = 2^n(\cos\theta+\cos\phi+\cos\psi)^n$$

except that an additional factor 1/2 is required on the left-hand side for terms in which any of r, s, t happens to be zero.

By the method employed for the determination of coefficients in a Fourier expansion it follows that

$$\pi^3 f_n(r, s, t) = 2^n \int_0^\pi \int_0^\pi \int_0^\pi (\cos\theta + \cos\phi + \cos\psi)^n \cos r\theta \cos s\phi \cos t\psi \, d\theta \, d\phi \, d\psi,$$

this being true even when some of r, s, t are zero, as compensating factors 2 arise on the left-hand side.

Remembering that the total number of different paths in this case is 6^n, the probability that (r, s, t) is an end point is given by

$$\mathrm{P}_n(r, s, t) = \frac{1}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \left(\frac{\cos\theta + \cos\phi + \cos\psi}{3}\right)^n \cos r\theta \cos s\phi \cos t\psi \, d\theta \, d\phi \, d\psi. \quad . \quad . \quad . \quad . \quad (2)$$

Behaviour of $P_n(r, s, t)$ when n is large.

When n is large the contributions from the integrand of equation (2) are only significant when $|\cos\theta + \cos\phi + \cos\psi| \simeq 3$, *i. e.* in the regions $\theta = \phi = \psi = 0$; $\theta = \phi = \psi = \pi$.

When $n - (r + s + t)$ is even the substitutions $\theta' = \pi - \theta$, etc., show that the contributions from each region are the same.

When $n - (r + s + t)$ is odd the contributions are equal and opposite which is to be expected.

In the first case

$$\mathrm{P}_n(r, s, t) \simeq \frac{2}{\pi^3} \int_0^\epsilon \int_0^\epsilon \int_0^\epsilon \left(\frac{\cos\theta + \cos\phi + \cos\psi}{3}\right)^n \cos r\theta \cos s\phi \cos t\psi \, d\theta \, d\phi \, d\psi,$$

where ϵ is small

$$= \frac{2}{\pi^3} \int_0^\epsilon \int_0^\epsilon \int_0^\epsilon \{1 - 1/6(\theta^2 + \phi^2 + \psi^2)\}^n \cos r\theta \cos s\phi \cos t\psi \, d\theta \, d\phi \, d\psi$$

$$= \frac{2}{\pi^3} \int_0^\infty \int_0^\infty \int_0^\infty e^{-n/6(\theta^2 + \phi^2 + \psi^2)} \cos r\theta \cos s\phi \cos t\psi \, d\theta \, d\phi \, d\psi$$

the smallness of the exponential term permitting the limits to be extended to infinity.

Now $$\int_0^\infty e^{-a^2x^2} \cos \lambda x \, dx = \frac{\sqrt{\pi}}{2a} e^{-\frac{\lambda^2}{4a^2}},$$

$$\therefore \qquad \mathrm{P}_n(r, s, t) \simeq \frac{2}{\pi^3} \left(\frac{1}{2}\sqrt{\frac{6\pi}{n}}\right)^3 e^{-\frac{3}{2n}(r^2+s^2+t^2)}$$

$$= \frac{1}{4} \left(\frac{6}{\pi n}\right)^{3/2} e^{-\frac{3}{2n}(r^2+s^2+t^2)}.$$

This value, like Silberstein's value for the two dimensional case, is just twice the value obtained by Lord Rayleigh for the more general problems [Theory of Sound, second edition, chapter II].

This is very satisfactory when it is realized that the probability function is zero for half the lattice points and will agree exactly with Lord Rayleigh's result if we " average out " so as to include these points also.

LXX. *Notices respecting New Books.*

Smithsonian Elliptic Functions Tables. By G. W. and R. M. SPENCELEY. [Pp. iv+366; $9\frac{1}{4}'' \times 6''$.] Smithsonian Miscellaneous Collections, Vol. 109. (Smithsonian Institution, Washington, 1947.).

THE title is perhaps misleading: these are Smithsonian tables of the Jacobian elliptic functions, the elliptic integral of the second kind, and two associated theta-function quotients. Four consecutive pages are allotted to each value of the modular angle $\theta=\arcsin k=1^\circ\ (1^\circ)\ 89^\circ$. On these pages eight successive columns give, for the argument $r=0^\circ\ (1^\circ)\ 90^\circ$, the functions $u=r\mathrm{K}/90$, $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, ϕ, $\mathrm{E}(\phi)$, $\mathrm{A}(r)$, and $\mathrm{D}(r)$; where (using the Weierstrass-Hancock notation for the theta-functions of parameter q)

$$\sin\phi=\operatorname{sn} u\,;\quad \mathrm{A}(r)=\theta_1\left(\frac{r}{180},q\right)\Big/\theta_2(q)\,;\quad \mathrm{D}(r)=\theta_0\left(\frac{r}{180},q\right)\Big/\theta_0(q).$$

Each of these eight functions is tabulated to 12 places of decimals, ϕ being in radians. In addition each set of four pages is headed with the values of the constants K, K′, E, E′, q, q', D(90), and 1/D(90); each to 16 significant figures with the exception of 1/D(90) which is to 15 places of decimals. Subsidiary tables at the end of the book give, for the argument $r=1^\circ\ (1^\circ)\ 90^\circ$, the functions r (radians), $\sinh r$, $\cosh r$, $\sin r$, $\cos r$; the first three being to 15 places of decimals and the last two to 25 places of decimals.

The typography of the tables is good and clear; but the explanatory appendix contains a number of minor misprints. J. M. HAMMERSLEY.

The Theory of Mathematical Machines. By Prof. F. J. MURRAY. [Pp. vii+116; $11'' \times 9''$.] (King's Crown Press, Columbia University, 1947.)

CALCULATING machines and devices cover so heterogeneous a field that it would prove extremely difficult to formulate a coherent theory for them. In this respect, Professor Murray's book does not realise the aspirations of its ambitious title: it comprises instead a rather hotch-potch collection of mechanical and electrical devices, that aid calculation, together with mathematical comments upon their operation. Although this collection is not exhaustive, it does serve a very useful purpose in bringing together a good deal of such information in a single volume. In editing his material the author has not achieved a very happy balance: he has dismissed a number of important topics, on the grounds that he has "not time to discuss them", in favour of several disproportionately long accounts of subjects of relatively minor importance, that may possibly have aroused his own personal interest.

The reviewer feels that some of the generalisations in this book are too sweeping, and that various definitions are not sufficiently specific. There are, however, more solid reasons why the book should not be read uncritically. Thus, it is not immediately apparent from the text that certain schematic diagrams and circuit diagrams have been incorrectly drawn (*e. g.*, on pages 25 and 33). There are also some bizarre errors: on page 76 a symbol, alleged (incorrectly) by the text to represent an operator, has been equated to a real variable; and on page 94 the functional determinant has been credited with Stuart lineage by being repeatedly called the "jacobean matrix".

J. M. HAMMERSLEY.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

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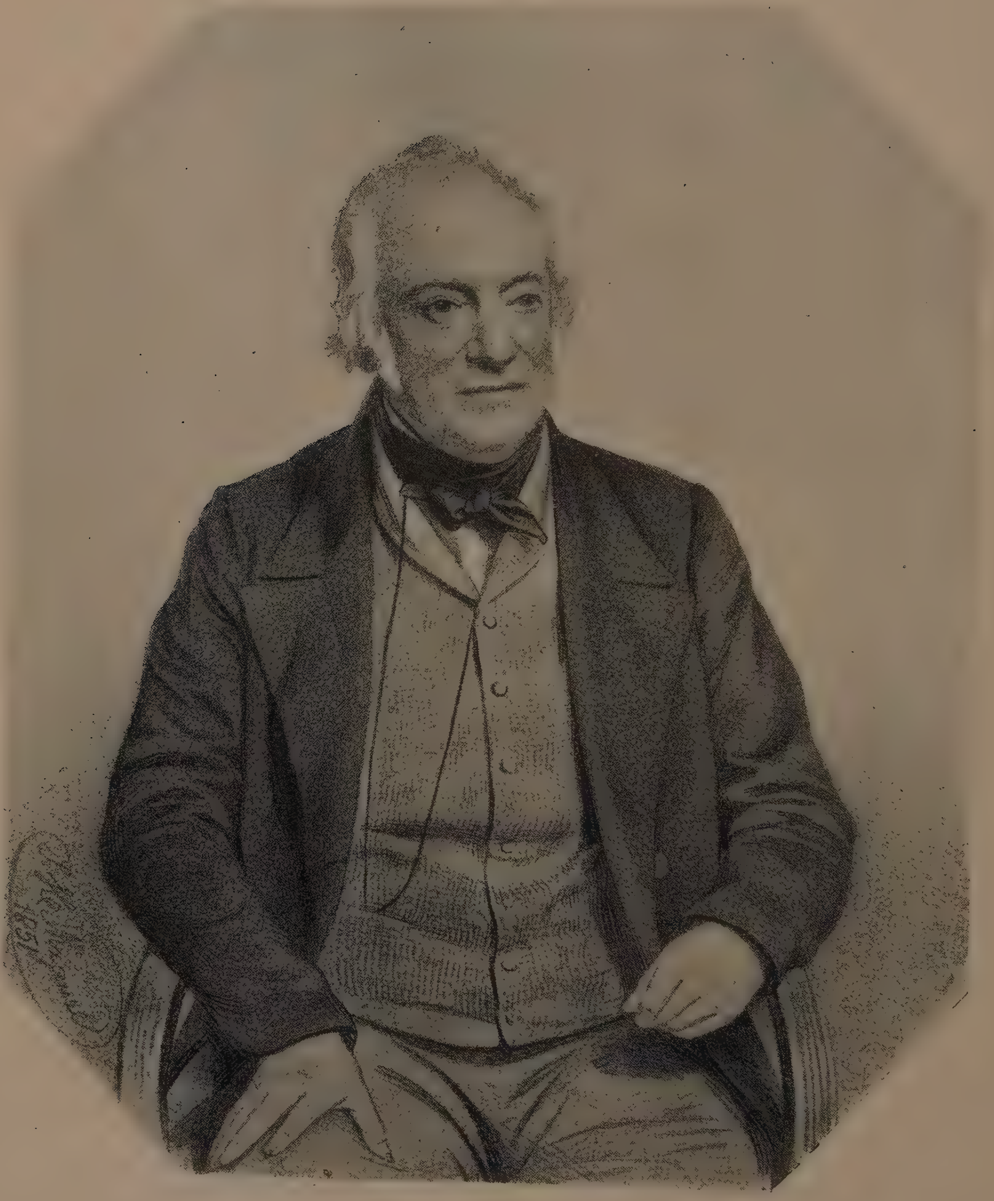

Richard Taylor

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Commemoration number to mark the 150th anniversary of the
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AND

ALLIED TOPICS

Edited by ALLAN FERGUSON

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>
> Just. Lips. *Polit.* lib. i. cap. 1. Net.

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——"Cur spirent venti, cur terra dehiscat,
Cur mare turgescat, pelago cur tantus amaror,
Cur caput obscura Phœbus furrugine condat,
Quid toties diros cogat flagrare cometas,
Quid pariat nubes, veniant cur fulmina cœlo,
Quo micet igne Iris, superos quis conciat orbes
Tam vario motu."

J. B. Pinelli ad Mazonium.

ALERE FLAMMAM.

PREFACE

The Philosophical Magazine is an independently owned journal; it is not the organ of any institution or scientific body. As a scientific journal it is therefore in a unique position.

The writer in his imagination can weave an historical romance through the years of the existence of the Magazine and of the firm of Taylor and Francis, its proprietors and printers. It is not, however, his intention to lapse to any length in that direction, as the reader will have ample opportunity to build a picture from the interesting articles contributed to this Commemoration number of one hundred and fifty years in the life of the Magazine.

As Chairman of the Board of Directors of the firm it is a privilege to be given the task of writing this note of introduction to the illuminating articles which follow. The writer would like to take this opportunity of expressing his admiration for the courage of the Executive Editor who has successfully steered the Magazine through the years of the war, and, despite physical disabilities, has planned and produced this Commemoration number. He has succeeded in obtaining most interesting articles from able writers whose names are familiar to the scientific world and who hold positions of eminence in the field of science.

Few readers can be aware of the character of the home of the Philosophical Magazine, and some may be unmindful of the many fond parents who have attended to its welfare throughout the years of its growth to maturity. During those years it has achieved an international reputation as a scientific journal in the realm of physical and mathematical science. Let the reader allow his imagination to recall the history of the times of the notorious Bishop Bonner who was resident in a house forming part of the premises where the Magazine is still printed, and let him visualise the many editors who have ascended the seventeenth century staircase and then allow his imagination free play as he wanders through the rooms, three of which have ornate ceilings attributed to Sir Christopher Wren. He cannot fail to be impressed by the surroundings of the building, the walls showing signs of their age and of additional scars from the recent war. Further, he will see among the workers many who have given loyal service to the firm from their early youth.

Editors and workers know too well the difficult times through which the magazine and the firm have passed and yet the firm has succeeded in producing journals and other publications for many academic and scientific institutions among whom they have earned a high reputation as scientific printers and publishers.

The Philosophical Magazine has indeed passed through the trials and viscissitudes of life but the editors feel with confidence that the past achievements of the magazine justify their greater confidence in its continued success.

The Editorial Board—Sir Lawrence Bragg, Sir George Thomson and Dr. Allan Ferguson—has been strengthened by the addition of Professor N. F. Mott, who takes on the duties of Editor. The Board of Directors wish them every success in the journey towards the 200th anniversary of the foundation of the magazine.

E. J. BURDON.

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ERRATA

Page 5, 1st line. *For* 1920 *read* 1919.

Page 125, second line of third paragraph. *For* ‘de Letterati di,’ *read* ‘de Letterati.’

THE

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AND

COMMERCE.

BY ALEXANDER TILLOCH,

MEMBER OF THE LONDON PHILOSOPHICAL SOCIETY.

“ Nec aranearum ſane textus ideo melior, quia ex ſe fila gignunt. Nec noſter vilior quia ex alienis libamus ut apes.” Just. Lips. *Monit. Polit.* lib. i. cap. 1.

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Paternoster Row; Bell, No. 148, Oxford-street;
Vernor and Hood, Poultry; Harding, No. 36,
St. James's-street; J. Remnant, High-street,
St. Giles's; and W. Remnant,
Hamburgh.

Reproduction of title-page to first volume.

THE PHILOSOPHICAL MAGAZINE

By ALLAN FERGUSON, M.A., D.Sc.
AND
JOHN FERGUSON, M.A., B.D.

There is an obvious danger in associating great movements with the turn of a century, but the years about 1800 are associated with scientific and social events of the first magnitude. Volta in 1800 was to announce his invention of the voltaic pile and the " crown of cups " ; these systems were the first examples of practicable forms of battery producing a continuous current. The discovery was used almost at once by Nicholson and Carlisle to decompose water by electrolysis. In 1799, Benjamin Thompson, Count Rumford, founded the Royal Institution of Great Britain. About the same time that versatile genius, Thomas Young, was carrying out those experiments on the interference of light which were to rehabilitate the wave theory, though their influence was to be delayed by a spate of abuse from Brougham (" the feeble lucubrations of this author in which we have searched without success for some traces of learning, acuteness and ingenuity "). In 1798 Cavendish published the result of his classic torsion balance experiment on the Earth's mean density—a result in which that most accurate of experimenters made a slip in calculating his final mean, and so put on record his final result as 5·48 instead of 5·448.

The social aspect of England was changing rapidly, and the era of steam was looming ahead. In 1812 the *Comet*, Henry Bell's steamboat, was to be seen making its smoky progress along the Clyde ; the canal and road system of the country was taking shape, and the stage was set for the vast industrial expansion of the nineteenth century, an expansion bringing with it problems of affluence and poverty for which it is taxing all the resources of the twentieth century to find a solution. Interest in matters of science, pure and applied, was spreading, and the audience to which such topics appealed was increasing in numbers and intelligence. It was in circumstances such as these that Alexander Tilloch founded the Philosophical Magazine, the first number bearing the date June 1798.

Alexander Tilloch was a prominent figure in the contemporary world of invention and printing. He was born at Glasgow in 1759, the son of John Tulloch, a tobacco merchant and magistrate. He was educated at the University of Glasgow. About the year 1787 he changed his surname from Tulloch to Tilloch. He became interested in printing and independently rediscovered the process of stereotyping, for which, in 1784 he took out a patent jointly with Andrew Foulis, the printer. Part of the story may be read in an article in the Philosophical Magazine (vol. x. p. 267, 1801) over Tilloch's initials. Tilloch migrated to London in 1787, where, among various printing adventures which included an unsuccessful attempt to persuade the Bank of England to adopt a method of printing notes which would make forgery impossible, he founded the Philosophical Magazine. In later life Tilloch became deeply interested in the subject of prophecy and joined the Sandemanian sect, a sect of which Michael Faraday was an elder. Tilloch died in 1825 in Islington, near London.

But, in the opening years of the nineteenth century, a new name appeared in the printing world—the name of one who was to contribute in no small degree to the setting of a high standard of printing, especially of scientific works, and to show that a critical appreciation of the latest advances in science was not incompatible with a generous knowledge of and an enthusiasm for the humanities. Richard Taylor (1781–1858) was the second son of John Taylor, wool-comber and hymn-writer, and was born at Norwich. He was educated at a private school in his native city and, at the suggestion of Sir James Edward Smith (the founder of the Linnean Society), was apprenticed to an eminent printer of Chancery Lane, London, Jonas Davis by name, whose press had been active in London since the middle of the eighteenth century. Richard Taylor's adventures in printing may be followed in the pages of the Philosophical Magazine where appear, on title-page or as colophon, the name of the printer and the place of his press. The data are collected in appendix I., where it will be seen that in 1800 the firm Jonas Davis became Davis, Taylor & Wilks, and in 1801 the name of Davis disappears. Richard Taylor set up the Black Horse Court press in partnership with his father; Arthur Taylor, who appears as a partner from 1814 to 1823, was a younger brother of Richard, and John Edward Taylor, partner from 1837 to 1851, was a nephew. In 1852, Dr. William Francis, of whom we shall have more to say later, became a partner in the firm.

Richard Taylor was a man of wide learning and of liberal views. He was a competent classical scholar, and had made a special study of mediæval and renaissance writers in Latin and in Italian. He was a student of the French, Flemish and Anglo-Saxon languages and literatures, and he published many works dealing with Anglo-Saxon literature. In 1810 Taylor was appointed under-secretary to the Linnean Society, a position which he filled with distinction for nearly half a century. In 1822 he became joint-editor with Tilloch of the Philosophical Magazine. In 1838 he initiated, with the help of William Francis, a journal which in 1841 became the Annals and Magazine of Natural History, which has continued to the present day, and is now chiefly concerned with systematics. His feeling for the needs of the times is well shown by his publication of Taylor's Scientific Memoirs, a series of translations of important scientific papers, which appeared in five stout volumes between 1837 and 1852.

Taylor was a good citizen; he played a large part in the foundation of the City of London School and of University College, and he did much for the corporation library.

The quotations from mediæval authors which are found on the obverse and reverse of the title-pages of ach volume of the Philosophical Magazine are characteristic of an age of scholarship rather than specialization; an age in which William Thomson, the 22-year-old professor of natural philosophy in the University of Glasgow, could compose and read in Latin his inaugural lecture, "De Caloris Distributione per Terræ Corpus," of which feat his friend and successor Andrew Gray remarks "it is unlikely that the form of the inaugural dissertation cost him much more trouble than the matter." Indeed, this feat, which would not be regarded by his contemporaries as anything out of the ordinary, affords material for an interesting commentary on Johnson's gibe against the Scots: "Their learning is like bread in a besieged town; every man gets a little, but no man gets a full meal."

The quotation from Justus Lipsius appears on the title-page of the first volume, and although Justus Lipsius was one of Richard Taylor's favourite authors in later years, it is barely possible that he had a hand in the selection of this passage *.

Richard Taylor broke down in health in 1851 and Dr. William Francis, who took as large a share in moulding the fortunes of the Philosophical Magazine during the second half of the 19th century as Richard Taylor had taken in the first half, became a partner in the firm and took over Taylor's share of the editorial work. Richard Taylor died at Richmond in 1858.

Dr. William Francis was born in London in 1817 and was educated at University College School and St. Omer. After a period of study at University College, London, he went to Berlin and later to Giessen, where he worked on the salts of molybdenum under Liebig, and graduated Ph.D. in 1842. He had been apprenticed to Richard Taylor and did good service to the Philosophical Magazine by translating important foreign papers for that journal before his name appeared on the editorial board in 1851. His name remained there till his death in 1904. During this long period, his wide knowledge, great ability, and his capacity for friendship had no small share in forwarding the influence of the journal.

Such, in brief, were the characters of the founders of the Philosophical Magazine and his successors. What of the journal itself? It began its career as a miscellany of scientific information and research, and its aims are set out in characteristic language in the preface to the first volume, which preface is reprinted here as appendix IV., retaining even a delightful printer's error, which has been indicated by (*sic* !).

"New or curious." A rummage amongst the early volumes discloses much that is both new and curious. Volume 9, for example, contains an article on "A Singular Case of Dropsy" and recounts the story of an unfortunate boy of 12 who was tapped twice and on each occasion

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* Translations of the passages on the obverse and reverse of the title-page appear in appendix II.

A note on the house-symbol appears in appendix III.

" 7–8 quarts of milky fragrant chyle " were withdrawn. Volume 11 contains a dissertation on the problem " Whether mixed metals can be distinguished by the Smell " ; and volume 14 has a paper on a " New Theory of the Constitution of Mixed Gases elucidated by J. Dalton, Esq." Volume 19 has a paper by C. Gordon, M.D. on the Antiquity of the Gealic (*sic*) language in which the learned author links the Gealic *Precept* with the Hebrew *Decalogue*, and proves to his own satisfaction that Gealic, not Hebrew, is the primitive language of mankind. The more serious scientific papers had, in many instances, appeared elsewhere before publication in the Philosophical Magazine. Davy's Bakerian lectures, many of which were reprinted in the Philosophical Magazine, are cases in point. Some of the titles of important or curious articles in the earlier volumes are given in appendix V.

As the century wore on the Philosophical Magazine tended more and more to become a journal of research. It had absorbed two journals—the Journal of Natural Philosophy, Chemistry and the Arts (1797–1814) commonly known as Nicholson's Journal, and Thomas Thomson's monthly scientific journal the Annals of Philosophy (1813–1827), and in the year of incorporation of Thomson's Annals (1827) the name of an external advisor, Richard Phillips, appears on the editorial board. In 1832 Sir David Brewster's name appears ; the subsequent changes in the board of editors are set out in appendix VI.

A journal of research necessarily admits papers of very different degrees of importance. There is much honest, interesting and accurate work which merits publication, but of which it cannot be said that its rejection would appreciably retard the advancement of science, such as happened in the rejection by the Royal Society of Waterston's fundamental paper on the kinetic theory of gases. The refusal to publish this paper delayed the development of the kinetic theory by ten or fifteen years. It is unfortunate that Waterston did not select the Philosophical Magazine as his medium for publication, as he did for many of his investigations. H. A. Rowland, one of the outstanding experimental physicists of the 19th century, could find no American journal which would accept his first scientific paper, sent it to Clerk Maxwell, who at once communicated it to the Philosophical Magazine where it appeared in 1873.

The Philosophical Magazine has indeed had its share of important papers. In 1851 Sir William (Rowan) Hamilton contributed a long series of papers on extensions of the notion of quaternions. William Thomson (Lord Kelvin) contributed many papers. The paper by Michelson and Morley on the relative motion of matter and ether appeared in the volume for 1887.

Sir Joseph Thomson was ever a faithful friend of the Philosophical Magazine. In his bibliography of 241 papers, no fewer than 85, including the first (1876) and the last (1939) appeared in the Philosophical Magazine, and the 85 include some of fundamental interest.

Rutherford's epoch-making papers on the artificial transmutation of the elements—a series which marks the close of his Manchester period—

appeared in the Philosophical Magazine for 1920. Papers involving significant advances in atomic physics have also been contributed by Bohr, Moseley, Chadwick and Millikan.

The years of the second world war brought their special problems. It became difficult to find papers to publish and to find paper on which to print them, but the journal continued to appear though each number was reduced from 8 sheets to 4½ sheets. And the problem of the paper supply has not become much easier since the end of the war. Nevertheless, in the last decade the journal has contained papers of outstanding interest which include Milne's paper on kinematical relativity, radio papers from Sir Edward Appleton, and papers from the research laboratories of the Scandinavian countries.

"Wonder breeds curiosity, curiosity investigation, investigation discovery," and from the relation of things new and curious has sprung the systematic application of the powers of the mind to the understanding of nature, and the collection and collation of the results thus obtained in orderly exposition. The 150 years of the Philosophical Magazine display the development of scientific method and achievement the more strikingly because unconsciously. In that development the journal has itself played no small part and continues to-day to provide a vehicle whereby may be spread abroad the products of the devoted scholarship of scientific men.

APPENDIX I.

PRINTERS OF THE PHILOSOPHICAL MAGAZINE.

1798. Volumes 1, 2 & 3.—No printer's impression.
Volumes 4 & 5.—Davis, Chancery Lane.
1800. Volume 6.—Davis, Taylor & Wilks.
1801. Volume 11.—Wilks & Taylor, Chancery Lane.
1802. Volume 14.—Taylor & Wilks (but not so Volumes 13 & 15).
1803. Volume 16.—Taylor, Black Horse Court, Fleet Street.
[Volume 17.—J. Taylor.]
1804. Volume 19.—R. Taylor, Black Horse Court, Fleet Street.
1805. Volume 20.—R. Taylor & Co., Black Horse Court, Fleet Street.
Volume 22.—R. Taylor & Co., 38 Shoe Lane, Fleet Street.
1814. Volume 43.—Richard & Arthur Taylor, Shoe Lane.
1823. Volume 61.—Richard Taylor, Shoe Lane.
1827. Volume 2, *Second Series*.—Richard Taylor, Red Lion Court, Fleet Street.
1837. Volume 10.—Richard & John E. Taylor, Red Lion Court.
1851. Volume 1, *Fourth Series*.—Richard Taylor, Red Lion Court.
1852. Volume 3.—Taylor & Francis, Red Lion Court.

APPENDIX II.

Translation of the passages appearing on the obverse and reverse of the title-pages of each volume of the Philosophical Magazine.

Justus Lipsius "Nec aranearum . . . ."

The way the spiders weave, you see, is none the better because they produce the threads from their own body, nor is ours the worse because like bees we cull from the work of others.

Hugo de S. Victore " Meditationis est . . . . "

Meditation means searching out that which is hidden, contemplation regarding with wonder that which is not. Wonder breeds curiosity, curiosity investigation, investigation discovery.

J. B. Pinelli " Cur spirent venti . . . . "

Why the winds blow, why fissures appear in the earth, why the sea swells, why the ocean is so salt, why the sun hides his head in ruddy darkness, what makes the comets blaze forth so often and so terribly, what gives rise to clouds, why thunderbolts appear in the sky, with what fire the rainbow flashes, who summons together the heavenly bodies in such varied courses.

The passage from Justus Lipsius appeared on the title-page of Volume 1 (1798) ; the passage from Hugo de S. Victore appeared on the reverse of the title-page in 1845 and the passage from Pinelli was added in 1849.

APPENDIX III.

The house-symbol, the lighted lamp with its motto " Alere flammam " appears first in 1823 on the reverse of the title-page. In 1835 it is transferred to the end of the volume, and in 1883 it reappears on the reverse of the title-page.

APPENDIX IV.

PREFACE TO VOLUME 1.

Philosophical Magazine.

1 1798

PREFACE

HAVING concluded our first Volume, we would be deficient in gratitude did we not return thinks (*sic* !) to the public, in general, for the favourable reception our labours have experienced ; and to those Scientific Gentlemen, in particular, who have assisted us with Communications, as well as Hints respecting the future conducting of the Work.

As the grand Object of it is to diffuse Philosophical Knowledge among every Class of Society, and to give the Public as early an Account as possible of every thing new or curious in the scientific World, both at Home and on the Continent, we flatter ourselves with the hope that the same liberal Patronage we have hitherto experienced will be continued ; and that Scientific Men will afford us that Support and Assistance which they may think our Attempt entitled to. Whatever may be our future Success, no Exertions shall be wanting on our part to render the Work useful to Society, and especially to the Arts and Manufacturers of Great Britain which, as is well known, have been much improved by the great Progress that has lately been made in various Branches of the Philosophical Sciences.

APPENDIX V.

Some titles of articles from the earlier volumes.

Volume 9.—" History of Astronomy for the year 1800," by Jerome de Lalande.
" Singular Case of Dropsy."

Volume 11.—" A dissertation on the History of Sugar," by Professor Beekmann.
" Observations on the Onager of the Antients, or the real Wild Ass," by Professor Pallas.

Volume 14.—Several articles by Kirwan.
" On Painting," by Mr. E. Dayes, Painter.

Volume 19.—" A Report of the State of His Majesty's Flock of Fine-woolled Spanish Sheep, for the year ending Michaelmas 1803," by Sir Joseph Banks, P.R.S.

Volume 23.—" Observations on the singular figure of the planet Saturn," by William Herschel, LL.D., F.R.S.
" Abstract of Observations on a diurnal Variation of the barometer between the Tropics," by J. Horsburgh, Esq., in a letter to Henry Cavendish, Esq., F.R.S.

Volume 32.—Davy's Bakerian lecture on Alkalis.
Memoirs of the late Erasmus Darwin, M.D.

Volume 33.—" Analysis of the Mécanique Céleste of M. La Place," by M. Biot.

Volume 36.—" On Crystallography," by M. Haüy. Translated from the last Paris edition of his ' Traité de Minéralogie.

Volume 38.—Memorandum on the subject of Lord Elgin's Pursuits in Greece.
" Some speculations on the Nature of Instinct," by Arthur Mower.
The reports of Mr. William Smith and Mr. Edward Martin to the Bristol and Taunton Canal Company on the State of the Collieries at and near Nailsea, in Somerset.

Volume 43.—" Experiments tending to prove that neither Sir Isaac Newton, Herschel nor any other Person, ever decomposed incident or impingent Light into the prismatic Colours," by Joseph Reade, M.D.
" On the Capacity of Heat or Calorific Power of Various Liquids," by Benjamin, Count Rumford.

Volume 51.—" Whether Music is necessary to the Orator—to what extent and how most readily attainable ? " by Henry Upington, Esq.
" On the Case of Miss Margaret McAvoy."
" On the Atomic Theory," by William Higgins, Esq.

Volume 54.—Report by the Select Committee of the House of Commons on Mr. Telford's Plan for hanging an Iron Bridge across the Menai Straits.

Volume 62.—Report of T. Telford, Esq., on the Effects which will be produced on the Thames by the rebuilding of London Bridge.
" On Fluid Chlorine," by Mr. Faraday.
" On the Condensation of several Gases into Liquids," by Mr. Faraday.

Volume 65.—" Description of an improved Cross for Land-Surveyors," by by Mr. Isaac Newton.
" An Experimental Inquiry into the Nature of the radiant heating Effects from terrestrial Sources," by Baden Powell, M.A.
" On the Osteology of Reptiles, and on the Geological Position of their fossil remains," by Baron Cuvier.

Volume 66.—" On new compounds of Carbon and Hydrogen," by M. Faraday, F.R.S.

New Series.

Volume 10.—Professor Whewell on Isomorphism.

Volume 11.—First report on British Association for the Advancement of Science. Various papers by Sir D. Brewster including one on " An Account of a Curious Chinese Mirror, which reflects from its polished face the figures embossed upon its back."

APPENDIX VI.

List of Editors.

Table I.

Alexander Tilloch	1798–1825
Richard Taylor	1822–1858
Richard Phillips	1827–1851
David Brewster	1832–1868
Robert Kane	1840–1889
William Francis	1851–1904
John Tyndall	1854–1863
Augustus Matthiessen	1869–1870
William Thomson	1871–1907
George Francis Fitzgerald	1890–1901
John Joly	1901–1934
William Francis (Jr.)	1904–1932
Oliver Joseph Lodge	1911–1940
Joseph John Thomson	1911–1940
George Carey Foster	1911–1919
Richard Taunton Francis	1921–1931
Alfred W. Porter	1931–1939
John R. Airey	1932–1937
Allan Ferguson	1937–
Lawrence Bragg	1941–
G. P. Thomson	1941–

Table II.

1798. Alexander Tilloch.
1822. Alexander Tilloch and Richard Taylor.
1825. Richard Taylor.
1827. Richard Taylor and Richard Phillips.
1832. Sir David Brewster, Richard Taylor and Richard Phillips.

[From 1835 to 1844, a note appears on the reverse of the title-page :– " The Conductors .... beg to acknowledge the editorial assistance rendered them by their friend. Mr. Edward William Brayley, F.L. & G.S., Librarian to the London Institution ; which commenced in the year 1823, in Vol. 61 of the First Series, and has been regularly continued from that time.]

1840. Sir David Brewster, Richard Taylor, Richard Phillips and Robert Kane.
1851 (Vol. 1). *Add* William Francis.
(Vol. 2). *Omit* Richard Phillips.

1854. *Add* John Tyndall.
1859. *Omit* Richard Taylor.
1864. *Omit* John Tyndall.
1868. Robert Kane and William Francis (*omit* Brewster).
1869. *Add* Augustus Matthiessen.
1870. *Omit* Augustus Matthiessen.
1871. Robert Kane, William Thomson and William Francis.
1890. *Add* George Francis Fitzgerald. *Omit* Robert Kane.
1901. *Add* John Joly. *Omit* G. F. Fitzgerald.
1904. William Francis (Jr.) replaces his father.
1908. *Omit* Lord Kelvin (William Thomson).
1911. Oliver Lodge, J. J. Thomson, John Joly, George Carey Foster and William Francis.
1919. *Omit* G. Carey Foster.
1921. *Add* Richard Taunton Francis.
1931. *Add* A. W. Porter. *Omit* Richard Taunton Francis.
1933. *Omit* William Francis. *Add* John R. Airey.
1934. *Omit* John Joly.
1937. *Add* Allan Ferguson.
1938. *Omit* J. R. Airey.
1939. *Omit* A. W. Porter.

Leaving before end of 1939 until end of 1940 :—Sir Oliver J. Lodge, Sir Joseph J. Thomson and Allan Ferguson.

1940. Sir Lawrence Bragg, Sir George Thomson and Allan Ferguson.
1948. *Add* N. F. Mott.

ASTRONOMY THROUGH THE EIGHTEENTH CENTURY

By Sir H. SPENCER-JONES, F.R.S.

Up to the beginning of the eighteenth century astronomy had been concerned primarily with a single great problem, the plan of the solar system. Observations had been largely restricted to the determination of the positions of the Sun, Moon, planets, and fixed stars, which provided the material for the investigation of this problem. The formulation by Newton of the principle of universal gravitation, enabling him to account for the motions of the planets according to Kepler's three laws and for the more important inequalities in the motion of the Moon, had at length provided in broad outline the solution of this problem. A beginning had been made in the application of the telescope to astronomical observation; poor in quality though the early telescopes were, the first-fruits of their use had been well worth gathering. The discovery of spots on the Sun and of the Sun's rotation, of the phases of Venus, of mountains and other formations on the Moon, and of the four major satellites of Jupiter had been made by Galileo with his primitive telescope. The grinding and polishing of lenses of larger aperture was undertaken by Hevelius in Dantzig, by Christiaan and Constantyn Huygens at the Hague, and by Campani in Rome. Hevelius, by his study of the surface of the Moon, had laid the foundations of selenography; Christiaan Huygens had recognized the ring structure of the appendages of Saturn, which had puzzled Galileo, and had discovered Titan, the brightest satellite of Saturn; J. D. Cassini, using lenses made by Campani, had discovered the rotation of Mars, four satellites of Saturn (Iapetus, Rhea, Dione, and Tethys) and the division in Saturn's ring system, known as the Cassini division. But the large aberrations of the simple lenses of that time necessitated greater and greater focal lengths as apertures were increased, so that telescopes became unwieldy and awkward to manage. The difficulties of observation were so great that the long focus telescopes had soon fallen into disuse; many years elapsed before the satellites of Saturn which Cassini had discovered were once again seen.

During the greater part of the eighteenth century, astronomical observations continued to be largely concentrated on measurements of position. The application of telescopic sights, improvements in the design and construction of astronomical instruments, and refinements in the methods and technique of observation enabled a progressive increase in accuracy to be attained. Hevelius, who died in 1687, was the last astronomer of importance to use open sights for observations of position. It is surprising that such an assiduous observer, who had employed the telescope to such good effect in his observations of the Moon's surface, should have steadfastly refused to use it on his quadrants and sextants.

The observations of Tycho Brahe had been sufficiently accurate to enable Kepler to deduce from them his laws of planetary motion. Though the observations of Hevelius were more accurate than Tycho's, they proved to be of little scientific value. The time had passed when any important discovery could be made by using measures of position with an accuracy limited to about one minute of arc.

The accuracy of astronomical observations had in fact by the end of the seventeenth century lagged behind what was required for essentially practical purposes. A reliable solution of the problem of determining the position of a ship at sea had become a matter of great practical importance. When Charles II appointed a committee to consider a proposal to determine the longitude at sea by a method which was essentially that of lunar distances, John Flamsteed reported that in theory the longitude could be determined in this way; but, in practice, because the positions of the stars in the best available star catalogue could be in error by as much as 5′ and the position of the Moon derived from the current lunar tables might be in error by one-third of a degree, the derived longitude could be as much as 300 leagues in error. Flamsteed saw clearly what was the most important need of astronomy at that time; it was the systematic observation, continued for many years and with the highest attainable accuracy, of the positions of the fixed stars, the Sun, the Moon, and the planets, so that the positions of the stars could be improved and data provided from which tables of the motions of the Moon and other bodies could be constructed. He recommended that an observatory should be built for this purpose and provided with instruments and with persons, skilled in mathematics and astronomy, to make the observations. The outcome was the foundation of the Royal Observatory at Greenwich by Charles II in 1675 and the appointment of Flamsteed as the first Astronomer Royal. Flamsteed laboured for 44 years at Greenwich, setting the tradition for systematic and accurate observation which has characterized the Greenwich Observatory throughout its long history. He introduced new methods of observation, some of which have become standard practice since his time, and he established the fundamental points of practical astronomy upon a new basis. His observations exceeded in accuracy those of his predecessors or contemporaries and are, in fact, the earliest observations from which the phenomenon of aberration is clearly deducible. His great catalogue of the positions of nearly 3000 stars was an important contribution to the astronomy of his day.

The Paris Observatory, commenced in 1667 and completed in 1671, a few years prior to the Greenwich Observatory, had a brilliant start under its first director, G. D. Cassini, whose observations with long focus telescopes have already been mentioned. The observations by Picard in Paris and by Richer in Cayenne of Mars at its close approach in 1672 provided the best determination up to that time of the Sun's distance. It was found that a pendulum of given length beat more slowly at Cayenne than at Paris, showing that the intensity of gravity was less near the equator than in higher latitudes and suggesting that the earth was not a perfect sphere. Roemer, a young Danish astronomer, who in 1671 had been brought to Paris by Picard, proved in 1675, from the study of the times of eclipses of Jupiter's satellites, that light travels with a finite velocity, which he was able roughly to estimate. But in spite of this auspicious beginning, the Paris Observatory produced little of real importance during the greater part of the eighteenth century. No special duties were assigned to the Director and no definite plan of observation was projected. Thus it came about that the greater part of the best observing work of the eighteenth century was done in Great Britain.

We have mentioned that Newton had provided in broad outline the solution of the problem of the motions of the solar system. He left to his successors the task of accounting in detail for these motions, a task of incredible difficulty, which in effect created a new department of astronomy, gravitational astronomy. But though the Newtonian theory of universal gravitation was slower in gaining acceptance on the Continent, and particularly in France, where Descartes' theory of vortices held sway, than it was in England, Newton's own countrymen contributed little during the eighteenth century to the development of gravitational theory. It was to a group of brilliant continental mathematicians and geometers that the foundations of celestial mechanics were due. Newton had written the ' Principia ' almost entirely in the language of geometry ; though he had doubtless first solved many of the problems by his method of fluxions, he translated the results into geometry. He probably preferred not to employ in the ' Principia ' mathematical methods that were not familiar. Excessive reverence for Newton on the part of his countrymen, together with the estrangement between British and Continental mathematicians, which developed from the quarrel between Newton and Leibnitz as to their respective claims to the invention of what Newton termed fluxions and Leibnitz termed the differential method, prevented British mathematicians from using the powerful analytical methods that were being developed on the Continent. British mathematicians were in consequence almost isolated from the main trend of mathematical development during the eighteenth century. The great names in the development of gravitational astronomy, which began about 50 years after the publication of the first edition of the ' Principia ' and continued to the end of the eighteenth century, were Euler (1707–83), Clairaut (1713–65), D'Alembert (1717–83), Lagrange (1736–83), and Laplace (1749–1827). This development was, however, closely interlinked with the improvements in the precision of observation, whereby a more accurate control over the results of theory was provided.

Improvements in the construction of clocks had by the last decade of the seventeenth century made it practicable to determine the right ascensions of celestial bodies by observations of the times of their transits across the meridian, using a telescope fixed in the meridian. Roemer had realized, in order that a telescope should move as nearly as possible in the meridian, that it was much better to support it from the two ends of an axis rather than to mount it, as in the quadrant, from a short pivot. In 1689 Roemer invented an instrument which he called the *machina domestica*, which was the first transit instrument. The construction first met with approval in England and in 1722 Halley installed the first transit instrument to be used at Greenwich. It became standard practice to use the transit instrument for the observation of right ascensions and the mural quadrant for the observation of declinations. Roemer in 1704 had invented another instrument, called the *rota meridiana*, which was the first transit circle and could be used to determine both right ascensions and declinations. But the transit circle seems to have been forgotten and did not reappear until it was revived by Ramsden at the end of the eighteenth century.

Improvements in the design and construction of positional instruments were due mainly to the celebrated English makers Graham, Bird, Ramsden, and Troughton, who supplied instruments to many of the principal observatories. Improved methods of dividing arcs were developed and by the end of the century the errors of graduation had been lowered to about 2″ or 3″. The magnificent series of observations made by Bradley at Greenwich, particularly those made between 1750 and 1762, amounting to about 60,000, formed the most important contribution of the century to meridian astronomy. They showed a considerable advance in accuracy over those of Flamsteed, which represented the best that had been previously made: they are in fact the earliest observations which are of sufficient accuracy to be combined with modern observations for the investigation of the proper motions of the stars. The accuracy was due in some measure to the good construction of Bradley's instruments, made by Graham and Bird, but even more to his skill in observation, to the care with which he controlled the adjustments of the instruments, and also to his improved tables of atmospheric refraction, in which account was for the first time taken of the effects of temperature and of barometric height.

Various attempts to determine stellar distances by measurement of annual parallax proved unsuccessful. It was not until 1838 that the distance of a star was first determined. Bradley was one of the astronomers who had tackled the problem, by the observation of the zenith distances of the star γ Draconis, using a 12½-foot zenith sector by Graham; the attempt, though unsuccessful, led Bradley to the discovery in 1729 of the aberration of light, thereby incidentally providing the first observational proof of the motion of the earth round the Sun. Some small residual effects, which Bradley suspected to be due to the Moon, induced him to continue his observations through the period of revolution of the Moon's nodes (about 19 years), and enabled him to announce in 1748 the discovery of the nutation of the earth's axis. Bradley gave the correct explanation of nutation as a variation in the apparent annual precession caused by the variation in the gravitational action of the Sun and Moon on the equatorial parts of the earth.

Newton had inferred from theoretical considerations that the earth was spheroidal, being flattened towards the poles. Richer's pendulum experiments at Cayenne had given evidence of a departure from the spherical form. But measurements of arcs of the meridian in different latitudes were somewhat contradictory. The Cassini school asserted that the earth was elongated towards the poles. In order to decide this question, the French Academy sent an expedition under Bouguer to Peru in 1735 and another under Maupertuis to Lapland in 1736, to measure the length of a degree of the meridian. The results of these expeditions, combined with measurements of the length of a degree arc in France, proved that the length of a degree of the meridian increased as the latitude increased and established the correctness of Newton's view that the earth was flattened towards the poles.

The first observations of the positions of stars in the southern sky were made from St. Helena by Halley, who in 1678 published a catalogue of the places of 341 southern stars. Much more extensive observations were made by Lacaille in the course of a scientific expedition to the Cape of Good Hope, 1750–54, organized by the French Academy. More than 10,000 stars were observed with care. A catalogue of 400 of the brightest stars was published in 1757; a larger catalogue of 2000 stars was published posthumously in 1763. The positions derived from the observations of the remaining 8000 stars were first published in 1845 by the British Association. Lacaille also observed and described a number of southern nebulæ, nebulous stars, and star clusters.

In 1705 Halley published his 'Synopsis of Cometary Astronomy.' He had collected all available observations of 24 bright comets, whose parabolic orbits he computed. Struck by the resemblance between the paths of the comets of 1531, 1607, and 1682 and by the approximate equality in the intervals between their respective appearances, he conjectured that the three comets were different appearances of the same comet and that comets really moved in elongated elliptical orbits under the gravitational attraction of the Sun. The inequalities in the intervals of appearance he correctly attributed to the perturbations produced by planets near which the comet had passed; making approximate allowance for the perturbations produced by Jupiter he predicted that the comet would return about the end of 1758. The comet, known as Halley's comet, was first detected by an amateur astronomer, Palitzch of Saxony, on Christmas Day, 1758. Great interest had been aroused by Halley's prediction, as the predicted time for the reappearance of the comet drew near. The return of the comet proved that the motions of these bodies were in accordance with the law of gravitation and not only shook the superstitions attaching to comets but also convinced those who still hesitated to accept Newton's theories.

The Greeks had distinguished between the fixed stars, which were attached to a crystal globe, and the wandering stars or planets. After the Copernican system was accepted, it was realized that the stars were scattered through space, at various distances from the earth. There was no longer any reason to suppose that they were fixed. In 1718 Halley showed, by comparing the positions provided by current observations with those obtained in Greek times, that a few of the bright stars had changed their positions; the stars were not, in fact, fixed. The derivation of the proper-motions of the stars, from the comparison of positions separated by an interval of years, became of importance. The improved accuracy in the determination of the positions of stars from the time of Flamsteed onwards was gradually providing material suitable for this purpose. In particular, Maskelyne at Greenwich made numerous careful observations of the positions of 36 bright stars, for which he derived proper-motions. Herschel, using the derived proper-motions of 14 bright stars, showed in 1783 that the motions were not distributed at random; they showed a systematic effect which could be attributed to a motion of the Sun in a direction towards the constellation of Hercules. Though this conclusion was based on scanty data, its general correctness has since been fully substantiated.

Another discovery made by Halley was to prove of great theoretical interest. From the study of a number of ancient eclipses of the Sun and Moon he came to the conclusion that they could be reconciled with the motion of the Moon derived from current observations only by supposing that the mean motion of the Moon was being accelerated. More than two centuries were to elapse before the causes of this secular acceleration of the Moon's mean motion were fully understood.

The observations of the positions of the Sun and Moon which were carried on systematically at the Greenwich Observatory, combined with observations by Lacaille, Tobias Mayer, and others, were providing the data required for the construction of improved tables of the motions of these bodies, important not only for astronomy but also for navigation. Lacaille in 1758 published new solar tables, in the construction of which the perturbations by the planets were for the first time taken into account. These tables were revised and improved by Tobias Mayer (1723–62), director of the Göttingen Observatory, and the most important astronomer produced by Germany during this period. Mayer's most important work was on the Moon; he studied its librations and the theory of its motion, and succeeded in calculating lunar tables which were a considerable improvement on any earlier tables. In proceeding from the mathematical theory of the Moon's motion to the construction of tables, various numerical quantities must be derived from observation. The superiority of Mayer's tables was due in large measure to the skill with which he deduced these numerical quantities. Mayer sent his tables to England in 1755, where Bradley reported favourably on their accuracy. Revised tables, prepared by Mayer shortly before his death, were also sent to England. His 'Theory of the Moon' and his improved Solar and Lunar Tables were published in England in 1770 at the expense of the Board of Longitude.

The urgency of the solution of the problem of finding longitude at sea had become so great that in 1713 the British Government had offered a prize of £20,000 for a method that was reliable to within half of a degree, together with smaller prizes for methods of less accuracy. A Board of Longitude was appointed to consider applications for the prizes. The prize of £20,000 was eventually won by John Harrison for his invention in 1759 of a time-piece that would keep accurate time at sea, which was the forerunner of the modern marine chronometer. Before the end of the century, largely through the work of John Arnold (1736–1799) and of Thomas Earnshaw (1749–1829), the chronometer had been brought to a form which is substantially identical with that of the present day.

The invention of the marine chronometer did not do away with the necessity of determining longitudes at sea by the method of lunar distances. The astronomical observations were still needed in order to provide a general check on the gaining or losing rate of the chronometer. Maskelyne, who became Astronomer Royal in 1765, took a special interest in the problems of navigation. In 1763 he had published the 'British Mariner's Guide,' a handbook for the determination of longitude at sea by the method of lunars. The year after his appointment as Astronomer Royal, he published the first number of the 'Nautical Almanac' (for the year 1767), designed for the use of seamen, in which the astronomical data were given in the most convenient form. Maskelyne continued to produce the Almanac for 44 years and it justly acquired a high reputation for accuracy. Its publication has continued uninterruptedly to the present day but necessarily with numerous changes in form and content. Amongst other work carried out by Maskelyne, mention may be made of his measurement, in 1774, of the deflections of the plumb-line on each side of the mountain of Schehallion in Perthshire, which provided, through calculations carried out by Charles Hutton, the first determination of the mean density of the earth or, what is equivalent, of its mass. Near the end of the century, in 1798, an improved determination was made by Henry Cavendish by means of the well-known Cavendish experiment.

In June 1761 and 1769 occurred the rare phenomenon of a transit of Venus across the face of the Sun. Of the previous pair of transits, in December 1631 and 1639, the second had been observed by Jeremiah Horrox, the first such transit ever to be seen. The next pair were not due to occur until December 1874 and 1882. Halley had pointed out in 1679 that the transits of Venus could be utilized for determining the distance of the earth from the Sun by observing from different parts of the earth's surface the apparent path of the planet across the disk of the Sun. Extensive plans were made by governments, academies, and private persons for the observations of the transits. Expeditions were sent from many countries to various parts of the earth, from Siberia to California, from Hudson's Bay to the Cape of Good Hope and Madras, from Norway to Tahiti. The occasions provided the first instance of international co-operation on a large scale in astronomy. Both transits were widely observed, but the results proved disappointing. It had been expected that the internal contacts of Venus with the Sun could be timed with high accuracy. This proved not to be the case ; disturbing effects were caused by the atmosphere of Venus, so that the times of the contacts determined by different observers were singularly discordant. Widely different values for the solar parallax were obtained by discussing the observations in different ways, indicating the presence of systematic errors. No real improvement in the knowledge of the distance of the Sun resulted from the immense trouble taken over the observation of these two transits.

The eighteenth century saw considerable developments in the design and construction of telescopes. In 1720 Hadley made the first really workable reflecting telescope. It had a 6-inch speculum mirror, of 63-inches focal length, mounted in a wooden tube and capable of motion in altitude and azimuth. Hadley cast, ground, figured and polished the mirror himself, taking successfully the essential step of parabolizing the mirror, which had never been achieved before ; he also developed methods for testing the figure as the work proceeded. Bradley tested Hadley's telescope against a Huygens objective of 123 feet focal length and found that it would bear as much power with equal definition. Hadley disclosed his methods of working to others ; Molyneux, Hawksbee, and particularly James Short soon acquired a high reputation for the excellent qualities of the reflecting telescopes which they made.

Attention had been turned to the reflecting telescope because of the optical aberrations of simple lenses, which compelled long focal lengths to be used, with all the resulting inconveniences. In 1733 the achromatic lens was invented by Chester Moor Hall, an Essex magistrate, who had some made for him by a London optician, George Bass. In 1758 it was independently invented by John Dollond, who took out a patent for it. Dollond was an excellent craftsman and his small achromatic telescopes of about 5 feet focal length had a ready sale.

Improvements were also made in the mountings of telescopes. The compact size of the telescopes with achromatic objectives made the equatorial form of mounting practicable. Sisson in 1760 devised the simple English mounting, with the polar axis supported by an upper and a lower bearing. Before the end of the century Ramsden had introduced the English yoke mounting, in which the polar axis is in the form of a yoke or cradle, the two sides of which support the declination axis and within which the telescope swings. The Shuckburgh equatorial, made by Ramsden in 1791, was a marked advance on earlier construction.

The achromatic objectives were limited in size, however, by the poor quality of the glass which was available. It was far from homogeneous, the flint particularly being full of faults. Pieces of glass suitable for lenses of apertures greater than 3-inches were rarely obtained. For greater light-gathering power, speculum mirrors still had to be used. William Herschel (1738–1822) was the first to make telescopes of large aperture. In 1774, after he had settled in Bath, he undertook the construction of his first reflector, which was of the Gregorian type. He had to learn by hard experience the difficult technique of casting, figuring, and polishing the mirrors. The figuring of the Gregorian secondary mirror proved difficult, so he soon gave preference to the Newtonian type. As his skill and technique improved, he was able gradually to increase the size of his mirrors, adhering usually to the rather large f-ratio, for a reflector, of about 12 or 13. The best of his mirrors were of extremely good quality; one of $6\frac{1}{4}$-inches aperture, of 7 feet focal length, was tested at Greenwich against a $9\frac{1}{2}$-inch mirror of Short's and found to be superior to it. On 13 March, 1781, when examining the heavens with his best 7-foot telescope, he detected an object which showed a minute disk. He thought at first that it was a comet, but it proved to be a new planet, the first planet to be "discovered," to which he gave the name of the *Georgium sidus*. This name was not in accordance with the practice of naming the planets by the deities of classical mythology, and the name Uranus, which had been suggested by Bode, was soon adopted in preference to the name Herschel had assigned.

The discovery of the new planet enlarged the bounds of the solar system and brought Herschel fame. He was appointed King's Astronomer by George III and moved to Slough, where he was able to devote himself entirely to astronomical observation and to the making of telescopes. After several unsuccessful attempts to cast larger mirrors, he succeeded by the autumn of 1783 in completing two good mirrors of $18\frac{3}{4}$-inches diameter and 20-feet focal length. The 20-foot telescope was his favourite instrument, with which most of his subsequent observations were made. It was too large to be mounted at that date as an equatorial, so that the less convenient altazimuth form of mounting was used; the telescope was mounted between the two parts of a light framework, supported from a circular turntable. It could be raised or lowered in altitude by tackle, the turntable being rotated to provide motion in azimuth.

Herschel made and sold a large number of telescopes, up to the 20-foot size, to amateurs in all parts of Europe, to kings, princes, and financiers; the receipts enabled him to carry out costly experiments on the mechanical grinding and polishing of large mirrors. He designed a large polishing machine, which was finished in 1788. Several unsuccessful attempts were made to cast a speculum disk larger than any yet made. At length success was achieved and a large telescope, of 48-inches clear aperture and 40-feet length, was constructed. Its aperture-ratio was the largest of any of the telescopes made by Herschel. An altazimuth mounting, which was a much elaborated form of that used for the 20-foot telescope, was employed. The telescope was not of the Newtonian type but was arranged as a front view telescope, the mirror being slightly tilted, in order to throw the image near to the top outer edge of the tube. The large f-number made it possible to tilt the mirror sufficiently without introducing too much coma, while the loss of light caused by reflection at a secondary mirror was avoided. With this telescope Herschel discovered the 6th satellite of Saturn, Enceladus, and also the 7th, Mimas. This great telescope proved to be of value in sweeping for nebulæ and for studying faint objects. But it was used relatively little, for it was rather cumbrous to manipulate; there was some trouble from flexure of the mirror under its great weight, the method of supporting it being rather primitive; also the rapid tarnishing of the mirror necessitated repolishing, which involved refiguring, every two years. In order to facilitate the casting, annealing and working of the large speculum disks, Herschel increased the copper content to 75 per cent., with the result that the surface tarnished easily.

Another instrumental development of importance was the invention of the heliometer by Dollond in 1755. In this instrument the objective is divided into two halves along a diameter, the two halves being capable of separation along the common diameter, the amount of separation being indicated by a scale. The objective and mount could be rotated into any position angle. The distance between two stars is determined by bringing the image of one star, formed by one half of the objective, into coincidence with the image of the other star, formed by the other half of the objective, when the common diameter has been set along the direction of the line joining the stars. Distances up to about 2° can be measured with accuracy, but the instrument was designed primarily, as its name implies, for the measurement of the Sun's diameter. Its principal application at a later date was for the determination of stellar parallaxes; the accuracy of the results obtained with it was not surpassed until the development of photographic methods caused the heliometer to be superseded.

Some outline must now be given of the more important developments in gravitational astronomy during the eighteenth century. The general problem of determining the motions and positions of three or more bodies under their mutual gravitational attractions is insoluble, except for a few special cases. In the most important cases which arise in the solar system there are special circumstances which enable solutions to be obtained by successive approximations. In planetary theory, the masses of the planets are very small compared with the mass of the Sun, so that the forces due to the other planets which perturb any particular planet are relatively small; also the eccentricities and inclinations of the planetary orbits are all small. In the more difficult lunar theory, the great distance of the Sun relative to that of the earth provides a simplification. Much attention was given to the theory of the Moon's motion because of its importance for the determination of longitude at sea. Euler, Clairaut, and d'Alembert occupied themselves with the problem and derived approximate solutions independently and nearly simultaneously. They all deduced from their respective analytical solutions a motion of the apogee that was equal to only half the observed motion. As Newton had encountered the same discordance between theory and observation, it began to be suspected that gravitation did not vary exactly as the inverse square of the distance. In 1749, however, Clairaut found that when certain small terms, which had previously been neglected, were taken into account, the computed and observed values were brought into agreement. Clairaut's lunar theory was published in 1752, Euler's in 1753, and d'Alembert's in 1754. The employment of analytical methods enabled the principal inequalities in the motion of the Moon, which were known from observation, to be explained with considerable accuracy quantitatively as well as qualitatively, with the one important exception of the secular acceleration of the mean motion, which had been discovered by Halley. Clairaut's most important work was his treatise on the figure of the earth, published in 1743. In this work he gave for the first time the general equations of the equilibrium of a fluid and applied them to the determination of the form which a rotating body like the earth would assume under the influence of the mutual gravitation of its parts, hypotheses of a very general nature as to the variations of density in its interior being made. He deduced a formula for the value of gravity in different latitudes which is in close agreement with the results of pendulum observations. Clairaut's treatise has formed the basis for all subsequent investigations of this subject, to which few results of fundamental importance have since been added.

The first attempts were also made about this time to account for the inequalities in the motions of the planets caused by perturbations by other bodies. The most important were the inequalities in the motion of the earth, which showed as irregularities in the apparent motion of the Sun; and certain irregularities in the motions of Jupiter and Saturn, which had been known to Horrox and which Halley had suggested were caused by their mutual perturbations. Clairaut, in 1757, discussed the perturbations of the earth by the Moon, Venus and Jupiter; by comparison of his theory with observation, he was able to estimate the masses of Venus and the Moon. His mass of the Moon was a considerable improvement on Newton's estimate from the tides, while the mass of Venus was not only new, but was in fair agreement with modern estimates. This was the first attempt to estimate masses of celestial bodies by means of the perturbations which they produced.

Euler made an important contribution to general theory by his method of the variation of the elements or parameters of an orbit, which, though developed in 1756, was not published until 1771. A planet moving in a perturbed orbit was represented as moving at any instant in an ellipse, whose elements are continually changing. The whole effect of the perturbations was in this way thrown upon the elements of the orbit; when these elements had been determined for any instant of time the position of the planet could be calculated by ordinary elliptic theory.

The important contributions made to gravitational astronomy by Lagrange and Laplace, the two greatest mathematicians of the next generation, commenced when the work of Clairaut, d'Alembert, and Euler was nearly finished. There was an important difference between these two great contemporaries. Lagrange valued elegance and generality of treatment; a particular problem was an occasion for the application of a general method. Laplace used mathematics as a tool, to be modified to suit each special problem. Lagrange made the more important contributions to general dynamical theory, whereas Laplace made the more important contributions to celestial mechanics in so far as the detailed explanation of the motions of the celestial bodies is concerned. They both assisted in the development of the mathematical treatment of perturbations, distinguishing between those which are *periodic* and those which are *secular*. When the perturbations of one planet by another are investigated, the mathematical expression for the disturbing force depends upon the positions of the planets at a given time and also upon the elements of their two orbits. It can be divided into two parts, one of which involves the positions of the planets while the other does not, though both involve the orbital elements. The first portion of the disturbing force changes relatively rapidly, going through a complete cycle of changes in an interval of time which is of the same order as the orbital periods of the planets. This part can be represented by a sum of periodic terms. The second part changes very slowly, as the elements of the orbits change: for long periods of time, this part can be represented by a power series, with the time as the variable quantity. It is generally sufficient to use only the first three terms, which represent the value at a particular epoch, the rate of change at that epoch, and the secular change or acceleration. In lunar theory the periodic inequalities are large and of great importance; in planetary theory the periodic inequalities are generally small and the secular inequalities are important.

The question arises whether the mutual perturbations of the bodies in the solar system could be cumulative to such an extent that the system would become unstable. Lagrange and Laplace both contributed to the solution of this problem. Laplace in 1773 proved that the axes of the planetary orbits were not undergoing secular change, nor could they have altered appreciably for a long time past; Lagrange in 1774 proved that the inclinations of the orbits could not pass outside certain limits. Then in 1775 Laplace obtained a similar result for the eccentricity. A further step forward was made by Lagrange in 1776, who proved that, provided the disturbing forces were small, the lengths of the axes could only fluctuate within certain limits, whatever the eccentricities, inclinations and masses might be. The final step was made in 1784 by Laplace, who proved that, whatever the relative masses of the planets, the eccentricities and inclinations, if once inconsiderable, would always remain so provided that the planets all revolved round the Sun in the same direction. The conclusion from this series of investigations is that the changes in lengths of axes, in eccentricity, and in inclination of any planetary orbit are all permanently restricted within certain definite limits. The solar system is stable.

The slow alteration in the rates of motion of Jupiter and Saturn, to which Halley had drawn attention, proved to be difficult to explain. Lagrange in 1766 attempted unsuccessfully to account for it as a secular inequality. Laplace, in 1773, by a more complete analysis showed that there was no secular change in the motions of these planets. He eventually found the real source of the irregularities in the near commensurability of the mean motions of the two planets, which are approximately in the ratio of 5 to 2. In consequence certain terms in the longitudes which would normally be small, being of the order of the cubes of the eccentricities, become larger because they are divided by the small factor $(5n-2n')^2$, where n, n' are the mean motions. An appreciable inequality, with a period of 929 years, is thus produced. The observations were satisfactorily accounted for and the irregularities of the two planets, which had seemed to be inexplicable by the law of gravitation, provided a striking proof of its truth.

Laplace succeeded also in finding the cause of the acceleration of the mean motion after both Euler and Lagrange had failed in their attempts to account for it. As a result of the action of the planets on the earth, the eccentricity of the earth's orbit is decreasing slowly. In consequence there is a gradual decrease in the mean action of the Sun upon the Moon, which causes a slow increase in the Moon's mean motion. Laplace calculated the rate of increase and obtained a value in close agreement with observation. His memoir on this subject was published in 1788 and appeared finally to have settled the question. But, in 1853, J. C. Adams showed that the calculations of Laplace were incomplete, and that the correct theoretical value was little more than half the observed value. Some seventy years were to elapse before G. I. Taylor and H. Jeffreys proved that the residual effect was attributable to the slowing down of the earth's rotation through tidal friction.

In his great work, the 'Mécanique Céleste,' published in five volumes over a period of 26 years (1799–1825), Laplace presented all his astronomical work as a reasoned whole, and exhibited the theory of the solar system in a state of almost complete development. This work was highly mathematical; to the layman Laplace is best known through his nebular hypothesis of the origin of the solar system, which was published in 1796 in his popular book, the 'Système du Mond.' Laplace put the hypothesis forward merely as a conjecture with, as he said, "all the distrust which everything which is not a result of observation or of calculation ought to inspire."

We have seen that great progress was made during the eighteenth century in positional astronomy and in gravitational astronomy. Equally great progress was made in descriptive astronomy, the branch of astronomical observation which is concerned with questions of the appearances and structure of the celestial bodies and of their mutual relations. We are here concerned almost entirely with the work of one man, William Herschel, whose most important researches fall into the last quarter of the century. Mention has already been made of his skill in the construction of telescopes and of his discovery of the planet Uranus. After settling at Slough, Herschel was able to devote himself assiduously to observation and for many years he observed throughout every suitable night. In 1811 he said that " a knowledge of the construction of the heavens has always been the ultimate object of my observations." Beyond the limits of the solar system, astronomers had hitherto concerned themselves only with the measurement of the positions and motions of the stars ; several attempts had been made to measure the distances of stars, but with no success. They had never studied the stellar universe *per se*, so that Herschel in these observations was opening a new chapter in astronomy.

Because the attempts of Bradley and others to measure stellar distances had failed, Herschel decided to attack the question of stellar distribution by an indirect method. For this purpose he developed his method of star-gauging, counting the number of stars visible in the field of his telescope (a circular area 15′ in diameter) in different regions of the sky. The variation in the numbers of stars seen might be due either to a real inequality of the distribution of the stars in space, or to the stellar system extending to greater distances in some directions than in others, or to both causes combined. Herschel assumed that the distribution of the stars was approximately uniform and was then able to estimate the relative extension of the system in different directions. The true nature of the Milky Way had been a matter of conjecture before he commenced his observations. He proved that it was an aggregation of a vast number of faint and therefore distant stars, which were individually too faint to be visible to the naked eye. He found that the space occupied by our stellar system was shaped roughly like a disk or grindstone, the plane of the disk corresponding with the central plane of the Milky Way. The diameter of the system was about five times its thickness and the Sun was somewhere near its centre.

Herschel was aware that the hypothesis of uniform distribution was at best merely an approximation. The existence of star clusters showed that it was not correct in detail. He therefore made use also of the brightness of the stars as a measure of their relative distances. If the stars have the same intrinsic brightness, a knowledge of their apparent brightness would enable their relative distances to be inferred. At that time stars were assigned a magnitude merely by comparison with adjacent stars to which magnitudes had already been assigned. Photometric observations to compare the relative brightness of stars of different magnitudes had not yet been made. Herschel made rough comparisons of the amount of light received from different stars by using telescopes of different apertures and also by measuring the aperture at which a star just remained visible.

In one of his latest papers Herschel (1817) made an attempt to compare the assumptions of uniform distribution and of uniform intrinsic brightness, and concluded that the fainter stars are either more closely packed in space than the brighter or that they are intrinsically less luminous.

In the course of his systematic surveys of the heavens, Herschel made a note of everything that was unusual. In this way he recorded a great number of nebulæ and star clusters. Messier had recorded a certain number of nebulæ in the northen sky and in 1781 published a list of 103 objects, his purpose being to prevent such objects being mistaken for comets by those who were engaged in comet searching. Lacaille had published a list for the southern sky. In 1786 Herschel published a list of 1000 new nebulæ and clusters; in 1789 a further list of 1000 objects; and in 1802 a third list containing 500. The appearance and position of each object was described.

Herschel gave much thought to the nature of the nebulæ. He was for some time inclined to the belief that they were aggregations of faint distant stars which his telescopes could not resolve into discrete stars, just as the naked eye cannot resolve the Milky Way into discrete stars. But the detection in 1791 of a star surrounded by a circular nebulous halo convinced him that some of the nebulæ were not aggregations of stars, but consisted of " a shining fluid, of a nature totally unknown to us." The possibility that other nebulæ might be distant clusters of stars was not excluded. He noted that objects of different types were differently distributed over the sky; nebulæ associated with clusters were common near the Milky Way and were often surrounded by a region of the sky relatively devoid of stars. (These are now known to be gaseous nebulæ, shining by the light of stars embedded in them, and associated with clouds of obscuring dust.) The other type of nebulæ, devoid of stars, were most frequent near the poles of the galactic circle. (These are the extra-galactic nebulæ, whose density decreases towards the Milky Way because of galactic absorption.) Up to the end of his life Herschel adhered to the view that some of the nebulæ were external universes, but that others belonged to our galactic system. The correctness of this view was not established until 1920.

The discovery of binary stars, or physical double stars, was made by Herschel. He had prepared a catalogue of pairs of stars lying nearly in the same line of sight, with angular separation not exceeding 2′. His intention was to keep these stars under observation in order to determine their relative parallax. Supposing the proximity of the stars to be merely accidental, it is probable that the fainter star of the pair is the more distant, so that the parallactic displacement, caused by the revolution of the earth round the Sun, should be less for the fainter star than for the brighter star. Herschel therefore measured the distance between the stars and the position angle of the line joining them. On repeating his observations at later dates, he found for a few of the double stars changes in distance and position angle which were not such as could be explained by a relative proper-motion, and could only be due to the two stars revolving round one another under the influence of their mutual gravitation. These stars must consequently be true binary systems. The binary nature could be conclusively established in Herschel's time for a few only of the double stars, though changes of position angle and distance were recorded also for others. No success in determining stellar parallaxes was achieved; we now know that the parallactic displacements were too small to be detected. But a more important discovery was made, for it is only through the gravitational interaction of stars in binary systems that any information about stellar masses can be obtained.

The discovery of the binary nature of some of the close pairs of stars verified the conclusion which had been reached by John Michell from considerations of probability. Considering the distribution of stars over the sky Michell showed in 1767 that the number of close pairs was far larger than was to be expected from a random distribution. In 1784, after some of Herschel's lists of double stars had been published, Michell returned to the question and concluded that many of the pairs must be physically connected.

Because of the difficulty of making accurate photometric measurements, Herschel devised the simple method of arranging groups of stars in sequences in order of relative brightness as a means of detecting at a future time any marked amount of variation of brightness. He used this method of sequences extensively and between 1796 and 1799 published four catalogues of comparative brightness, detecting in the course of the work a number of variable stars.

Besides the stellar observations, Herschel added considerably to the knowledge of the solar system. In addition to discovering Uranus and two satellites of Saturn, as already mentioned, he discovered two satellites of Uranus, showed that their orbits are nearly perpendicular to the ecliptic and that their motion is retrograde; he determined the period of rotation of Saturn, discovered its spheroidal figure, and measured its ellipticity; he detected the rotation of the ring system, discovered the faint inner crape ring, and showed that the thickness of the rings did not exceed a few hundred miles; he found that the variations in brightness of Iapetus were periodic, the period being equal to the period of revolution of the satellite; he also noted the seasonal changes in the polar caps of Mars.

The researches of Herschel opened up so many new lines of investigation that he has justly been termed "the father of modern astronomy." When the nineteenth century began, there were already living several who were destined to make important contributions to astronomy—Gauss, Bessel, F. G. W. Struve, Argelander, Hansen, Fraunhofer, and John Herschel. Gauss was already developing his method of determining the orbit of a planet or comet from three observations. The first of the minor planets, Ceres, was discovered on January 1st, 1801, only to be lost after having been under observation for a few weeks. Gauss applied his new method to determine its orbit, with the result that the planet was found again at the end of the year. His method of least squares was to prove of the greatest value in the analysis and adjustment of astronomical observations. Bessel was to systematize the reduction of meridian observations, to construct much improved tables of refraction, and to set a new standard for precision of observation. The accurate observations of double stars continued by Struve for many years at Dorpat and Poulkova were to establish the foundations of double-star astronomy on a secure basis. Bessel, Struve, and Henderson were each to determine the distance of a star and, practically simultaneously, to succeed where so many of their predecessors had failed. Argelander was to prepare the great catalogue of some 324,000 stars and the star maps of the 'Bonner Durchmusterung,' a work of the greatest reference value to astronomers. Hansen was to make a great advance in lunar theory and to construct tables of the Moon that were a great improvement on all previous tables. Fraunhofer, by his careful examination of the dark lines in the Sun's spectrum, was to show that new knowledge was to be gained by the application of the spectroscope to astronomical observation, thereby paving the way for the birth of a new branch of astronomy, called astrophysics, which is concerned with the physical conditions, the composition and the constitution of the celestial bodies. His collaboration with the Swiss artisan, Guinand, was to lead to important developments in the manufacture of optical glass. Guinand had commenced in 1784 to improve the quality of optical glass, and by 1799 had succeeded in making flint disks of high quality, as large as 6 inches in diameter, and also in producing denser types of flint. For some years from 1805 he joined forces with Fraunhofer: Guinand produced the glass, Fraunhofer devised the methods of working it on automatic machines. These developments were to make it possible for achromatic objectives of larger size to be constructed. Fraunhofer was, in addition, to perfect the equatorial mounting and its clockwork drive, and to bring the refracting telescope substantially into its present form, the reflecting telescope being almost entirely superseded for about half a century. John Herschel was to extend his father's survey of the heavens to the southern hemisphere and to add greatly to the knowledge of star clusters, nebulæ, and double stars.

Besides the great names we have mentioned there were many others who were born before the nineteenth century opened and whose work was to have important significance. Among them were Schwabe, who was to discover the periodicity of sunspots, and Sabine and Gautier, who were to show that there was a relationship between the frequency of the appearance of sunspots and various magnetic phenomena on the earth. Lord Rosse, who in 1845 constructed and erected at Birr Castle the great reflector of 6-foot aperture and 54-feet focal length, was born in 1800. This great instrument, with its cumbrous method of mounting, belongs in a sense to the eighteenth century, for the mirror was the last large speculum mirror to be used. But it has to its credit one important and significant discovery, that of the spiral structure of many of the nebulæ which are now known to lie beyond our galaxy, and thereby it forms a link between the observations and speculations of William Herschel in the eighteenth century and the new knowledge that has been gained about these objects in the twentieth century.

One of the most sensational triumphs of astronomy in the nineteenth century was the discovery in 1846 of the planet Neptune in the position that had been indicated by Leverrier and Adams. Both of these astronomers belong to the nineteenth century; but it was because the work of Lagrange and Laplace during the eighteenth century had placed the law of gravitation on so firm a basis that it became reasonably certain that the discordances between the observed and computed positions of Uranus (itself a discovery of the eighteenth century) must be due to the perturbing action of an unknown planet, and both Adams and Leverrier were thereby induced to attack the difficult problem of inverse perturbations.

PHYSICS IN THE EIGHTEENTH CENTURY

By Prof. HERBERT DINGLE, D.Sc.

At the time when the 'Philosophical Magazine' was founded, a reflective natural philosopher, looking back on the century which was closing, would have beheld a scene of some complexity. The achievement of the seventeenth century had been precise, profound and almost immeasurably great. At the beginning of that great epoch the conception of the universe which, though not unchallenged, was still predominant, was essentially mediæval in spirit; at the centre were the earth and the baser elements, subject to change and decay and tending to move in straight lines towards their own places, while surrounding them the crystalline spheres, incorruptible in the heavens, performed eternally the perfect circular motions proper to their nature. By the end of the century the whole scheme had vanished and the Newtonian system reigned in its stead. The spheres had disappeared; matter and motion were the same everywhere, on the earth and in the heavens, and were indissolubly linked with one another through the universal force of gravitation; and a technique had been created for the conquest of other phenomena by means of Newtonian forces yet to be discovered. After a change so radical it was only to be expected that a period of development would set in, a period on one hand of full exploitation of the field of mechanical motion for which Newton had provided the means, and on the other of the application of his essential principles to other physical phenomena such as heat, electricity, magnetism and the rest which he had left practically untouched. The former effort belongs mainly to astronomy, which provides almost completely perfect examples of the operation of Newtonian forces; the latter lies chiefly in the realm of physics and is the subject of our present concern. It lasted, in fact, for two centuries, and midway in that long period the 'Philosophical Magazine' came to birth. Of the achievements of the nineteenth century it is itself no bad record. Let us place ourselves at the beginning of that time and survey the heritage which the eighteenth century had bequeathed to it.

I have said that the view which faces us would have appeared to a natural philosopher of the time to be a complex one. That is true, and if today we are able to disentangle its threads and see a simple pattern running through it, the reason is that we have now left the Newtonian epoch sufficiently far behind to understand in large measure what its true significance was. We see it in a light which did not shine for those who worked therein, and this fact, fortunate though it is for our understanding of science, has compensating disadvantages when we try to understand scientists. The power to clear one's mind of knowledge and of tendencies that have become almost instinctive and to live the mental life of a dweller in days long past is not a possession of the present writer, and he will not make the vain effort to acquire it. This survey of eighteenth century physics is a distant retrospect, an interpretation as well as a description, and it makes no pretence to be anything else.

It should not be without profit, however, to begin by reminding ourselves of a peculiar difficulty which besets our understanding of the early post-Newtonian natural philosophers and makes them in one sense more alien from us than the men of the Middle Ages. Mediæval science, strange and indelibly coloured with enchantment as it appears to us now, was at least a unity. Modern divisions, such as those separating astronomy from theology, and both from psychology, did not exist. The Abode of the Blessed was outside the remotest stellar sphere, and the planets influenced men's temperaments and terrestrial fortunes while at the same time declaring the glory of God. Such progress as was made was all within the framework of one unifying idea, and if one succeeds in becoming mediæval in one sphere of thought he is not far from entering completely into the spirit of the age. But with Galileo and Newton this unity disappeared for ever. The history of thought, which could formerly proceed along a single line though at an irregular pace, becomes broken up into separate courses, like a river entering its delta. One branch—the study of mechanical motion—far outstrips the others, which themselves advance at various speeds. To understand the thought of a particular time we must take a cross-section, and this reveals the most incongruous associations. At the end of the eighteenth century, for instance, we find Sir William Herschel entertaining views of the stellar system and the distant nebulæ very similar to our own, and at the same time holding that the Sun was an inhabited globe whose people were protected by a cloudy veil from the hot outer shell ; while in France, Laplace was calculating how the curvature of space was related to the matter in it and the Academy was refusing to consider evidence for the fall of meteorites on the ground that such events could not possibly happen. Who will undertake to recapture the outlook in which these are co-existing elements ?

The dominant fact that directed the whole of eighteenth century science was that Newton had succeeded in formulating laws of motion so complete and comprehensive as to be applicable to phenomena large and small throughout the universe. This having been achieved, not only dynamical astronomy, but also the whole field of terrestrial mechanics lay at the mercy of the natural philosopher, who had need of mathematical, rather than experimental, skill to reap the richest harvests. His success was great and lasting, and is on this account better represented in modern text-books than the now largely superseded pioneer work of the students of other departments of physics. For this reason it would be an unprofitable use of a limited space to describe the contributions of the eighteenth century to the study of the general mechanical properties of matter and of sound, and I shall, therefore, confine myself to those subjects in which the

Galilean-Newtonian philosophy stood as an example to be copied rather than a fully efficient tool to be directly applied. Its essence lay in its isolation of a phenomenon—motion—exhibited by bodies of all kinds, in whatever other characteristics such as shape, colour, temperature and what not they might differ, and the choice of concepts—space, time, mass and force—in terms of which the laws of motion could be described. If, then, other phenomena were to be dealt with in the same way, the first step was clearly the choice of the appropriate concepts, and this was the problem whose solution, though not then quite completely reached, was pre-eminently the work of the eighteenth century. The formulation of general laws in terms of those concepts was left to the nineteenth century.

HEAT.

Let us begin with heat. Galileo had already conjectured that since, in his view, external bodies possessed no qualities other than size, figure, number and motion, the physical origin of our sensation of heat was " a multitude of minute corpuscles thus and thus figured, moved with such and such a velocity." But he made no experiments and hazarded no speculations concerning the figure and velocity of such corpuscles, nor did he make even the preliminary measurements necessary to suggest the kind of concepts which would have been appropriate to their description. He did, indeed, devise a "thermometer"—the first such instrument of which there is any record—but it appears to have been used only for purely practical ends. In his day such ideas as one can find concerning heat are of the vaguest.

We can perhaps realise the state of affairs by imagining ourselves as looking at our present clear-cut trilogy of thermal concepts—heat, temperature and entropy—through a sort of mental telescope which we can gradually put out of focus. The first effect is the merging together of heat energy and entropy into *caloric*, the puzzling character of which was largely due to the unrealised incompatibility between the conservation of the one and the continual growth of the other—a contradiction from which Carnot was saved by his choice of conditions of reversibility in which entropy also is conserved. A further turn of the screw takes us back through the eighteenth century and shows us a single blur in which caloric and temperature are indistinguishable. Nevertheless, to continue the metaphor, it is not a uniform blur. Two suggestions of foci are discernible, interpreted as opposite qualities of " heat " and " cold." Such was the view of Morin, a younger contemporary of Galileo, who, apparently uninfluenced by Galileo's inspired but unfruitful conjecture, regarded heat and cold as separate but inseparable entities. What we would now call a temperature of x degrees was to Morin a mixture of a degrees of heat and b degrees of cold, where $a+b=8$ (apparently a conventional figure resulting from the custom of dividing the range of the thermometers of the time into eight parts); and with this picture in his mind he discusses the effect of mixing together quantities of water at different temperatures. For this purpose he adopts certain principles (apparently purely hypothetical) governing the interaction of the opposite qualities, and obtains, in fact, surprisingly good results *.

It is fortunate that Newton's example of empiricism here supervened to stimulate accuracy of measurement and that, without waiting to know exactly what they were measuring, physicists turned their attention to the improvement of the "thermometer." By the end of the seventeenth century, liquids, such as water or alcohol, had taken the place of Galileo's air as the expanding substance, and the use of two "fixed points" and a standardized "degree" had been suggested though not practised. It was Fahrenheit who constructed the first thermometer worthy to be counted as a measuring instrument in the modern sense of the term. By 1721 he had substituted mercury for alcohol and had established the principle of the scale of what we now call temperature which still bears his name. His work was apparently suggested by a visit in 1708 to Römer, who had constructed a thermometer according to which the temperatures of melting ice and normal human blood were numbered respectively $7\frac{1}{2}$ and $22\frac{1}{2}$. Fahrenheit adopted this scale, with each division subdivided into four parts so that the afore-named temperatures became 30 and 90 degrees, respectively, and adhered to it until 1717, when, finding the fractions awkward, he increased the readings by one-fifteenth, so that the temperatures of melting ice and blood became 32 and 96. There appears to be no foundation for the common statement that Fahrenheit's zero defined the temperature of a mixture of ice and salt. By means of his thermometer he was able to verify the constancy of the readings corresponding to the melting of ice and the boiling of water (212), but he never regarded these as fixed points, and succeeded, indeed, by the use of his instrument, in showing that the boiling point of water varies with the atmospheric pressure.

Simultaneously, but independently, Réaumur in France was devising an instrument of comparable precision. Preferring alcohol to mercury because of its greater expansibility, he differed from Fahrenheit also in a much more fundamental, but practically less important, respect by fixing only one point (the zero, corresponding to the freezing point of water) and choosing as the degree the "heat" which made the alcohol expand by a definite fraction (generally $\frac{1}{1000}$) of its volume at the zero point. This entailed the laborious process of graduating the tube in terms of its volume, and introduced the principle, as we should express it nowadays, of making the dimensions of temperature those of volume instead of primary, or fundamental. This is of some interest in view of de Luc's improvement of the instrument later in the century, an account of which he published in 1772. By that time the idea of heat as a substance had become much more clearly established, and it was possible to determine

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* Morin's statement is reproduced in 'The Discovery of Specific and Latent Heats,' by D. McKie and N. H. de V. Heathcote, to which I am much indebted.

whether Réaumur's degrees " really " corresponded to equal increments of " heat." Assuming that when two equal quantities of water at different " temperatures " (*i. e.* containing different amounts of " heat ") are mixed, the average of the initial readings should be recorded by a " true " thermometer, de Luc showed that none of the thermometric liquids he tested reached this ideal when the graduations were made on Réaumur's principle, but that mercury came nearest to it. Réaumur, as we have seen, needed no upper fixed point, but he did, in fact, stipulate, as a standard of purity of the alcohol to be used, that it should expand from 1000 to 1080 units of volume when heated from the zero to its boiling point. This would, of course, indicate 80 degrees as the boiling point of alcohol, but in course of time this figure became attached to the boiling point of *water*. This matter also was cleared up by de Luc, and thenceforward, as now, the direct relation of expansion to volume was abandoned and the Réaumur degree was obtained by dividing the thermometer stem between the points corresponding to freezing and boiling water into 80 equal parts.

The third of the familiar standard liquid thermometer scales—that of Celsius—dates from 1742. There were two fixed points—the " temperatures " of melting ice and boiling water—and the interval was divided into 100 equal parts on the thermometer stem, but in the original instrument the higher point was zero and the lower was " 100." Christin in the following year reversed this order and introduced our present " centigrade " scale.

So much for measuring instruments ; but what did they measure ? This was not a primary concern in the eighteenth century, and men like Black, for instance, were content to discover relations between their measures without asking themselves what was the nature of the things they measured. This mental discipline, learnt from Newton, is one which we are trying to restore in these days, after the Victorian striving for " models of nature," and we are less inclined than our grandfathers to think patronisingly of the earlier students of heat because they placed more emphasis on devising an accurate formula for calculating the thermometer readings when substances of different temperature were mixed than on the description of underlying physical processes.

Nevertheless, some mental picture was necessary even for this, and the purest empiricist could not avoid the choice of words to describe what was happening. To Morin, as we have seen, heat and cold were " qualities " measurable by their degree and not their amount, and the thermometer measured a sort of resultant quality like the " lukewarmness " with which we regard something that at the same time attracts and repels us. The eighteenth century, however, thought more in terms of one entity, variously conceived as a substance or a form of motion. To those holding the former view, the thermometer naturally measured the amount of heat in a body, but at the same time there was an inarticulate feeling that heat, whatever it might be, had both intensive and extensive aspects. Thus Richmann in 1747–48, in a memoir dealing with what we should call "temperature" alone but which he entitled "On the Quantity of Heat" (De Quantitate Caloris), assumes that when the same "degree of heat" (idem caloris gradus) is uniformly distributed through different quantities of water, the "heat produced" (calor hinc generatus) is inversely proportional to their amounts. Two distinct ideas represented by the same word are here clearly evident.

Much the same confusion is found in Boerhaave, but in Black, who worked in the period 1759–63, we find the clearest possible distinction between the two ideas. He points out that Boerhaave fails to distinguish between quantity and intensity of heat and adds "very soon after I began to think on this subject (anno 1760) I perceived that . . . the quantities of heat which different kinds of matter must receive, to reduce them to an equilibrium with one another, or to raise their temperature by an equal number of degrees, are not in proportion to the quantity of matter in each, but in proportions widely different from this, and for which no general principle or reason can yet be assigned." Here clearly what the thermometer records is "temperature," and "quantity of heat" is something other. From this, Black goes on to define "capacity for heat" and "latent heat"—the name as well as the discovery of the fact is his. He concerned himself as little as possible with the problem of what "heat" might be, but inclined more towards the "substance" than the "motion" theory, as indeed the term "capacity" indicates. In his later years he tended to favour the theory advanced by Cleghorn in 1779, according to which the particles of heat, or "fire," repel one another but are attracted by different kinds of matter in different degrees. This accorded well with his experiments. Wilcke, who independently but later and less satisfactorily repeated Black's discoveries, also favoured the "substance" theory and conceived clearly enough that the "quantity of heat" in a body was something different from the reading of the thermometer when placed in contact with it. As to terminology, "fire," "heat," "matter of heat" and such terms were in use up to 1787, when the name "caloric" was coined by certain French chemists. This is the name by which the material theory of heat is generally known today, but it is considerably younger than the theory itself.

In so far as the pioneers of calorimetry were guided by theoretical conceptions, therefore, it is the material theory of heat that served their purpose. But there was another range of phenomena which forced itself on the attention of physicists in the latter half of the century, which showed at least that the question was not a simple one. Several experimenters demonstrated beyond question that when a red-hot body was placed at a focus of an optical system (generally a system of concave mirrors), a thermometer placed at the other focus showed a rise of temperature. Hence, something exhibiting the reflecting powers of light could appear as heat. It was not ordinary light because a plate of glass placed in the presumed course of the heating agency allowed light to pass but destroyed the heating effect, though this difference was not observed when rays from the sun were substituted for those from the hot body. When a cold instead of a hot body was used the thermometer showed a fall of temperature; hence apparently " cold " could be reflected also. Speculations concerning the nature of this " radiant heat," " liberated fire," " obscure heat " as it was variously termed, were divided between those which represented it as some modified form of light and those which gave it a separate identity; here most theorists suspended judgment. There was, however, a balance of opinion in favour of the view that it was some material emanation rather than vibrations in a " heat fluid " for which some arguments were advanced. That the apparent " reflection of cold " did not revive the seventeenth century dualism is due mainly to Prévost, who in 1791 put forward his celebrated " theory of exchanges." Following de Luc in regarding heat as a discrete fluid whose particles were in constant motion and which was similar to, though different from, light, he held that all parts of space radiated this " perfectly free fire " to one another in amounts which increased with the temperature.

The material theory of heat was thus held to be adequate to deal with all the phenomena known. That it came to be challenged was due less to the demands of empiricists for a satisfactory guiding conception than to certain apparent inconsistencies of which the absence of any indubitable evidence that the " matter of heat " possessed weight was one of the most intriguing. Numerous experiments on this subject were made, culminating in an elaborate series by Count Rumford, which produced an entirely negative result. Rumford's experiments on the production of heat by boring cannon are well known, and with these, published in 1798, and the " weight of heat " investigations, published in 1799, we come to the date at which the Philosophical Magazine took up the task of assisting the prosecution of science. It remains to add only that, contrary to what is often stated, neither Rumford's experiments nor the later hypothetical ones of Davy destroyed the material theory of heat, which persisted far into the nineteenth century until the doctrine of the conservation of energy was able to take over its useful elements and reject the materialistic embroidery.

LIGHT.

In light, the eighteenth century achieved far less than in heat. This was due very largely to the fact that it started with a much greater stock of knowledge, and further advance demanded a correspondingly greater degree of experimental skill. The laws of reflection and refraction, the facts of diffraction and dispersion, the optical characteristics of thin films, and the finite velocity of light were all known in the seventeenth century. Geometry had been applied as far as it would go, and for the outstanding phenomena the mainly corpuscular theory of Newton and the undulatory theory of Huygens offered alternative and at least partly adequate explanations. There was an obvious need for a definite decision between these hypotheses, but the crucial experiments did not become obvious until a century later. All that the eighteenth century could do was to compare the requirements of the hypotheses with facts already known and seek to discard one or the other accordingly. On the purely experimental side the greatest deficiency was in the field of photometry, and some of the most important work of the period lay in the establishment of the foundations of this science.

A discovery of major importance— which, however, was so fundamental that it had no bearing on the purely physical problems of the time—was that of the aberration of light by Bradley in 1725. That this was an optical discovery is, in a sense, an accident. Its significance lies in its relation to motion in general, and that the moving thing which revealed it was light is a consequence of practical rather than theoretical necessities. Clearly, it offered no means of deciding between waves and corpuscles. For that the criterion seemed to be of the same nature as that applicable to the " substance " and " motion " theories of heat. If light were corpuscular its production should result in a gradual wearing away of the luminous body ; whereas there seemed no reason why waves should not go on being emitted for ever. This consideration led the celebrated Euler, for example, to favour the wave-theory since there was no evidence that the sun was being consumed in spite of its exceeding brilliance. True, luminous terrestrial bodies are burnt up, but that Euler attributed to the emission of smoke, not light. Again, if all the stars, as well as the sun, sent out particles in all directions, these particles would interfere with one another and bodies would have hazy outlines. Euler's theory, of course, required an all-pervading ether, but he used this to assist his explanations of magnetism and electricity and gravity also. He attributed differences of colour to differences of frequency of vibration, and amplified his theory in other ways, some of which brought it close to the nineteenth century wave-theory.

The predominant view, however, was the corpuscular one, which could claim the authority of Newton as well as more substantial support. The answer to Euler was given by Priestley. He attempted to estimate the amount of substance lost in the emission of light by measuring the pressure which light exerted on the surface on which it fell. He analysed experiments by Michell in which sunlight was reflected from a two-foot concave mirror on to a copper plate fixed to the end of a horizontal wire which was delicately balanced so as to rotate about a pivot if pressure was applied to the plate. The wire did indeed rotate, and, supposing this to be a genuine effect of light pressure, Priestley was able to calculate from its magnitude that the sun must be losing just over two grains a day, from which, if the density of the sun were equal to that of water, it would follow that its radius must have decreased by about 10 feet since the Creation. This was, of course, an effective answer to one argument of Euler's, though we know now that Priestley had not detected the pressure of light and if he had done so he would not have invalidated the wave-theory. It was one form of that theory, in fact, that made it possible later to estimate how great the pressure of light should be. To meet Euler's "interference" difficulty it was necessary to suppose that the particles of light were so small that their dimensions were negligible in comparison with the distances between them—a hypothesis that, if one adopted the discontinuous view of matter, was consistent with the fact that light could pass through transparent bodies.

Clearly hypothesis was running ahead of experiment in this matter of the nature of light, and more lasting work was done by Bouguer and Lambert in placing the measurement of the intensity of light on a solid basis. To Bouguer we owe the first trustworthy photometers, and to Lambert the foundation of the theory of photometry. The inverse square law was already common knowledge, but Lambert added other principles such as that the eye could judge if two sources of light were equal in brightness or not, but not by how much the brightness of one exceeded that of the other; that "if the same surface is illuminated at one time by *m* and at another time by *n* sources of light, each of which has the same intensity and sends its light to that surface in exactly similar circumstances, then the respective degrees of brightness are to each other as *m* to *n*"; and that "the brightness of the illumination of a surface falls off in the same proportion as the sine of the angle of inclination of the incident beam to the surface." (The quotations are from Wolf's 'History of Science, Technology and Philosophy in the Eighteenth Century.') Lambert's most thorough treatment of the subject included a discussion also of atmospheric scattering, the intensities of images and other aspects of photometry. He is deservedly commemorated in the name of the unit of brightness.

Although the foundations of the theory of colour had been well laid by Newton, little advance was made in spectroscopy until the nineteenth century. A misconception of Newton's that deviation without dispersion of light was impossible, was corrected independently by Chester Moor Hall in 1733, Euler in 1747 and Dollond in 1758, the first named arguing simply enough that since the lens of the eye produced images without colour, presumably through the combination of the effects of the lens and the surrounding humours, artificial achromatic combinations should be possible. He followed up the argument by making one, but did not publish his discovery. Good refracting telescopes thus again became a possibility, though they were not made on a large scale until much later owing to the difficulty of manufacture of satisfactory flint glass. Phosphorescence created much interest but little knowledge, and one discovery of outstanding but unrealized importance was made by Melvill who, on examining the light emitted from a spirit flame in which various salts were put, noticed that different colours predominated in the different experiments and, in particular, that the bright yellow colour which stood out when sea-salt was used was so sharply distinguished from the surrounding colours that it must be "of one determined degree of refrangibility." Here was the germ of spectrum analysis, but it was not to develop in our period.

On the whole, however, the most significant contribution of the eighteenth century to the science of optics was not in discoveries but in the generation of men from whom discoveries and ideas were later to come. In the year in which the first number of the Philosophical Magazine appeared, Fraunhofer was a boy of 11, Young was 25 but almost unknown, and Fresnel 10. The age which produced these pioneers is not fairly judged by its matured fruits alone.

MAGNETISM.

Of the various physical phenomena, none bore a closer superficial resemblance to gravitation than magnetism. Both were manifested by their effects on the movements of bodies, both appeared on the small scale and on the large, for Gilbert's hypothesis that the earth was a magnet was universally accepted in the eighteenth century, and both clearly were associated with effects that decreased with increase of distance between the bodies concerned. In spite, therefore, of obvious differences, it was impossible not to suppose that magnetic phenomena were manifestations of forces of the Newtonian type. On the other hand, whereas Newton had specifically limited his endeavours to the formulation of the law of gravitational attraction and had disclaimed all intention of speculating on the *cause* of gravitation, the investigators of magnetism combined research into the law of magnetic force with theories of the nature of magnetism itself. As Newton would probably have told them, the former labours yielded the more valuable dividends.

The original form of the magnet was, of course, the naturally occurring lodestone, but the eighteenth century made considerable progress in the manufacture of artificial magnets, using purely magnetic methods since the connection of magnetism with electricity was unknown. Several analogies between the phenomena indeed were noted, and much speculation was aroused by the curious fact that iron could be magnetized by lightning which, by the middle of the century, Franklin had shown to be an electrical discharge ; but the nature of the relation, and the circumstances necessary for its utilization, were not even suspected. Any method, however, of producing artificial magnets greatly facilitated the investigation of the subject, and although the strongest fields producible were weak when judged by modern standards, they had the outstanding merit of being controllable so that precise measurements could be made.

The idea that the magnetic force varied as the inverse square of the distance from the pole of a magnet must have occurred very early, but its experimental establishment was not easily achieved. Among others, Hauksbee and Musschenbroek sought to derive a general law, but without success. Michell, however, in 1750 was able to show that the most probable explanation of the work of these experimenters and of his own was that the inverse square law was the law of magnetic force. He found that, at the same distance and in the same direction, the two poles of a magnet exerted exactly the same force but in opposite senses, and that this force decreased proportionately to the square of the distance from a pole. With true scientific caution, however, he added: "I do not pretend to lay it down as certain, not having made experiments enough yet, to determine it with sufficient exactness." Others obtained the same result, notably Lambert, who compared the field of a magnet, as plotted with a small compass needle, with the field calculated on the assumption of an inverse square law.

It was Coulomb, however, who made the final indubitable experiments. These were of two distinct kinds, roughly corresponding to the two commonest methods used nowadays in elementary laboratories for determining the moment of a magnet or the intensity of a magnetic field. First, he measured the period of oscillation of a small compass needle under the influence of the earth's field alone and when modified by the presence of a pole of a long thin magnet placed at various distances from it. The possible disturbing factors were taken into account and a definite result arrived at. In the second series of experiments he used a torsion balance of the kind which he had already employed to investigate the law of electrical repulsion. This had in all essentials the same form as the modern instrument, and is too well known to need description. Coulomb's results were published in 1785, and although Gauss was to confirm them later by a still more precise method, the law of magnetic force may be said to have been known from that time onwards.

For a considerable part of the century the origin of magnetism was, strangely enough, generally attributed to Cartesian vortices which, so far as gravitation was concerned, had been thoroughly discredited by Newton's achievements. Euler pictured the vortices as passing through pores in the magnetic body, but always in one direction only, thus producing opposite polarities at entrance and exit. Later, there came "one-fluid" and "two-fluid" theories, the former having the advantage of simplicity in the explanation of either attraction or repulsion, but becoming decidedly embarrassed when both had to be accounted for. Even a two-fluid theory did not readily explain the *inseparability* of the magnetic poles, and it was probably this awkward fact that demanded recourse to the discontinuous theory of matter—a hypothesis which seems in the eighteenth century to have been called upon when needed and at other times ignored. The ultimate particles of a potential magnet were conceived as being themselves magnets, and the process of magnetization consisted simply in their alignment. This view, which has a quite modern appearance, has in fact a very respectable antiquity. It has been suggested that Gilbert may have held it, though the evidence is not convincing, but it was certainly no stranger to eighteenth century thought. Among others, Swedenborg, applying his own theoretical ideas to the experiments of Musschenbroek, gave an account of molecular magnetism with diagrams that might find a place in modern text-books. The remarkable work of this philosopher, in this as in other fields, had little influence on the general progress of science, but the molecular theory of magnetism found independent expositors who could claim a considerable following.

The phenomenon of magnetism is often classified today under three headings—ferro-, para- and dia-magnetism. An example of the first of these was known to the early Greeks; the others first came to light in the eighteenth century. Cobalt and nickel were discovered during this period and shown by their discoverers to possess relatively feeble magnetic properties. Coulomb made an extended investigation of the possibility of magnetization, and added other substances to the list. The more startling fact of diamagnetism was discovered by S. Brugman in 1778: he found that both bismuth and antimony repelled the poles of a magnetic needle. The discovery seems to have been without influence on theories of magnetism at this time.

The chief contributions to the subject of terrestrial magnetism lay in the accumulation of data concerning the variation (as declination was then called) and dip in different parts of the earth, and the diurnal and annual changes in the variation at one place. Wilcke, in 1768, published the earliest chart of a considerable region of the earth's surface (including most of the earth except Asia and North America) which showed isoclinic lines. This was a considerable achievement, though the accuracy was admittedly not of a high order. A variation chart had been published by Halley at the very beginning of the century and was frequently revised thereafter. Towards the end of the century, the studies of distribution of the magnetic elements, which had previously been mostly confined to variation and dip, were extended to the intensity of the earth's field. The measures were relative, the first absolute determination of magnetic intensity being reserved for Gauss in the next century. Speculation on the origin of terrestrial magnetism, though not entirely non-existent, was fortunately restrained in view of the scantiness of the data.

ELECTRICITY.

Of all the phenomena with which physics is concerned, there is none to which the contribution of the eighteenth century is so fundamental as it is to electricity. Heat, light and sound, and even magnetism, were familiar enough before; one could hardly escape them in ordinary, everyday experience. But electricity was not normally encountered. To observe it special things had to be done, and the results attending these were so slight as to escape the notice of all but the most curious. The subject was not completely unknown; the ancient Greeks, of course, had observed the peculiar property of rubbed amber, and Gilbert and von Guericke during the previous century had given some attention to it, but beyond the construction of a simple frictional machine and the recognition that there was a distinction between what we now call conductors and non-conductors, we may say that in the year 1700, complete ignorance existed of what was later to become the most comprehensive department of physical knowledge.

Throughout the century, the basic phenomena progressively came to light—the basic phenomena, that is, of what we now call electrostatics. A current of electricity was unknown. The eighteenth century had to discover the various manifestations of electricity, to prove that they were manifestations of the same thing, to devise instruments for their measurement, and to form concepts for their understanding. Its success was very great and very hard won. A striking feature of its attainment was the fearlessness with which the investigators used themselves as instruments of detection, despite the unknown dangers lurking in this strange effluvium. The experience of a shock seemed to be the standard evidence for the presence of electricity, and even when sufficient knowledge had been obtained to adopt safer methods, it continued to be used. Musschenbroek, reporting to Réaumur his discovery of the Leyden jar in 1746, wrote: "I wish to report to you a new but terrible experiment, which I advise you on no account to attempt yourself." He had, however, evidently repeated it a number of times, for he described its dependence or otherwise on the shape and material of the vessel used. Musschenbroek's letter was read to the French Academy, whereupon Nollet and Le Monnier immediately repeated the experiment. Richmann, whose views on heat we have already considered, was killed in 1753 while experimenting on the electrical nature of lightning. Nevertheless, the electric shock continued to be regarded as almost an essential part of an electrical experiment.

Early electrical machines, of which Hauksbee's in the first decade of the century is an elementary type, consisted of a rotating body, usually glass, rubbing against the human hand or some other material. The charge was collected by a "prime conductor" and transferred where needed. Such instruments, however, did not become generally used until halfway through the century, and in Gray's pioneer experiments on electrical conduction, which were made about 1730, only the simple manual operation of rubbing was used. Gray discovered that when one part of a system was rubbed, not only that part but a distant unrubbed part could also attract light bodies. This suggested that the electrical influence was being conducted from one place to another, and by an extension of his experiments, Gray was able to conduct it over a distance of some 800 feet. Difficulties with the supporting materials for such long conductors revealed the existence of two classes of bodies, which Desaguliers later called *conductors* and *electrics per se* and which evidently corresponded to what Gilbert had called non-electrics and electrics.

The discovery that there were two kinds of electricity as well as two kinds of electrifiable bodies was made by du Fay shortly afterwards. He found first that an electrified body attracts non-electrified bodies, and after communicating electricity to them, repels them; and secondly, that there are two opposite kinds of electricity which he called *vitreous* and *resinous* (the names *positive* and *negative* were given later by Franklin, and indicate the transition from a purely descriptive to a theoretical account of the phenomena). Neither terminology is satisfactory, as Wilcke and Canton independently found about 1754. The same body, whether "vitreous" or "resinous" in character, may develop either kind of electricity according to the material with which it is rubbed; and the difference between the two kinds is not that between excess and defect as Franklin supposed).

The introduction into electrical experiments of the Leyden jar, which was soon, in battery form, to become the chief instrument of research with what were regarded as large charges of electricity, gave a great impetus to the study of the science. This was due independently to von Kleist of Pomerania and Musschenbroek of Leyden, who almost simultaneously made the discovery accidentally. It led, among other things, to attempts to discover the speed of propagation of the electric fluid. Several observers discharged Leyden jars over great distances, but no appreciable interval was noticed between the shocks or sparks observed at widely separated points. Franklin was openly sceptical of the value of such experiments. Thinking of a discharge as a movement of a fluid which filled the whole circuit, he held in effect that the interval between the passage of electricity across two separate points of the circuit no more indicated the velocity of electricity than the interval between pushing one end of a rod and the movement of the other end would indicate the speed of movement of the rod.

Franklin's best known contribution to electricity is his famous demonstration of the electrical nature of lightning by means of his kite. This experiment was performed in 1752, but he had already, in 1749, set out the evidence, in the form of a number of analogies, for his belief in this identification, and the kite experiment was rather a confirmation of a belief already held than an original source of conviction. The kite, made of silk on a cedar-wood frame, was flown when a thunderstorm seemed to be approaching, and to the end of the cord near the hand was attached a piece of silk which was to be kept dry and by which the apparatus was held. A key was placed at the junction of the cord and silk, and when the kite and cord were electrified by an approaching thundercloud, electricity could be drawn off from the key and shown to possess all the properties of laboratory electricity. By repeated experiments, Franklin was able to show that thunderclouds were sometimes positively and sometimes negatively charged.

Franklin's experiment was in essence performed earlier under the supervision of Dalibard at Marly, near Paris. Dalibard had been impressed by Franklin's arguments, and sought a direct proof, using a pointed iron rod about 40 feet high instead of a kite. Having to be absent during the thunderstorm, he left the observations to Coiffier, an ex-dragoon, and Raulet, the Curate of Marly. Sparks about $1\frac{1}{2}$ in. long were obtained at intervals which Raulet estimated at between a Pater and an Ave (freedom of choice of time scales is not a new idea). Raulet received the spark in person, sustaining a wounded arm and a charge of having acquired a smell of sulphur, but the theological implications of the accusation were not established.

The practical application of his discovery for the protection of buildings and ships soon occurred to Franklin, and as early as 1750 he described the principle of the lightning conductor. It was immediately taken up and some thirty years later became the occasion of a controversy which, in its confusion of scientific facts with political doctrines, seems more characteristic of twentieth century Europe than eighteenth century England and America. Franklin had prescribed pointed upper ends for lightning conductors, but Wilson and others thought that these would attract lightning that might otherwise take another course and that the ends should therefore be blunt. The ensuing controversy dominated the contributions on electricity to the 'Philosophical Transactions' for a considerable period, and was said to have led to the resignation of Sir John Pringle from the Presidency of the Royal Society. The following passage appears in the biography of that gentleman in the 'English Cyclopaedia':— "About the year 1778 a dispute arose among the members of the Royal Society relative to the form which should be given to electrical conductors so as to render them most efficacious in protecting buildings from the destructive effects of lightning. Franklin had previously recommended the use of points, and the propriety of this recommendation had been acknowledged and sanctioned by the Society at large. But, after the breaking out of the American Revolution, Franklin was no longer regarded by many of the members in any other light than an enemy of England, and, as such, it appears to have been repugnant to their feelings to act otherwise than in disparagement of his scientific discoveries. Among this number was their patron George III, who, according to a story current at the time, and of the substantial truth of which there is no doubt, on its being proposed to substitute knobs instead of points, requested that Si John Pringle would likewise advocate their introduction. The latter hinted that the laws of nature were unalterable at royal pleasure ; whereupon it was intimated to him that a President of the Royal Society entertaining such an opinion ought to resign, and he resigned accordingly." 'The Dictionary of National Biography,' however, makes no reference to the controversy and merely says that Pringle felt compelled to resign after his health began to fail.

All the work so far described was purely qualitative—an indication of the backwardness of the subject of electricity compared with heat, say, where we were able to start with the thermometer already in existence and the question of giving a meaning to its readings ripe for consideration. Even the first primitive electroscopes did not come into use until towards the middle of the century. They usually took the form of suspended threads, with or without pith balls at the ends. The next stage in precision was the control of the discharge so that it did not exceed the danger limit—a precaution especially necessary after the discovery of the Leyden jar and the use of electrical treatment in medicine. This was achieved by including a gap in the circuit, the length of which determined how strongly the jar could be charged before discharging through the patient. The earliest actual measuring instruments were simply electroscopes with a scale of degrees attached to indicate the angle of separation of the threads. Henly's "quadrant electrometer" (not to be confused with the modern instrument of that name), produced in 1770, was of this type. Volta, who had created a valuable instrument in the *electrophorus*, by which successive, presumably equal, quantities of electricity could be transferred from a cake of resin to a conductor, wished to establish an absolute scale of electrical measurements, and it had already been suggested that this could be achieved by balancing the attraction of a charged balance pan by weights placed in the other pan. Instruments of this sort were, in fact, made, but the unit of electric charge had to be defined before the desired result could be achieved, and the existing ideas of electricity were too undeveloped for this problem to be satisfactorily solved.

The variation of electrical attraction or repulsion with distance, however, was a soluble problem, irrespective of the measurement of electricity itself. The inverse square law was, of course, suspected long before it was established, and experiments indicating that it was the actual law of electric force were not wanting before Coulomb's definitive demonstration. Thus Priestley, in 1766, showed the absence of electric force from the inside of a charged vessel, and suggested the inference that the law of attraction must be the same as that of gravitation, since it was known that if the Earth were a shell, a body inside it would not be attracted to one side or another. Cavendish followed up this work with much more precise experiments of the same kind, in which he proved that the charge on a conductor lay entirely on the surface. From the delicacy of his experiments he concluded that the electric force varied inversely as some power of the distance lying between $2+\frac{1}{50}$ and $2-\frac{1}{50}$.

Coulomb's investigations with the torsion balance were carried out from 1784 onwards. As with magnetism, two types of experiment were performed. For electrical repulsion the force was measured by the angle through which the torsion head had to be turned to maintain a chosen distance between the electrified balls. For attraction a small charged body was oscillated in the neighbourhood of an oppositely charged sphere, and its frequency was noted. In these experiments, of course, the (then immeasurable) magnitude of the charge did not matter so long as it remained constant. Coulomb investigated the leakage of charge and made corrections for it. He also obtained Cavendish's result concerning the distribution of charge over the surface of conductors, and realized that this followed from the inverse square law.

The introduction of measurement into electrostatics inevitably raised questions about the nature of the thing measured. All theories supposed electricity to be some sort of substance, so that the idea of quantity of electricity as the object of measurement was a natural one. It was not, however, adopted automatically. Thus, Priestley, in his 'History of Electricity,' first published in 1767, wrote: "By *electricity* I would be understood to mean, only those *effects* which will be called electrical; or else the *unknown cause* of those effects"—indicating that he was prepared to admit at least the possibility that electricity was an intensive, rather than an extensive, quantity. Cavendish went further and likened the "degree of electrification" to the pressure in a fluid. This "degree of electrification" determined in which direction electricity would flow when two charged conductors were connected with one another. He was evidently approaching the idea which later came to be known as " potential " and has played so important a part in electrical theory. He also conceived the modern notion of "capacity," which he measured in " inches of electricity," and to some extent anticipated Faraday's work on specific inductive capacity. All this is very remarkable at so early a stage, when both conceptions of electricity and measuring instruments were of the crudest. True to his character, he kept most of his results to himself, but it may be doubted whether ideas so far in advance of their time would have had much general effect had they been published.

Another modern conception which was foreshadowed in the eighteenth century was that of electrical resistance. Aepinus, about 1759, conceived that the difference between conductors and non-conductors was simply a matter of degree, the former offering less opposition to the passage of electricity than the latter. Priestley some ten years later showed that the discharge of a battery of jars was independent of the shape of the external circuit, but did depend on its length and cross-section. His criterion for the intensity of the discharge was the length of spark gap that would be bridged in preference to a fixed alternative circuit. Cavendish extended the experiment to solutions of various strengths and studied the effect of temperature on resistance. He showed, further, that the resistance of a conductor did not depend on the strength of the discharge, and described how the discharge was distributed among several conductors in parallel. We are so accustomed to associating these ideas with the phenomena of steady electric *currents* that we are apt to overlook how much it is possible to learn from discharges of static electricity. Cavendish, however, was a genius of the highest order, who, but for his extreme secretiveness, would figure much more prominently in the history of science than he does.

All this work was of the nature of prophecy rather than of principles for immediate guidance ; it awaited its Faraday for its proper understanding. So far as working hypotheses were concerned, the eighteenth century, in electricity as in everything else, looked to effluvia, or fluids, or ethers for the explanation of the curious things it was bringing to light. Since the phenomena of magnetism and electricity were so similar it was natural that explanations of the same type should be called forth. There were supporters of both one-fluid and two-fluid theories, the general arguments for their reality and descriptions of their modes of application being similar to, though by no means entirely identical with, the corresponding accounts in magnetism. There was, it is true, the simplifying feature that the two kinds of electrification, unlike the two kinds of magnetism, were not inseparable, though Wilcke showed in 1757 that in electrification by friction both were invariably produced. There was also a general idea that a charged body was surrounded by an " electrical atmosphere "—a notion which offered a ready, if only partial, explanation of electrification by induction which was discovered by Canton and studied more completely by Wilcke and Aepinus about the middle of the century. But something more definite was required by the intellectually curious, and it was inevitable that a variety of views should be held, according to the particular phenomena among the growing number that were considered to be most evidential.

Euler, whose luminiferous ether has already been mentioned, gave what is probably the first electromagnetic theory of light by making his one all-pervading ether serve for all physical phenomena that seemed to require such a medium. If a body had ether forced into it, it acquired one kind of electrification ; if it suffered a loss from its normal quantity it acquired the other kind. He evidently wished to avoid the multiplicity of ethers which threatened to drag science back to the "Aristotelian" philosophy of "occult qualities" against which Galileo and Newton had contended. Franklin, confining his attention to electricity, also postulated an all-pervading ether, an excess of which gave a body a positive charge while a defect gave it a negative charge. This ether was a sort of matter, but its particles were extremely fine and did not attract but repelled one another. Ordinary matter and electrical matter were mutually attractive, and a material body normally gathered within itself as much electrical matter as it could accommodate, so that when more was added it lay on the surface and formed an atmosphere around the body. This theory was regarded favourably, though without enthusiasm, by Priestley. A somewhat similar theory was held by Aepinus and Cavendish, with, however, one important difference ; namely, that the particles of ordinary matter, as well as those of electricity, repelled one another, the law of repulsion being the same for both and involving a force varying inversely as some power of the distance less than the cube. The body was thus held together by the attraction between its material and its electrical constituents. Substitute for matter, atomic nuclei, and for electricity, electrons, and this theory becomes identical with that of the present day. This was a remarkably bold hypothesis to hold in view of the known gravitational attraction of ordinary uncharged matter.

The weight of authoritative opinion thus seemed to favour a one-fluid theory, but the other view was very much alive also. It had the advantage of the one-fluid theory in the explanation of electrical repulsion, and it was this phenomenon that led du Fay to introduce it on his discovery of the difference between "vitreous" and "resinous" electricity. Nollet, who in his early days collaborated with du Fay, seems to have held to a single fluid, but provided bodies with two sets of pores through one of which electricity could enter bodies while through the other it could leave them. As with heat, much labour was spent in an attempt to discover the weight of electricity, and since no success was obtained, theorists were at liberty to suppose the weight positive or negative—a great advantage in dealing with awkward phenomena. On the whole it might be said that neither theory led to any outstanding discovery which could not have been made by means of the other, and the various views, though doubtless serving to coordinate the facts in the minds of the experimenters, could claim little other intrinsic merit.

It was reserved for the last years of the century to reveal the facts which were to give an entirely new turn to the subject. In 1780 Galvani noticed that a frog's leg became agitated in the neighbourhood of an electrical discharge. This small beginning led him to further experiments culminating in the observation that similar convulsions could be produced in frogs attached to a brass hook and an iron trellis when the metals came into contact. The agitation seemed to require no special external conditions for its occurrence, and this made it clear that here was no manifestation of atmospheric electricity or any other known electrical action, but something hitherto unknown. Galvani ascribed it to "animal electricity," a property peculiar to organic bodies, but Volta—with, as it turned out, truer insight—thought the contact of the metals was the source of the electricity, the frog serving merely to show it. Acting on this idea—to which he was converted from an original agreement with Galvani's hypothesis—he proceeded through a series of experiments to his fundamental discovery of electrification of two different metals by contact, without the presence of any third substance. This he described in 1797, and in such an atmosphere of promise the 'Philosophical Magazine' in the following year drew its first breath.

Two things strike us in comparing the scientific work of this century with that of our own. In the first place the sciences were completely isolated from one another. Each was still very close to its basic phenomena and had not advanced far enough to encounter the others. Hence the tendency to multiply fluids, ethers, forces and what not. This was not a fault but a sign of immaturity. The facts had not been discovered which would have justified the use of the same concept for a variety of purposes, and such a unifying conception as, for instance, the ether of Euler, was not of any real value. Much more efficacious for further advances were the separate hypotheses which dealt more completely each with its own branch of study. The day had not arrived when such sciences as electro-magnetism, thermo-dynamics, photo-chemistry and so on could profitably be conceived.

Secondly, despite the separation of the sciences, the same men studied them all. The gradual unification of physics has paradoxically been attended not by a diminution but by an increase of specialization. In reviewing the various branches of physics, chemistry, astronomy and mathematics in the eighteenth century we find the same names occurring everywhere—Priestley, Cavendish, Wilcke, Michell, Lambert, Euler, to mention only a few. It was the spring-time of modern physics, when something new appeared every day and the man who had a mind for such things could take his pleasure where he would. That this is impossible today is doubtless a necessary result of progress, to be welcomed rather than deplored. Yet, looking back on an age now gone for ever, one might ask pardon for a faint feeling of regret.

CHEMISTRY THROUGH THE EIGHTEENTH CENTURY

By Prof. J. R. PARTINGTON, D.Sc.

In the year 1700 Chemistry had gone a long way towards becoming a science. It had become enriched by the acquisition of a large body of experimental knowledge, and serious attempts had been made to reduce this to some kind of system. The principles underlying this systematization were, it is true, often incorrect, but they were conceived in terms of contemporary scientific theory and were very different from the views of the alchemical school a century before. The rationalization of outlook owed much to the active progress in mathematical astronomy and physics which followed the work of Newton, and his influence may be traced in some parts of chemistry during the whole of the eighteenth century. Interesting views were put forward in Newton's letter to Boyle (1) in 1679, and more particularly in the " Queries " in the later editions of the 'Opticks.' The latter, although containing some important material previously published by Mayow and by Stahl, introduced the idea that chemical changes might be explained in terms of atoms with short-range attractive forces acting between them, and suggested that chemistry might turn out to be a branch of mechanical philosophy. This view was the foundation of a theory proposed by John Keill (2), which assumed a power in matter superadded to gravity, but diminishing in a greater ratio than the inverse square of the distance, and an attempt was made to explain the phenomena of cohesion, elasticity, crystallization, solution, fermentation, effervescence, etc. The Newtonian theory was also used by John Freind in his Oxford lectures (3) to explain the solution of metals in acids, algebraical relations being given. The construction of tables of affinity was attempted by E. F. Geoffroy (4), but the antagonism of the French Academy to Newton's ideas caused him to replace the word " attraction " by the neutral term " rapport," which, in fact, expressed the old view that affinity acts most strongly between substances having the greatest relation to one another. These affinity tables persisted through the century, finding their most elaborate form in Bergman's tables (1775) (5). They assumed that affinities are absolute forces; if the attraction of a substance A for another B is greater than the attraction of A to a third substance C, then the substance B will act upon a compound AC so that the change $B+AC=AB+C$ takes place completely. The tables are the result of chemical experiments, and no use was made of any theory of attractive forces. That the results could be different in solutions and in dry reactions, and that a large excess of a substance is sometimes necessary to cause a reaction to go to completion, did not escape Bergman's notice, but the chemical theory of affinity had to await the acute criticisms and ingenious experiments of Berthollet, about 1800, before the idea of the action of mass was introduced. Berthollet's views were also founded on Newton's, but again the actual results were based on purely chemical experiments.

Boyle and Newton made much use of the atomic theory, which seems to have come to them from the old Greek source of Epicurus by way of Gassend (6). In the form in which Newton had left it, the atomic theory was unsuitable for later chemical applications, since air in Newton's time was still an element, and he could not, therefore, discuss the problem of the structure of mixed gases, the intensive study of which led Dalton to his chemical atomic theory (7).

Of great influence in the early eighteenth century was the famous text-book 'Elementa Chemiae' (1732) of Hermann Boerhaave, professor of chemistry at Leyden from 1718 to 1729. Boerhaave's definition of chemistry was given (long after it had become antiquated) in the 'Dictionary' of Samuel Johnson (who greatly admired Boerhaave, and wrote a biography of him): it would apply equally to cookery :—

> 'Chemistry is an art which teaches the manner of performing certain physical operations, whereby bodies cognizable to the senses, or capable of being rendered cognizable, and of being contained in vessels, are so changed by means of proper instruments, as to produce certain determinate effects; and at the same time discover the causes thereof, for the service of various arts'.

The first volume of the book, dealing with theory, neglects Stahl's phlogiston theory (see below), although Boerhaave's own theory, that combustibles contain a "food of fire" (*pabulum ignis*), is very like Stahl's. He describes experiments of Boyle, Du Clos, and Homberg on the increase in weight of metals on calcination, and he proved that the matter of fire has no weight by showing that a mass of 8 lb. of iron has the same weight red-hot and cold.

Parts of the chapter on air are in contradiction, some having been written before the appearance of Hales's 'Vegetable Staticks' in 1727; Boerhaave adopted to some extent the doctrine of the fixation of air in bodies, but (perhaps because he did not think this sufficiently proved) he did not retract what he had previously said to the contrary.

In a section on menstruums, he says the solvent action depends on motion, the small particles of the menstruum strongly cohering with the particles of the solute (*arctissime adhaerentium ad corpusculi soluti per Menstruum corporis*); it is not easy to understand the physical manner by which this motion is excited, but it seems that it is not due to impulse, gravity, elasticity, magnetism, etc., but to a particular cause, not common to all bodies, exerted between the solvent and solute (*singularis enim est inter solvens et solvendum, non communis omnibus corporibus*). Boerhaave, although using mechanical analogies, and quoting Newton, did not think chemical changes could be explained mechanically, but thought they depended on some other cause.

The name gas, and the recognition that there are several gases different in properties from atmospheric air, appear in the writings of Van Helmont

(1648) in the preceding century (8). Van Helmont, although sharing the credulity of his time and under the influence of alchemy, marks a transition to Boyle's outlook, because he sought rational explanations, carried out quantitative experiments, and taught that substances preserve their individuality in chemical changes; a metal dissolved in an acid, for example, is recoverable from the solution.. Van Helmont's rationalism, although not so complete as they would have wished, appealed to Boyle and Newton, the former frequently referring to him with approbation. Alchemy had had an independent philosophy, which was used by Francis Bacon, and in its rationalized form with marked success during the seventeenth century by Boyle, Tachenius, Glauber, and Kunckel. This method, which will be called chemical to distinguish it from the mathematical or quantitative method of astronomy and physics, was to be used with brilliant results by Scheele and Priestley. Whatever views Newton may have held in private on alchemy, they do not intrude in his published works, on which alone his attitude to chemistry should be judged (9). He had a good knowledge of the new method in chemistry, and did not, as some of his followers did, believe that chemistry could make progress by the use of the mathematical method alone.

Boyle, and especially Mayow (1674), had collected some gases and the latter had shown that they had the same compressibility as common air, but Boyle called them "factitious airs," or "air generated *de novo*," and the existence of several gases with chemical properties different from those of common air was first clearly recognized in the second half of the eighteenth century. Stephen Hales, in his 'Vegetable Staticks' (1727), had collected several gases but although he paid great attention to the quantities of these "airs" obtained from given weights of different materials, following the quantitative method, he completely neglected their chemical properties and reached very faulty conclusions. He says little more than that "air abounds in animal, vegetable and mineral substances," and that it is:—

> 'very instrumental in the production of the growth of animals and vegetables, both by invigorating their several juices, while in an elastick active state, and also by greatly contributing in a fix'd state to the union and firm connection of the several constituent parts of bodies, viz. their water, salt, sulphur and earth.'

Hales's apparatus was clumsy, but his work is important in having attracted Cavendish and Priestley to the study of gases.

The earlier issues of the Philosophical Transactions contain several papers on "airs" and "damps," mostly unimportant. A notable paper is that of William Brownrigg (1765) on "An Experimental Enquiry into the Mineral Elastick Spirit, or Air, contained in Spa Water." This had been read twenty-four years before it was published, and was known to Hales. It deals with carbon dioxide, but the chemical properties are not adequately described. A primitive form of pneumatic trough, with a shelf above the water in which bottles were secured by wedges, was used.

Publication on gases really began with Cavendish's paper (10) in 1766. It describes the properties of two known gases, fixed air (carbon dioxide) and inflammable air (hydrogen), the determinations of their densities, and the solubility of fixed air in water and spirit of wine. The properties of hydrogen were first clearly described by Cavendish, who is sometimes regarded as its discoverer. Cavendish collected and stored fixed air over mercury, a method later much used by Priestley. He also recognized (1766, 1788) the existence of equivalent weights of reacting substances and used the name equivalent. Some important purely chemical work he did not publish. He did not discover any gas not previously known. Inflammable air had been collected and examined by Boyle, and fixed air (already recognized by Van Helmont) had been shown to play a part in chemical changes by Black.

Joseph Black was born on 16th April, 1728, at Bordeaux, where his father, of Scots descent, but born in Belfast, had a wine business. Joseph was educated in medicine at Glasgow University (1746), where he came under the influence of Dr. William Cullen (1710–1790), then lecturer in Chemistry, and worked in Cullen's laboratory. He succeeded Cullen in 1756 as lecturer in chemistry at Glasgow, and in 1766 as professor of chemistry in Edinburgh, occupying the last position until his peaceful death, seated in his chair, on 6th December, 1799. Black was one of the most notable teachers of chemistry in the century ; ill-health prevented him from prosecuting much original research, and apart from the work to be mentioned he published practically nothing. His lectures were taken down by students, and manuscript copies are still available ; they were published by his pupil Robison in 1803, and German translations appeared in 1804–5 and 1818.

Black's systematic and critical mind, and his wide knowledge of chemistry, enabled him to form correct judgments of new discoveries, and he was one of the first anywhere to accept Lavoisier's new theory. His influence in Edinburgh was very great, and led to a keen interest in Lavoisier's work, the 'Traité Élémentaire de Chimie' (1789) of the latter being translated and published in Edinburgh in 1790. Black had accepted the new theory at least as early as 1784.

Black's inaugural dissertation for the M.D., 'De Humore Acido a Cibis Orto, et Magnesia Alba,' was presented in 1754, and an extended version of the chemical part was published (11) in 1756. In this he showed by careful quantitative experiments that "mild" magnesia, limestone, and the so-called "mild" alkalis, contain what he called *fixed air* (carbon dioxide), which was the same as Van Helmont's *gas sylvestre*. Calcined magnesia, quicklime, and caustic alkalis contain no fixed air, and owe their caustic properties to the deprivation of the mild forms of the fixed air contained in them. The mild substances effervesce with acids because the acid has a greater affinity for the caustic form than the fixed air has, and hence it displaces the fixed air, which escapes in the elastic state.

Black's quantitative work marks a turning point in chemical method. He was not the first to use the balance in chemistry, but he drew far-reaching theoretical conclusions from its use. These conclusions, simple as they now seem, were not obvious ; in 1764 Johann Friedrich Meyer, an apothecary of Osnabrück, published a book in which caustic alkalis and quicklime were alleged to contain an otherwise unknown fiery acid, *acidum pingue* (giving their solutions a soapy feel), and since they were already saturated with this acid, they did not effervesce with other acids. Meyer's theory was taken seriously by Lavoisier in 1772, and there was a prolonged controversy between the supporters of Black and of Meyer.

Meyer's *acidum pingue* was like, if not identical with, another figment of the imagination. Johann Joachim Becher (1635–1682) in his 'Actorum Laboratorii Monacensis, seu Physicae Subterraneae' (1669) stated that the constituents of bodies are air, water, and three earths, the inflammable (*terra pinguis*), mercurial, and vitreous. The *terra pinguis*, regarded as a constituent of combustible bodies, was re-named *phlogiston* by the famous Georg Ernst Stahl (1660–1734), from 1716 physician to the King of Prussia in Berlin. He gave a sketch of the phlogiston theory in his small 'Zymotechnia Fundamentalis' (1697), based on his lectures in Jena, and expanded it in a supplement ('Specimen Beccherianum') to his edition of Becher's book (1703) and in two versions of a text-book of chemistry, 'Fundamenta Chemiae' (1723 and 1746–7), a somewhat altered English version of the 1723 issue of the latter being published in 1730.

According to Stahl, combustible bodies and metals contain a principle of inflammability, phlogiston, which is identical in them all and can be transferred from one body to another. When bodies burn, or metals are calcined, phlogiston escapes. When the calx of a metal is heated with charcoal or oil, phlogiston is restored and the metal is formed. A calx is an element, a metal is a compound of its particular calx and phlogiston ; sulphur is a compound of sulphuric acid (an element) and phlogiston. The last proposition was proved by showing that when the compound of sulphuric acid and potash is heated with charcoal (rich in phlogiston) the same liver of sulphur (potassium sulphide) is formed as is obtained on fusing potash with sulphur :—

$$\begin{aligned} &(\text{sulphuric acid}+\text{potash})+\phi=\text{liver of sulphur},\\ &\qquad\text{sulphur}+\text{potash}\qquad=\text{liver of sulphur},\\ &\therefore\qquad\text{sulphuric acid}+\phi=\text{sulphur}. \end{aligned}$$

Stahl recognized the existence of sulphurous acid intermediate between sulphur and sulphuric acid, and, following Mayow (1674), he gave clear examples of the displacement of one metal from its salt by another metal, so arriving at an affinity series which is reproduced by Newton.

Stahl's definition of Chemistry is much better than Boerhaave's :—

> 'Chemistry, or Alchemy, or the Spagiric Art, is the art of resolving mixed, compound, or aggregated bodies into their principles, and of composing such bodies from those principles.'

After the work of Black and Cavendish, the chemistry of gases was greatly advanced by the researches of Scheele and Priestley, carried out independently about the same time, and both leading to the discovery of oxygen.

The great extension of chemical knowledge represented by the discoveries of gases by Scheele and Priestley was the result of following the chemical experimental method. They made little use of the quantitative procedure which, too rigidly followed, had sometimes proved more of a hindrance than a help. Scheele and Priestley, by qualitative methods, made discoveries from which a new science of chemistry could be created by Lavoisier by the use of the quantitative method, but Lavoisier made no discoveries of new substances and, without the facts made available to him, there would have been no new quantitative science.

Carl Wilhelm Scheele, born in 1742 at Stralsund, then the chief town of Swedish Pomerania, was an apothecary. Throughout his life he was in modest circumstances and his work in Malmö, Stockholm, Uppsala, and Köping, was carried out in very unfavourable conditions which accelerated his early death in 1786. Scheele belongs to the few really great chemists, and as Hoefer said: "avec de petites ressources, il fit de grandes choses." His steady devotion to chemistry, undeflected alike by disappointment or success, was the guiding star in his short life. In a letter to his friend Gahn he says: "How glad is the enquirer when discovery rewards his diligence; then his heart rejoices," and if Scheele's reward in discoveries was rich, it was abundantly deserved. In recognizing the early genius of Scheele, giving him encouragement, and helping to bring his discoveries to the notice of contemporary chemists, great credit is due to Bergman, then the eminent professor of chemistry at Uppsala.

Scheele's researches began before 1770; those leading to the discovery of oxygen seem to have been made about 1772, if not earlier (12). They were announced in a book, 'Chemische Abhandlung von der Luft und dem Feuer,' which was sent to the printer in 1775 but did not appear until 1777, when oxygen had been independently discovered by Priestley, and Lavoisier was just announcing his new theory. Scheele's laboratory journal has been preserved, and his priority is without question. Priority in discovery need not depend on date of publication if written records of unquestionable date are available.

Although Scheele's discovery of oxygen is best known, his other discoveries are no less notable. They include chlorine, manganese and barium (in its compounds) (1774), molybdenum (1778) and tungsten (1781) as oxides, arsenic hydride and arsenic acid (1775), silicon fluoride and hydrofluoric acid (1771 and 1781), copper arsenite (1775–8), and a large number of organic compounds: tartaric and pyrotartaric (1770), oxalic (1784), mucic (1780), lactic (1786), uric (1776), prussic (1782–3), citric (1784), malic (1785), gallic and pyrogallic (1786), acids, glycerol (1783–4), murexide (1776), several esters and aldehyde (1782), and casein (1780).

About 1772, if not earlier, Scheele had concluded that air is a mixture of two gases, fire air which supports combustion and respiration, and foul air which does not. He had extracted fire air, which is oxygen, from nitric acid, nitre, mercuric oxide, manganese dioxide, and silver carbonate, discovered its most important chemical properties, and showed that it is rather heavier than air. This involved some quantitative work. Scheele was guided in these researches by a peculiar form of the phlogiston theory. He assumed that heat is a compound of fire air and phlogiston (ϕ) : heat=fire air$+\phi$. If a substance presented to heat has a greater attraction for phlogiston than fire air has, such as red precipitate of mercury or calx of mercury (mercuric oxide), this will rob the heat of phlogiston and set free the fire air :

$$\underbrace{(\text{fire air}+\phi)}_{\text{heat}}+\text{calx of mercury}=\underbrace{(\text{calx of mercury}+\phi)}_{\text{mercury}}+\text{fire air}.$$

Scheele's views on phlogiston (which he called the "inflammable principle") were not quite consistent ; in some cases he identified it with inflammable air (hydrogen), as had been done by Cavendish and Kirwan, and when he discovered chlorine by heating marine acid (hydrochloric acid) with pyrolusite (manganese dioxide), he regarded the product as (marine acid$-\phi$)=(marine acid$-$hydrogen), which Davy afterwards showed is correct. Lavoisier and the French school, misled by a theory that all acids contain oxygen, regarded chlorine as oxidized marine (or muriatic) acid.

Joseph Priestley (13) was born near Leeds in 1733 and died, a refugee from political persecution, in America in 1804. His main interest was theology, in which he was heterodox, but he became interested in science and, as literary companion to Lord Shelburne and later as a friend of Wedgwood, he was able to carry out his experimental work in easy circumstances. His chemical experiments began about 1767 with fixed air formed in vats in a brewery, and he continued work with gases, becoming acquainted with the earlier work of Hales and Cavendish, which served as a model for his own. His apparatus and technique were much neater than theirs, and his trough with a shelf for collecting gases over water, and his use of mercury for collecting soluble gases (already applied by Cavendish in 1766), enabled him to discover many new gases. Priestley's method was largely but not entirely qualitative, and by pressing on with such work he acquired a technique in the manipulation of gases which made him an acknowledged expert in this difficult field. He was little interested in theory, some rather crude forms of the phlogiston theory, which he changed as he thought fit, sufficed for his needs, and he was, in consequence, quite unable to draw correct conclusions from his results. He can be blamed for his obstinate retention of the phlogiston theory, since his views were taken seriously by other chemists. Much time and effort expended in traversing Priestley's erroneous theories could have been better used in other directions.

In March 1772, Priestley read to the Royal Society a paper "Observations on Different Kinds of Air." This was later published in the Philosophical Transactions, with additions made up to the end of 1772, and describes the discovery of nitrous air (nitric oxide, discovered in June, 1772), acid air (hydrogen chloride), and (less definitely) nitrogen, and carbon monoxide. The confusion of the last gas with hydrogen was the source of most of his mistakes. His later work is largely contained in his book, 'Experiments and Observations on Different Kinds of Air' (3 vols., 1774–77) and its continuation, 'Experiments and Observations relating to Various Branches of Natural Philosophy' (3 vols., 1779–86), the six volumes being summarized and systematized in the three volumes of 'Experiments and Observations on Different Kinds of Air and other Branches of Natural Philosophy' (1790). In these he announced the discovery of oxygen (1775), alkaline air (ammonia) (1774), nitrous oxide (1774–7), vitriolic acid air (sulphur dioxide) (1775), and nitrogen peroxide (1777). His experiments on the oxides of nitrogen are important.

Antoine Laurent Lavoisier (14) was born in Paris in 1743. He was well educated in physics and mathematics as well as in chemistry. He entered the Academy in 1768 and became a *fermier général* in 1780. He carried out many researches of national importance and was a member of the commission for the introduction of the metric system. He had ample means and had a liking for elaborate and expensive apparatus. His work was carefully planned and strictly quantitative in character. In a large number of memoirs (over sixty in the period 1768–87) he developed the new theory that combustion and calcination depend on the union of the combustible or metal with dephlogisticated air, which he called oxygen, this theory being definitely stated in 1777. It should not be forgotten that Hooke in 1665 and Mayow in 1674 had made some striking anticipations of the theory that the atmosphere contains a substance which exists in nitre, although they did not isolate this "nitre air" (oxygen). The Russian scientist Lomonossov in 1744–7 had also made experiments on calcination, showing that there was no increase in weight with sealed retorts.

Lavoisier, in collaboration with other French chemists, introduced a new nomenclature (1787), and his text-book, 'Traité Élémentaire de Chimie' (1789) systematized the new theory and made it well-known. It was rapidly adopted. Lavoisier's work on the analysis of organic substances by combustion in oxygen laid the foundations of quantitative organic analysis, and his work on respiration, showing that oxygen is absorbed and carbon dioxide evolved, the "combustion" of carbonaceous material being the source of animal heat, was fundamental. Lavoisier's death by the guillotine in 1794 ended a brilliant career, still full of promise.

Lavoisier's general method was to show that when substances burn in air or oxygen, the weight of the product is equal to the sum of the weights of the combustible and the oxygen gas used up. In the calcination of metals, the weight of the calx is equal to the weight of the metal and of the oxygen absorbed. In his most famous experiment, he showed that when mercury is heated in a confined volume of air it slowly forms red calx (oxide) of mercury, the increase in weight being equal to the loss in weight of the air. On heating the calx, pure oxygen, the weight of which was equal to the loss in weight of the air, was evolved and metallic mercury formed.

Lavoisier defined an " element or principle " as " the last point which analysis is capable of reaching," and he emphasized that " they act with regard to us as simple substances, and we ought never to suppose them compounded until experiment and observation have proved them to be so." On this basis he drew up a fairly extensive table of chemical elements, some of which were later proved to be compounds.

Priestley's work (1772–9) on the part which green plants play in the purification of the atmosphere vitiated by respiration (later shown by others to be due to the conversion of carbon dioxide into oxygen under the influence of light), and that on respiration (1776), were obscured by his use of the phlogiston theory.

Priestley obtained oxygen on 1st August, 1774, at Calne. He was heating " a great variety of substances " in glass tubes over mercury by a burning glass, and observed that an " air " was evolved from red precipitate of mercury (oxide of mercury) which supported combustion much better than common air. He had obtained this gas in 1772 by heating nitre, but had not followed up the observation. The discovery seems to have been accidental, since he says that " so far from having formed any hypothesis that led to the discoveries I made . . . they would have appeared very improbable to me had I been told of them." In March, 1775, he knew that the " dephlogisticated air " (*i. e.*, common air *minus* phlogiston) was different from nitrous oxide, which also supported combustion better than common air.

The gas left after combustions had occurred in common air, and any fixed air (carbon dioxide) formed had been extracted by alkali, was called phlogisticated air by Priestley. He regarded it as common air loaded with phlogiston. This was nitrogen. Nitrogen was discovered about 1772 by Scheele, who had recognized it as a distinct gas present in air; by Cavendish, who communicated his result to Priestley (as the latter says in his paper of 1772); and by Daniel Rutherford (' Dissertatio inauguralis de Aere fixo dicto, aut mephitico,' Edinburgh, 12th September, 1772).

In October, 1774, Priestley told Lavoisier in Paris of his discovery of the " air " obtained by heating red precipitate, saying that it supported combustion much better than common air. By 1776 Lavoisier had found that common air is formed by mixing dephlogisticated air and phlogisticated air. In 1777 he was able to state that air is composed of two different elastic fluids, one respirable and capable of combining with metals to form calces, and the other (which he then called a *mofette*, but afterwards named azote) incapable of sustaining respiration and combustion. This was published in 1778, and in 1777 exactly the same statement had appeared in the belatedly published book of Scheele.

The phlogiston theory gave an explanation of the fire and light disengaged in combustion processes. These were due to the escape of phlogiston, the principle of fire, which streamed out of the burning body in a rapid whirling or vortex motion. Lavoisier, in showing that combustion involves a gain of oxygen and not a loss of phlogiston by the body, had told only half the story; the production of fire had still to be explained. He did not distinguish sharply between heat and light, and in place of the weightless phlogiston (sometimes regarded as having a negative weight) he introduced the weightless material of fire, which he called *caloric*. Gases are compounds of material bases and caloric, just as, according to Black, steam is a compound of liquid water and latent heat. Oxygen gas is a compound of oxygen base and caloric; in combustion the oxygen base combines with the combustible, such as phosphorus, sulphur, or carbon, and caloric is set free in the form of fire, or heat and light. The phlogiston theory regarded the combustible body as the source of the fire, Lavoisier considered that this came from the oxygen gas.

The caloric theory of heat was very generally adopted. In Lavoisier's work with Laplace (15) on the ice calorimeter the alternative theory that heat was some form of molecular motion was proposed, but almost certainly by Laplace, who was then engaged in adapting the principles of dynamics to an explanation of physical changes, and hence favoured a dynamical theory of heat (which goes back to Francis Bacon). Lavoisier was more attracted by the material, or chemical, theory of heat, which regarded it as an imponderable element, and at that time it certainly seemed more plausible. He develops this idea quite dogmatically, and the caloric theory was generally accepted. One of Davy's first researches (16) was an attempt to refute Lavoisier's caloric theory experimentally; his later work on the alkalis and chlorine provided further corrections to Lavoisier's views.

In its later phases the phlogiston theory took many curious forms (17). The necessity of explaining the heat and light in combustion remained an important part of chemical theory. The statement of the various forms of the phlogiston theory is a highly technical part of the history of chemistry. One phase of the theory is concerned with the supposed levity of phlogiston. In this, a distinction was drawn between relative levity, phlogiston being regarded as lighter than air (Guyton de Morveau, 1772), and absolute levity, phlogiston being regarded as absolutely light, and hence decreasing the weight of a body with which it unites. The absolute levity of phlogiston had been entertained by Stahl (1747), and was held for some time by Gren (1786). Its consequences engaged the attention of mathematical physicists (J. T. Mayer, 1790; C. F. Hindenburg, 1790), whose criticisms forced Gren to abandon the theory.

A third phase of the phlogiston theory was held after the general adoption of Lavoisier's theory, since the explanation of the source of fire given by Lavoisier was not always felt to be satisfactory. This form of the phlogiston theory seems to have been first proposed by Gadolin (1788); it was developed by Richter (1793) and by Gren (1794). It started from Macquer's theory (1778) that phlogiston "is nothing but the proper substance of light, immediately or mediately fixed in a large number of compounds," *i. e.*, the matter of light, now distinguished from the matter of heat or caloric. In the development of this theory many took part, but its most complete form is presented by Richter, the less known theories of Crawford (1779–88), Elliot (1780–82), Hopson (1781), Lubbock (1784), and Scott (1786) differing from his only in points of detail.

Lavoisier supposed that combustion is an example of simple affinity, the combustible taking the oxygen-base from oxygen gas and letting the heat-matter (caloric) of the oxygen gas escape. In this case, light must be considered to be a modification of heat-matter, which had not been proved, or else the source of the light must be regarded as different from that of the heat; yet both the cause of the light and the matter of heat are, on this theory, located in the oxygen gas. It is more likely that combustion depends on double affinity, in which case something is present in the combustible body which is called combustible matter or phlogiston, the combustible being a compound of an unknown substrate and phlogiston. Light is a compound of phlogiston and the matter of heat. Every combustion in atmospheric air or oxygen is an action of double affinity; the oxygen-base combines with the combustible substrate, and the matter of heat of the oxygen gas combines with the phlogiston of the combustible to form heat and light, the relative proportions of which depend on the amounts of matter of heat and phlogiston available. In the decomposition of oxide of mercury by heat, the substrate which is combined with oxygen takes up phlogiston from the fire, in which phlogiston is combined with the matter of heat, and at the same time the oxygen-base combines with the matter of heat to form oxygen gas. Thus: combustible=base+phlogiston; oxygen gas=oxygen base+matter of heat; fire=phlogiston+matter of heat.

Richter gave a searching criticism of Lavoisier's theory, and answered several objections to the phlogiston theory. It was not until the theory of energy was fully recognized, in the middle of the nineteenth century, that chemists could feel that the problem of the production of fire in combustion no longer concerned them directly. Until then, chemistry included the imponderable elements heat, light, electricity, and magnetism, the disentanglement of which into energy and other entities is still proceeding.

Lavoisier's new theory of combustion left an important chemical change unexplained. A metal such as zinc dissolves in dilute sulphuric acid to form sulphate of zinc and an inflammable gas, called inflammable air, is evolved. Sulphate of zinc is formed when calx of zinc (oxide of zinc) dissolves in dilute sulphuric acid, but no inflammable air is evolved. On the phlogiston theory these two reactions were easily explained.

Inflammable air had been assumed by Cavendish (1766), Kirwan (1781), and Priestley (1782), to be identical with phlogiston, and zinc is a compound of calx of zinc and phlogiston, hence :

$$\text{I. } \underset{\text{zinc}}{(\text{calx of zinc}+\phi)}+\text{acid}=\underset{\text{zinc salt}}{(\text{calx of zinc}+\text{acid})}+\underset{\text{inflammable air}}{\phi}$$

$$\text{II. calx of zinc}+\text{acid}=\underset{\text{zinc salt}}{(\text{calx of zinc}+\text{acid})}$$

Lavoisier gave no explanation of this puzzle, the solution of which awaited the discovery of the composition of water.

In 1781 Warltire, an itinerant lecturer on science, wrote to Priestley saying that he had exploded a mixture of dephlogisticated air (oxygen) and inflammable air (hydrogen) in a copper vessel by an electric spark, and had found a small loss in weight ; also that he had exploded the mixed gas in a glass vessel (as Priestley had done in 1775) and observed, as he says Priestley had done, that the glass, though clean and dry before, became bedewed with moisture. Priestley in reporting this says it confirmed an opinion long entertained by Warltire, that common air deposits its moisture when it is phlogisticated. This " random experiment," as Priestley calls it, was communicated to Cavendish (18), who began work on the subject in 1781. He soon showed that there was no loss in weight, and then, by exploding a mixture of dephlogisticated air and inflammable air in a closed glass vessel by a spark, he obtained water, the weight of which was equal to the weight of the two gases taken. The formation of an acid in the water was a discovery which protracted Cavendish's experiments until he had shown that it was nitric acid, formed from a little nitrogen contained as an impurity in the gases, and his paper was not read until January, 1784. Cavendish stated his results in a peculiar form. He says :

> " I think we must allow that dephlogisticated air is in reality nothing but dephlogisticated water, or water deprived of its phlogiston ; or, in other words, that water consists of dephlogisticated air united to phlogiston ; and that inflammable air is either pure phlogiston, as Dr. Priestley and Mr. Kirwan suppose, or else water united to phlogiston ; since, according to this supposition, these substances united together to form pure water . . . there seems the utmost reason to think that dephlogisticated air is only water deprived of its phlogiston, and that inflammable air, as was said before, is either phlogisticated water, or else pure phlogiston ; but in all probability the former."

In a footnote he explains the reason for his preference. Whereas common or dephlogisticated air will combine at the ordinary temperature with nitrous air (nitric oxide), taking the phlogiston from it (*i. e.*, oxidizing it) and forming red fumes of " phlogisticated nitrous acid " (nitrogen peroxide), dephlogisticated air will not act upon inflammable air below a red heat. Hence, inflammable air can hardly be pure phlogiston, which would be expected to unite at once with dephlogisticated air.

Cavendish thus represented his results as follows :

$$\text{inflammable air} = \text{water} + \phi,$$
$$\text{dephlogisticated air} = \text{water} - \phi,$$
$$\text{inflammable air} + \text{dephlogisticated air} = 2 \text{ water}.$$

He seems to have regarded water as pre-existing in both gases, and to be deposited as the result of a redistribution of phlogiston. In the paper, however, Cavendish says his experiments could also be explained by the assumption that :

> ". . . water consists of inflammable air united to dephlogisticated air ; . . . and indeed, as adding dephlogisticated air to a body comes to the same thing as depriving it of its phlogiston and adding water to it, and as there are, perhaps, no bodies entirely destitute of water, and as I know of no way by which phlogiston can be transferred from one body to another, without leaving it uncertain whether water is not at the same time transferred, it will be very difficult to determine by experiment which of these opinions is the truest ; but as the commonly received principle of phlogiston explains all phaenomena, at least as well as Mr. Lavoisier's, I have adhered to that."

Lavoisier at first thought the product of combustion of inflammable air in oxygen should be an acid, which he tried in vain to find. In June, 1783, Blagden (who had been Cavendish's assistant) visited Lavoisier in Paris and told him that Cavendish had shown that on exploding inflammable air and dephlogisticated air, water equal in weight to the two airs was the only product. A hurried experiment was made by Lavoisier and some other French scientists, and on the next day, or within a few days, an account of it was sent to the Academy. A summary was published in December, 1783, in Rozier's 'Observations sur la Physique,' in which Cavendish's results are most inadequately mentioned. The paper published in the memoirs of the Academy in 1784 contains the remarkable statement that Blagden :

> 'told us that M. Cavendish had already tried, in London, to burn inflammable air in closed vessels, and that he had obtained a very sensible amount of water.'

Lavoisier did not show that the weight of water formed was equal to the weights of the two gases, merely remarking that, since :

> 'it is no less true in physics as in geometry that the whole is equal to its parts . . . we think we are right in concluding that the weight of this water was equal to that of the two airs which served to form it.'

However true this might be in geometry and physics, chemists could feel that it was no adequate substitute for an experiment in which the result had been demonstrated, and such an experiment had been made by Cavendish. Blagden afterwards (19) pointed out that Lavoisier should have said in his memoir that he had been told of Cavendish's experiments and that Cavendish had obtained, not merely a very sensible quantity of water, but a quantity of water equal in weight to the sum of the weights of the two gases, adding that Lavoisier had discovered nothing beyond what had been pointed out to him as having previously been done in England.

Blagden, a close friend of many French chemists, lived much in Paris, and must have been very sure of his ground before publishing this statement. In a later report by Lavoisier, Brisson, Meusnier, and Laplace, it is, in fact, stated that :

> 'Cavendish seems first to have noticed (*remarqué*) that the water formed in this combustion is the result of the combination of the two aerial fluids, and that its weight is equal to theirs. Many experiments made on a large scale by Lavoisier, la Place, Monge, Meusnier, and Lefebvre de Gineau, have confirmed this important discovery.'

In his first publication in December, 1783, and in his memoir of 1784, Lavoisier gave, for the first time, a clear statement of the composition of water. Although it is necessary, he says to prove that the matter of heat and light which escapes from the vessel when the two gases combine has no weight, and this was engaging his attention, it could hardly be doubted that water :

> 'previously regarded as an element, is not a compound body . . . formed by the union of the oxygen principle with the inflammable aqueous principle.'

With the discovery of the composition of water, the source of the inflammable air formed when a metal dissolves in a dilute acid is clear. The metal withdraws oxygen from the water to form the oxide, which dissolves in the acid to form a salt, and the hydrogen of the water is set free as the inflammable air :

$$\underset{\text{water}}{(\text{hydrogen}+\text{oxygen})}+\text{zinc}=\underset{\text{oxide of zinc}}{(\text{zinc}+\text{oxygen})}+\text{hydrogen}$$

$$\text{oxide of zinc}+\text{acid}=\text{zinc salt.}$$

It was truly said by Davy that the explanation of the phenomena of combustion, calcination, and (in part) of respiration by Lavoisier was only part of the advance in chemical knowledge made in the eighteenth century. Another, and no less important part, was concerned with the quantitative laws of chemical combination, and the investigation of the laws of affinity.

It is a mistake to suppose that Lavoisier introduced the use of the balance and quantitative method into chemistry. Quite apart from the quantitative experiments of Black, there had been some good quantitative work in the seventeenth century, and in the eighteenth century there was careful quantitative work by Cavendish and Bergman before Lavoisier, and by Wenzel, Richter and Kirwan after he had begun his experiments. The law of conservation of mass, formally stated by Lavoisier (1789), had been assumed in quantative analyses for a long time, and no one would have spent time and trouble in making quantitative analyses of salts unless these were supposed to have definite compositions. The law of fixed proportions was formally stated by Proust in 1797. The law of equivalent proportions was first stated by Richter (21) in 1791, although Cavendish in 1766 and 1788 had recognized particular cases of it, and had used the name equivalent. Owing to a slip by Berzelius, the law of equivalents was attributed to Wenzel (1777), who had actually denied its validity.

Jeremias Benjamin Richter was born in Silesia in 1762. He studied under Kant at Königsberg, and graduated M.D. in 1789 with a dissertation 'De Usu Matheseos in Chymia.' Before this he was in an engineer corps in the army, and had been studying chemistry. He failed to obtain an academic position, and in 1798 became chemist in the Berlin porcelain factory. He carried out, mostly at night, his own experimental and literary work, and made hydrometers for sale. His health was poor, and during his short life he received little encouragement; even after his death his discoveries were mostly credited to others. Besides his work on combining proportions, he made important investigations on uranium and other subjects.

Richter was permeated with the idea, which he may have got from Kant, that chemistry is a branch of applied mathematics. His style is generally tedious, but now and again statements of great beauty and power stand out from the dull background of mathematical irrelevance. Richter was a man of great independence and originality. Arithmetical errors in his publications drew upon him very unfavourable and wounding comments, the great importance of his generalizations being quite overlooked.

In 1791 Richter found that solutions of calcium acetate and potassium tartrate remain neutral on mixing, although calcium tartrate is precipitated and potassium acetate remains in solution. From this and similar experiments he drew the conclusion that:

> 'there must be a definite ratio between the masses of each neutral compound and that the terms of ratio are of such a character that they can be determined from the masses of the neutral compounds.'

In 1792 he expressed the same idea as follows:

> 'when two neutral solutions are mixed, and a decomposition follows, the new resulting products are almost without exception neutral also . . . the elements must therefore have among themselves a certain fixed proportion of mass.'

The composition of the products can be calculated from those of the reacting substances:

> 'This rule is the true touchstone of the experiments instituted with regard to the ratios of neutrality; for if the proportions empirically found are not of the kind that is required by the law of decomposition by double affinity, where the decomposition actually taking place is accompanied by unchanged neutrality, they are to be rejected without further examination as incorrect.'

Richter drew up tables of the combining proportions (equivalents) of acids and bases, and attempted to find non-existent mathematical regularities in them. For some reason he failed to see that all these numbers can be combined in a single table, between the members of which ratios of equivalence hold good. This important step was taken by Ernest Gottfried Fischer (1754–1831) of Berlin, in his German translation of Berthollet's 'Recherches sur les Lois de l'Affinité' (Berlin, 1802), in which he gave an account of Richter's little known researches and drew up a table of equivalents of acids and bases, taking 1000 parts of sulphuric acid as standard. Fischer's table, and part of his note, were reproduced in a French translation in the first volume of Berthollet's 'Statique Chemique' (1803, I, 134), which made Richter's work known, his own writings being extremely scarce even at the beginning of the nineteenth century. The only nearly complete set of them in England, found, with much trouble, by Angus Smith, is in the Manchester University Library.

Another set of quantitative experiments on combining proportions was made by Richard Kirwan (22), born in Galway in 1733. Educated in law and for a time practising as a barrister in London, Kirwan turned to science, in 1799 becoming President of the Royal Irish Academy in Dublin. He was a man of great brilliance, versatility and originality. He was at first a defender of the phlogiston theory, his 'Essay on Phogiston and the Constitution of Acids' (London, 1787) being translated into French (1788) with refutations of the various parts by Lavoisier, Berthollet, Fourcroy, de Morveau, and Monge. In 1791 Kirwan announced his conversion to the antiphlogistic theory.

Kirwan also drew up the first table of specific heats (1780), and wrote 'Elements of Mineralogy' (1784; second edition 1794–6), containing a scale of hardness based on Werner's, and very like that afterwards called Mohs' scale.

In a series of papers (23) Kirwan published measurements of the specific gravities of salt solutions and the compositions of salts. These really include tables of equivalent weights, although he overlooked this fact. In his later papers he refers to Richter. He also determined the densities of gases.

The explanation of the laws of chemical combination, the law of constant proportions, the law of equivalents and the law of multiple proportions (implied in the statements of William Higgins in 1789) was first adequately given by Dalton early in the nineteenth century, but some interesting approximations to the chemical atomic theory were made by Bryan Higgins, born in Sligo about 1737, who conducted a school of practical chemistry in Greek Street, Soho (which Priestley attended), and his nephew William Higgins, born in Sligo about 1768, who was assistant to Beddoes at Oxford and later professor in Dublin. Bryan Higgins (24) believed that the particles of elements attract one another, and all of them except earth and water are also endowed with repulsive forces. The repulsion is due to fire, which is a material elastic fluid pervading all bodies. In gases, the particles stand at some distance from one another, and combination (or "saturation") cannot occur until the atmospheres of fire round the particles are broken. By means of diagrams he shows how ammonia and hydrogen chloride (which he calls "alkaline fluid" and " acid fluid," respectively) unite particle with particle to form solid sal ammoniac ; this combination is in " determinate proportion," and the "molecules" of the compound contain one atom of each "element," because atoms of like nature repel each other. This was an extension of Newton's theory of gases in which it is combined with Newton's theory of chemical combination.

William Higgins was one of the first British chemists to accept Lavoisier's views, his book ' A Comparative View of the Phlogistic and Antiphlogistic Theories ' (1789, second edition 1791) being a reply to the ' Essay on Phlogiston ' of his friend Kirwan. He said (25) :

> ' For a considerable time I stood alone in England, where I then resided, being the first who adopted the antiphlogistic doctrine, and the only man who had expressly written in favour of it in the English language.'

This is perhaps literally correct, since Black, who had accepted Lavoisier's theory before 1784, lived in Scotland, and the spirited attack on phlogiston and defence of the antiphlogistic theory written in 1784 by John Lubbock (' Dissertatio de principio sorbile,' Edinburgh, 1784) was in Latin.

William Higgins recognized that the existence of several oxides of nitrogen proved that the assumption that combination (or " saturation ") occurs only by the union of one particle of each element is too restricted. He recognized a decrease in stability as more and more particles of one element are added to a particle of another, and explained this by assuming that the force of attraction between a particle A is equally shared between successive particles of B until, in the end, the attractive force becomes too small to hold the particles together. This force is quite different from gravitational force, and cannot be said to be satisfactory. The law of multiple proportions follows at once from the description and diagrams. In all but one case, he assumed that the particles or atoms (which he calls " molecules ") of different elements are equal in weight ; the equivalents of common elements (except hydrogen) are, in fact, not very different (carbon 6, nitrogen 7, oxygen 8, sulphur 8). Thus he did not anticipate Dalton's clear statement that particles of different elements have different weights, and he says that equal volumes of gases contain unequal numbers of particles (as Dalton also believed).

We have now reached the end of the eighteenth century, and a summary may be given of some of the important discoveries in chemistry made during that century and the position of the science at its close.

One of the most important contributions was the clear recognition of several kinds of gases, different in physical and chemical properties. Hydrogen, oxygen, nitrogen, chlorine, the oxides of nitrogen, carbon dioxide, carbon monoxide, ammonia, sulphur dioxide, and silicon fluoride were all known. In their discovery, Scheele and Priestley had played the most prominent part. Closely connected with the discovery of oxygen was the development by Lavoisier of a new theory of combustion, which replaced the phlogiston theory held earlier in the century.

Lavoisier's theory also required a revision of the list of elements, since many substances regarded as elements by the phlogiston theory were shown to be compounds, and *vice versa*. This made it necessary to introduce a new nomenclature in chemistry, which was done by Lavoisier and his associates. The chemical nature of respiration, and the changes taking place in the atmosphere in the growth of green plants in light, were also fairly clearly stated in the later years of the century.

Elements discovered in the eighteenth century are chlorine (Scheele, 1774), chromium (Vauquelin, 1797), cobalt (Brandt, 1735), manganese (Gahn, 1774), molybdenum (Hjelm, 1790), nickel (Cronstedt, 1751), nitrogen (Scheele, Cavendish, Priestley, Rutherford, 1772), oxygen (Scheele, 1772; Priestley, 1774–5), platinum (Brownrigg, 1750), strontium (as compounds, Crawford, 1790; the element by Davy, 1808), tellurium (Müller von Reichenstein, 1782), titanium (Gregor, 1789), tungsten (brothers d'Elhujar, 1783), uranium (Klaproth, 1789), zirconium (as compounds, Klaproth, 1789, the element by Berzelius, 1824).

The mathematical-quantitative method introduced into astronomy and physics by Newton, and applied by him and his followers in chemistry, had proved unfruitful in several lines of research in which it was too closely followed, and the chemical method, which had arisen from alchemy but was freed from mysticism and made rational by Van Helmont and Boyle, had led to very important discoveries in the seventeenth century. In the eighteenth century it proved very fruitful in the hands of such experimenters as Scheele and Priestley, who either made good use of qualitative and incorrect theories, or (in Priestley's case) had very little interest in theory of any kind. Lavoisier, a master of quantitative method, made no discoveries of chemical substances, and his main contribution to the science consisted in drawing correct conclusions from the discoveries of others, and reinforcing them by his own experiments.

Although some quantitative experiments on the composition of chemical compounds had been made in the seventeenth century, this branch of chemistry was greatly extended in the eighteenth century by the experiments of Cavendish, Wenzel (who failed to draw correct conclusions from his results), Bergman, Richter and Kirwan. In this way the laws of fixed, equivalent, and multiple proportions were perceived (the last by William Higgins). Richter's tables of equivalents were combined later into one by Fischer (1802). Bryan and William Higgins speculated on the combination of particles in a way which could have led them to the chemical atomic theory, but did not.

Although the phlogiston theory was abandoned after Lavoisier's work, it left the explanation of the evolution of heat and light in chemical changes still undecided, and several modifications of the theory were proposed in which the existence of phlogiston was assumed alongside that of oxygen. This problem still remained until the middle of the nineteenth century, when the theory of energy was accepted.

The eighteenth century was perhaps the most important one in the development of chemistry. It saw the introduction of a correct view of the chemical elements, a true explanation of the most important chemical changes, and the discovery of a large number of important substances. With this foundation, the science could develop during the nineteenth century on a plan laid down by the atomic theory of Dalton. The eighteenth century was the heroic age of chemistry.

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MATHEMATICS THROUGH THE EIGHTEENTH CENTURY

By J. F. SCOTT, Ph.D.

A. Developments in Pure Mathematics.

The close of the seventeenth century marks a turning point in the history of mathematics. Within a space of less than fifty years, two weapons of enormous power had been forged; these were the analytical geometry of Descartes and Fermat and the calculus of Newton and Leibniz. But these achievements, momentous though they were, represent but a fraction of the mathematical contributions of the seventeenth century. A workmanlike system of notation had been evolved, and side by side there had grown up an increased knowledge of the nature of equations. The important theory of probability had been foreshadowed by the investigations of Fermat and Pascal. The foundations of geometry had been critically examined, and a new type of geometry, the projective geometry of Desargues, had made its appearance. In mechanics the fundamental ideas on motion had been enunciated by Galileo and clarified by Newton, and with the formulation of the law of gravitation at the hands of the latter, the subject had begun to assume the appearance of a well-coordinated science. It is hardly too much to say that in regard to its mathematical development, no century starting from so little had achieved so much.

The eighteenth century proved a worthy heir to its mathematical heritage. Mathematical discovery continued to advance even with accelerated pace and in more diverse directions. But of the vast fields now thrown open to the mathematician, none proved so attractive as the new calculus, and for the first few decades mathematicians seemed to have little care beyond improving its technique and widening the scope of its application. The new instrument was wielded with ingenuity and with power and this led to a marked expansion of the mathematician's horizon. Within a few years such offshoots as the calculus of finite differences and the calculus of variations began to appear. Specialized branches as, for example, the study of differential equations, elliptic integrals and the theory of probability were established and prosecuted with vigour. Not the least important aspect of this awakening of mathematical interest was the application of the new methods to problems in mechanics, especially celestial mechanics. Interest in the geometrical methods of Descartes waned somewhat during the latter half of the seventeenth century; the calculus was thought to prove a richer mine, and it was not until the eighteenth century was well established that attention commensurate with their importance began to be directed to them.

In this mathematical renaissance many countries took part. The English school at first appeared vigorous and fruitful, but interest was short-lived and after the middle of the century Englishmen had little share in the developments which were taking place all over Europe. This was largely a result of an excessive reliance upon the geometrical and fluxional methods of Newton. The great German philosopher, Leibniz, had fashioned his calculus with less regard to rigour and strict logic, but he had framed it in a simpler and more workmanlike notation. Moreover, it had found able and zealous disciples in the two Bernoullis, James and John, to whose industry in popularizing the new methods on the continent the calculus owes much. The English school, unwilling to emancipate itself from the trammels of a clumsy notation, lagged behind, and it was not until a century had elapsed that it began to take its rightful place in the front rank. Nevertheless, its contributions were not to be despised. Names such as De Moivre, Cotes, Taylor, Stirling, Maclaurin, Simpson and others remind us of its virility before the decline set in.

But it is to the continent that we must turn for the most spectacular advances. France had taken the lead in the later years of the seventeenth century and she did not relinquish it until well on into the eighteenth. Varignon, L'Hôpital, Parent, Clairaut, D'Alembert were the pioneers; they were succeeded by names no less illustrious in the history of mathematics: Monge, Laplace, Legendre, Poncelet, La Croix, Cauchy. Italy's contributions were far from insignificant, and names such as Ceva, Fagnano, Agnesi, Ruffini bear eloquent testimony to the restless quest for mathematical adventure under southern skies. The greatest mathematician of the century, Lagrange, though of Italian birth, spent his best years outside his native land, and the same may be said of Euler, whose industry and genius have left a permanent impression upon every department of mathematical knowledge. Born at Bâle, Euler spent most of his time at St. Petersburg, and it was thence that he made contributions to almost every branch of mathematics. Bâle may also claim the Bernoullis; of this gifted family it was James and John who gave the greatest impulse to the study of the new methods and made them understandable.

Germany did not show her real strength until the century was well established, even though much of Lagrange's best work was compiled during his residence in Berlin. Gauss, the greatest of a brilliant line, was unknown until nearly the close of the century. It was the particular glory of Gauss that he impressed conceptions of rigour upon the new methods; prior to him even the greatest mathematicians had not paid sufficient attention to the foundations upon which they were erecting such an imposing structure.

A tendency to increased specialization was an inevitable consequence of the rapid widening of the mathematician's domain. We pass, therefore, to a brief account of the main branches which were cultivated during the eighteenth century. It will be convenient to group them under the following headings :—

(1) The extension and systematization of the calculus, and its application to physical problems.

(2) The establishment of the calculus of variations.

(3) The development of the theory of probability.

(4) Progress in algebra and related subjects.

(5) Progress in geometry and trigonometry.

The development of applied mathematics will be treated in a separate section.

1. *Extension and Systematization of the Calculus.*

The calculus as it left the hands of Newton and Leibniz was admired but imperfectly understood, and before any solid development could take place the new methods had to be popularized and made intelligible. In this task the two Bernoullis, James and John, combined indefatigable zeal with uncommon ingenuity. It was probably James who was the first to realize the enormous scope of the new calculus, and his persistent advocacy of the new methods is not the least of his services to mathematics. In a collection of essays published in 1691, he explained the principles of the new analysis in a form which could be understood. He applied the calculus to the solution of a number of physical problems and, by his investigation of isoperimetrical problems (1701), he laid the foundations of the calculus of variations. In 1698 he published an essay on the differential calculus and its application to geometry ; in this he investigated the properties of a number of curves, notably the equiangular spiral. His great work, ‘Ars Conjectandi’ (1713), published eight years after his death, contains much that is of the highest importance in mathematics. Though its main contribution is to the theory of probability, there is much in it that is still of value in other branches of mathematics. Hardly less influential in developing the new methods was his brother John, whose ‘Lectiones Mathematicae de Methodo Integralium’ (written 1691, published 1742), was a landmark in the development of the calculus. The quadrature of surfaces, the rectification of curves, the solution of different types of differential equations are all to be found in its pages ; the work, too, is replete with examples of different artifices in integration. He also attacked the problem of the vibration of strings. The exponential calculus has been claimed as one of his discoveries. Like his brother, he studied isoperimetrical curves. It was a pupil of John Bernoulli who provided the first treatise on the subject of the calculus. This was the ‘Analyse des Infiniment Petits’ (1696) of l’Hôpital, a work which did much to give currency to the notation and methods of the calculus.

The Bernoulli tradition was continued by Daniel, son of John, whose 'Exercitationes quaedam Mathematicae' (Venice, 1724) still further advanced the study of differential equations. His best known work, however, is his 'Hydrodynamica' (1738). In this he reduced all his results to a single principle, the conservation of energy.

In 1734 there appeared 'The Analyst: or, A Discourse addressed to an infidel Mathematician,' by Bishop Berkeley. This contained a devastating, yet able, attack upon the principles of the new methods. To this onslaught, Benjamin Robins replied with 'A Discourse concerning the Nature and Certainty of Sir Isaac Newton's Method of Fluxions' (1735), which was designed to shatter the objections raised by the learned bishop. A more vigorous refutation came from Maclaurin, who published in 1742 his 'Treatise of Fluxions,' the first systematic exposition of the method of fluxions to appear. This contains his well-known series. The work also contains many mathematico-physical problems, the solution of which Maclaurin effected by employing the Newtonian methods. Maclaurin also contributed a number of important papers to the 'Philosophical Transactions.' He was one of the most gifted men of the age, but his adherence to geometrical methods was, perhaps, unfortunate, inasmuch as it encouraged a certain indifference amongst his countrymen to the analytical methods which were being pursued with conspicuous success on the continent. Among other British mathematicians, one of the best known is Brook Taylor, a man of profound inventiveness. Like Maclaurin, he was an enthusiastic admirer of Newton. In his 'Methodus Incrementorum Directa et Inversa' (1715), he laid the foundations of the method of finite differences. He also solved a number of difficult problems in physics, notably on the transverse vibration of strings. Roger Cotes widened the resources of integration by some elegant applications of logarithms and circular arcs ('Logometria: A Treatise on Ratios,' Phil. Trans. xxix.). Stirling and the French born De Moivre extended the scope of the calculus by their use of series. The industrious Thomas Simpson made important contributions in his 'New Treatise of Fluxions' (1737). Finally, John Landen tried to remove the metaphysical difficulties inherent in the conception of fluxions by substituting purely algebraical methods. This was the main purpose of his 'Residual Analysis,' which appeared in 1764. Landen is best remembered by his 'Mathematical Lucubrations' (1755). After Landen, the mathematical genius of the English school slumbered. Writing towards the end of the century, Lalande ('Life of Condorcet') affirms that by 1764 there was not a single first-rate analyst in the whole of England (1).

The Italian mathematicians contributed materially to the development of the new methods. The Riccatis, father and son, and Manfredi, gave solutions of differential equations of the first degree which exhibit no little skill. Fagnano, in his 'Produzioni Matematiche' (1750), showed himself a mathematician of power. His attempts at the rectification of the ellipse and the hyperbola led him to the study of elliptic functions, and his paper 'Teorema da cui, si diduce una nuova misura degli Archi Elittici, Iperbolici et Cyclidali' (1716) marks the first contribution to this important subject. Fagnano's work proved an inspiration to Euler, who from 1733 occupied the chair of mathematics at St. Petersburg. Despite serious physical handicaps Euler was one of the most prolific writers of any age, and there was scarcely any branch of mathematics which his researches did not enrich. In addition to a vast number of memoirs and other works, Euler produced four monumental treatises in pure mathematics alone :—

(1) 'Introductio in Analysin Infinitorum' (1748),
(2) 'Institutiones Calculi Differentialis' (1755),
(3) 'Institutiones Calculi Integralis' (1768–70),
(4) 'Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes, sive Solutio Problematis Isoperimetrici' (1744).

The first of these was designed as an introduction to methods purely analytical. It deals with the theory of equations and the geometry of curves. The idea of a function is developed; logarithms are defined as exponents and many of the difficulties with regard to the logarithms of negative and imaginary numbers which had embarrassed a number of mathematicians were clarified. The second and third works constitute the most complete and systematic exposition of the calculus that had yet appeared. They contain Euler's investigation of differential equations of the second order and a treatment of the beta and gamma functions which were invented by him. Euler turned to elliptic integrals, but his researches thereon are not distinguished by any marked originality. This was reserved for Legendre. Following his preceptor, John Bernoulli, Euler directed his attention to the problem of isoperimetric curves and this led him to the calculus of variations. His contribution to this important subject formed the subject of the 'Methodus Inveniendi Lineas Curvas,' a work which no doubt Lagrange found very much to his taste. Euler also contributed a work on algebra ('Vollständige Anleitung zur Algebra,' 1770). This contains his proof of the binomial theorem, which, however, is not rigorous. In spite of this defect, Euler's work was of a very high standard and there were few branches of mathematics upon which he did not leave his mark.

As we have indicated, it was the French school which cultivated the new methods with the greatest assiduity in the early years of the century, and although many years were yet to elapse before we meet any name of the first rank, nevertheless there were many whose powerful advocacy of the new methods brought them into prominence. A contemporary of L'Hôpital, Pierre Varignon, assisted materially in this direction; so did Pierre Rémond de Montmort, who contributed to the theory of finite differences, and later to the theory of probability. These were followed by Clairaut, who showed unusual skill in solving differential equations. Clairaut turned aside from analytical to geometrical methods in his 'Théorie de la Figure de la Terre' (1743), nevertheless he continued to publish important mathematical papers dealing with maxima and minima and the calculus. Contemporary with Clairaut was D'Alembert, whose researches into mechanics and astronomy are his chief title to fame. In one of his later works he investigated the problem of vibrating strings, and was led to the differential equation of the form

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2},$$

the solution of which he gave in a memoir to the Berlin Academy (1747) as

$$u=\phi(x+t)+\psi(x-t),$$

ϕ and ψ being arbitrary functions. D'Alembert's main contributions however, are to applied mathematics and will be mentioned later.

In 1736 was born Joseph Louis Lagrange at Turin. By the time he was twenty he had established himself as one of the foremost mathematicians of even that illustrious age. Although Lagrange showed himself a master of almost every branch of analysis, it was a general solution of certain isoperimetrical problems which he communicated to Euler that first brought him into prominence. In the course of his demonstration, Lagrange developed the calculus of variations. This was followed by a series of memoirs in which he covered a wide range of subjects including partial differential equations, the theory of numbers, elliptic integrals. He also applied his calculus to the solution of a number of physical and astronomical problems, notably that of the moon's libration. Most of these contributions are to be found in the 'Miscellanea Taurinensia.' Lagrange deduced formulæ from purely analytical considerations and he endeavoured to erect a science on strict mathematical principles entirely divorced from ideas of infinitesimals or of vanishing ratios. This was the subject of his 'Théorie des Fonctions Analytiques' (1797, 2nd edn., 1813). Not the least important of his services to the rapidly expanding subject of mathematics was the inspiration he gave to his successors. His 'Leçons sur le Calcul des Fonctions' (1801, 1804, 1806), by the emphasis it placed upon rigorous methods, undoubtedly guided Cauchy in his 'Cours d'Analyse' (1821). But besides rigour, Lagrange impressed generality upon the subject. He made a careful study of the contributions of his predecessors and showed that all their artifices could be reduced to a uniform procedure. Before Lagrange, individual problems had been solved by the employment of a particular device. Each solution was something of a *tour de force*, and the discovery of the appropriate technique was usually regarded as no mean achievement. With rare insight Lagrange systematized the procedure and substituted for a series of attacks upon isolated problems a unified method of approach. His treatment of partial differential equations led to important solutions during the next fifty years of complicated problems in physics. In his monumental treatise, 'Mécanique Analytique'

(Paris, 1788), Lagrange established the principle of virtual work; from this, with the aid of the calculus of variations, he deduced all the theorems of mechanics and hydrodynamics.

Hardly less important in the development of eighteenth-century mathematics was Pierre Simon Laplace. His contributions to mathematics and mechanics, especially celestial mechanics, show him to have been a man of exceptional gifts. His 'Exposition du Système du Monde' appeared in 1796. In this work Laplace applied the calculus to astronomical problems with consummate skill. This was followed by the 'Mécanique Céleste' (1799–1825), which is usually considered his masterpiece. In addition, Laplace contributed many important memoirs on various branches of higher mathematics, such as the theory of equations, determinants, differential equations, and the solution of complicated physical problems. Much of his most enduring work (*e.g.*, 'Théorie Analytique des Probabilités, 1812') lies outside the period under review. The same is true of the great German mathematician, Karl Friedrich Gauss. But before the end of the century Gauss had already given evidence of his exceptional mathematical power by his 'Disquisitiones Arithmeticae,' much of which had already been made public in the decade preceding its appearance in 1801. Gauss's work proved an inspiration to his successors. As a result of his labours to establish the subject upon sound foundations, mathematics left his hands a very different instrument from what it was when he first directed his attention to it.

Before the close of the century there appeared the 'Traité Élémentaire du Calcul Différentiel et du Calcul Intégral' (1797 and 1819), of La Croix, a man who attained a position of repute through perseverance rather than unusual brilliance. He has no outstanding discovery to his name, nevertheless his works were deservedly popular. They were translated into English and in the hands of the Cambridge mathematical school they were instrumental in introducing the continental notation and methods into this country.

2. *The Establishment of the Calculus of Variations.*

The calculus of variations has proved one of the most fruitful outgrowths of the calculus. Though its development was stimulated by the rapid growth of mechanics during the eighteenth century, its origin is much older and can be traced in the investigation of the curve of quickest descent and the determination of the form of a hanging chain, problems which had exercised the ingenuity of Galileo and his contemporaries. The problem reappears in Leibniz in his method of differentiating a curve whose equation itself is presumed to undergo the minutest alteration.

Euler, who may be regarded as the creator of the subject, defines its scope thus: "The calculus of variations is the method of finding the variation undergone by an expression involving any number of variables, when the values of some or all of these are varied" (2). Thus the problem is reduced to making a given integral a maximum subject to a certain restriction.

As an illustration of the nature of the problem involved we may quote the one which brought the subject into prominence during the seventeenth and eighteenth centuries : "There being given two points, not in the same vertical plane, it is required to find the curve along which a body will slide from one to the other in the least possible time." This is the so-called *brachistochronic path*. Mathematically, the problem is similar to the familiar isoperimetrical problem, namely, the finding of the curve of given length which will enclose the maximum area.

In the development of the theory during the eighteenth century we recognize four stages, and these are associated with the names of the Bernoullis, James and John, Euler, Lagrange, and finally Legendre and his successors.

The foundations of a general theory were laid by James Bernoulli when he solved the problem of finding the curve of quickest descent, which had been proposed by his brother John in 1697. The latter had proposed six questions of considerable difficulty, *e. g.*, "of all the semi-ellipses described on a given horizontal axis, to determine the one which will be traversed by a body rolling along its concave surface in the shortest possible time." These appeared in the 'Journal des Sçavans' (Dec. 2nd, 1697). James' solution is to be found in the 'Acta Eruditorum' for June, 1700 (Jacobi Bernoulli : 'Solutio Propria Problematis Isoperimetrici'). This was followed in May, 1701, by another, Jacobi Bernoulli : 'Analysis Magni Problematis Isoperimetrici.' John had already discovered an elegant solution of the brachistochrone problem ; his brother's solution, however, covered a much wider range and was in fact a masterpiece of clarity and skill. The generality of his methods was recognized by Euler, whose attention had for some time been directed to the problem. It was by solving these that he was led to the calculus of variations. In his treatise of 1744 ('Methodus Inveniendi,' etc.), he displayed all his customary skill. But it is Lagrange who must be regarded as having placed the subject of firm foundations. In 1755 he sent Euler a solution of a certain isoperimetrical problem which had been engaging the latter's attention. Euler at once recognized the superiority of Lagrange's method and he did not hesitate to proclaim as much in a memoir, "Elementa Calculi Variationum" (Novi Commentarii Acad. S. Petropolitanae, Tom x). Within five years Lagrange had acquired complete mastery over the new method. It was he who greatly extended the subject and introduced the notation for variation whence the theory has taken its name. His method was strictly analytical and he employed differential equations freely. His methods and results are to be found in the second volume of the 'Miscellanea Taurinensia' (1760–2), and in his 'Théorie des Fonctions Analytiques' as well as his 'Leçons sur le Calcul des Fonctions.'

Euler in his treatise of 1744 had found the necessary condition for the case of one variable. Eleven years later Lagrange found it for two. But neither of these established criteria for distinguishing between maxima and minima. This was the contribution of Legendre, who in

1786 studied for the first time what is called the second variation of the quantity to be minimized. This was the purpose of his "Mémoire sur la Manière de distinguer les Maxima et les Minima" (Hist. de l'Acad. Royale des Sciences, 1786, pp. 7–37). Legendre's work was not free from blemish, nevertheless it inspired others, notably Gauss, Poisson, Cauchy, Jacobi, to remove the obscurities and to develop the subject further. After Jacobi, no further progress was made until Weierstrass directed his attention to it half a century later.

In mechanics the calculus of variations has played a significant part. Lagrange showed its power when he was able to establish by its aid the whole of mechanics on a single principle. It has also inspired later writers and has given rise to a new type of problem, namely that of Least Action, foreshadowed by Fermat in his optical researches and developed by Hamilton.

[See Todhunter: 'History of the Calculus of Variations.' London, 1861.]

3. *The Development of the Theory of Probability.*

It is customary to regard Fermat and Pascal as joint founders of the mathematical theory of probability. But attempts to forecast the results of games of chance are much older. Many mathematicians of the fifteenth and sixteenth centuries—Pacioli, Cardan, Tartaglia, Galileo—to name but few, had applied their skill to the problems involved. Nevertheless, it would be too much to say that the origin of the mathematical theory is discernible before the publication of the celebrated problem submitted by the Chevalier de Méré to Pascal and sent by him to Fermat:

"Two players of equal skill leave the table before the completion of the game. Given the total stakes and the score of each player the problem is to discover how the stakes should be divided."

The problem opened up a new world to Fermat and to Pascal, each of whom gave a solution. For his solution Fermat employed the method depending upon combinatorial analysis, *i. e.*, the finding of the number of ways in which a given operation can be performed, or in which an event, completely specified, can take place. Pascal, who arrived at the same solution as Fermat, relied upon his "arithmetical triangle." Pascal's methods were followed by Huygens, from whom the next contribution came; this was the 'De Ratiociniis in Ludo Aleae,' which Schooten inserted at the end of his 'Exercitationes Mathematicae' in 1658. In spite of the widespread interest which the subject had aroused, nothing further appeared for half a century, when Pierre Rémond de Montmort, a pupil of Malebranche, published a work on chances which contained some novelties ('Essai d'Analyse sur les Jeux de Hasard,' 1708). Meanwhile, James Bernoulli had made some trivial observations upon the subject in the 'Journal des Sçavans' (1685) and the 'Acta Eruditorum' (1690). These were elaborated and developed in the 'Ars Conjectandi' (1713), published posthumously. This may be regarded as the first serious contribution to the theory of the subject. The treatise is divided into four parts :—(i) A statement of the principles of the theory, together with a reprint of the problems of Huygens, (ii) The theory of the doctrine of permutations and combinations, (iii) The solution of a large number of problems relating to games of chance, (iv) Application to moral and economic questions. Despite the comparative novelty of the subject, Bernoulli succeeded in placing it on a sound mathematical foundation, and much that is in the 'Ars Conjectandi' is of value today. Nicolas Bernoulli was familiar with the above treatise and in 1709 he published his 'Specimina Artis Conjectandi ad Quaestiones Juris Applicata,' which is replete with references to his uncle's researches In this work Bernoulli attempted to apply mathematical calculations to various questions relating to the expectation of life.

The next mathematician to cultivate the subject with any degree of success was De Moivre, whose 'Doctrine of Chances, or a Method of Calculating the Probability of Events in Play' appeared in 1718. De Moivre had already communicated a paper to the Royal Society on the subject seven years earlier under the title 'De Mensura Sortis.' His interest in it seems to have been stimulated by a perusal of De Montmort's contribution to the same subject in 1708. In this treatise De Moivre first elaborated the theory of permutations and combinations; then he proceeded to discuss games of chance depending upon cards and upon dice; finally he gave solutions of a number of problems found in the treatise of Huygens. The 'Doctrine of Chances' contains seventy-four problems dealing with games of chance as well as a number of questions relating to life insurance. In his demonstrations De Moivre made free use of the method of recurring series (finite differences); by the application of this, he was able to solve a number of difficult and complicated problems. In a later work ('Miscellanea Analytica de Seriebus et Quadraturis,' 1730) he investigated problems on duration of play, of which he gave a general solution. By his employment of recurring series and his introduction of the normal distribution curve, De Moivre did more to establish the theory on a firm mathematical foundation than did any other writer with the sole exception of Laplace.

Meanwhile the Bernoulli tradition was being carried on by Daniel. His "Specimen Theoriae Novae de Mensura Sortis," which is to be found in the Commentarii Acad. Petrop. v., 1730–1 (pp. 175–192), is remarkable for its boldness and sagacity. In this work, Bernoulli applied his researches to life insurance and health statistics. He solved some of the problems proposed by Huygens and he gave examples of duration of play, a branch which was later to exercise the highest skill of Lagrange and Laplace. 'Disquisitiones Analyticae de Novo Problemate Conjecturali' (1769) was designed to illustrate the power and the scope of the differential calculus.

'The Nature and Laws of Chance' (1740), from the pen of Thomas Simpson, had some interesting observations on the subject. This is, however, little more than an abridgement of De Moivre's work and there is little in it that was new. Two years later, Simpson published a work on annuities (‘Doctrine of Annuities and Reversions,’ 1742). Another Englishman, the Rev. Thomas Bayes, turned his attention to the subject. He published in the Phil. Trans. for 1763 (pp. 370–418) a work on probability in which the notion of inverse probability is plainly discernible. Bayes' fundamental theorem may be stated thus: If an event has happened p times and failed q times, the probability that its chance at a single time lies between a and b is

$$\int_a^b x^p(1-x)^q\,dx \div \int_0^1 x^p(1-x)^q\,dx^{(3)}.$$

Other writers who contributed to the subject were the Swiss, Jean Trembley (‘Disquisitio Elementaris circa Calculum Probabilium,’ 1793), Edward Waring (‘Meditationes Algebraicae,’ 1770), and Condorcet (‘Essai sur l'Application de l'Analyse à la Probabilité,’ 1785). The contributions of D'Alembert are not without interest. More than once he had criticized the fundamental principles of the calculus; now he challenged the views of his contemporaries on the subject of probability. His ‘Doutes et Questions sur le Calcul des Probabilités,’ which is to be found in the ‘Mélanges de Philosophie,’ I (Paris, 1821), consists mainly of efforts to refute doctrines which were commonly accepted.

The versatile Euler turned his attention to the subject, (“Calcul de la Probabilité dans le Jeu de Rencontre”: Hist. de l'Acad. Berlin, 1753, pp. 255–270), but his contributions were of no permanent value. With Lagrange, however, it was different. The development of physical science had now reached such a stage that the application of the doctrine of probability became an important aspect of scientific method. Lagrange's contributions are to be found in the fifth volume of the ‘Miscellanea Taurinensia’ (1770–83, pp. 167–232). In his exposition he made free use of the calculus; he discussed theories of errors and he first showed how to obtain the best value from a series of non-concordant observations of the same physical quantity (‘Mémoire sur l'Utilité de la Méthode de prendre le Milieu entre les Résultats de plusieurs Observations . . . . par le Calcul des Probabilités’). But it was Laplace who made the most outstanding contributions. His masterpiece on this subject did not appear until a dozen years after the close of the period we are considering, but it is clear that the subject had been occupying his attention from about 1770. Some of the memoirs he contributed to the subject are still regarded as classical expositions of the principles. His treatment was purely analytical and he used partial differential equations consistently and extensively. His interest in the subject was stimulated by his researches on celestial mechanics. By his ‘Théorie Analytique des Probabilités’ (1812) Laplace ranks as the true founder of the theory of probability. Gauss and Legendre also made important contributions; the extensive application of their investigations is not to be detected until the nineteenth century was well established, when Poisson, De Morgan, Sylvester and others turned their attention to it. [See Todhunter: ‘History of the Mathematical Theory of Probability,’ London, 1865).

4. *Progress in Algebra and Related Subjects.*

In the latter half of the seventeenth century, and during the course of the eighteenth, algebra continued to advance. Although we can detect no outstanding change, it acquired greater flexibility and perfection in all its branches. The construction of cubic and biquadratic equations which Descartes had effected by combining a circle with a parabola (' La Géométrie, Book III,' 1637) was improved. By a skilful application of the binomial theorem Leibniz had succeeded in mastering the so-called irreducible case of Cardan. Euler solved biquadratics by a method which marked an improvement upon that employed by Descartes. His attempts to solve an equation of the fifth degree proved abortive (the impossibility of this was hinted at by Ruffini in 1803, and demonstrated by Abel in 1824). The outstanding developments in the method of solving equations came from Lagrange. He contributed two erudite papers to the Berlin Academy, (' Sur la Résolution des Équations Numériques,' 1767, and ' Reflexions sur la Résolution Algébrique des Équations,' 1770) ; these were followed by his ' Traité de la Résolution des Équations Numériques ' (1798–1808). In these Lagrange made extensive use of the method popularized by Descartes and later employed by Euler and Etienne Bezout ; this consists of finding a resolvent. Lagrange also made important contributions by considering the general conditions of solvability. Fourier also wrote on the solution of numerical equations and, although his results did not appear until 1831 (' Analyse des Équations Determinées '), the substance of this work had been made public as early as 1796 in the course of his lectures at the École Polytechnique. Euler's failure to solve the quintic led to the study of approximate methods of solution ; this branch was successfully prosecuted by Halley and by Taylor.

The nature of equations came in for renewed attention. The formulæ for the computation of the symmetric functions of the roots had been established by Vieta, Harriot, and Newton. A rigorous demonstration of Descartes' Rule of Signs was given by Segner in 1728 and again by de Gua in 1741. Euler added notably to the knowledge of equations by his ' Vollständige Anleitung zur Algebra ' (1770), and so did Waring, an able but obscure analyst, whose ' Meditationes Algebraicae ' appeared in 1770 and in 1782. Étienne Bezout's ' Théorie Générale des Équations Algébriques ' (1779) is a résumé of all that was then known on the subject.

Series were cultivated with vigour. Montmort, in 1714 and again in 1717, and James Bernoulli employed them freely in their investigation of the laws of chance. De Moivre carried these researches much further in the study of the calculus of probability. In 1731 he published an original work explaining the properties of recurring series, already suggested by Daniel Bernoulli in 1728, which he employed freely in his ' Doctrine of Chances.' About the same time, Stirling brought out a treatise on series (' Methodus Differentialis sive Tractatus de Summatione et Interpolatione Serierum Infinitarum,' 1730) in which he considered their convergence, summation and interpolation. Though it is difficult to detect any original plan in his writings, Stirling showed considerable dexterity in transforming one series into another. Euler's work on series led to the development of the integrals known by his name. Though he recognized the need for convergence, he had no satisfactory criterion for detecting it and his somewhat casual employment of infinite series occasionally led him into errors rarely found in a scholar of his sagacity. Waring's views on convergence (1757) were well in advance of his day. He was aware of the well-known test for convergence usually attributed to Cauchy. After Stirling the subject was developed by Simpson, Maclaurin, Landen ('Miscellanea Analytica,' 1762), and later by Pfaff and Krampf. Algebra received a further development by the application of combinatorial analysis, which may be regarded as an important extension of the binomial. Vieta, Mersenne, Schooten, Pascal, Wallis, James Bernoulli, De Moivre, Euler, had already contributed to this subject. The greatest expansion, however, came from Germany. Hindenburg published a tract in 1779 which threw much light upon the subject.

Many writers of the period busied themselves with the compilation of treatises on algebra and some of these reached a very high standard. Euler's 'Algebra' has already been mentioned. Though not free from blemish it was a work of considerable merit. Hardly less important were Maclaurin's 'Elements of Algebra' (1748), which, however, was left incomplete, and Clairaut's 'Élémens d'Algèbre' (1749 and 1760). Vandermonde contributed a work in which occurs the first systematic exposition of determinants. He also gave the theorem still known by his name. Simpson's 'Elementary Treatise on Algebra' (1745), Sanderson's 'Treatise on Algebra' (1740), Fantet de Lagny's 'Nouveaux Élémens d'Algèbre' (1727) are important as reflecting the interest which was now being devoted to the subject.

The subject of imaginaries began to attract increasing attention. They first appear in Cardan ('Ars Magna,' 1545) and again in Bombelli, Girard and Wallis. Leibniz developed the subject in his study of Cardan's rules when he showed $\surd(1+\sqrt{-3})+\sqrt{}(1-\sqrt{-3})$ to be equal to $\surd 6$. Newton's work on imaginaries is confined to his consideration of the number of roots of an equation. Fagnano gave an expression for the value of π which involved the use of imaginaries. Some attempt at graphic representation is faintly discernible in Wallis ('Algebra,' 1685). It was, however, Caspar Wessel who, in 'An Essay on the Analytical Representation of Direction' (1797), first gave the modern treatment. Argand extended the subject in 1806, but it was left for Gauss to establish the principles clearly and systematically. In his dissertation for the doctorate (1799) Gauss showed that every algebraic equation has a root of the form $(a+b\surd{-1})$. The introduction of i for $\surd{-1}$ is due to Euler. Cauchyu sed imaginaries freely; to him are due some of the improvements in notation which are still current.

D'Alembert (Mem. de l'Acad. Berlin, 1747) showed that imaginary quantities could be reduced to the form $a+b\sqrt{-1}$, a and b being real. De Moivre and Cotes had already detected the way in which $\sqrt{-1}$ provides a link between the logarithmic and trigonometric functions.

The theory of numbers, inseparably associated with the name of Fermat, was revived by Euler. The second volume of his 'Algebra' contains solutions of some of the problems proposed by Fermat, *e. g.*, every number whatsoever consists of not more than four squares. Waring ('Meditationes Algebraicae,' 1770) also has some contributions on the subject. But the most notable advances came from Lagrange. These are contained in the third volume of the 'Miscellanea Taurinensia.' Several of his earlier memoirs contain solutions of problems upon this subject, for which he appears to have had a peculiar gift. The subject was taken up by Gauss ('Disquisitiones Arithmeticae,' 1801) and later by Legendre, whose 'Théorie des Nombres' (1830) remains a standard work on the subject. This contains the first satisfactory proof of the theorem of quadratic reciprocity. Legendre treated the subject as a branch of algebra and he carried it as far as the current algebraical methods allowed. The researches of Gauss and Legendre provided a starting point for the investigations of Jacobi, Dirichlet, Riemann and others during the nineteenth century.

5. *Progress in Geometry and Trigonometry.*

As we have already noted, progress in the new geometrical methods was slow during the latter half of the seventeenth century. The first exposition of the science of coordinate geometry was Descartes' 'La Géométrie' (1637). In this work Descartes deliberately left many obscurities which made it unintelligible to many. By the time De Beaune and van Schooten had elucidated these the new calculus was well established, and for nearly a century mathematicians were too busy exploring the new treasures unfolded before their eyes to pay much heed to the seemingly less fruitful geometrical methods. Nevertheless the possibilities of the subject were recognized by a few writers later in the century; of these the most important were Jan de Witt ('Elementa Curvarum Linearum,' 1659) and La Hire ('Construction des Équations Analytiques,' 1679). Newton developed the subject still further in his "Enumeratio Linearum tertii Ordinis," 1668 ('Opuscula Newtoni,' I, pp. 235–270) and, by the time the 'Principia' appeared, the main principles were well established. Newton's contributions were still further advanced by Stirling, whose 'Lineae tertii Ordinis Neutonianae sive Illustratio Tractatus D. Neutoni de Enumeratione Linearum tertii Ordinis' appeared in 1717. With the publication of this work the new methods may be said to have come into their own.

Descartes, in his treatise, had restricted his exposition to curves of two dimensions. At the end of the second book of 'La Géométrie' is a hint—the only one in all his writings—of the possibility of extending the investigation to three dimensions. He did not pursue the matter, and it was not until the turn of the century that the hint was followed up. In 1700 (July 24th and Aug. 23rd) Antoine Parent submitted a paper to the Académie des Sciences (" Des Affections des Superficies ") in which Descartes' principles were extended to solids. This paper, which is included in Parent's ' Essais et Recherches de Mathématiques et Physiques ' (1713), proved an inspiration to Clairaut, whose ' Recherches sur les Courbes à Double Courbure ' appeared in 1731. Substantial contributions were also made by De Gua " Démonstration de la Règle de Descartes " (Hist. de l'Acad. des Sciences, 1741, pp. 72–96), but the next important development came from Euler in his ' Introductio in Analysin Infinitorum,' 1748. The second part of this work deals with analytical geometry both plane and solid, each being treated with complete mastery. Euler investigated tangents and tangent planes, normals, areas and volumes, and he gave an exhaustive account of curves defined by equations of the second degree. In the same year there appeared a work hardly less important. This was the ' Instituzioni Analitiche ad uso della Gioventù Italiana ' (1748) of Maria Agnesi. This was translated into French in 1755 and into English (' Analytical Institution,' Colson) in 1801. The ' Introduction à l'Analyse des Lignes Courbes algébriques ' (1750) of Gabriel Cramer was also a work of merit, and so was an earlier work, ' Nouvelles Pensées sur le Système de M. Descartes ' (1730), from the pen of John Bernoulli.

Meanwhile the projective geometry of Desargues had suffered an even more humiliating fate. Published in 1639, the ' Brouillon Proiect ' contains the fundamental theorems on involution, poles and polar, and the theory of perspective which Desargues employed to investigate the properties of conics. Desargues' methods seemed to have impressed Pascal, whose ' Essay pour les Coniques ' (1639) appeared when he was but sixteen. Despite this, however, nearly a century was to elapse before projective geometry again began to attract notice. The revival began with the publication of Maclaurin's ' Geometria Organica ' (1720), which contained an elegant investigation of the curves of the second order by regarding them as generated by the conditional intersection of lines and angles turning about fixed points or *poles*. But the real extension did not come until a group of French mathematicians, Monge, Carnot, Poncelet, turned their attention to the subject. Monge is usually regarded as the inventor of descriptive geometry. In his ' Géométrie Descriptive, Leçons données a l'École Normale 1794–95 ' (Paris, 1799) he elaborated the main principles. He developed the theory of perspective and he derived the properties of surfaces from perpendicular projections on to three surfaces. He also solved many important problems relating to the intersection of planes. Monge also contributed a number of papers on the same subject; many of these are embodied in his ' Application de l'Algèbre à la Géométrie ' (2nd edn., 1850). Carnot, in his ' Géométrie de Position ' (1803), opened up new prospects. He also revived and extended the work of Maclaurin under the title of transversals (‘Essai sur les Transversals,’ 1806). Poncelet turned aside from analysis and infused new vigour into purely geometrical methods in his ‘Traité des Propriétés Projectives des Figures’ (1822). Many of his results had already appeared in Crelle’s ‘Journal.’ Poncelet freely employed projection and reciprocation to deduce many important properties of figures.

In elementary geometry there had been little development since the Greeks. In the eighteenth century many writers published works founded on the ancient geometry. David Gregory brought out an edition of Euclid in 1703. Thomas Simpson’s ‘Elements of Plane Geometry’ appeared in 1747; this was followed by Robert Simson’s ‘Elements of Euclid’ in 1756, and later by Clairaut’s ‘Élémens de Géométrie’ in 1765. Euler gave a proof of the theorem giving the relation between the faces, vertices and edges of a polyhedron ($E+2=F+V$); this, however, occurs in a little-known work of Descartes. Of considerable importance was the fact that the foundations of the subject began to be critically studied and serious attempts were made to prove the famous parallel postulate of Euclid. Gauss maintained, and Beltrami subsequently demonstrated, that the postulate could not be proved. This led to the development of non-Euclidean geometries. The first contributions appear to have been made by Saccheri (‘Euclides ab omni naevo Vindicatus,’ 1733), a work from which Lobachevsky no doubt drew inspiration for his ‘Ueber die Principien der Geometrie’ in 1829. Legendre’s ‘Élémens de Géométrie’ (1794) was designed to replace the ‘Elements’ of Euclid.

We now turn to the developments in trigonometry. During the eighteenth century trigonometry was simplified and considerably extended. This was due in no small measure to the improvements in notation which had been made in the seventeenth. Oughtred (‘Trigonometrie,’ 1657), Caswell (whose contributions are to be found in Wallis’s ‘Algebra,’ 1685), Newton and Cotes all shared in this development. Newton gave formulæ for $\sin nx$ and $\cos nx$ and the Frenchman de Lagny did the same thing for $\tan nx$ and $\sec nx$. John Bernoulli treated trigonometry by analytical methods and his son Daniel did useful work in computing the trigonometrical functions. He was the first to employ the present notation for the inverse trigonometrical functions (1729). Euler (1798) gave the formula for $\tan^{-1} x$ which was an improvement upon that by James Gregory. But the main developments lay in the direction of the use of imaginaries in trigonometry. Cotes (Phil. Trans., 1714, and ‘Harmonia Mensurarum,’ 1722) showed that $ix=\log(\cos x+i\sin x)$. De Moivre established the theorem known by his name, viz., that $\cos nx+i\sin nx$ is one value of $(\cos x+i\sin x)^n$, n being rational. Euler in his ‘Introductio in Analysin Infinitorum’ (1748), established the relation $e^{ix}=\cos x+i\sin x$. The work also contains a number of formulæ in spherical triangles. Euler also introduced the notion of indicating the sides of a triangle by the letters a, b, c.

Lagrange, in 1761 gave a rigorous proof that π is irrational, so also did Legendre. In 1768 the Prussian, Johann Heinrich Lambert, published a paper in which he developed the results established by De Moivre. In this he introduced the hyperbolic sine and cosine, though he did not employ the symbols $\sinh x$, $\cosh x$ until three years later. Gauss, in a work on celestial mechanics (‘Theoria Motus Corporum Coelestium,’ 1809), obtained the formulæ now known as Gauss's Analogies, which are used in spherical trigonometry. Simpson (‘Trigonometry, Plane and Spherical,’ 1748), and later La Croix, wrote on trigonometry, but there is nothing original in their writings.

B. DEVELOPMENTS IN APPLIED MATHEMATICS.

The progress made in applied mathematics during the seventeenth century was in some respects even more spectacular than that in pure mathematics. Yet the two advances were intimately related. The increased attention to mechanics during the latter half of the century reacted powerfully upon the development of mathematics. It had been a period of great experimental activity and this had given rise to the appearance of fresh problems in statics and, more particularly, dynamics. But the dawn of the eighteenth century found men lacking the mathematical equipment needed to cope with the problems raised. To supply this need was the task to which men like Euler, D'Alembert, Lagrange, applied themselves. Not the least important aspect of their work lies in the elaboration of principles and methods by which many of the problems bequeathed by their predecessors could be successfully attacked.

During the eighteen hundred years which separated Archimedes and Stevin, progress in mechanics had been unaccountably slow. The elucidation of the fundamental problems of statics and dynamics attracted the attention of singularly few men of science, and even these seemed to be content with nothing more than prolix commentaries upon the writings of Aristotle. Some few indeed did turn to the subject; such are Nicolas of Cusa, Cardan, Tartaglia and others, but their writings reveal little originality. The researches of Stevin had opened up a new world and in his ‘Hypomnemata Mathematica’ (1608) he had established the important principle of the composition of forces. In the meantime, whilst Francis Bacon was philosophizing about the new experimental methods, Galileo was putting them into practice, and by his study of the motion of falling bodies had already challenged the omniscience of the great Stagyrite philosopher. The researches of Galileo had stimulated others—Castelli, Viviani, Baliani in Italy, and Roberval and Mersenne in France—to turn their attention to this important subject. Descartes, in an attempt to make his vortex theory a workable system (‘Principia Philosophiae,’ Part II, 1644), had devised a completely new system of mechanics. But it was founded upon an entirely erroneous conception of motion and it did nothing to advance the subject. Much more enduring were the contributions of John Wallis, whose ‘Mechanica sive De Motu, Tractatus Geometricus' (1670) did much to establish the subject upon firm foundations and thus blaze the trail for Newton. With the publication of the first two books of the 'Principia' it is not too much to say that Newton transformed mechanics into an entirely new science.

The investigation of the principles relating to the composition of forces and of the laws governing motion provided a convenient starting point for the development of mechanics during the eighteenth century. Pierre Varignon had studied the composition of forces about 1687 and had simplified the proofs of many of the theorems already established. John Bernoulli furthered the science of statics by proposing the principle of virtual work which Varignon had already applied in his 'Nouvelle Mécanique ou Statique' (1725). Bernoulli also gave greater generality to the important principle of the conservation of living forces (*vires vivae*) suggested earlier by Huygens. By noting that the same body can receive different gravitational accelerations, Bernoulli was first led to the distinction between mass and weight ('Meditatio de Natura Centri Oscillationis: Opera Omnia,' II, p. 168)⁽⁴⁾. James Bernoulli contributed to statics by first giving a demonstration of the form assumed by a hanging chain, a problem which had already been considered by Galileo. He also studied the behaviour of beams under various loads and he tried to establish a relation between deflection and load. Another of this gifted family, Daniel, gave one of the earliest demonstrations of the parallelogram of forces. He also discovered, simultaneously with Euler, the principle of the conservation of areas (1746).

An important contribution was made by Maclaurin ('Complete Treatise of Fluxions,' 1742). He employed three mutually perpendicular axes for the resolution of forces. This was seized upon by D'Alembert, who devised a means of reducing dynamical problems to problems in statics. Of the other great mathematicians of the period, Euler, Clairaut, Lagrange, Laplace, etc., their contributions were mainly to dynamics; nevertheless, they enriched the science of statics by considering special problems, as for example, elasticity, the vibration of strings, etc. In the investigation of the latter problem the names of Taylor, John and Daniel Bernoulli, D'Alembert and Euler are conspicuous. Euler also devoted some attention to the bending of beams, and although he left many things unexplained, his work forms an important link between that of Daniel Bernoulli and Coulomb, to whom much of the modern theory is due. Monge, better known for his geometrical researches, published a treatise on 'Statics' in 1786 which exhibits his usual thoroughness and skill.

Dynamics proved a much more fruitful source of enquiry. The law of falling bodies had been correctly enunciated by Baliani in his 'De Motu Naturali Gravium Solidorum et Liquidorum,' Prop. V (Genoa, 2nd Edn., 1646). Eight years earlier, Galileo's 'Discorsi e Dimostrazioni Matematiche' had initiated a complete reform in what had hitherto been a neglected branch of science. Nearly fifty years later Huygens had extended Galileo's investigation to motion in a circle. His pendulum experiments led James Bernoulli to attempt to explain the compound pendulum on the principle of the lever, but his observations were not free from error. In his papers to the 'Acta Eruditorum' (1691) and the Proc. Paris Acad. (1703), he appears to have been more successful. His brother John also attacked the question. More important, however, were the latter's investigation of the impact of bodies. Following Descartes' observations on the subject, in which he rarely stumbled across the truth, Wallis, Wren and Huygens had made contributions of value. In 1724 the Académie des Sciences proposed for their prize essay the laws of impact of bodies. John Bernoulli's solution is contained in a work 'Discours sur les Loix de la Communication du Mouvement,' which was written in the style of Leibniz. But it was a disciple and fellow-countryman of the Bernoullis, J. Hermann, who produced the first treatise on dynamics fashioned upon the analytical model. This was his 'Phoronomia' (1716). Despite a certain clarity and conciseness, the work lacks completeness and it is not distinguished by any great novelty of conception.

The task of compiling a work on dynamics, complete and original in all its branches, fell to Euler who directed his energies to the production of his 'Mechanica, sive Motus Scientia Analytice Exposita' (1736). Earlier contributions to mechanics from his pen are to be found in the Transactions of the St. Petersburg Academy. Euler considered that the new problems that were arising were beyond the scope of the geometrical methods favoured by Newton and his followers, and he tried to replace them by the more powerful analytical methods. He was not entirely successful. That was left for Lagrange. But by his exposition, Euler certainly advanced a long way in that direction. He also enriched dynamics with the equations still known by his name concerning the motion of a rigid body about a fixed axis. These are to be found in the Memoirs of the Royal Academy of Science, Berlin, 1718. It has been said, and not without reason, that the 'Mechanica' of Euler did for mechanics what 'La Géométrie' of Descartes did for geometry.

In 1743 there appeared D'Alembert's 'Traité de Dynamique,' an important landmark in the development of the subject. Like almost every other prominent mathematician of the period D'Alembert studied the pendulum, and his researches on the centre of oscillation led him to a series of simple observations which ultimately he generalized and embodied in the principle which goes by his name, viz., that the internal actions and reactions of any system of rigid bodies in motion balance. Following Bernoulli, D'Alembert showed that in any system of bodies acting mutually their several motions at any instant are decomposed into two portions, one of which is retained in the next instant and the other spent, and since an equilibrium must obtain among the lost motions an expression is hence derived for the motions that are preserved. By the application of this principle the most complex problems in dynamics are thus reducible to simple statical problems and solved by generalized methods. D'Alembert, by use of this principle, gave an elegant and complete solution in 1747 of the difficult problem of the precession of the equinoxes. He also attacked the *three bodies* problem. The application of his method was further extended by Maclaurin's use of coordinates to represent forces. Later (1744 and 1751), D'Alembert extended his principle to hydrodynamics.

About this time George Atwood studied the laws of falling bodies. In 1784 he published a description of Atwood's Machine in a treatise 'On the Rectilinear Motion and Rotation of Bodies.'

Clairaut, whose main contributions lie in his improvement of the mathematical technique, also devoted some attention to celestial mechanics. He investigated the form which a mass of fluid rotating about its centre of mass would assume. His 'Théorie de la Figure de la Terre' (1743) contains his formula for the acceleration due to gravity at different latitudes. In a later work, 'Théorie de la Lune' (1752), Clairaut abandoned analysis in favour of the geometrical methods of Newton and Maclaurin. Clairaut's work was taken up by Boscovitch, whose 'Theory of the Constitution of the Universe,' purged of its antiquated metaphysics, contains some contributions of importance. He postulated that atoms were nothing more than centres of attractive and repulsive forces acting at a distance. Another writer of this period whose works deserve notice is Segner, an original thinker who occupied the Chair of Mathematics at Göttingen. In 1755 he published a short essay on rotating bodies ('Specimina Theoriae Turbinum'), in which are contributions of importance to physical astronomy. It was Segner's invention of the water wheel which led Euler to the study of machines operated by running water.

The science of dynamics reached its highest point of perfection in the 'Mécanique Analytique' of Lagrange (1788). In this work Lagrange combined D'Alembert's principle with that of virtual velocities and thus converted the whole into an absolute analytical system of great power and flexibility. His method was to refer the efforts of every particle of a moving system to three mutually perpendicular axes and thence derive three separate differential equations which on integration would give the final solution of the problem. By applying the calculus of variations he showed how the whole of statics could be reduced to a single principle, that of virtual work. Although his main contributions are contained in the monumental work referred to above, there are numerous memoirs in the 'Miscellanea Taurinensia' (Vol. III), in which Lagrange illustrates the generality of his methods by a number of examples. Lagrange also made an attack on the celebrated *three bodies* problem; in this, the motion of three bodies each of which attracts the other two in accordance with the law of gravitation is investigated. For his 'Essai sur le Problème des trois Corps' Lagrange was awarded the prize offered by the Paris Academy in 1772. The same problem was attacked by Laplace, who succeeded in obtaining the general equations of motion. Laplace was interested primarily in celestial mechanics, which overshadowed all his other work. In his great treatise, ‘Mécanique Céleste’ (1799–1825), he determined the attraction of a spheroid on a particle outside it, an investigation which had already been foreshadowed by Legendre. Laplace introduced the idea of spherical harmonic analysis. The notion of potential, which was to play such a vital part in nineteenth century physical science, was also developed by him. His researches made possible an attack on physical problems which otherwise would have been unapproachable.

Projectiles.—Besides the general developments considered above, certain specific problems attracted the notice of many writers. One of the most important of these was the motion of projectiles. Galileo and others, neglecting the resistance of the air, had shown the path of a projectile to be a parabola. Galileo was never of the opinion that the resistance of the air played no part in the motion; he had no means of attacking the problem mathematically. The contributions of John Wallis are important if only because of the emphasis he placed upon an appeal to experiment. With his usual thoroughness he tried to place the fundamental notions on a sound basis. He was agreed that “the random of a bullet is very different from that of a parabola,” and he promised, in response to a request from Halley, to investigate the problem and communicate his results to the Royal Society. In this he was forestalled by Newton.

Newton declared that the resistance offered to a projectile moving through the air was proportional to the square of its speed. After Newton, John Bernoulli was the next to pay serious attention to the problem. Although he, too, lacked the necessary mathematical ability, he nevertheless, by a wise combination of calculation and experiment, obtained some conclusions of value. The subject was taken up by Benjamin Robins who, by his ‘New Principles of Gunnery’ (1742), added notably to the theory of the flight of projectiles. In the course of his researches he devised an ingenious method of determining the impulse of a shot by firing it against a heavy loaded pendulum, the deflection of which correctly indicated the momentum thus communicated. From his observations he was able to deduce the velocity of the shot at any stage of its flight. He concluded that Newton’s statement that the resistance was proportional to the square of the velocity was only true for low velocities. At higher velocities the resistance increased at a much faster rate. Robins’ ballistic pendulum was adapted by Hutton, Professor of Mathematics at the Royal Military Academy, Woolwich, in an elaborate series of experiments on the velocity of rifle bullets, in 1790 and 1791.

The investigation of the motion of projectiles was taken up by an able geometer, J. C. Borda, who tried to determine the nature of the curve described by a projectile moving through a resisting medium, and also to determine the elevation needed to produce the maximum range. His results are contained in the Hist. de l'Acad., 1769 (p. 121). He found that the elevation to produce the maximum range depended upon several factors, *e. g.*, weight of the projectile, its initial velocity, but that in no case was it 45°. A projectile of weight 140 pounds, with an initial velocity of 600 feet per second, was found to have a maximum range when the elevation was 37° 15′ ; a similar projectile with an initial velocity of 1100 feet per second produced a maximum range at 33° 20′. Borda's investigations on the motion of projectiles were continued by Templehof in 1781.

Euler, in an Addendum to his ' Methodus Inveniendi Lineas Curvas,' studied the question of the motion of projectiles (' De Motu Projectorum '). In this he considered the motion of a particle moving in a resisting medium. In his treatment Euler employed the calculus of variations, and his investigation led to an important principle which was further developed later in the century. This was the Principle of Least Action. The origin of this principle is to be found in antiquity when Euclid and others assumed the rectilinear propagation of light. Hero of Alexandria showed that in reflection at plane surfaces the path taken by a ray of light was a minimum. Nearly fifteen centuries later Fermat showed that a ray of light passing from a point in one medium to a point in another would so adjust its path that it described it in the least possible time. It was, however, Maupertuis who brought the subject of Least Action into prominence. He declared (' Recherches des Loix du Mouvement ' : Acad. Roy. des Sciences, Apl. 15, 1744) that whenever any change occurs in nature the quantity of action employed in this change is always the least possible, and he maintained that from this one fundamental principle all the laws of mechanics could be deduced. The principle as enunciated was not free from obscurities, and led to endless disputes as to what was involved in the term *action*, Maupertuis maintaining that it was the continued product of mass, velocity and space. Eventually these were clarified during the nineteenth century, mainly by Hamilton, and the theory has since played an increasingly important part in the development of physical science.

Side by side with the rapid advances which had been made in statics and dynamics is to be observed an awakened interest in hydrodynamics, or the science of the motion of fluids. The conditions of equilibrium of a fluid had been discovered by Archimedes but little further was contributed until Stevin traced the effects of fluid pressure and obtained results which were subsequently verified by Pascal (' Récit de la Grande Expérience de l'Équilibre des Liqueurs,' Paris, 1648). Pascal's treatise is probably the first in which the laws of hydrostatics are simply and clearly demonstrated. The genius of Galileo is reflected in two of his pupils, Castelli, whose ' Della Misura dell'Acque Correnti ' (Rome, 1628) created a new department of hydraulics, and Torricelli, whose ' Opera Geometrica ' (Firenza, 1644) is remarkable for some noteworthy contributions to the science of hydrostatics.

There were two problems in hydrostatics which occupied the attention of the mathematicians of the eighteenth century. These were, first, the resistance encountered by a solid moving through a fluid, and secondly, the efflux of liquids from an aperture in the base of the containing vessel.

The first of these appears to have been first undertaken by Torricelli in the treatise mentioned above. His work was mainly experimental. Newton undertook the examination of Torricelli's results in the second part of the 'Principia,' but he does not appear to have been entirely satisfied with his conclusions.

Daniel Bernoulli ('Exercitationes quaedam Mathematicae,' Padua, 1724), investigated the motion of fluids by the principle of *vis viva.* He studied the problem by a series of experiments in which he observed the flow of water in canals, the width and depth of which could be varied. Meanwhile, his father ('Joh: Bernoulli. Hydraulica nunc primum detecta et directe demonstrata ex principiis pure Mecanicis, anno 1732') made observations on the same problem.

Clairaut, in his treatise on the figure of the earth (1743), contributed to the subject; so did Euler and D'Alembert. The former, in a series of memoirs 1755 to 1772, reduced the subject to strictly analytical form. D'Alembert's contribution was his 'Traité de l'Équilibre et du Mouvement des Fluides' (1744). In 1750 the problem of the theory of the resistance of fluids was proposed by the Berlin Academy. To the questions raised, D'Alembert ('Essai d'une Nouvelle Théorie sur la Résistance des Fluides,' 1752) submitted a general solution in which he tried to found the subject of sound mathematical principles with the aid of analytical methods. His work was taken up some years later by Condorcet and by Bossut. The latter published the results of some observations on the flow of water in tubes ('Traité d'Hydrodynamique,' 1786). Condorcet made observations on the resistance encountered by vessels dragged through water by forces of known magnitude.

Meanwhile, Lagrange in 1781 examined the chief difficulties which surrounded the subject. Seven years later, in his 'Mécanique Analytique,' he still further simplified and generalized the subject. He united the two branches, the mechanics of solids and the mechanics of fluids, which had hitherto pursued separate courses, into one whole. In the first part of his treatise the problems relating to the equilibrium of bodies, fluids as well as solids are treated as branches of one and the same science and brought within the scope of his powerful analytical methods.

The investigation of the discharge of fluids had exercised the sagacity of Newton, who studied the flow of water through a small hole in the base of a cylinder. Twenty years later Varignon attacked the same problem. But a more complete investigation came from Daniel Bernoulli ('Hydrodynamica, seu de viribus et motibus Fluidorum Commentarii,' 1738). His work was largely experimental, and the principle of *vis viva* is freely employed. D'Alembert, in his treatise on the equilibrium and motion of fluids (1744), also studied the question.

But it was in Italy that the problem attracted the greatest notice. Michelotti, a distinguished Italian doctor, described his observations in a memoir: 'Sur la détermination des vitesses de l'eau, etc. (Mem. Torino vii.). In this he rejected the conclusions already reached by Newton. Others who contributed to the subject were Gabrielle Manfredi, who enriched with excellent notes the edition of Guglielmini's 'Della Natura dei Fiumi,' and Giovanni Poleni, Professor of Astronomy at Padua. His 'De Castellis per quae derivantur Fluviorum Aquae,' which was published in Padua in 1717, is especially important. To him also is due a new edition of Frontinus, entitled: 'Frontini, de aquae ductibus Urbis Romae Commentarius,' to which he added valuable notes.

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(2) Wolf, 'History of Science, Technology and Philosophy in 18th Century' (p. 52).
(3) Cajori, 'History of Mathematics' (1924), p. 230.
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ENGINEERING AND INVENTION IN THE EIGHTEENTH CENTURY

By ENGINEER CAPTAIN EDGAR C. SMITH, O.B.E., R.N.

I.

When the first issues of the Philosophical Magazine appeared, the effects of Tudor policy in fostering trade and manufactures had shown themselves plainly; Great Britain had become the foremost commercial nation in the world and the country was fast becoming a great workshop. Many enterprising and ingenious men, some of foreign origin—merchants, manufacturers, inventors and engineers—had contributed to the remarkable progress of the century and their work was to be seen in a hundred directions. The population, which at the time of Elizabeth had been about five million, had at least trebled, and our mines, mills, factories and workshops were active as never before. Communications and transport had been improved, commerce both at home and overseas had reached new levels, and both civil and mechanical engineering had become recognized professions. Such was the wealth created by the combined activities of the people that Lecky wrote that "the triumphant issue of the Great French War was largely, if not mainly due to the cotton mill and the steam engine. England might well place the statues of Watt and Arkwright by the side of Wellington and Nelson for had it not been for the wealth they created she could never have supported an expenditure which during the last ten years of the war averaged more than £84,000,000, nor could she have endured without bankruptcy a national debt which had risen in 1816 to £884,000,000." The national wealth might have been still greater had not so much been squandered on a foolish war, for as Benjamin Franklin wrote to Sir Joseph Banks in 1783, "I join with you most cordially in rejoicing at the return of peace . . . What vast additions to the convenience and comforts of living might mankind have acquired if the money spent in wars had been employed in works of public utility. What an extension of agriculture, even to the tops of our mountains, what rivers rendered navigable or joined by canals, what bridges, aqueducts, new roads, and other public works, edifices and improvements, rendering England a complete paradise, might have been obtained by spending those millions in doing good, which in the last war have been spent in doing mischief, in bringing misery into thousands of families and destroying the lives of many thousands of working people who might have performed the useful labour."

The history of any century can be viewed from many angles and the eighteenth century is no exception. The century of Newcomen, the Darbys, Brindley, Smeaton, Arkwright, Wedgwood, Boulton and Watt was certainly a century of material progress, but it is notable for numerous other reasons. It was a century of famous land and sea battles, and in between the victories of Marlborough and Nelson were the events of the Seven Years War, the campaigns of Clive and Hastings in India, and the struggle in North America. One notable feature of the time was the spread of colonization and the discoveries of explorers, among whom none stands higher than Captain James Cook. It was also a century of political and social progress. If corruption and rotten boroughs existed there was an antidote in the liberty of the press and persons. Side by side with widespread indifference, scepticism and cynicism there was a general toleration in politics and religion, likewise a revival of the latter and a genuine philanthropy. John Wesley's long and fruitful life covered three-quarters of the century, the end of which saw the self-sacrificing labours of John Howard for the reform of prisons and the work of Clarkson and Wilberforce for the abolition of the slave trade, although England had gained much wealth thereby. The callousness of the times was reflected in the terrible punishments, the crowds at executions, and the public exhibition of the unhappy inmates of Bethlehem Hospital. As late as 1774 the 'Morning Post' reported the burning at the stake of a woman. If a glance at society does not reveal the same great contrasts between the luxury and extravagance of the few and the hardships and misery of the many, as was to be seen in France, notorious evils and abuses were common. As for education, this was for the fortunate only, but even they were condemned to the study of dead languages. For the generality of children there was little but drudgery from early childhood. Of schoolmasters, Steele, in one of his essays wrote: "I have very often with much sorrow bewailed the misfortune of the children of Great Britain, when I consider the ignorance and undiscerning of the generality of schoolmasters . . . Many of these stupid tyrants exercise their cruelty without any manner of distinction of the capacities of children, or the intentions of parents in their behalf." Few, indeed, of the pioneers in engineering and industry during the eighteenth century owed much to their educators.

Yet withal it was a period of great literature. A century, the beginning of which could count an Addison, a Pope and a Swift, and at the end could boast of Wordsworth, Coleridge and Lamb, and which saw the writings of Fielding, Hume, Johnson, Gibbon, Bentham and Adam Smith, must be counted a great century. Then, too, newspapers and periodicals multiplied and literary and scientific societies came into being in Edinburgh, Dublin, Newcastle, Leeds and Manchester. The present Royal Society of Arts and Manufactures was founded in 1758, and Rumford's Royal Institution in the closing years of the century. Among the many periodicals published, of special interest here, was the Ladies' Diary, a forerunner of the Philosophical Magazine. It was both an almanac and a scientific journal and was contributed to by many men afterwards distinguished. Founded by John Tipper in 1704, on his death in 1713 it was edited in turn by Henry Beighton, Robert Heath, Thomas Simpson, Edward Rollinson and Charles Hutton. Beighton was one of the improvers of the steam engine while Simpson and Hutton both held successively the chair of mathematics at the Royal Military Academy, Woolwich. The Gentleman's Diary, first published in 1741, was amalgamated with the Ladies' Diary a hundred years later, and the combined magazine continued to appear till 1871.

Many travellers of the eighteenth century left accounts of their journeys through England and from their writings can be gained pictures of the conditions of our roads and water ways, the growth of our manufactures and the state of agriculture. One of these was Celia Fiennes (1662–1741), who between 1685 and 1703 rode on horseback through the English counties from Northumberland to Lands End; another was Daniel Defoe (1661–1731) who in 1722, not long after he had published 'Robinson Crusoe,' set out on the first of his seventeen "tedious and vastly expensive tours." Forty years later the young French metallurgist Gabriel Jars (1732–69) spent a year visiting British mines, ironworks, factories and the like, his reports on which provide the best view of industrial England at that time.

The object of Defoe was to see how "the fate of things gives a new face to things; produces changes in low life and innumerable incidents; plants and supplants families; raises and sinks towns; removes manufactures and trades; . . . great rivers and good harbours dry up and grow useless; again new ports are opened; brooks are made rivers; small rivers navigable pools and harbours are made where there were none before, and the like." Had Defoe been able to repeat his journeys in 1822 he would have seen plenty of examples of the fate of things giving a new face to things. His first tour took him through the eastern counties and he arrived at Cambridge when the now almost forgotten Stourbridge Fair was being held on what is today Stourbridge Common. Here indeed was a "Britain Can Make It" Exhibition. In Defoe's view the Fair was bigger than those at Leipzig, Frankfort, Nuremberg or Augsburg. Every possible kind of manufacture and ware was there and in abundance, from north, south, east and west. Fifty hackney carriages had been sent from London to ply between Cambridge and the Fair, and some of London's hundreds of wherries had been carried by waggon from the Thames to the Cam. On the river were the barges by which much of the merchandise had been conveyed from King's Lynn, to which place it had been shipped from many a river estuary. Our streams were the channels for the carriage of heavy and bulky goods and one of the great achievements of the eighteenth century was the improvement of inland water transport and of the terminal ports.

The leisurely and all embracing plans of Defoe had no place in the programme of Gabriel Jars. When the Seven Years War was over the French Government sent him to report on our industries, of which they had heard much. Though but half the age of Defoe he was admirably fitted for his task for he had passed through the École des Ponts et Chaussées, and had learnt a great deal about mining, having made several tours to the mines and ironworks of Saxony, Bohemia, Austria, the Tyrol, Styria and Carinthia. He arrived in England in July 1764, and stayed till September 1765, visiting collieries, blast furnaces, foundries, file and saw factories, lead mines, paper works, chemical works, salt works and porcelain factories, taking in his stride Carron, Edinburgh, Glasgow, Liverpool, Manchester, Sheffield, Birmingham, and so down to Cornwall. His six original reports are preserved in the French Archives Nationales, and these M. Jean Chevalier recently said " indicate all the most arresting features which British industry could show in 1765, to a French engineer who had become an extremely well-informed metallurgist as a result of his three years study of the mines and iron works of Central Europe."

It has often been pointed out that at the beginning of the eighteenth century England, in common with other countries of Europe, was largely an agricultural country. So she remained in spite of ironworks and cotton mills, but she exhibited at the end of the century an agriculture vastly improved through the efforts of Jethro Tull (1674–1741), Viscount Townshend (1674–1738), Robert Bakewell (1725–95), Coke of Holkham (1752–1842), and Arthur Young (1741–1820). Young was an unsuccessful farmer, it is true, but he was an indefatigable traveller, an acute observer, and a prolific writer. On the foundation of the Government sponsored Board of Agriculture in 1793, with Sir John Sinclair (1754–1835) as its president, Young became the secretary. With improvements in tillage, cropping, manuring and grazing, came the first of a long series of improvements in agricultural implements, and the invention by Andrew Meikle and others of that great labour-saver, the threshing machine. Agriculture, however, was one of the last industries to be affected much by the inventors and engineers of the eighteenth century, whose work it is the object of this article to review.

II.

Steam Engineering.

If ever there was an invention which satisfied an urgent need it was that of the steam engine as introduced by Thomas Newcomen (1663–1729) of Dartmouth, and improved by James Watt (1736–1819). It is true that Newcomen's was but an " atmospheric " engine in which steam played second fiddle to the pressure of the atmosphere, but it was nevertheless the first heat engine to be used on any considerable scale. Moreover, it was the first notable example of the application of scientific discovery to industry linking the work done in Italy, Germany, the Netherlands and France by Galileo, Torricelli, von Guericke, Huygens, Papin and others with the dewatering of British mines, and this as the result of the genius of an English tradesman.

In the early days of Newcomen power was obtained from horses, water and wind. There were water-wheels on nearly every stream, and windmills formed an arresting feature in the landscape of this and other countries, especially Holland. At one time 500 could be counted in Kent and 2,000 in the Fen districts. The millwright—the predecessor of the mechanical engineer of the present day—worked mainly in wood, but was skilled in the use of the forge and the lathe as well as the axe and the hammer. The windmill as applied to grinding corn or to pumping represented the acme of his art. When neither wind nor water was available, horses provided the power. Some collieries maintained scores of horses for working their pumps, and as late as the end of the eighteenth century the dry docks in the Royal Dockyards were emptied by means of chain pumps driven through horse gears. If it was desired to see the greatest example of millwrighting of the time of Newcomen, it was necessary to visit France and inspect the "Machine de Marli" which raised water from the Seine to the dry plateau 500 ft. above it, on which Versailles is built. This huge machine was constructed for Louis XIV by the Belgian mechanician Swalm Ranniquin, or Renkin, during the years 1675–82. It comprised 14 water-wheels, 253 separate pumps, and literally miles of iron transmission rods, the noise of which when working was "like that of a number of waggons loaded with bar iron running down hill with axles never greased." Like the Versailles palaces it was just one of the extravagances of the king, and was paid for by money wrung from his long-suffering subjects.

In this country it was the much more useful and prosaic object of draining mines that stimulated the inventors of pumping plant, and especially of pumps worked by steam power. Many were the individuals who applied themselves to the problem which was incompletely solved by Captain Thomas Savery (1650–1715) in 1698, and completely by Newcomen almost contemporaneously. Savery's ingenious apparatus was used on a very limited scale; Newcomen's engine, however, came into use in 1712, and atmospheric engines, not differing in principle but only in size, were working two hundred years later. Of Newcomen we know little, and still less of how he came to invent his engine, for neither sketch, plan, calculation nor model of his has survived. His engine included an overhead wooden beam or lever supported at the middle. From one end hung the heavy pump rods; from the other hung a piston, 2 ft. or so in diameter, which worked in an open-topped cylinder of brass. Beneath the cylinder was a boiler of copper such as brewers used fired at atmospheric pressure. When not in use the beam rested with the pump rods end down. The object of the engine was to raise the rods; when released their weight did the rest. Three operations were necessary to start and keep the engine in motion. Firstly, steam had to be admitted to the cylinder; secondly, a spray of cold water had to be injected to condense the steam and thus produce a partial vacuum; thirdly, the water of condensation had to be drained away.

On the formation of the partial vacuum, the pressure of the atmosphere forced the piston to the bottom of the cylinder and the pump rods were thereby raised. In its early form the necessary valves or cocks on the cylinder were probably worked by hand, but all extant illustrations show the engine to be fully automatic. Here was something entirely new to the world, an engine which would work wherever there was fuel to burn and water to boil. Thermodynamically the engine was extremely inefficient, but where coal was cheap this mattered little. The engine could exert more power than horses could be concentrated to do and that sufficed. Known generally as "fire engines" many were erected in Newcomen's lifetime, and by the time of Gabriel Jars' visit in 1764 it is known that about a hundred were installed, one of them with a cylinder of over 6 ft. diameter. Such engines had been introduced by English engineers into various European countries and just when Jars was setting out on his tours of inspection here, Ivan Ivanovitch Pulzunow, a Russian mechanic, was constructing a Newcomen engine at Barnaoul in the mining district of western Siberia. With the passage of time improvements were made in these fire engines; brass cylinders gave place to those of cast iron, and boilers were constructed of wrought iron. The year 1769 was memorable because Smeaton, as the result of experiments with a model, placed the design of such engines on a sound basis. The engines Smeaton was responsible for were among the largest and best of their kind. Though it would be incorrect to say that Newcomen's name had ever been forgotten, it was, however, not till 1921 that any memorial to him was erected. A year before this was unveiled at Dartmouth, his name had been given to the Newcomen Society for the Study of the History of Engineering and Technology, the 'Transactions' of which now form the most valuable corpus of information available to the student on those subjects.

Of James Watt, usually acclaimed as the inventor of the steam engine, but more correctly as its greatest improver, everyone has read. There are more books about him, more memorials to him, than to any other engineer. His correspondence, his sketches, his drawings, his models are all familiar. Of his upbringing and his environment, his ill-health, fits of despondency, want of self-confidence and his inability to deal with men, as well as his versatility and marvellous fertility of invention much has been written. Like his senior, Smeaton, he began his career as an instrument maker in London, during the year 1755–56. There was probably no lonelier youth in the city but assuredly no more assiduous pupil to the trade. Back in Glasgow he became instrument maker to that University, and as such enjoyed intercourse with mathematicians and philosophers Simson, Robison, Adam Smith and Black. The University owned a model of an atmospheric engine, but though it had been overhauled by one of the best instrument makers of the day it would not work. In 1763 it was put into Watt's hands. This proved to be the turning point in his career, and it is probable that for the next thirty years he seldom went to bed without thinking of steam engines. After prolonged thought, calculation and experiment, in 1765 came the flash of inspiration which led to the application to the engine of a separate condenser and air pump, the greatest single improvement ever made. A patent for this, and trials with a full-scale engine seemed out of the question. First Watt borrowed money from Black; then Roebuck, busy at the recently founded Carron Ironworks, took over Watt's debt to Black for a share of the patent which was secured in June 1769. In secrecy at Roebuck's residence at Kinneil an engine was constructed, but with indifferent success. Then Roebuck got into financial difficulties, but the buoyant businesslike Birmingham manufacturer, Matthew Boulton, came to Watt's rescue. To Birmingham went both Watt and the Kinneil engine; in 1775 success was achieved, an extension of the patent for twenty-five years was secured and the famous partnership of Boulton with Watt begun. Soon "Beelzebub," first of Watt's engines, was completed; in 1776 a Watt engine was supplied to John Wilkinson for working a blast furnace blowing engine at Broseley, and that year when Boswell visited Soho, Boulton, on showing him round, exclaimed, "I sell here, Sir, what all the world desires to have. Power!" Much less extravagant in coal consumption than the older "fire" engines, the Watt pumping engines were soon ousting the Newcomen engines, especially in Cornwall where coal was dear; hence by 1780 there were no fewer than twenty at work in that county. What the world needed now was an engine for turning mill work, for by 1781 Boulton was describing London and Manchester "steam mill mad." To the seventeen-seventies belongs the early development of the Watt pumping engine; to the seventeen-eighties the early development of the Watt rotative engine. Inventions flowed one after the other from Watt's prolific brain. Substitutes for the crank, methods of working steam expansively, double-acting cylinders, the parallel motion, the centrifugal governor, the throttle valve, indicators and gauges, etc., all belong to this period. The works at Soho became the centre of the mechanical world, and a great engine factory, Soho Foundry, came into being. This was not only the home of engine-making, but a training ground and a technical school for the rising generation of mechanical engineers. By the end of the century, when Watt retired, some five hundred engines had been constructed.

When Watt was so absorbed with the inventions of the seventeen-eighties there were many engineers experienced in the erection and operation of engines who thought themselves capable of adding their contributions to steam engineering. These men often found their path blocked by the unduly prolonged patent of 1769; yet they pushed ahead. One of these engineers was Jonathan Hornblower (1753–1815), who in 1781 secured a patent for an engine in which the steam was used successively in two cylinders, the forerunner of the later compound engine. Hornblower erected several engines in the West Country, but they proved in no way superior in economy to the Watt engine, and after litigation his clients had to pay royalties to Watt for using his condenser patent. Other men were thinking of the engine as a means for driving boats and vehicles, and others were dreaming of high-pressure engines. An uncle of Hornblower's, Josiah Hornblower (1729–1809), in 1753 had erected a Newcomen engine in New Jersey, North America, but nothing more was done there till just after the American War, when John Fitch (1743–98) and James Rumsey (1743–92) made experiments with steam boats, the one on the Delaware and the other on the Potomac River. There were also contemporary experiments in France by the Marquis de Jouffroy (1751–1832), and in 1788 the Scottish engineer, William Symington (1764–1831), in collaboration with the Edinburgh banker, Patrick Miller, constructed the marine steam engine preserved in the Science Museum, South Kensington. Fourteen years later Symington made a far superior engine for the tug *Charlotte Dundas*, which in 1802 towed barges on the Forth and Clyde Canal. No one has a better claim than he to be called the "father" of marine engineering. As early as 1769, the French military engineer Nicolas-Joseph Cugnot (1725–1804) had constructed the first of all mechanically driven vehicles, and in 1784 William Murdock (1754–1839), Boulton and Watt's most valued assistant, built his model steam carriage. Much of the post-Watt success of the Soho Foundary was due to Murdock. Through him it was the first factory in the world to be lighted by gas, and the first in which compressed air was used for the transmission of power.

By the time of Watt's retirement the stage was set and the actors ready for the next scene in the drama of early mechanical engineering, and the nineteenth century saw engine factories spring up on the banks of the Thames, the Clyde and the Tyne, in Liverpool, Leeds, Hayle and elsewhere. Some of the works then established exist today. If it were possible to carry this review further it would introduce the reader to the work, among others, of John Hall (1764–1836) of Dartford, Matthew Murray (1765–1826) of Leeds, Arthur Woolf (1776–1837) of Camborne, Richard Trevithick (1771–1833), by many regarded as the greatest mechanical genius of his age, William Fawcett (1771–1845) of Liverpool, and Henry Maudslay (1771–1831) of Lambeth.

III.

Iron and Steel Manufacture.

Though the introduction of steam power must ever render the eighteenth century memorable in the annals of material civilization, scarcely less far-reaching in their effects were the contemporary advances made in this country in the manufacture of iron for castings and forgings and of steel for tools, all so essential for the new mechanical engineering. The histories of industries are inextricably interwoven, the demands of one leading to advances in another and these in turn proving of benefit to still others and creating new ones. It was the needs of the collieries and mines which led to the introduction of the steam engine, and this by its capacity and adaptability assisted both miners and ironworkers. The iron industry in its turn made further demands on the miners and engineers, the result being that coal production, iron output and machinery manufacture increased simultaneously.

At the beginning of the eighteenth century when all iron was smelted with charcoal as fuel, and when the production of bar or wrought iron was almost the sole aim of the ironmaster, output was small and iron was expensive. By the end of the century the production of iron had multiplied between thirty and forty times, coal output had increased several-fold, and steam engines were installed at nearly all collieries and ironworks and in many factories. Moreover, the small furnaces and forges situated in the well-wooded counties of England and Wales had been superseded by large establishments situated on the coal fields of the north of England, the Midlands, Scotland, and South Wales.

Outstanding among the many individuals who played their part in this great transformation were the three Darbys, Abraham I (1677–1717), Abraham II (1711–63), and Abraham III (1750–91), whose work was done in Shropshire; the three Wilkinsons, Isaac (d. 1784), John (1728–1808) and William (1738–1808), whose principal blast furnaces and foundries were in Denbighshire, Staffordshire, and Shropshire; Dr. John Roebuck (1718–94), the founder of the Carron ironworks, Stirlingshire; Henry Cort (1740–1800), whose experiments which revolutionized the production of wrought iron were made in Hampshire; and Benjamin Huntsman (1704–76), the founder of the crucible cast steel industry of Sheffield. It must not be overlooked, of course, that other countries contributed to the advancement of the iron and steel industries, or that during the whole century the best wrought iron came from Sweden, always to the fore in mining and metallurgy, or from Russia where Peter the Great early in the century had founded a mining academy and a mining school on the model of that in Saxony.

An ironworks at the beginning of the century consisted perhaps of one small blast furnace for the production of pig iron and a forge with a finery or chafery in which the pig iron was converted into bar iron. These works were situated in isolated spots where iron ore, timber and water power were available. One of such places was Coalbrookdale, some miles south of Shrewsbury, the stream down the valley of which falls into the river Severn. It is " a very romantic spot, it is in a winding glen between two immense hills, which break into various forms, and all thickly covered with wood, forming the most beautiful sheets of hanging wood. Indeed too beautiful to be much in unison with that variety of horrors art has spread at the bottom; the noise of the forges, mills etc. with all their vast machinery, the flames bursting from the furnaces with the burning of coal, and the smoak of the lime kilns, are altogether sublime . . . " So wrote Arthur Young in 1776. To this romantic spot sixty-eight years earlier had come the Quaker brassfounder of Bristol, Abraham Darby, to lease one small furnace there and to start his patented process for making bellied pots in cast iron which should be cheaper than brass pots, and the material for which he succeeded in producing by smelting iron with coke. Unfortunately death overtook him in 1717, when he was only forty years of age and his son but six. The business was then formed into a company and managed for the time by his widow and her son-in-law Richard Ford (1717–45). On coming of age Abraham Darby II. became the head, and under him the works flourished. Smelting with coke was perfected and new furnaces and forges were set up at Horsehay and Ketley to the north of Coalbrookdale. Already in 1724 the firm was using cast iron largely in the production of castings of all kinds such as "cillinders," "pisterns," pump barrels and pipes for "fire engines," and about 1740 a "fire engine" was installed for pumping back water to drive the water-wheel when the stream was low. In 1763 death again intervened, Abraham Darby II. dying at the age of fifty-three, and this time his son-in-law, Richard Reynolds (1735–1816) assumed charge. To him was due the introduction about 1767 of the use of cast iron plates on the wearing surfaces of the many miles of wooden tramways connecting the various works. In 1766 Reynolds assisted two of the leading men of the works, Thomas and Robert Cranage, to secure a patent for "making pig-iron or cast-iron malleable in a reverberatory furnace with pit-coal only." The invention was not brought to a successful issue or it would have anticipated Cort's invention of eighteen years later.

The progressive spirit of the third Abraham Darby, who became the head of the business in 1768, was shown in many ways, and by 1785 the company besides working many iron mines, limestone and coal pits, had ten blast furnaces, nine forges and sixteen steam engines. His most remarkable undertaking was the construction of the cast iron bridge over the Severn between Madeley and Broseley, which still stands and has been scheduled as an ancient monument. Its semi-circular arch has a span of 100 ft., the main members being five ribs with a cross-section of 9 in. by 6 in., each rib being cast in two lengths. The weight of the whole structure is 378 tons. It was by far the largest and finest cast iron structure the world had ever seen, and paved the way for other notable bridges of the same material. Coalbrookdale thus will always be remembered as the birthplace of the use of coke for smelting, the cast iron cylinder, the iron tramroad and the cast iron bridge.

It was but natural that the new method of smelting with coke should gain acceptance first in the Midlands, and it was in this district John Wilkinson became famous. His reputation was perhaps partly due to his association with Abraham Darby III. on the one hand, and with Boulton and Watt on the other. He was, however, the architect of his own fortunes. His father Isaac is first heard of in Cumberland farming and charcoal iron-making. In 1738 when John was ten, the family removed to North Lancashire where Isaac started iron-founding. The same year he took out the first of his four foundry patents, that for a box smoothing iron. In 1753 he migrated again, this time to Bersham, near Wrexham, Denbighshire, John having meanwhile established himself as an ironmaster at the Bradley furnace, near Bilston, Staffordshire, adopting the methods of the Darbys. After being assisted for some years at Bersham by John, Isaac removed to Bristol, and about 1762 John, then thirty-four, with his brother William, founded the New Bersham Company, the partnership lasting till 1795 when the works were closed. To add to his business John also founded works at Broseley on the Severn, not far from Coalbrookdale. In 1774, the memorable year Watt removed to Birmingham, he took out a patent for an improved boring machine, the accuracy of which appealed so greatly to Boulton and Watt that from 1775 onwards for the next twenty years he had the monopoly of casting and boring cylinders for the firm. In 1776 he obtained from Soho an engine for driving a blowing cylinder for his blast furnaces, thus obtaining the higher air pressure which removed the final objection to smelting with coke. Ever ready to stimulate the iron trade, he assisted Darby in the erection of the cast iron bridge already mentioned, and in 1787 he launched on the Severn the first of several barges or lighters having a sheet iron skin, and thus became the father of the iron ship. His brother William was equally busy though in other ways. After obtaining contracts for the cast iron pipes for the water supply of Paris he established the ironworks at Le Creusot, where he installed blast furnaces, kilns, a foundry, a cannon boring mill, and introduced the use of both cast iron rails and the cupola. He afterwards helped to re-organize the iron industry of Silesia. There was little in common in the outlook of the Wilkinsons and the Darbys, save in their views on the importance of iron, for John certainly possessed nothing of the Quaker spirit.

It is some measure of the influence of the pioneering work of the Darbys that Dr. John Roebuck and his partners, who founded the Carron ironworks, near Falkirk, Scotland, should from the first have used coke and not charcoal. The works, laid out entirely for the production of cast iron, were built in 1759, and on January 1, 1760, with ceremony befitting the launching of a ship, the first furnace was blown in and the casting of pig iron begun. Roebuck, then forty-two years of age, had qualified at Leyden as a physician, but becoming interested in industrial chemistry, about 1749 had set up as a sulphuric acid manufacturer at Prestonpans, using lead chambers instead of glass bells. From chemicals he turned to iron, and by his work at Carron introduced the large scale making of iron into Scotland. When Gabriel Jars visited the Carron works he found there two blast furnaces 30 feet high yielding about two tons of iron every twelve hours, the iron being of excellent quality. Here as elsewhere a difficulty was the insufficiency of the blast and Jars noted that a bellows of prodigious size, 22 ft. in length and incorporating pieces of oak 10 in. thick, was being made. Some years later Smeaton designed a blowing engine for the works. The manufacture of castings of all kinds was undertaken, and as time went on the firm became famous for fire grates, railings, balustrades, urns, panels, plaques, etc., many of which were decorated with designs by architects and artists. From Carron too came the naval "carronade" sometimes known as "the smasher," and early in the nineteenth century the company made experiments with and supplied large numbers of the spherical case shots invented by Colonel Henry Shrapnel.

All this progress in the making of cast iron had little effect on the production of the sister material, wrought iron, and the country remained largely dependent on Sweden and Russia for the best material for anchors, chains, tyres, boiler plates, and smiths' work generally. This situation, however, was completely changed through the work, not of an ironmaster, but of the Navy agent Henry Cort. Through marriage he had become related to one Attwick who had a contract for supplying Portsmouth Dockyard with iron naval stores. In 1772 the contract was assigned to a Mr. Morgan who had a forge at Fontley, two miles from Fareham, Hampshire. The work done there was the reforging of old scrap iron from the dockyard. To Morgan Cort advanced money, but the former's business languishing he took over control, and it was at Fontley he made the experiments which led to his two notable patents of 1783 and 1784, describing (1) the use of rolls with grooves and collars for the production of bar iron, and (2) a method of producing wrought from cast iron in a reverberatory furnace, heated with coal, a process subsequently known as "dry puddling." Cort's inventiveness consisted in the combination of existing techniques to establish new ones, and his methods brought about a revolution in the wrought iron trade comparable to the earlier one in the cast iron trade, and except for iron of special qualities the country became independent of Sweden. The new technique spread rapidly and Cort should have realized a fortune as great as that of John Wilkinson, had it not been that through the death of his partner Adam Jellicoe, who had, it turned out, defrauded the Navy of a large sum, Cort lost everything and he died poor. Before his methods were superseded in the nineteenth century by the inventions of Bessemer, Siemens, Martin and Thomas there were more than eight thousand puddling furnaces in the country.

Into the history and working of blast furnaces, fineries, chaferies, air furnaces, reverberatory furnaces and cupolas, all in use in the eighteenth century, there is no need to enter. The iron industry from primitive times to the nineteenth century is the subject of many scholarly and critical papers by Jenkins, Ashton, Hulme, Hall, Dickinson and others in the 'Transactions' of the Newcomen Society. Neither is it intended to trace the development of the process by which the pure wrought iron of Sweden is converted by cementation into blister steel, from which by "faggoting" is produced the shear steel used for many of the tools sold by ironmongers. These processes were carried on by many manufacturers in or near Sheffield at the beginning of the eighteenth century, and Jars found each of them "ceaselessly occupied in inventing new means of cutting down labour costs and thereby increasing his profit."

Had Jars by chance gained admittance to a small works at Handsworth close by, he would have found Benjamin Huntsman, behind closed doors, trying, not to reduce costs, but to make a steel superior to any Sheffield could produce. A Lincolnshire man by birth, and a Quaker by persuasion, as a watchmaker at Doncaster he was dissatisfied with the steel available for springs, and experimented with a view to finding something better. In 1740 he removed to Handsworth, and it was there he found what he sought by smelting piece of shear steel in crucibles heated by burning coke and casting it in the form of ingots. It took some time to overcome the conservatism of steel users, but by the end of the century Sheffield was producing the finest razors and pocket knives in the world. The works Huntsman founded at Handsworth in 1751 were moved in 1770 to Attercliffe to the north of Sheffield, where the firm of B. Huntsman, Ltd. still makes steel by the founder's method.

IV.

Textile and Other Industries.

The tide of invention which set in with the eighteenth century affected all trades and occupations, as can be seen by the patent records, which show that the number of patents issued in the four successive quarters of the century were 116, 176, 458 and 1349. These included master patents such as those of Watt and Arkwright taken out in 1769, but also minor patents such as that for making blackcurrant lozenges for curing sore throats. But patents cost over £100, and many improvements are not found recorded in any Patent Office. The increasing interest in invention was due to many causes, one of which was the work of the Royal Society. As early as 1663 it was laid down that the business of the Society included the improvement of all useful "Arts, Manufactures, Mechanick practices, Engines and Inventions by Experiments." Gunnery, shipping, the diving bell, Papin's digester and Savery's pump aroused almost as much interest as did air pumps, telescopes and microscopes. In a pamphlet issued in 1718 during the presidency of Newton, giving a list of subjects suitable for contributions to the Society, was an item referring to "New Inventions or Improvements in Mechanicks with descriptions of Machines, Engines, Instruments and the like . . ." In 1738 the Society awarded the Copley Medal to James Vauloue for his pile-driver used in the construction of old Westminster bridge. In 1746 it was given to Benjamin Robins for his experiments on projectiles and the invention of the ballistic pendulum. Three years later John Harrison received it for his work on chronometers, and in 1759 it was awarded to Smeaton for his "Curious experiments concerning Water-wheels and Windmill-Sails." No other engineer received the Copley Medal till it was awarded to Sir Charles Parsons one hundred and sixty-nine years later. What the Royal Society did in the first half of the eighteenth century was followed up by the Society of Arts, ever ready to encourage useful invention. The work of the British clock and instrument makers in the eighteenth century was outstanding, and the names of Tompion, Graham, Harrison, Mudge, Ramsden and Troughton are familiar to all. There are no finer examples of their art than the chronometers of Harrison and the dividing engine and theodolites of Ramsden, some of whose practices were adopted by mechanical engineers. Contemporary with their work came the invention of stereotyping by Ged and its development later by Alexander Tilloch, the founder of the Philosophical Magazine, the invention of the compound-lever weighing machine by John Wyatt, of the hydraulic ram by Whitehurst, and of the hydraulic press by Bramah. The circular saw of Miller, the screw-cutting lathe of Maudslay, and the woodworking machines of Marc Isambard Brunel and Sir Samuel Bentham, like the boring machines of Smeaton and Wilkinson, all belong to the eighteenth century. Invention appealed to men of all degrees. The ingenious Irish landlord Richard Lovell Edgeworth contrived and worked a telegraph, devised a track-laying vehicle, a dynamometer for ploughs, and wrote on roads and carriages. His contemporary, Charles, third Earl of Stanhope, was equally versatile, making experiments with fireproof buildings, lime kilns, calculating machines and steam boats, and inventing the Stanhope printing press. Perhaps in this list of benefactors of the arts should be included the pioneers of our ceramic industry, of whom Josiah Wedgwood was recognized not only as a great master potter, an improver of his art, but also as a promoter of canal construction, and " a pioneer and populariser of Watt's steam engine in its application to industry installing an engine at Etruria as early as 1782."

No branch of manufacture benefitted more from invention than the textile industry, which in a hundred years was transformed from a purely domestic industry carried on in homes in towns and country by means of the centuries-old distaff, spinning wheel and hand-loom into a large-scale factory industry with machines driven by water or steam power. Defoe, in his account of the Stourbridge Fair, wrote of the prodigious trade in woollen goods which came to Cambridgeshire from places as far apart as Norwich, Exeter and Halifax, and of the display of cloths, kerseys, pennistons and fustians from Lancashire and Yorkshire. A good deal of technical history centres on these fustians, goods made with warp of flax and weft of cotton. The advent of the new fibre, cotton, had a marked effect on the textile trade and its spinning and weaving had their home in Lancashire, this being partly due to the humid climate of the district. The manufacture of cotton goods met with considerable opposition from the woollen and silk trades, and early in the eighteenth century legislation was passed prohibiting the living from wearing certain cotton goods and the dead from being buried in them. Fustians, however, were in a different category, and by the Manchester Act of 1735 it was definitely laid down that they were exempt from the restriction. In the first Manchester directory, issued in 1772, the names of no fewer than 132 fustian manufacturers were given. Bolton was another centre of the fustian industry. It was easier to spin cotton yarn suitable for weft than for warp, but by the inventions of Hargreaves, Arkwright and Crompton all difficulties were removed.

Looking back it is not a little surprising that spinning by machines came so late, for as early as 1589 the Rev. William Lee (d. 1610) had invented the stocking frame which could do the work of a dozen hand knitters, while the Italians had long possessed machines for doubling and twisting silk. Copies of their machines were to be found in the silk mill of Sir Thomas Lombe (1683–1739), erected in 1718 at Derby. This factory employed 300 workpeople, and its machines for drawing, spinning and twisted silk included 97,746 wheels, movements and individual parts all driven by a water-wheel. The doubling and twisting of silk filaments, however, is fundamentally different from the drawing out and building up of cotton fibres into a yarn, yet this exemplification of factory production preceded by half a century the cotton spinning factory of Arkwright.

Lombe's patent, granted in 1718, expired in 1732, when the silk manufacture was thrown open to all, with benefit to the trade, and it was at this juncture of affairs that the weaving industry received its first notable improvement by the invention in 1733 of the "fly shuttle" for looms, an invention due to the unfortunate John Kay, of Bury, not to be confused with the John Kay of Warrington who afterwards assisted Arkwright. From thence onward invention followed invention and by the end of the eighteenth century the revolution in the technique of the cotton industry had been accomplished. It is impossible in a short space to give an account of the progress in detail, but it may be of interest to set down in chronological order the names of the principal inventors and what improvements they introduced:—

1733. John Kay, of Bury, patented the "fly shuttle" for looms, more than doubling the speed of weaving.

1738. Lewis Paul (d. 1759) patented a machine for spinning wool incorporating principle of drafting by rollers, the technique of which was successfully achieved by Arkwright in his "water frame." Paul was assisted by John Wyatt (1700–66).

1748. Paul patented a carding engine, the technique of which was likewise adopted by Arkwright.

1758. Jedediah Strutt (1726–97) of Derby patented ribbed stocking frame.

1760. Robert Kay, son of John Kay, patented "drop box" for looms, enabling two or more shuttles to be used successively on the same fabric.

1764. James Hargreaves (1720 ?–1778) of Blackburn invented hand-worked "spinning jenny," an intermittent multiple drafter, twister and winder on. Not patented till 1770.

1769. Richard Arkwright (1732–92) of Preston, with assistance of John Kay of Warrington, patented his "water frame" incorporating improved rollers for drafting, and adapting water power to drive it; the first power driven spinning machine.

1774–79. Samuel Crompton (1753–1827) of Bolton perfected hand-driven multiple-spindle spinning " mule " capable of spinning the finest counts of yarn. Not patented.

1785. Rev. Edmund Cartwright (1743–1823) of Goadby Marwood, Leicestershire, patented power driven loom. Other patents secured in 1786, 1787 and 1792.

1793. Eli Whitney (1765–1825) in Georgia, U.S.A., invented hand driven " cotton gin " for removing seeds from the cotton bolls. Patented in 1794.

In this list may be added the French inventor Joseph-Marie Jacquard (1752–1834) who, by bringing together the inventions of Falcon, Bouchon and Vaucanson perfected the loom that goes by his name, capable of weaving figured fabrics of the utmost complexity.

Of the great difficulties under which these men laboured, of the opposition they met with from operatives who believed their livelihood to be threatened, and of their unequal rewards much has been written. Of them all Arkwright, who was knighted in 1786, was the only one to realize a fortune. The first power driven cotton spinning factory was that he set up in 1771 at Cromford, on the river Derwent in Derbyshire, with the assistance of Jedediah Strutt. Arkwright erected others at Belper, and Chorley, Lancashire, and entered into partnership with the philanthropic industrialist David Dale (1739–1806) in establishing a mill at New Lanark, near the Falls of Clyde.

The cumulative result of the inventions and enterprises referred to was to place Great Britain far ahead of any other in the manufacture of cotton goods and to establish at the hands of mechanical engineers a flourishing textile machinery industry. In the story of trade there is no more remarkable chapter than that which traces how a natural fibre grown in hot climates has been brought year after year thousands of miles by sea to the damp valleys of an English county and then after manufacture taken back those thousands of miles for the use of the peoples who have grown the cotton plant. Engineers, however, today manufacture and export air-conditioning plant which enables cotton mills to be operated anywhere.

V.

Progress in Civil Engineering.

A review of the civil engineering in the eighteenth century reveals the same remarkable progress as in industry, the constructors endeavouring to meet the ever-growing demands of the day. There was ample scope for improvement, for the state of the roads, waterways, bridges and harbours of England was far below the standard necessary for rapid communication and cheap transport. The Normans had built great castles, the Plantagenets magnificent cathedrals and the Tudors stately mansions, but all undertakings, classed today as public works, were almost completely neglected. The inhabitants of London had been content for five centuries with the single inadequate bridge over the Thames constructed in 1209, and when in 1710 Wren's son placed the last stone on the summit of the lantern of St. Paul's Cathedral the streets of London were still undrained. Thirty-five years later when the House of Commons discussed a bill for the better paving of Westminster it was said that the streets were a disgrace to the nation and abounded in such filth as a savage would look at with amazement. The country roads were no better than the streets and every traveller, whether he went on foot or horseback, by waggon, post-chaise, stage-coach or the later mail-coaches, grumbled. Thanks to numerous Turnpike Acts there were stretches of road in fairly good condition, and it was on one of these that Johnson was riding in a post-chaise in March 1776 when he remarked to Boswell that " Life has not many better things than this." The experiences of most travellers were more like those of Sydney Smith who wrote that during a nine hours journey from Taunton to Bath he had " suffered between 10,000 and 12,000 severe contusions." Journeys by road were tedious, expensive, uncomfortable and sometimes dangerous, and it was not till the end of the eighteenth century that engineers seriously addressed themselves to a study of road making.

With water transport it was a different story. The rivers of a country are its natural highways, and England was fortunate in possessing numerous navigable streams. Schemes for improving these were often brought forward, and a correspondent of the Chancellor of the Exchequer wrote in 1697 that " they had been greatly plagued in the house with petitions about rivers being made navigable." The House was plagued a good many more times in the eighteenth century, in which canal building began in a small way, grew into a great business and rose to a mania. Side by side with the making of canals, with their embankments, locks, aqueducts and tunnels, came the straightening and deepening of rivers, the building of bridges and the construction of docks and harbours. The state of the Fen district had engaged the attention of both James I and Charles I, and in the eighteenth century nearly every civil engineer was concerned with Fen drainage schemes. With these activities, for the first time came into being a class of men who were wholly engaged in carrying out surveys, drawing up plans, writing reports and supervising constructions, thus ushering in the new profession of civil engineering. The most famous of these early civil engineers included John Perry (1670–1733), James Brindley (1716–72), John Smeaton (1724–92), Robert Mylne (1734–1811), William Jessop (1745–1814), Thomas Telford (1757–1834), and John Rennie (1761–1821). There were others almost equally active, though the biographers have overlooked them. Something of their careers, however, can be gleaned from two papers in the 'Transactions' of the Newcomen Society, Vols. XVII. and XVIII. (1936–38). The first of these papers was entitled " The Society of Civil Engineers (Smeatonians) " and was by Mr. S. B. Donkin; the second was on " The Early Smeatonians " and was by Mrs. E. C. Wright, Ph.D. The first was based on the records of a society still active, which was founded in 1771 and reconstituted in 1792; Mrs. Wright's paper was the outcome of painstaking research at the British Museum and elsewhere. It is from these studies that information is obtained of the labours of Smeaton's contemporaries, John Grundy, Senr. (d. 1748), John Grundy, Junr. (1719–83), Thomas Yeomans, F.R.S. (d. 1781), John Golbourne (d. 1783), Robert Whitworth, "who levelled for that great Genius Mr. Brindley," Joseph Nickalls, and others. Mrs. Wright pointed out that in the early part of the century some of these individuals called themselves engineers, but that the term "civil" was not added till later on. Most of their work was connected with private undertakings, financed by private individuals, and the engineers were in no way government servants. It was otherwise in France, where the famous Corps des Ponts et Chaussées was founded in 1716, followed by the opening in 1747 of the École des Ponts et Chaussées, where a grounding in scientific principles was given. These institutions boast of a long line of eminent engineers such as Perronet, Chezy, Gauthey, Lamblardie, Prony and Girard, who were responsible for the public works of France in the eighteenth century.

Of the many hundreds of undertakings the British civil engineers mentioned were concerned with, only a very few can be referred to, but some conception of the advances made can be gleaned from brief notices to which attention must be confined, of one or two works carried out at the beginning, in the middle and towards the end of the eighteenth century.

Two early works on a considerable scale were the construction of the first dock at Liverpool by Thomas Steers from which that city's great dock system has been developed, and the repair of the breach in the banks of the Thames at Dagenham by John Perry, after the unsuccessful attempt by another "undertaker," William Boswell. None of the eighteenth century civil engineers was trained as such and Perry was no exception. He started life as a naval officer, but having fought an unsuccessful action against French privateers was court-martialled, fined £1,000 and committed to the Marshalsea for ten years. This was in 1693. Five years later he secured his release and entered the service of Peter the Great, becoming Comptroller of Maritime Works. He spent three years on the cutting of a canal from the Volga to the Don, another eight years in the Voronezh district constructing docks and improving the navigation of the River Voronezh, and another period surveying routes for canals to join Lake Ladoga with the Neva. He returned to England in 1714 and related his experiences in his entertaining book 'The State of Russia under the Present Czar' (1716). The great breach of the banks of the Thames of 1707 had flooded several thousands of acres of land, and a contract was given to Boswell in 1714 for the necessary repairs. The task proving too much for him it was given to Perry. The work took five years and cost about £25,000. Perry was also known for his proposals for improving the harbours of Dover and Dublin, and at the time of his death at Spalding he was engineer to the Adventurers of Deeping Fen, Lincolnshire, a post he was succeeded in by John Grundy, Senr.

The most considerable undertaking after Perry's repair of the Dagenham breach was the building of the old Westminster bridge over the Thames by the French engineer Charles Labelye (1705–81 ?) during the period 1738–50. This was followed a decade later by the building of the Blackfriars bridge to the designs of Mylne. Both bridges were of stone, but whereas Labelye retained the use of circular arches, Mylne adopted elliptical arches over which there was much controversy. The two bridges remained in use for about a century, by which time three others had been thrown across the Thames. In between the building of the Westminster and Blackfriars bridges came the erection during 1756–59 of Smeaton's noble tower on the Eddystone rock. Winstanley's curious lighthouse had been swept away in the great storm of 1703, and John Rudyerd's fine wooden tower was destroyed by fire in 1755 after standing for forty-six years. When consulted about a new lighthouse Smeaton had had no experience as a civil engineer, although he had become an authority on all kinds of mill work; his work on the lighthouse was therefore the more remarkable. Constructed of stone, with every block dovetailed and secured to all adjacent ones, and with the lower part completely solid, it was completed in 1759 and was a model for other lighthouses in exposed positions. Today the stump still stands on the rock, but the tower has been re-erected on Plymouth Hoe, as fine a memorial as any engineer could wish for. It was not the state of the tower, but the undermining of the rock which led to the building of another one hundred and fourteen years after Smeaton and his friend Dr. Zachariah Mudge had together sung the Old Hundredth in its lantern, open to visitors today.

The building of the Blackfriars bridge coincided with the construction of the Bridgewater Canal by Francis Egerton, the 3rd Duke of Bridgewater (1736–1803), assisted by Brindley whose native talents had gained for him the reputation of one of the most capable millwrights in the Midlands. The first section of the canal connected Worsley with Manchester. This was only seven miles long, but it had to cross the valley of the Irwell, and it was due to the representations of Brindley that the use of locks was abandoned in favour of the Barton Aqueduct, the first of its kind in the country. Begun in 1759, the canal was brought into use in 1761, but its extension to Runcorn on the Mersey took more than ten years, and by the time of its completion Brindley had passed away. In those ten years or so he had acted as engineer for many other canals including the Great Trunk Canal connecting the Mersey with the Trent. To the Duke of Bridgewater canals brought great wealth, to Brindley fame, and to the country an urgently needed means of transport for its coal, iron, manufactures and agricultural produce.

When Smeaton died in 1792 leaving many monuments to his genius, such as the Perth, Banff and Coldstream bridges, Ramsgate Harbour and the Forth and Clyde Canal, Telford was thirty-five and Rennie but thirty-one, and many of their most important works belong to the first quarter of the nineteenth century. The construction of the Ellesmere Canal by Telford and of the Kennet and Avon Canal by Rennie, however, occupied the closing years of the eighteenth. Both Telford and Rennie were of Scottish birth, Telford beginning his career as a stonemason, and Rennie as a millwright, but whereas Telford was almost entirely self-educated Rennie was fortunate enough to spend three winters at Edinburgh University, being the earliest British engineer to enjoy such an advantage. After working at his trade in Edinburgh Telford was employed on the building of Somerset House, London, and the Commissioner's House at Portsmouth, where his talents and industry soon attracted attention. In 1788 he was appointed Surveyor to the County of Shropshire, and five years later engineer to the Ellesmere Canal. This waterway had to cross the river Ceriog at Chirk and the river Dee at Pont-Cysylltau, and in both instances Telford used aqueducts instead of locks. The larger of the two, the Pont-Cysylltau aqueduct, has a length of 1007 ft., and is carried on 19 arches, the piers of which are 121 ft. high. The most remarkable feature of the work, however, was not its size, but its waterway formed of cast iron girders and plates, all firmly bolted together, the ironwork being supplied by William Hazeldine (1763–1840), who afterwards forged the great chains for Telford's suspension bridge over the Menai Straits. Combining the skill of the architect, the civil engineer, the mechanical engineer and the ironfounder, the Cysylltau aqueduct was described during its construction as " among the boldest efforts of human invention in modern times." It was begun in 1795 and completed ten years later.

Rennie's first work in England was the fitting up of the flour-milling machinery and its gearing at the great Albion Mills erected by Boulton and Watt, and others during 1784–88 on the banks of the Thames near the southern end of Blackfriars bridge. There were no fewer than twenty pairs of millstones, the whole being driven by two Watt engines. In size and capacity the mill far surpassed anything hitherto erected, and Boulton looked upon it as a great advertisement. Unfortunately, in March 1791, it was completely destroyed by fire. It had, however, established Rennie's reputation, and he then set up in business for himself, carrying out both mechanical and civil engineering work. His Kennet and Avon Canal opened in 1799 had no such remarkable features as the Pont-Cysylltau aqueduct, but it was pronounced one of the best executed in the country. With a length of fifty-seven miles it had seventy-nine locks and several aqueducts, and allowed barges to pass for the first time from the Thames to the Avon at Bath, a matter of some strategic importance.

With the opening years of the nineteenth century the operations of both Telford and Rennie extended rapidly, and their works are to be seen in many parts of the United Kingdom. From Shropshire Telford was sent to Scotland, where he built some hundreds of bridges and hundred of miles of roads, and also the Caledonian Canal. His Menai Straits bridge was built during 1819–25, by which time he was the recognized head of his profession, and had become the first president of the Institution of Civil Engineers. Rennie's later works were largely connected with the Navy and the City of London. He was the trusted adviser of the Admiralty, and constructed both Sheerness Dockyard and the Plymouth Breakwater. He also designed the London and East India Docks, and the old Waterloo and Southwark bridges, and the present London bridge, built under the supervision of his son Sir John Rennie. Closely associated with Rennie in many of these undertakings were Sir Edward Banks (1770–1835) and his partner William John Jolliffe (1774–1835), the contracting firm of Joliffe and Banks being the largest of its day. Jolliffe and Banks it may be recalled are buried and commemorated in the neighbouring Surrey parish churches at Merstham and Chipstead, but Rennie, like Wren and Mylne, is buried in the crypt of St. Paul's Cathedral, and Telford in the nave of Westminster Abbey, fit resting places for two great representatives of the eighteenth century civil engineers, "a self-created set of men, whose profession owes its origin, not to power or influence, but to the best of all protection, the encouragement of a great and powerful nation . . ."

SCIENTIFIC INSTRUMENTS IN THE 18th CENTURY

By ROBERT S. WHIPPLE, M.I.E.E., F.Inst.P.

THE 18th century witnessed a great development of experimental work which necessitated the use of scientific instruments. Frequently the experimenter himself designed and made the apparatus ; more frequently, he obtained the co-operation of an instrument maker who, working with him, designed and made the desired equipment. The century produced a galaxy of famous scientific men and also a group of instrument makers of outstanding merit who worthily supported their scientific colleagues.

At the commencement of the century the astrolabe, the backstaff and cross-staff were used for determining the latitude of a ship at sea. These instruments were all of them cumbersome and fairly inaccurate in their results. At the end of the century, Hadley's Octant (1731), which at the suggestion of John Campbell (1757) was enlarged into the sextant so as to measure angles up to 120°, was in universal use. This is not surprising when it is considered that in all but the smoothest seas it must have been impossible to obtain readings of any accuracy with the backstaff. On the other hand, with the octant, the reflected image of the sun remains bisected by the horizon during a large movement of the instrument, so that a good reading of the altitude is possible, even in a fairly heavy sea. The scales of the instruments were generally divided on ivory or boxwood to a third of a degree and were fitted either with diagonal scales, or a vernier, so that readings could be taken to one minute of arc. The instruments were sometimes fitted with low-power telescopes in the place of simple pin-hole sights.

The determination of longitude is a more difficult problem in that the local time at some instant must be compared with the standard time corresponding to that of the prime meridian. The difference between these two times gives the longitude from the prime meridian.

As the Astronomer Royal mentions in his paper " Astronomy through the Eighteenth Century," the British Government in 1713 offered a large award for a method of determining a vessel's longitude at sea to within an accuracy of 30′ at the end of a voyage to the West Indies. This corresponds to an error of two minutes in time at the end of a six weeks' voyage. A chronometer, or marine timekeeper, was designed and made by John Harrison, a Yorkshire carpenter. Between 1728 and 1759 he designed and constructed four instruments and with the fourth obtained the award.

In 1766 a Frenchman, Pierre Le Roy, produced his " montre marine " which was fitted with the first compensation balance and this was almost universally adopted by other makers. Arnold and Earnshaw, both ingenious craftsmen, developed the making of chronometers on a successful and a commercial scale, so that the instrument remained practically unchanged until the end of the 19th century.

The invention of the pendulum clock by Huygens in 1657 and of the anchor escapement by Hooke a few years later made possible a great advance in timekeeping. For astronomical work, however, the variations in timekeeping owing to changes in length of the pendulum with temperature were serious and Graham in 1721 was the first to attempt to overcome the difficulty. He used a steel pendulum rod which carried a bob consisting of a steel tube containing mercury and the quantity of mercury was so adjusted that when heated its upward expansion compensated for the downward expansion of the rod.

A few years later Harrison introduced his "grid-iron" pendulum in which use was made of the differential expansion of brass and steel rods to which the bob was attached. About 1800, zinc and steel were used, as owing to the greater expansibility of zinc, fewer rods were required. Until the invention of invar (an alloy of nickel 35·7 per cent., steel 64·3 per cent.) by C. Guillaume in 1895 the standard observatory clock remained practically unchanged since the middle of the 18th century. At his private observatory at Kew, George III. had a collection of such clocks made by the leading clock-makers of the day. Several of these clocks are still in use at the present time.

The Astronomer Royal, in his article previously mentioned, has dwelt on the evolution of the telescope regarded from the point of view of the astronomer. He points out that the compact size of the telescopes with achromatic objectives made the equatorial form of mounting practicable. For the same reason the invention had a great influence on the development of surveying instruments. This is understandable, because the difference in principle between a telescope mounted to survey the heavens and one used to measure distances upon the earth's surface is a small one.

It must also be remembered that both groups of instruments, astronomical and terrestrial, were made by the same men and that in many details the construction was the same. As in the case of the early astronomical instruments, the surveying instruments of the same period were fitted with pin-hole sights, but about 1715 the better instruments were being fitted with telescopes (unachromatic) provided with cross-wires. Some surveying instruments made about the middle of the century are in existence and are examples of excellent workmanship.

As the demands of the astronomer and the surveyor for accuracy increased, so the need for improved divided circles became more insistent. The accurate dividing of circles has always been one of the more difficult tasks of the instrument maker. For many years there was no alternative but to bisect, or trisect, with a beam compass or other tool, the spaces set out on the scale or circle, and to continue this operation until the scale was completed. The accuracy obtained by some of the makers at this time was extraordinary. It is stated that John Bird (1709–1776) was able to obtain an accuracy of five minutes of arc on his 8-ft quadrant by continued bi-section of the arc.

Henry Hindley of York, about 1739, completed a small machine for cutting the teeth in clock wheels and for dividing instrument circles. In 1766 Jesse Ramsden (1735–1800) made his first circular dividing engine and in 1775 his second one which was the pioneer of the modern dividing machine. It was with this machine that Ramsden divided the circles of the 3-ft. theodolites used in the principal triangulation of Great Britain and Ireland. They were divided to 10 minutes of arc and were read to one second by three micrometer microscopes. One of these instruments is exhibited in the South Kensington Museum.

With the exception that in 1826 William Simms invented a self-acting mechanism by means of which the dividing machine became completely automatic, no fundamental improvements were introduced into the machine for nearly a century.

Mention must be made of the telescope eye-piece invented by Ramsden and universally known by his name. It gives a real, or positive, image and for this reason is generally used in optical measuring machines.

The thermometer, or thermoscope, invented by Galileo early in the 17th Century had become a fairly satisfactory article by the use of two fixed points and the division of the interval between the points into a number of equal parts. It was, however, to Fahrenheit (1686–1736), a scientific instrument maker in Amsterdam, that the modern form of thermometer owes its inception. In 1721 he constructed the first mercury thermometer and with it determined the boiling points of a variety of liquids. He standardized his thermometer by taking as his zero the point he defined as " the greatest cold, " obtained with a mixture of salt and ice, and his highest point, or blood heat, obtained by inserting the thermometer in the mouth or armpit of a human body; this corresponded to 96°. It is not known how he obtained his scale of temperature, but it is thought that he divided the distance between these two points into 24 parts, and later subdivided these into four parts, thus arriving at a scale of 96°. On this scale the freezing point of water was 32° and when he extended the scale to the boiling point of water it became 212°. He does not appear to have used the boiling point as a " fixed point " on his scale.

The French naturalist, Réaumur (1683–1757), unaware of the work of Fahrenheit, published a paper in 1730 in which his " rules for constructing thermometers with comparable graduations " were stated. He used alcohol instead of mercury as his thermometric liquid owing to its higher coefficient of expansion. He stressed the importance of calibrating the thermometer tubes and also the purity of the alcohol employed. On Réaumur's scale the melting point of ice was taken as 0°, and the boiling point of water at standard pressure as 80°.

In 1742 the Swedish astronomer Celsius described his method of making a mercury thermometer. In this the scale had two fixed points, that of melting snow and that of boiling water, the barometric pressure having its mean value at the time of the experiment. The stem between the two points was divided into 100 parts. Celsius placed his zero at the boiling point and the 100° at the freezing point. The centigrade scale in which the values were reversed was introduced by Christin of Lyons in 1743.

The well-known Six's maximum and mimimum thermometer was introduced by James Six in 1782. In the modern form of his thermometer the indices containing a small piece of steel and a fine glass spring are re-set by means of a magnet just as he described in his original paper. The modern meteorologist, for accurate work, still uses the maximum and mimimum thermometers of John Rutherford, invented in 1790. The two thermometers were mounted horizontally. The maximum temperature was shown by a mercury thermometer; an index, consisting of a conical piece of ivory, rested on the surface of the mercury. When the temperature rose the index was pushed before it, when it fell the index was left behind, thus recording the maximum temperature. In the modern instrument a thread of mercury separated by an air bubble has taken the place of the ivory index. The mimimum temperature was recorded by an alcohol thermometer containing a small glass index completely immersed in the spirit. If the temperature fell, the alcohol contracted, pulling the index down with it. If the temperature rose, the spirit passed over it and the index recorded the mimimum temperature.

The community owes a great debt to De Luc (1727–1817) for the work he did in perfecting the barometer. He was primarily interested in using it for the measurement of altitudes and came to the conclusion that the reason for the discordant results he was obtaining was due to the unreliability of his instruments. About 1750 he found that the chief source of error was due to air and moisture included in the mercury column. He boiled a series of barometer tubes, thus driving out the air and moisture contained in them, and found that he obtained agreement in their readings. He also determined accurately the effect of temperature on the length of the mercury column.

As a result of this work he found that the thermometers he was employing were as unsatisfactory as the barometers. He investigated a number of liquids and decided that mercury was the best thermometric substance. His method of calibrating the thermometer tubes and his method of filling and boiling the mercury in them are identical with those followed by the modern maker. It is not incorrect to state that the technique of thermometer manufacture at the close of the 18th century remained practically unchanged until John Welsh in the middle of the 19th century pointed out the importance of prolonged annealing of the thermometer bulb, and the continental physicists introduced about fifty years later the new hard glasses from which to make the bulbs and stems.

De Luc's long and thorough investigation of the sources of error in the barometer had so much advanced the manufacture and accuracy of this instrument that barometers made by Newman at the commencement of the 19th century are still in use at the Greenwich and Kew Observatories.

A distinguished geologist, De Saussure (1740–1799), published an account of his hair-hygrometer in 1783, an instrument which he had long been developing. A taut hair lengthens when wetted and contracts when dried, over a range which may be about one-fortieth of its length. De Saussure devised simple instruments by means of which the changes in length were caused to move a needle over a dial. He devoted a great deal of time to the selection of his hairs, to freeing them from grease and to an exhaustive study of their performance. All this was the work of an able experimenter and has stood the test of time. The latest application of his hygrometer is in the little meteorograph attached to the balloons used for determining the humidity of the air at high altitudes.

The calorimeter developed by Lavoisier and Laplace and described by them in 1784 was devised for the determination of the specific heat of substances. They defined the unit of heat as the quantity of heat required to raise the temperature of a pound of water by one degree. The specific heat of a substance was expressed as the ratio of the heat required to raise the temperature of a given mass of the substance to the quantity required to raise an equal mass of water through the same range of temperature. In their calorimeter the specific heat was determined by noting how much ice a known quantity of the substance heated to a definite temperature would melt. The quantity of water so obtained was collected and weighed. This mass of water, divided by the product of the mass of the body and the number of degrees of temperature it was originally above zero, is proportional to the specific heat of the body. In its general form the calorimeter remains the same to-day although in detail it has been much elaborated.

John Ellicott's " instrument " (1736) and John Smeaton's " pyrometer " (1754) for the determination of the coefficient of expansion of metals should be mentioned. In Ellicott's instrument the expansion of the metal under test was compared with that of a standard iron bar, and in Smeaton's pyrometer with that of a brass rod, the expansion of which had been determined against that of a bar of wood. In both instruments great ingenuity was shown in designing the details of the measuring mechanism.

In connection with the geodetic survey previously mentioned, Ramsden in 1785 devised a piece of apparatus, generally known as " Ramsden's pyrometer," for measuring the expansion of a metal bar against standard bars maintained at a definite temperature. The observations were required to define accurately the length of a geodetic base line on Hounslow Heath. Three parallel troughs were taken, over 5-ft. long, the outer ones each containing an iron bar kept at a constant temperature by a packing of pounded ice, and the inner one, suitably heated, containing the bar to be tested which was fixed at one end. The change in length of the bar was determined by means of a simple but efficient optical system *.

The development of the telescope and its mountings enormously aided astronomy and geodesy and the concurrent evolution of the microscope had a considerable influence on the biological sciences.

The publication of *The Micrographia* by Robert Hooke (1665) was a great milestone in the development of the microscope. In this book he described a variety of objects he had studied and of which he gave illustrations. The book aroused much interest both in this country and on the Continent and created a demand for microscopes. The introduction of the field lens by Hooke to enlarge the field of view, although originally due to Monconys, was such an improvement that it was generally adopted.

At the commencement of the 18th century the Hooke form of microscope was being made by John Marshall, one of the great English opticians. The first description of his instrument was published in Harris' *Lexicon Technicum*, 1704, and there are in existence several instruments made by him.

The circulation of the blood had been announced by Harvey in 1628 and its demonstration in the tail of a fish or in the web of a frog's foot was of great interest to users of the microscope. Marshall advertised his instrument as "New Invented Double Microscope For Viewing the Circulation of the Blood." Technically the microscope was a great improvement on its predecessors; it was provided with six objectives ranging from a power of four diameters to that of one hundred and was fitted with a screw fine-adjustment for focusing.

About 1738 Lieberkühn (1711–1756) devised a combination of a simple lens, mounted in a polished metal reflector, as a means of illuminating opaque objects. The construction was frequently modified so that the Lieberkühn could be applied to various objectives and it is so used to the present day.

The invention of the solar microscope is generally attributed to Lieberkühn although there is reason to think that it may have been due to Fahrenheit †. In the early instruments a beam of sunlight was thrown directly, into the microscope, by a lens mounted in a fitting placed in an aperture in a shutter, the image of the object was thus projected on to a white screen. About 1743, John Cuff (1708–1772), the inventor of the Cuff microscope, originated the form of solar microscope in which the tube was fixed and the motion of the sun was allowed for by projecting a beam of sunlight, by means of a movable mirror, into the microscope.

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* I am indebted to Prof. A. Wolf's 'History of Science Technology, and Philosophy in the Eighteenth Century' (George Allen & Unwin, Ltd.) for this reference and also for a good deal of information used in this article.

† See 'The History of the Microscope,' by R. S. Clay and T. H. Court, (Charles Griffin and Company Ltd.), p. 213.

A great many forms of such microscopes were made by various makers and excellent instruments made by Benjamin Martin, George Adams, Ramsden, Dollond and others are still in existence.

Benjamin Martin (1704–1782) was one of the great pioneers in the development of the microscope during the 18th century. He designed and made the first drum microscope in which the mirror was fitted in the cylinder base. This instrument was the forerunner of the cheaper form of microscope so largely made by Continental manufacturers during the 19th century. Although Martin did not claim to have invented the idea of embodying in his microscope the "between-lens" which made his objective into a compound one, thus greatly improving the definition obtained, there appears to be little doubt that he was the first to do so.

He was the first maker to fit a screw-micrometer (1742) in the eye-piece of the microscope so that the magnifying power of the object-glass could be determined. He appears to have suggested the rotating multiple lens-carrier nose-piece in which he placed different powers to rotate successively into the axis of the microscope. His "Grand Universal Microscope" made about 1780 was provided with a mechanical stage having rectangular movements made by fine-threaded screws the heads of which were graduated.

It is not generally known that Martin was the first to construct an achromatic microscope objective. In his *New Elements of Optics*, London, 1759, in the year following Dollond's communication to the Royal Society of his construction of an achromatic telescope objective, Martin dealt with the principle of achromatism and compares the result he obtained with an achromatic object glass and "Speculum, applied as an Object-Glass in a compound Microscope" *. From the comparison he decided that the results obtained from the speculum were much to be preferred to those from the achromatic objective. An interval of about sixty years was to pass before Tully in London and Chevalier in Paris produced satisfactory achromatic microscope objectives.

Martin specialized in making cabinets of optical instruments containing a solar microscope, a drum microscope, a scioptic ball, a large variety of accessories and a telescope. Some of these cabinets are excellent examples of craftsmanship. It should also be mentioned that Martin published a large number of books for popularizing science which had a considerable vogue and which undoubtedly helped in the sale and use of scientific instruments, globes, etc. The two George Adams, father (1708–1773) and son (1750–1795), did a great deal to develop the microscope and its uses. The book by the father, *Micrographia Illustrata*, 1746, shows the great variety of instruments made by the firm, practically covering all the types in general use and it also gives illustrations of many objects suitable for the microscope.

To Adams, senior, also is due the suggestion of screwing two or three microscope objectives in series with one another so that the power of the combination could be varied. This method of varying the power of a microscope is still in use in inexpensive instruments. Adams was the first to introduce the method of supporting a microscope on trunnions at its centre of gravity and also developed a micrometer capable of reading to 1/10,000 inch. He made a great variety of instruments for instructional purposes. Many of these for teaching the principles of mechanics were based on J. T. Desagulier's translation of the work of Gravesande, *Mathematical elements of Natural Philosophy. Confirmed by Experiments : Or an Introduction to Sir Isaac Newton's Philosophy*, sixth edition, 1747, and those for teaching Optics on Robert Smith's *Compleat System of Opticks in Four Books*, 1738. Excellent examples of these instruments can be seen in the King's Collection (George III.) now at the South Kensington Museum.

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* See 'Cantor Lectures on The Microscope,' by John Mayall, junior, Society of Arts, London, 1885, p. 48.

The Adams also specialized in the manufacture of globes. In 1766 Adams senior published the first edition of his book, *A Treatise describing and explaining the construction and use of new celestial and terrestrial globes.* The book had a great vogue, passing through thirty editions.

The French physicist Pierre Bouguer (1698–1758) and his German contemporary Johann H. Lambert (1728–1777) developed two simple photometers and carried out, independently, a series of experiments on photometry, the results of which have remained almost unchallenged till the present day. The photometers have been greatly improved in detail, but the underlying principles have remained.

Great interest was shown in optical instruments and in optical illusions towards the end of the 18th century. Magic lanterns, the illuminant generally consisting of one or more candles, with crude hand-painted slides—frequently of astronomical subjects—were the playthings of the well-to-do. Spy-glasses of exquisite workmanship were to be found in the opticians' shops of London and Paris. Spectacles, as we know from the trade cards of various opticians, were slowly coming into use *.

The 18th century saw the birth of the first efficient glass electrical machine by Francis Hauksbee, who described his machine and experiments made with it in his book *Physico-Mechanical Experiments on Various Subjects, etc.*, 1709. Several experimenters developed machines and an excellent example of one made by Edward Nairne (1726–1806) stands in the hall of the Royal Institution.

The invention of the Leyden jar by E. G. von Kleist in 1745 led to an intense interest in electrical experiments. Franklin published a theory of the Leyden jar and his experimental results during the years 1747–1755.

Charles Augustin Coulomb (1736–1806) about 1784 invented his torsion-balance which was to become so fruitful in obtaining electrical and magnetic data. He showed that the repulsion between two similarly electrified spheres varies inversely as the square of the distance apart of their centres. By use of practically the same apparatus he determined the law according to which the force of a magnet pole varies with the distance.

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* The following is an extract from the trade-card of Edward Scarlett (1677–1743) in which it states that he " Grindeth all manner of Optick Glasses, makes Spectacles after a new method, marking the Focus of the Glass upon the Frame, it being approved of by all the Learned in Opticks as ye Exactest way of fitting different Eyes."

Various experimenters had worked at the problem of an electrometer, in which two moving members were mutually repelled from each other by an electric charge, but the first satisfactory one was made by William Henly in 1770. In this instrument a cork ball was hung by a light rod from the centre of a graduated quadrant over which the rod turned. The charge was measured by the deflection of the rod.

Abraham Bennet (1750–1799) appears to have been the inventor of the well-known gold-leaf electrometer, the instrument which in the hands of the workers at the Cavendish Laboratory was to become, in the early days of the 20th century, such a valuable tool.

Alessandro Volta (1745–1827), who had previously invented his electrophorus and condenser, described in 1800 in a letter to Sir Joseph Banks, his invention of the crown of cups—the first battery. The first man in England to make a battery in accordance with Volta's letter was a London surgeon, Sir Anthony Carlisle, and later William Nicholson co-operated with him. It is of particular interest at this time to note that Nicholson described the experiments in *Nicholson's Journal* of July 1800, pp. 179–191, later absorbed by the *Philosophical Magazine*.

There are in museums many cases of drawing instruments made in the 18th century, especially those made by French makers for artillery officers and engineers, which show that the form and construction of many of the instruments have remained practically unchanged till the present day. Some of the ivory and metal scales made during the 18th century are examples of excellent workmanship.

The slide rule was developed primarily as an aid for Custom's officers for determining the volumes of casks, etc. John Robertson (1712–1776) Librarian to the Royal Society, designed a slide-rule 30 in. long and 2 in. wide, having 12 logarithmic scales of natural numbers and trigonometrical functions engraved on one face. He also invented the cursor for use on the rule.

In 1787, William Nicholson (see *Phil. Trans.* 1787, Pt. II. p. 246, and *Nicholson's Journal*, 1797, p. 372) urged the use of concentric circular, or spiral, scales so as to obtain the equivalent of long rules. This has now become standard practice and in the case of the Fuller rule has been developed to a high degree of accuracy.

In a short article it has only been possible to touch upon the development of a few representative instruments. In chemistry, engineering and the biological sciences instruments were made for the solution of special problems and this experimental work has been continuously maintained until the present day.

THE SCIENTIFIC PERIODICAL FROM 1665 TO 1798.

By DOUGLAS McKIE, D.Sc., Ph.D.

Before the foundation of the scientific academies in the middle of the seventeenth century, there were no scientific periodicals; and "natural philosophers" conveyed their ideas and accounts of their work and experiments to one another by means of letters which, from their scope and length, might be more appropriately described as dissertations. Much of this voluminous correspondence has survived; and in it is to be found the give-and-take, and sometimes the cut-and-thrust, of scientific argument and debate. We need remind ourselves only of the extensive and historically important correspondence of Mersenne (1), from which today we obtain, or shall obtain when it has been completely published and studied, a real, or at least a deeper, knowledge of the history of physical science in the first half of the seventeenth century.

But the epistolary dissertation was not an ideal method for the communication of scientific fact and theory, even when it transcended frontiers. It was too personal. Men write to their friends, and not always, or not so often, to those who dispute their facts and reject their theories. Questions of priority were so easily raised; and ciphers were used for secrecy. Moreover, the method could not spread new knowledge and new ideas either rapidly or widely: it was too slow and too limited within narrow personal circles.

With the foundation of the Royal Society of London and the Royal Academy of Science in Paris, the need for some better and wider means of communication became obvious and urgent. Conditions in England differed, however, from those in France. In England interest in scientific matters, in the new "philosophy," was actively centred in one group, which during the struggle between King and Parliament, met first in London, and then partly in London and partly in Oxford in separate assemblies, until their reunion in London after the Restoration to form "The Royal Society of London for the Improving of Natural Knowledge." But the *virtuosi* formed one group only (2). In France, the very opposite conditions prevailed. In many French cities other than Paris there were independent groups of men interested in natural philosophy and meeting and joining in *conférences* and *sociétés* (3). It was the fashion of the hour.

The foundation in 1666 of the Royal Academy of Science in Paris increased this interest in matters scientific, but there had already appeared in France in 1665 the first scientific journal, the 'Journal des Sçavans,' ancestor of all its kind (4). Its editor, Denis de Sallo (5)

(1626–69), was a friend of Colbert, Louis XIV's Minister and prime mover in the establishment of the Académie Royale des Sciences in Paris. Descended from an ancient and noble family of Poitou and educated at the Collège des Grassins, he became a lawyer. He was an omnivorous reader of all kinds of books and he kept two secretaries to compile notes and transcribe extracts from the books that he read. Overwork brought ill-health and enforced inactivity; and de Sallo then thought that he might incorporate such material as he was led to abstract into a weekly journal of literary and scientific news, supposing that what had interested him would likewise interest others, especially the ever widening circle of amateurs of the "new philosophy."

The first number of the 'Journal' was published in Paris on Monday, January 5, 1665. The reader was advised that the journal would contain details of new books, obituaries, news of experiments in physics and chemistry, discoveries in the arts and in science, such as machines and the useful and curious inventions afforded by mathematics, astronomical and anatomical observations, legal and ecclesiastical judgments from all countries, and "enfin, on taschera de faire en sorte qu'il ne passe rien dans l'Europe digne de la curiosité des Gens de lettres. qu'on ne puisse apprendre par ce Journal." The founders, so the reader was informed, had given much consideration to the problem of whether the journal should appear annually, monthly or weekly. They had concluded in favour of weekly publication in view of the fact that news ages quickly: "Mais enfin ils ont creu qu'il devoit paroistre chaque semaine; parce que les choses vieilliroient trop, si on differoit d'en parler pendant l'espace d'un an ou d'un mois." They had been influenced also by the complaints of certain "persons of quality" that frequent issues of a journal were agreeable, whereas a whole volume of such items, from which the passage of time would have effaced the charm of novelty, might prove fatiguing (6).

De Sallo appeared as editor of the 'Journal' under the pseudonym of the Sieur de Hédouville, but it is not certain whether he sought anonymity or whether the name was taken from an estate he is said to have owned in Normandy. He had help in his editorial labours from Chapelain and probably from the Abbé Gallois, and possibly from others; and he allowed them complete freedom of opinion. However, this freedom led to criticism and raised an opposition that brought trouble upon the venture. After the appearance of the thirteenth number on March 30, 1665, de Sallo's privilege was withdrawn and the 'Journal' was suppressed, this step being said to be due, however, not to the complaints of authors who considered themselves aggrieved, but to Jesuit irritation at disrespectful Gallican comment on a decree of the Inquisition. De Sallo declined to continue publication with a censor. However, his resolute stand for independence had impressed Colbert, who remained friendly towards him and continued to consult him on the many subjects on which he was well informed. Towards the end of his brief life of forty-three years, de Sallo was given a financial post by Colbert because his private means had become seriously depleted through his generosity, his love of gaming and his haphazard life. Such was our first scientific editor.

After its suppression in 1665, the ‘Journal’ resumed publication on January 4, 1666, under the editorship of the Abbé Gallois, and was continued in 1685 by the Abbé de la Roque, in 1687 by Cousin and from 1702 to July 1792 by a committee. It was suppressed during the Revolution and revived in 1816. From 1665 to 1792, a series of one hundred and eleven volumes were published. The ‘Journal’ was reprinted in Amsterdam and also in Paris. It was imitated in other countries and even in France.

Although we are considering here the rise of the scientific periodical, and necessarily excluding the serial publications of the scientific academies, some reference must be made to the ‘Philosophical Transactions’ of the Royal Society of London. The Society had seen a copy of the ‘Journal des Sçavans’ and had discussed its contents and the problem of publishing a similar work; and they had concluded that something more “philosophical” was needed, excluding legal and theological items, but including more particularly accounts of the experiments made before the Society (7). The first number of the ‘Transactions’ appeared in London on March 6, 1665. It was edited by Henry Oldenburg, one of the two Secretaries of the Society. It was almost wholly scientific in content; it did not cater for the interests of a widespread public of amateurs; it was a monthly and not a weekly. It was the official organ of the Society and thus the first of its kind. Its appearance marked a new development; for it was a medium for the publication of new observations and original work in science, mostly carried out by the Fellows of the Society, and it became the model on which all other published proceedings of the scientific academies have been fashioned. It reviewed books and gave space for the publication of differing scientific opinions by those engaged in similar experiments and studies. It was less amateur and more professional, if the latter term may be applied to the productions of an age when the professional scientist had not yet appeared on the scene.

However, although the ‘Philosophical Transactions’ was the Royal Society's official organ, it was financially the private venture of Henry Oldenburg and it cannot be said to have yielded him much in the way of mere profit. Its continuance was no more secure than that of the ‘Journal des Sçavans,’ although for different reasons. It is also to be remembered that some stimulus towards its production is definitely to be ascribed to the appearance of the ‘Journal des Sçavans,’ since a discussion of the ‘Journal’ by the Society led to the decision to publish the ‘Philosophical Transactions’. The part played by Frenchmen in the foundation and development of the scientific periodical is inadequately recognized, as we shall see later on.

A word might be said in passing about Henry Oldenburg. He was born about 1615 in Bremen and died in Charlton, Kent, in 1677. He came to England about 1640 and, it is thought, remained here until 1648. In 1653 he came to England again to represent his native city on a matter of dispute with Cromwell and did not return to the Continent until 1657, when he acted as tutor and guardian to the Hon. Robert Boyle's nephew, Robert Jones, during a continental tour lasting until 1660. Returning to England, Oldenburg was appointed one of the two Secretaries of the Royal Society, the other being John Wilkins. Oldenburg carried on an extensive foreign correspondence on behalf of the Royal Society. Suspicion fell upon him during the Dutch War of 1665–67 and he was imprisoned in the Tower of London for two months. He was devoted to the new learning and enthusiastic in his labours to further the work of the Royal Society by "a commerce in all parts of the world with the most philosophical and curious persons to be found everywhere (8)."

The 'Journal' and the 'Transactions' gave two distinct models to scientific literature; the former long influenced the development of the scientific periodical, until the rise of the journal specially devoted to one science only, while the latter became the pattern for the publications of the scientific academies that arose in greater numbers throughout Europe during the eighteenth century.

The 'Journal des Sçavans' was imitated in Italy by the 'Giornale de Letterati di' (Rome), which continued to appear with a change of title from 1668 to 1697. The 'Philosophical Transactions' seem to have been the model for the famous 'Acta Eruditorum,' the learned scientific monthly which first appeared at Leipzig in 1682. Leipzig was the centre of the German trade in books and one of the most important functions of the 'Acta' was the announcement of new works of scholarship. It contained articles on the work of Europe's leading scientists; it included in its field medicine and mathematics, law and theology; it was the first German scientific periodical. Fifty volumes were published from 1682 to 1731, with ten supplementary volumes, followed by the 'Nova Acta Eruditorum' in forty-five volumes from 1732 to 1782 with eight supplementary volumes. The first editor was Otto Mencke (1644–1707), and he was succeeded by his son and then by his grandson until 1754, when the editorship passed out of the family (9). The 'Acta' will be remembered for many papers by Leibnitz, especially those dealing with his work on the calculus, which led to the controversy with Newton.

Mention might also be made of the 'Nouvelles de la république des lettres,' modelled on the 'Journal des Sçavans' and edited by Pierre Bayle (1647–1706) during its first three years (Amsterdam, 1684–87). It was Holland's contribution to the cause of popular scientific periodical literature and continued until 1718.

The turn of the century in which the two types of scientific periodical had been established witnessed an important consequence of the reorganization of the Paris Academy of Science, namely, the publication of the first volume of the long series of 'Histoire et Mémoires' of the Academy, with many supplementary volumes, a series modelled on the 'Philosophical Transactions.'

The eighteenth century brought great developments in the scientific periodical. Bolton (10) lists seventy-four new journals originating between 1725 and the close of the century, excluding purely medical journals but including all other periodicals containing scientific material. However, it is clear from his bibliography, that by far the greater number of these originated after the middle of the century. Indeed, only five of them appeared before 1750, namely, 'Raccolta, d'opuscoli scientifici e filologici' (Venezia, 1728–57, 52 vols., continued as 'Nuova raccolta,' etc., Ferrara, 1755–87, 42 vols.). 'Le Pour et Contre' (Paris, 1733–40, 20 vols.), 'Bibliothèque Britannique, ou Histoire des ouvrages des savans de la Grande Bretagne' (La Haye, 1733–47, 25 vols., followed by the 'Journal Britannique,' La Haye, 1750–57, 24 vols.). 'Göttingische Zeitung von Gelehrten Sachen' (Göttingen, 1739–52, 14 vols., continued from 1753 to 1801 in 117 volumes under another title in association with the Göttingen Academy), and the 'Hamburgisches Magazin' (Hamburg, 1747–67, 26 vols., continued as the 'Neues Hamburgisches Magazin,' Hamburg and Leipzig, 1767–81, 20 vols.).

These new periodicals doubtless contributed to the development of the scientific journal, but their number, namely five, in a half-century, or, if we take the actual first years of publication, five between 1728 and 1750, does not indicate any striking change as compared with the developments in this kind of literature during the pioneering second half of the seventeenth century. However, the pace rapidly quickened. Of the remaining sixty-nine periodicals to appear between 1750 and 1798, nine appeared between 1750 and 1759, six between 1760 and 1769, and nine between 1770 and 1779; but from 1780 to 1789, there were twenty new scientific periodicals and from 1790 to 1798 a further twenty-five. It was therefore only during the last two decades of the eighteenth century that the number of journals was greatly increased; for forty-five of the sixty-nine new publications originated between 1780 and 1798.

Of the nine new periodicals originating between 1750 and 1759, probably the most important was the 'Commentarii de rebus in scientia naturali et medicina gestis,' a most useful and successful journal giving news of the work of the academies and details of new books. It was published in Leipzig, the first number appearing in 1752, and continued until 1798 in a long series of forty-one volumes, including supplementaries. With the 'Acta Eruditorum,' already mentioned, the 'Commentarii' are a mine of valuable information for the historian of science. In passing, it might be noted that of the nine new periodicals published in this decade six were German, two French and one Dutch. The predominance of the German contribution is again evident during the period from 1760 to 1769, when of the six new periodicals that appeared, five were German, the other being Italian. None of these six calls for any special comment here.

From 1770 to 1779, nine further new periodicals were published, seven of which were German, one French and one Italian. This decade witnessed a most important and significant development in the first appearance of the specialized scientific journal both in physics and in chemistry, namely, the 'Observations sur la Physique' (11), edited by the Abbé Rozier, and the 'Chemisches Journal' of Lorenz Crell. The 'Observations,' however, although largely concerned with physics, dealt extensively with chemistry as well and included sections on natural history and the arts. But the contents tended to become more physical, although in the early years of the journal it was the medium for the publication of many of Lavoisier's classical papers in chemistry in their original unrevised form before they were re-presented and re-read to the Paris Academy of Science (12). In the history of chemistry the journal is a most important source. In the development of the scientific periodical, its appearance is a landmark and the circumstances merit some consideration.

The full title of the journal was 'Observations sur la Physique, sur l'histoire naturelle, et sur les arts'; it was a monthly and the first number appeared in July, 1771. Eighteen numbers were published from that date to the end of 1772 in 12°. From January 1773, the format became 4° and continued so until the final issue of 1793. The first eighteen numbers are very rare and it had already become necessary to reprint them in 1777 in two 4° volumes under the title of 'Introduction aux Observations sur la physique,' etc. The volume for 1773, comprising the issues from January to June of that year, is numbered volume one; and in all forty-three volumes, usually two every year, appeared up to 1793.

The "Avis" to the reader prefacing volume one (1773) (13) explained that the publications of the academies were written in the languages of the different nations and printed several years after the memoirs included in them had been read before the academies; that during that interval facts that might prove of the greatest usefulness remained unknown; and, moreover, that the publications of the Academies, having become very numerous and therefore very costly, were often beyond the means of those to whom they might be advantageous. The writer went on to draw attention to other causes of delay: "Il semble qu'à mesure que le nombre des Savans s'est accru, la Correspondance, entre ceux des Nations différentes, a été rallentie. Chacun a cru sans doute que les Académies nationales seroient suffisantes, & qu'on en tireroit tous les secours nécessaires. La Constitution de ces Compagnies, formées par les Souverains, ayant admis des Correspondans étrangers, sembloit prévenir cette illusion, & remédier à l'inconvénient qui en est la suite; mais cette précaution, si sagement prise, n'a pas été justifiée par le succès.

"Il résulte de ce peu de communication, que les progrès des Sciences sont très-lents, que des Savans de deux Nations différentes, travaillent long-tems sur la même matiere, & qu'ils perdent un tems précieux pour acquérir une gloire qui devient à la fin problématique. Cet inconvénient est moindre que celui de travailler sur une matiere éclaircie par des travaux déja publiés, dont on n'a aucune connoissance. L'Auteur perd un tems qu'il auroit mieux employé pour le bien & la gloire de sa Patrie, si, en entrant dans la carriere qu'il vouloit parcourir, il eût eu sous les yeux le tableau actuel des connoissances physiques, & le terme où elles sont restées."

For these reasons it appeared necessary to publish a new journal: " Ces motifs ont fait desirer qu'un Ouvrage périodique, d'un débit sûr & animé, annonçât les découvertes que se font chaque jour dans les différentes parties des Sciences, soit par des Notices abrégées, soit par des Mémoires très-étendus, qui continssent le développement de toutes les preuves de ces découvertes, en traçant même la marche de l'esprit inventeur. On a pensé que ce moyen, le plus prompt pour la publication des découvertes nouvelles, accéléreroit également le progrès des Sciences, qui ne sont autre chose que la somme de ces découvertes.

" Telles ont été les raisons qui nous ont engagés à entreprendre ce Recueil; & nous les présentons avec d'autant plus de confiance, aux savans Etrangers, que ce sera leur ouvrage. Il est écrit dans une Langue, aujourd'hui celle de tous ceux qui ont reçu quelqu'éducation en Europe. L'Académie Royale des Sciences de Paris sentoit depuis long-tems l'importance de ce Recueil. Plusieurs de ses Membres le proposerent l'année derniere à peu-près suivant le plan auquel nous nous sommes attachés. Des raisons particulieres en ont empêché l'exécution totale."

But the new journal was not for " leisured amateurs " seeking entertaining reading under the illusion that they were initiated into scientific pursuits of which, in fact, they had no real knowledge: " Malgré l'accueil favorable que cet Ouvrage a reçu; malgré les éloges que les Savans lui ont donné, nous nous croyons obligés de circonscrire nos limites pour le rendre encore plus digne d'eux. Nous rejetterons en conséquence ci qui ne seroit que compilation indigeste, & dénuée de vues neuves & utiles. L'importance des matieres, la maniere dont elles seront présentées, nous décideront sur le choix des morceaux qui doivent être insérés dans ce Recueil. Nous n'offrirons pas aux Amateurs oisifs, des Ouvrages purement agréables, ni la douce illusion de se croire initiés dans les Sciences qu'ils ignorent." It was clear that the aims of the journal were serious and immediate and that popular science was not its object.

Moreover, the historical development of science was included in the plan: " Nous nous occuperons sur-tout de l'histoire des Sciences que nous embrassons dans notre plan; & c'est dans ces vues, que nous nous attacherons à rapporter les faits de la même espece, & les raisonnemens différens qu'ils auront fait naître. Cette maniere de voir & de comparer, présente un fond inépuisable d'instructions, que nous saisirons avec le plus grand soin; de sorte qu'on verra au premier coup d'œil, la suite des faits qui auront concouru à l'établissement d'une vérité importante (14).

" On ne sauroit trop inviter ceux qui veulent faire des progrès dans les Sciences, à rapprocher les connoissances transmises par les Savans de tous les siecles & de tous les pays. C'est un préalable nécessaire pour parvenir à de nouvelles découvertes, & ils doivent considérer, comme le premier pas qu'ils aient à faire, celui où les grandes Hommes, qui les ont précédés, ont terminé leurs travaux. La continuité des efforts des uns & des autres, forme cette union, & cet accord qui doit regner entre les Savans de tous les Pays pour étendre les limites des connoissances.

" C'est donc de cette réunion de travaux, de cette somme de connoissances que nous devons partir pour en acquérir de nouvelles, & pour donner à ce Recueil la consistance que son objet semble lui assurer. L'étendue de ce travail surpasseroit nos forces, si une société formée par des personnes uniquement occupées des Sciences utiles, ne daignoit concourir avec nous pour remplir le but que nous nous sommes proposé.

The journal was intended also to increase the circulation of scientific information of the first importance and the editor took Mersenne for his model : " Les vrais Savans n'ont pas la manie de faire des secrets de leurs découvertes ; amis de l'humanité, leur gloire est de lui être utile ; aussi, c'est à eux que nous offrons ce Recueil, comme un dépôt où ils ont droit de prendre acte de leurs découvertes. Nous les invitons à regarder notre Cabinet, comme celui du Pere Mersenne." And general and experimental physics, chemistry, medicine, agriculture and the arts and crafts were to be included in its province : " en un mot . . . . tout ce qui a rapport à l'observation & à l'expérience (15)." Memoirs written by authors in other languages would be translated into French or extracts would be published until further details would be forthcoming.

We have quoted at length from this historically important document because it reveals the state of affairs at this time with regard to the scientific periodical in general ; and its contents apply to the current situation and explain for a particular instance the circumstances that led to such an extraordinary increase in and development of scientific periodical literature during the last quarter of the eighteenth century.

The editor of the ' Observations ' deserves some mention. François Rozier was born at Lyon in 1734 and died there on the night of September 28–29, 1793. He was educated for the priesthood and revealed an unusual interest in scientific observation. Anxious to preserve his freedom, he resisted the attempts of the Jesuits to persuade him to join their Order. After his father's death in 1757, he managed an estate for his elder brother, agriculture being one of his constant interests and a subject on which he wrote extensively and became an authority as a result of the long experiments made on the estate that he superintended.

He became a friend of Rousseau and in 1771, now an Abbé, he acquired from Gautier d'Agoty an earlier journal called ' Observations sur l'histoire naturelle, sur la physique et sur la peinture,' which had first appeared in 1752 and had continued under changed and expanded titles not too successfully. Rozier was also a friend of the economist and physiocrat, Turgot, the Minister who in his short term of office brought about the reform of the French gunpowder-office under the organizing genius of Lavoisier. Rozier approved of the Revolution of 1789. He fell in the siege of Lyon, a devoted priest seeking by personal example to maintain the courage and assuage the sufferings of his people (16).

Until December of 1778, Rozier was sole editor of the 'Observations.' In 1779 he took as his assistant his nephew, J. A. Mongez, who was later sole editor from 1780 to 1785, when De la Métherie became editor. The 'Observations' proved a striking success, because it achieved all its aims. Under the last editor the phlogiston theory was favoured and the papers on chemistry were not of a high standard, as the phlogiston theory was already being superseded by the new chemistry of Lavoisier except among some of the German chemists (17).

Much the same circumstances led Lorenz Crell to produce his various journals of chemistry; and since, strictly speaking, the 'Observations sur la physique' included subjects other than physics, Crell may be regarded as the founder of the first specialized scientific periodical. From 1778 to 1781 he published six volumes of his 'Chemisches Journal,' then thirteen volumes of 'Die neuesten Entdeckungen in der Chemie' (1781–86), and two volumes of his 'Chemisches Archiv' (1783), eight volumes of 'Neues Chemisches Archiv' (1784–91) and one volume of 'Neuestes chemisches Archiv' (1798), while his 'Chemische Annalen,' etc. ran to forty volumes from 1784 to 1803 with a 'Beitrage' of six volumes from 1785 to 1799. A tireless correspondent and editor. In the meantime, however, the 'Annales de Chimie,' to be referred to presently, had been founded in 1789.

From 1780 to 1789 twenty new journals appeared. Some of these edited by Crell we have already mentioned. The others included eleven German and two French periodicals and the well-known English periodical, 'The Botanical Magazine, or Flower Garden Displayed' (London, 1787, and in continuation), under the editorship of William Curtis (1746–99), who said of it that, whereas his 'Flora' had brought him praise, the 'Magazine' had brought him pudding (18). This beautifully illustrated journal with its one hundred and fifty volumes and nearly ten thousand coloured plates continues to the present day and is a splendid example of our periodical literature. Its editors have included Jackson, Hooker, Prain, Stapf and A. W. Hill.

But the most important journal to appear between 1780 and 1789, was undoubtedly the 'Annales de Chimie,' edited by de Morveau, Lavoisier, Monge, Berthollet, de Fourcroy, de Dieterich, Hassenfratz and Adet, the representatives in France of the new chemistry of Lavoisier. Born in the year of the Revolution, 1789, the 'Annales' outdistanced all its competitors. The 'Observations' under De la Métherie favoured the old theory and in any case became more physical in content: Crell's periodicals survived for some years. But the 'Annales,' of which ninety-six volumes appeared in Paris between 1789 and 1815 together with three volumes of ' Tables des matières ' (1801, 1807, 1821), became the leading journal of chemistry. It continued under the title of ' Annales de chimie et de physique ' in seventy-five volumes from 1816 to 1840 and later resumed its original name to continue to our own day, oldest survivor among chemical periodicals and a lasting monument to the great editors who launched it in the shadow of revolution. Lavoisier, its most distinguished editor and its enthusiastic founder, saw only the first few years of its success as he fell a victim to the Terror in 1794.

The last decade of the century was remarkable for the publication of twenty-five new journals, thirteen German, three French and five English. Notable among these was Gren's ' Journal der Physik,' which first appeared in 1790 in Halle and Leipzig and which, strictly speaking, is the first journal specially devoted to physics, since Rozier's ' Observations ' included material from other scientific fields. The English journals included ' The Repertory of Arts and Manufactures,' first published in London in 1794, Nicholson's ' Journal of Natural Philosophy, Chemistry and the Arts,' which first appeared in 1797 in London and continued until 1813. The most notable journal to appear in this decade was, however, ' The Philosophical Magazine,' edited by Alexander Tilloch, the first number of which was published in London in 1798. In the physical sciences it was to become second in importance only to the ' Philosophical Transactions ' of the Royal Society of London. This month of June, 1948, marks the one hundred and fiftieth anniversary of its foundation. Its distinguished services to science throughout the world and its long and eventful history are described in another article in this number.

REFERENCES.

(1) *Correspondance du P. Marin Mersenne*, ed. Tannery and de Waard, Paris, 2 vols. (1933–37).

(2) Even in the Cambridge of Newton's time it was found impossible to establish a "Philosophical Meeting," as Newton reported in a letter to Aston of Feb. 23, 1684/5, "for want of persons willing to try experiments" (C. R. Weld, 'History of the Royal Society,' etc., London, i. 305–6 (1848).

(3) See M. Ornstein, ' The Rôle of Scientific Societies in the Seventeenth Century, Chicago ' (1928), and Harcourt Brown, ' Scientific Organizations in Seventeenth Century France (1620–1680) ', Baltimore (1934). Attention is drawn to a recent valuable study by Miss R. H. Syfret, ' The Origins of the Royal Society ' (Notes and Records of the Royal Society of London, v. p. 75 (1948).

(4) Ornstein, *op. cit.*, pp. 198–209, and Brown, *op. cit.*, pp. 185–207.

(5) *Nouvelle Biographie Générale*, Paris, vol. 43, col. 189–91 (1864).

(6) Quoted from the Amsterdam reprint (1685) of the first volume of the *Journal* (1665). The writer has not been able to consult the original Paris issue of 1665.

(7) See a letter from Moray to Huygens, quoted by Harcourt Brown (*op. cit.*, p. 201).

(8) 'Dic. Nat. Biog.,' article on Oldenburg.

(9) *Nouvelle Biographie Générale*, Paris, vol. 34, col. 925–6 (1861).

(10) H. C. Bolton, 'A Catalogue of Scientific and Technical Periodicals (1665–1882),' etc., Washington, (1885), published in 'Smithsonian Miscellaneous Collections,' Washington (1887), vol. xxix. The number of journals listed is 5105. The second edition (Washington, 1897), carried the work to 1895 and included 8603 titles. The chronological tables given on pp. 1081ff of the second edition have been used in the present study. Bolton included in his bibliography all journals containing scientific material, but in the chronological tables he restricted the entries to purely scientific journals or seems to have done so, since he excluded the *Astronomisches Jahrbuch* (Berlin, 1774), the first successful astronomical journal and still surviving to our own time, presumably because the earlier volumes were ephemerides.

(11) The details of the title varied slightly from time to time, as may be seen by consulting Bolton (*op. cit.*, 2nd edn., p. 267, no. 2201).

(12) See Meldrum, 'The Eighteenth Century Revolution in Science—The First Phase,' Calcutta (1929), and McKie, 'Antoine Lavoisier,' London (1935).

(13) 'Observations,' etc., 1773, i. iii–vii.

(14) Many original memoirs from previous years were translated into French and given a greater currency in accordance with this policy.

(15) The influences of the *Encyclopédie* are evident in this passage.

(16) *Nouvelle Biographie Générale*, Paris, vol. 42, col. 827–30 (1863).

(17) See J. R. Partington and D. McKie, 'Historical Studies on the Phlogiston Theory,' Annals of Science, ii. 361 (1937); iii. 1 and 337 (1938), and iv, 113 (1939).

(18) See Samuel Curtis, 'William Curtis,' 1746–1799, etc., Winchester (1941), and an article by Hunkin in 'Endeavour,' v, 13 (1946).

SCIENTIFIC SOCIETIES TO THE END OF THE EIGHTEENTH CENTURY

By DOUGLAS McKIE, D.Sc., Ph.D.

THE first scientific societies (1) were founded in Italy in the sixteenth and seventeenth centuries. They were short-lived because of clerical opposition or suppression. The pioneer in their foundation was Giambattista della Porta (1538–1615), who in 1560 organized the Academia Secretorum Naturæ from a group of friends interested in experiments and meeting in his house in Naples. Membership was restricted to those who had a discovery to report or some advance in physics to communicate: and the members called themselves the "Otiosi." However, in a short time, Della Porta was accused of magic and sorcery, and summoned to Rome to defend himself; he was acquitted, but ordered to close the academy.

The second scientific society, the Accademia dei Lincei, was formed in Rome about 1600 or 1603 and reorganized in 1609. It was founded by the Prince Fredrigo Cesi and its members included Della Porta, Galileo, Pieresc, Fabio Colonna and others. After 1630, when its patron died, and especially after Galileo's condemnation in 1633, the continuation of the work of the academy became difficult, although it did not come to an end until 1657. This second academy had, however, succeeded where the first had failed; for a number of works were published under its auspices, notably Galileo's 'Il Saggiatore' (Rome, 1623) (2).

Italy, cradle of the Renaissance, saw also the foundation of the third scientific society, the famous Accademia del Cimento, or Academy of Experiment, at Florence in 1657, under the patronage of the Grand Duke Ferdinand II of Tuscany and his brother Leopold, of the Medici family, and friends and pupils of Galileo, Viviani and Torricelli. After ten years of brilliant activity, this third academy came to an end in 1657; in that same year, Leopold de Medici received the cardinal's hat that he had long coveted, and clerical opposition is said to have succeeded once again in terminating the work of those who sought to engage in the study of natural philosophy by means of experiments. The researches of the academy were published in Florence in 1666 under the title of 'Saggi di Naturali Esperienze, etc.', a second edition following in 1667. This first classic of modern experimental physics enshrines the work of the nine members of the academy (3); it was translated into English and into Latin, and later into French, and has been reprinted several times (4).

With the extinction of the Accademia del Cimento, leadership in physics passed from Italy to England and to France; it was a time marked also by the transition from Galileo to Newton. Meanwhile, developments had occurred in Germany. The Societas Ereunetica was founded by Joachim Jungius (1587–1657) at Rostock in 1622 for the study of experimental science, but survived for no longer than two years. The Collegium Naturæ Curiosorum, founded by the physicians of Schweinfurt in 1652 at the instance of Lorenz Bausch (1605–65), burgomaster and town-physician, was mainly concerned with medicine as related to science: in 1672 it received official recognition by the Emperor Leopold and became the Academia Caesareo-Leopoldina naturæ curiosorum. The second purely scientific society in Germany was the Collegium Curiosum sive Experimentale, founded at Altdorf in 1672 by Johan Christopher Sturm (1635–1703), professor of mathematics in the University of Altdorf, on the model of the Accademia del Cimento and for the experimental study of physics. Its importance diminished with the rise of the Prussian Academy in Berlin and the Bavarian Academy in Munich in the course of the eighteenth century.

Seventeenth-century France had shown a widespread interest in the "new philosophy" in which experiment and observation took the place of text and authority. Many societies and groups were formed both in Paris and in other cities for the purpose of indulging an interest in scientific matters and problems. Much of this interest was of an amateur kind, but it was active and general; and the various groups and societies attracted many cultured and scholarly members. In the period from 1620 to 1680, so thoroughly and admirably studied by Harcourt Brown (5), we find accounts of the groups in Paris associated with de Thou and with the brothers Pierre and Jacques Dupuy, the meetings held by Descartes in his lodgings about 1626, the conferences of Renaudot held in the Bureau d'Adresse which began in 1633 and continued until 1624, the meetings in the cell of Marin Mersenne (1588–1648), the Minorite father, the assemblies from 1654 to 1664 in the house of Montmor, the conferences of Henri Justel (1620–93) beginning in 1664, and the academy of the Abbé Bourdelot (1610–85), while in the provinces there were many societies and academies.

In England a very different state of affairs prevailed. Instead of many groups, there was only one. Its meetings from about 1645 onwards led eventually to the formation of the Royal Society of London on November 28, 1660, and its incorporation by Royal Charter on July 15, 1662 (6). The successful foundation of the Royal Society proved to be one of the factors in the formation of the Académie Royale des Sciences of Paris in 1666. Both societies, or, as they should be more properly called, national academies, prospered. The close of the seventeenth century found them securely established and actively pursuing their aims. Their history from their formation to the end of the seventeenth century is too well-known to detain us here.

The eighteenth century therefore opened with two active national academies concerned with science and with the publication of new facts and discoveries, the Royal Society of London and the Académie Royale des Sciences of Paris. The Italian societies had died away; the German societies had not been entirely successful and Germany needed the enthusiasm and genius of a Leibnitz for the foundation of a national academy of science. However, the seventeenth century had given two great models to the eighteenth century and national academies soon increased and multiplied: and independent and specialist societies were to appear as the century progressed in many countries in Europe, while a beginning was made in America.

In 1700 the Paris Académie opened with a new constitution and was completely reorganized. Its membership was increased from twenty to seventy. Official and continuous publication of its work was now undertaken and the long series of 'Histoire et Mémoires' began to appear in 1702, when the volume for 1699 was published as the first of the series. The 'Philosophical Transactions' of the Royal Society of London and the 'Histoire et Mémoires' of the Paris Royal Academy of Science were the outstanding official organs of the national scientific academies of the eighteenth century. The Académie and all other academies of the old régime were suppressed during the Revolution in 1793, and the Institut National was founded to replace them in 1795. In 1816 the Académie des Sciences was revived as part of the Institut.

In Germany in this same year of 1700 a national academy of science was at last established. Its foundation was essentially the result, not of the interest of a group of natural philosophers, but of over thirty years of compaigning and perseverance on the part of Gottfried Wilhelm von Leibnitz (1646–1716). Modelled on the academies of London and Paris, the new society received its charter on July 11, 1700, as the Societas Regia Scientiarum. Its official organ, 'Miscellanea Berolinensia,' first appeared in 1710, twelve of the sixty memoirs in it being contributed by Leibnitz. However, Frederick I had been succeeded by Frederick William I, and Leibnitz was not popular with the new monarch, who regarded scientific work as mere trifling. Moreover, relations between Leibnitz and the Academy had deteriorated. The second volume of the 'Miscellanea' did not appear until 1723, seven years after his death. Eight volumes were published between 1710 and 1744.

Leibnitz had a hand in the formation of other societies. He tried to found an academy in Dresden, but a war intervened. After his experience in Berlin, he may have become aware of the difficulty of obtaining adequate financial support for such societies; and his proposals for an academy in Saxony included the suggestion that money was to be provided from the tobacco monopoly, a source of revenue which has sometimes been tapped increasingly elsewhere for purposes that do not always arouse much enthusiasm. Leibnitz achieved a greater success in Russia. In 1711 he met Peter the Great, who was much impressed with his very ractical ideas for the improvement of national education and for the liffusion and increase of scientific knowledge. Although the Academy of St. Petersburg was not founded until twelve years after Leibnitz had died, it was derived directly from his conversations with Peter the Great. In Vienna, too, in 1710–12, Leibnitz, but for Jesuit opposition, would have succeeded in instituting an academy, the foundation of which was delayed until 1846.

The plan for the St. Petersburg Academy was approved by the Emperor early in 1724, but his sudden death arrested further progress. In December of 1725, the Empress Catherine I formally established the Academy and the first meeting was held on the 27th of the month. Throughout the eighteenth century, much valuable work was carried out by the academicians, especially on the natural resources of Russia. The history of these activities is still largely unknown. On the strictly scientific side, it need only be mentioned that many members of other nations were invited to join the Academy, which during this period included among its members Daniel Bernoulli, Krafft, Richmann, Euler and many others. Michail Lomonossov (1711–65), a versatile genius and forerunner of Lavoisier in rejecting the phlogiston theory in chemistry, was a member from 1741 until his death. The first publications of the Academy appeared in 1728 as 'Commentarii Academiæ Scientiarum Imperialis Petropolitanæ' and continued until 1747, when the title was changed to 'Novi Commentarii, etc.', which in turn became 'Acta Academiæ, etc.', in 1777. As in the Paris Académie des Sciences, the members, whose number was limited to fifteen, were salaried; in addition, they received the title of professor in the various subjects in which they worked.

In Sweden, the Kongliga Svenska Vetenskaps Akademien arose from a society, the Collegium Curiosorum, which first met in 1739 and included Linnæus among its six members; in 1741 it was formally established as the Royal Swedish Academy, and its work and publications became well-known during the eighteenth century, forty volumes of the first series having appeared up to 1779. In Denmark, the Kongelige Danske Videnskabernes Selskab, the Royal Academy of Science, was constituted by Christian VI at Copenhagen in 1743.

Before the middle of the century Benjamin Franklin had urged the formation of a scientific society in America in a pamphlet entitled 'A Proposal for Promoting Useful Knowledge among the British Plantations in America,' dated from Philadelphia on May 14, 1743 (7). It contained detailed suggestions of a practical kind, as might be expected from its author, and among these it recommended "That One Society be formed of Virtuosi or ingenious Men residing in the several Colonies, to be called *The American Philosophical Society* who are to maintain a constant Correspondence. That *Philadelphia* being the City nearest the Centre of the Continent-Colonies, communicating with all of them northward and southward by Post, and with all the Islands by Sea, and having the Advantage of a good growing Library, be the Centre of the Society. That at *Philadelphia* there be always at least seven Members, *viz.*, a Physician, a Botanist, a Mathematician, a Chemist, a Mechanician, a Geographer, and a general Natural Philosopher, besides a President, Treasurer and Secretary." It concluded with the statement that "*Benjamin Franklin*, the Writer of this Proposal, offers himself to serve the Society as their Secretary, 'till they shall be provided with one more capable." The American Philosophical Society Held at Philadelphia for Promoting Useful Knowledge was formed shortly afterwards. It celebrated its bicentenary in 1943. Franklin was President from 1769 to 1790 and Thomas Jefferson from 1797 to 1815. The Society's. 'Transactions' began to appear in 1771; its 'Proceedings' date from 1838. Another scientific society was formed in America during this period, namely, the American Academy of Arts and Sciences, founded at Boston in 1780. Its 'Memoirs' date from 1785, its 'Proceedings' from 1846.

Shortly after the middle of the century a private scientific society was again formed in Italy at Turin in 1757; and 1783 it became the Reale Accademia della Scienze, or Royal Academy of Science. The famous Accademia dei Lincei was not revived until the nineteenth century, when the Reale Accademia dei Lincei was constituted in 1870. The Lisbon Academy, the Academia Real das Sciencas, originated in 1779 and was reorganized in the nineteenth century. In Spain an academy was formed in 1657 in Madrid modelled on Della Porta's Academy at Naples; it was reorganized in 1847 with a constitution corresponding to that of the Paris Academy.

Provincial academies were formed in France in considerable numbers during the eighteenth century, namely, in Montpellier, Bordeaux, Dijon and other cities. The earliest society in Brussels dates from 1769 and its interests were literary; scientific academies in the Low Countries date from the nineteenth century. In Germany academies were formed in Göttingen in 1750, Erfurt in 1754, Mannheim in 1755 and Munich in 1759.

Enough has been said in the space of this short paper to indicate the rise and development of the scientific academies in the eighteenth century. Space prevents a full account of the proliferation of provincial scientific societies throughout Europe, but we shall give some brief account of some of the interesting and important, and some of the less important but none the less interesting, scientific societies and specialized scientific societies that arose in Great Britain during the eighteenth century. The detailed history of this phase of the history of science in Europe awaits further study.

Attempts had been made in 1683 to form a society in Dublin after the pattern of the Royal Society of London. A year later the society was established with Sir William Petty as President and the members numbered thirty-three at the close of that year. Though in correspondence with the parent society in London, the Dublin Society did not prosper; Ireland was disturbed by the political changes that occurred shortly afterwards (8). The Royal Irish Academy was formed late in the eighteenth century from a society in Dublin organized about 1782; its 'Transactions' first appeared in 1788.

In Scotland affairs went differently despite the political uncertainties of the eighteenth century. There was a philosophical society in Edinburgh in 1705 in communication with the Royal Society of London. Later the Earl of Morton supported the plans of Maclaurin, the mathematician, for extending the activities of a society formed in Edinburgh in 1731, and in 1739 the Philosophical Society of Edinburgh was accordingly instituted. Its publications appeared in three volumes as 'Essays and Observations, Physical and Literary, Read before a Society in Edinburgh, and Published by them' (Edinburgh, 1754–71). The second volume, published in 1756, contained the classical researches of Black on the mild and caustic alkalis, an extended study read to the Society in 1755 of work submitted for the degree of M.D. (Edinburgh) in 1754. In 1783 the Society became by charter the Royal Society of Edinburgh (9). Its roll of Fellows contains many names famous in the history of science. Mention might also be made of the society or club formed by the professors in the University of Glasgow during the second half of the eighteenth century. It was to this society on April 23, 1762, that Joseph Black announced his classic discovery of the latent heat of fusion of ice. Black refers to this body as "the Philosophical Club, or Society of Professors and others in the University of Glasgow" (10).

As the century drew towards its last two decades, Richard Watson (1737–1816), absentee Bishop of Llandaff and Regius Professor of Divinity in the University of Cambridge, lately Professor of Chemistry in that university, wrote in his 'Chemical Essays' in 1781:—"The Philosophical Transactions at London, the Histoire de l'Academie Royale des Sciences at Paris, the Saggi d'esperienze di Acad. del Cimento at Florence, the Journal des Scavans in Holland, the Ephemerides Academiæ naturæ curiosorum in Germany, the Acts of the Academy of Copenhagen, and the Acta eruditorum at Leypsic; all these works began to be published within the space of twenty years from 1665, when our Royal Society first set the example, by publishing the Philosophical Transactions. To these may be added, the works of the academies of Berlin, Petersburgh, Stockholm, Upsal, Bononia, Bourdeaux, Montpelier, Gottingen, and of several others which have been established within the course of the present century. Near a thousand volumes have been published by these learned societies within less than 120 years" (11). Societies and academies had multiplied greatly; but, as there was still a need for further periodicals in this period and for the specialized periodical, so also was there need for further societies and for specialized societies.

Recently Kendall (12) has drawn attention to some early chemical societies in Scotland. He showed that the Chemical Society mentioned in Joseph Black's papers was formed from students attending Black's lectures on chemistry in Edinburgh in 1785; that there was a Chemical Society in the University of Glasgow as early as 1786; and that in the early months of 1800 another chemical society was formed in Edinburgh. Wilson ⁽¹³⁾ had already drawn attention to the formation of a second chemical society in Glasgow in 1798. Until Kendall brought these facts to light, it was generally considered that the oldest chemical society was the Chemical Society of Philadelphia, which was founded by James Woodhouse in 1792.

A Botanical Society was founded in London in 1721 by Johann Jakob Dillen or Dillenius (1681–1747), Sherardian Professor of Botany in the University of Oxford, and John Martyn (1699–1768), son of a London merchant and amateur botanist and entomologist. The Society met at the Rainbow Coffee House in Watling Street, and later in a private house, at six o'clock on Saturday evenings, but it lasted for only five years. In 1745 the Aurelian Society for the study of insects began to meet in the Swan Tavern in 'Change Alley, but the fire in Cornhill in 1748 destroyed its library and its collection, and the Society appears to have perished also in the disaster. A second Aurelian Society lasted from 1762 to 1766. From 1780 to 1782 there was a Society of Entomologists of London. In 1782 on October 21, "at Mr. Dean's, the Corner House by the Turnpike, Pimlico," the Society for Promoting Natural History was founded. Later the members met in the Black Bear in Paccadilly, in "Greenwoods" or Leicester House in Leicester Square, in the house of one of its members, George Spence, in Pall Mall, and in its own rooms at 19 Warwick Square. Later it met in the York Coffee House in St. James's Street. Meetings were held at seven in the evening every month "on the Monday next following the Full Moon," this precaution being taken on account of the darkness and insecurity of the streets of eighteenth-century London. The historian of the Linnean Society records that the charges at the York Coffee House were invariably 10*s*. 6*d*. for the rooms and £1. 1*s*. for sundries, and adds that "it is not mentioned what the sundries were, but it is improbable that they consisted purely of ink." The general dissatisfaction of the abler members of the Society with its methods and its work led to the formation of the Linnean Society in 1788 by James Edward Smith (1759–1828), Samuel Goodenough (1743–1827) and Thomas Marsham (1747 or 8 –1819). Smith will be remembered for acquiring in 1784 the Linnean Herbarium, library and manuscripts, which subsequently became the property of the Society and remain its proudest possession. The foundation of the Linnean Society in 1788 marks the successful establishment of the first specialist scientific society in England. The Society for Promoting Natural History generously handed its funds, books and collections to the Linnean Society some years later. The history and reputation of the Linnean Society since its foundation are too well-known to need any further reference here ⁽¹⁴⁾.

The most important provincial scientific society in England was formed in Manchester in the later part of the eighteenth century. For several years before its foundation, a weekly club had been organized for those with an interest in literature and philosophy, *i. e.*, natural philosophy; the club became in 1785 the Literary and Philosophical Society of

Manchester. The first volume of the 'Memoirs' of the Society was published in Warrington in 1785. The Preface to this volume brings vividly before us the thought and feeling of another age (15): "The numerous Societies, for the promotion of Literature and Philosophy, which have been formed in different parts of Europe, in the course of the last and present centuries, have been not only the means of diffusing knowledge more extensively, but have contributed to produce a greater number of important discoveries, than have been effected in any other equal space of time.

"The progress that has been made in Physics and the Belles Lettres, owes its rapidity, if not its origin, to the encouragement which these Societies have given to such pursuits, and to the emulation which has been excited between different academical bodies, as well as among the individual Members of each institution. The collecting and publishing the more important communications which have been delivered to them, have saved from oblivion many very valuable discoveries, or improvements in arts, and much useful information in the various branches of science. These their modest authors might have been tempted to suppress, but for the respectable sanction of societies of men of the first eminence and learning in their respective countries, and the easy mode of publishing, which their volumes of Transactions afford."

But Manchester, then as now, had no love for what might be called metropolitan parochialism: "Though, in France, Societies for these purposes have been instituted in several of the provinces, in England, they have almost been confined to the Capital; and however great have been the advantages resulting from the researches of the learned bodies, who are incorporated in London, it seems probable, that the great end of their institutions, the promotion of arts and sciences, may be more widely extended by the forming of Societies, with similar views, in the principal towns in this kingdom.

"Men, however great their learning, often become indolent, and unambitious to improve in knowledge, for want of associating with others of similar talents and acquirements: Having few opportunities of communicating their ideas, they are not very solicitous to collect or arrange those they have acquired, and are still less anxious about the further cultivation of their minds.—But science, like fire, is put in motion by collision.—Where a number of such men have frequent opportunities of meeting and conversing together, thought begets thought, and every hint is turned to advantage. A spirit of inquiry glows in every breast. Every new discovery relative to the natural, intellectual or moral world, leads to a farther investigation; and each man is zealous to distinguish himself in the interesting pursuit.

"Such have been the considerations that have led to the institution of the Literary and Philosophical Society of Manchester."

We have quoted this at length, because it is the vindication of all such societies in the period under consideration and above all in the expanding years with which the century closed. This great provincial society has had a long and distinguished history; its roll of members has borne many famous names and we need recall only one, that of John Dalton, who read to the society his first papers on the atomic theory and exhibited at its meetings the first table of atomic weights. The whole world of science regrets that war brought complete destruction to the buildings and historic treasures of the oldest learned society in England outside London.

Birmingham, too, had its society, the Lunar Society, whose members were sometimes referred to as the "Lunatics" (16). The name was derived from their practice of meeting when the moon was full, in order to have light for the journey home, as we have seen was the custom in London in the Society for Promoting Natural History. The Lunar Society existed as early as 1768 and was founded by Matthew Boulton. It included among its members Erasmus Darwin, Samuel Galton, Richard Lovell Edgeworth, Thomas Day (author of 'Sandford and Merton'), James Keir, William Withering, James Watt and Joseph Priestley; it met at two in the afternoon for dinner and the meetings lasted until eight; many distinguished guests were entertained, and among them were Wedgwood, Banks, William Herschel, Smeaton, Samuel Parr, Afzelius, Solander, De Luc and others. The influence of the society in the stimulation of scientific inquiry was outstanding in that remarkable age; and it persisted long after the meetings had to be abandoned on account of there being too few survivors to continue them.

A different kind of society was the Spitalfields Mathematical Society (17), founded in 1717 by John Middleton and originally said to be formed of Spitalfields weavers, later of tradesmen and professional men of the neighbourhood. Little is known of the early history of this group, but it seems that every member, on penalty of being fined twopence, had to answer any mathematical or philosophical question that was put to him. The members were provided with pipes and porter to improve the occasion and doubtless to help them to concentrate on their problems. Philosophical lectures were given and charges for admission to them were made. In 1799 this activity brought the society almost to extinction; for a gang of informers laid an information against the members that they had taken money for an unlicensed entertainment, namely, a philosophical lecture. A voluntary guarantee fund of £254 was raised among the forty or fifty members in a few days and one of them took over the defence without fee; but scandal had been brought upon them and their lectures were not so well attended in future, while the informers were legally free to repeat the charge when they wished. After this, the supply of pipes and porter had to be left to the members; and it was found that "the small number attending on Saturday evenings arises in great measure from their not drinking in common," whereupon the custom was resumed. Every member had to lecture in turn and medals were awarded. Much apparatus was collected and also a library of over 2500 volumes, including many rare books. There were no distinguished names among its members. In 1846, after a life of one hundred and thirty years, the society was absorbed by the Royal Astronomical Society, who received the library and other items, including several valuable portraits.

In London in 1793 Bryan Higgins, encouraged by a number of friends with " a taste for Philosophical Experiments and Conversations," founded a new society for such experiments and conversations. The first meeting was held in Higgins's House on January 25, 1794, at 8 o'clock, and the Society for Philosophical Experiments and Conversations was formally instituted. Weekly meetings were held at the same place " during the Session of Parliament." Higgins lent his apparatus for the experiments. The annual subscription was five guineas. Fifty intending members had subscribed before the first meeting. Others were elected later. The Chairman was Field-Marshal Conway ; the " Didactic Experimenter " was " Dr. Bry. Higgins " ; and " Mr. Thomas Young " was one of three " Assistant Experimenters." The procedure at the meetings consisted in the performance of experiments followed by discussion of their results. Many of Lavoisier's classic experiments were repeated ; and the members appear to have been mainly interested in the new chemistry of Lavoisier, especially the problems of combustion and respiration, of the composition of the air and the properties and measurement of caloric. Twenty-one meetings were held, the last on June 21, 1794. A committee of publication met on July 3 of that year ; and at this point the activities seem to have ended (18).

Towards the close of the century two societies were formed in London, the Askesian Society in March 1796 and the British Mineralogical Society in April 1799. The latter combined with the former in 1806, and some of the members of the combined society instituted the Geological Society in 1807 (19). In the nineteenth century many more specialist societies arose throughout the world ; but it was the eighteenth century that gave the vital impetus to a movement that is characteristic of our time.

In closing, we can make only brief reference to two other developments at the very close of the century. Between 1794 and 1798 Beddoes had succeeded in establishing with the aid of subscriptions, mostly from Wedgwood and Watt, the Pneumatic Institution at Clifton for the experimental study of the medicinal effects of gases. The institute for research had unobtrusively made its first appearance. The name and genius of Humphry Davy is associated with this beginning and with the other innovation that followed shortly in the foundation in London in 1799 by Sir Benjamin Thompson, Count Rumford, of another institution, chartered in 1800 as the Royal Institution for the purpose of diffusing scientific knowledge and its applications to everyday life. Its subsequent history has been one of research and classic discoveries in chemistry and physics.

REFERENCES.

(1) See M. Ornstein, 'The Rôle of Scientific Societies in the Seventeenth Century,' Chicago, 1928, and 'Encyclopædia Britannica,' 11th ed., article " Academies " (section I.).

(2) Other works published were Galileo's book on sun-spots, 'Istoria e Dimostrazioni intorno alle Macchie Solari, etc.', Rome, 1613, and Oviedo's 'Thesaurus Mexicanus,' Rome, 1651.

(3) The nine members were Borelli, Candido and Paolo del Buono, Magalotti, Marsili, Oliva, Redi, Renaldini and Viviani.

(4) An English translation of the "Saggi" by Richard Waller was published in London in 1684 as 'Essayes of Natural Experiments made in the Academie del Cimento, etc.' A Latin version by van Musschenbroek, 'Tentamina experimentorum, etc.' was published in Leyden in 1731; and a French translation appeared in 1755. The work was reprinted in Italian in the 'Atti e Memorie inedite, etc.' in Florence in 1780 and again in 1841.

(5) Harcourt Brown, 'Scientific Organizations in Seventeenth Century France (1620–1680),' Baltimore, 1934.

(6) We draw attention to a recent valuable study on "The Origins of the Royal Society" by Miss R. H. Syfret in 'Notes and Records of the Royal Society of London' (v. p. 75, 1948).

(7) Quoted from the reproduction of this very rare pamphlet in Proc. Amer. Phil. Soc. lxxxvii. pls. iii. and iv. (1943). This number contains an interesting paper by van Doren (pp. 277 ff) on the beginnings of the American Philosophical Society, with a reproduction of the beautiful portrait of Franklin by Peale (pl. ii.).

(8) C .R. Weld, 'History of the Royal Society, etc.', London, i. pp. 299–302, 383–84 (1848).

(9) *Ibid.* i. p. 383, and ii. p. 24. See also Kendall, 'Endeavour,' v. p. 54 (1946).

(10) McKie and Heathcote, 'The Discovery of Specific and Latent Heats,' London, pp. 19 and 35 (1935).

(11) "Chemical Essays', 1st edn., London, i. pp. 35–36 (1781). Watson refers to the 'Journal des Scavans' as appearing in Holland, and presumably he was aware only of the Amsterdam reprint, which would explain why he states that the Royal Society "first set the example" by publishing the 'Philosophical Transactions,' whereas the 'Journal' preceded the 'Transactions' by two months (January to March, 1665).

(12) "Some Eighteenth-Century Chemical Societies" ('Endeavour,' i. p. 106, 1942). This paper mentions also a chemical society in London in 1782 meeting once a fortnight in the Chapter Coffee-house. See also Sir Wm. Ramsay's 'Life and Letters of Joseph Black, M.D.', London, pp. 110–11 (1918); and Kendall, J. Chem. Education, xii. p. 565 (1935).

(13) 'Annals of Science,' ii. p. 451 (1937).

(14) For the details of this paragraph the writer is indebted to A. T. Gage's 'History of the Linnean Society of London,' London, pp. 1–10 (1938).

(15) Mem. Lit. & Phil. Soc. Manch. i. pp. v–vii (1785).

(16) See T. E. Thorpe, 'Joseph Priestley,' London, 94 ff (1906), and Anne Holt, 'A Life of Joseph Priestley,' London, 127 ff. (1931).

(17) For information about the Spitalfields Society the writer is indebted to R. A. Sampson's contribution to the 'History of the Royal Astronomical Society 1820–1920,' ed. J. L. E. Dreyer and H. H. Turner, published by the Royal Astronomical Society, London, pp. 99–104 (1923).

(18) 'Minutes of the Society for Philosophical Experiments and Conversations,' London (1795). The publication committee of the Society resolved to publish this account of the work done during 1794. No further volumes appeared.

(19) See H. B. Woodward, 'The History of the Geological Society of London,' London, pp. 6–10 (1907).

THE TEACHING OF THE PHYSICAL SCIENCES AT THE END OF THE EIGHTEENTH CENTURY

By F. SHERWOOD TAYLOR, Ph.D.

An examination of scientific education at any period resolves itself into two enquiries, concerning the manner in which scientists of that period were in fact educated, and concerning the means of education that were available. Scientists today pass, with rare exceptions, through the officially provided channels of secondary school, or its equivalent, and university, but in the eighteenth century a considerable proportion of the great men of science received no formal instruction in the subject they made their own, or attended a course in it as a discipline ancillary to some other subject, normally medicine.

(1) *The manner of Education of the Scientists born between* 1760 *and* 1800.

For the purpose of this paper, the author selected the names of 68 persons who were born between 1760 and 1800, and ascertained the steps by which they took up the study of the subject in which they became famous. The results may be expressed in the form of the tables, given below. It is not claimed that the classification of these scientists is unimpeachable, for in some cases there is doubt concerning the category to which they may be most properly assigned.

Table I.—Men of science who received no formal higher education.

Name	Year of Birth	Remarks
Ampère, A. M. ..	1775	Educated himself privately in mathematics.
Baily, F.	1774	Became an astronomer in middle life.
Bessel, F. W. ..	1784	Went as super-cargo to study mathematics and navigation.
Brunel, M. I. ..	1769	Educated for priesthood.
Chevreul, M. E. ..	1786	Enters Vauquelin's manufactory of chemical products.
Daguerre, L. J. M. ..	1789	Painter.
Dalton, J.	1766	Self-educated.
Daniell, J. F. ..	1790	Business career ; sugar refining.
Faraday, M.	1791	Attended some private lectures.
Loudon, J. C. ..	1783	Apprenticed to a nursery-man.
Niépce, J. N. ..	1765	Sub-lieutenant in army.
Smith, W.	1769	Became a surveyor, learnt geology by observation.
Stevenson, G. ..	1787	No education.
Sturgeon, W. ..	1783	No education ; became a private soldier.
Wheatstone, C. ..	1802	Musical instrument maker.

TABLE II.—Men who received a higher education in non-scientific subjects.

Name	Year of Birth	Remarks
Amici, G. B.	1786	Engineer-architect at Bologna.
Avogadro, A. ..	1776	Jurisprudence.
Brewster, D.	1781	Studied for ministry and incidentally heard scientific lectures at Edinburgh.
Buckland, W. ..	1784	Theological studies.
Leslie, J.	1766	Theological studies.
Schweigger, J. S. C. ..	1779	Philosophical and classical studies.

TABLE III.—Men who received a predominantly mathematical education, including those whose training was military, for artillery or engineers.

Name	Year of Birth	Remarks
Arago, D. F. J. ..	1786	École polytechnique.
Biot, J. B.	1774	Artillery.
Bolyai, W. (pére) ..	1775	Mathematical education at Göttingen.
Fourier, J. B. J. ..	1768	Military School.
Fresnel, A. J. ..	1788	École polytechnique: became an engineer.
Gauss, K. F.	1777	Mathematical education at Göttingen.
Herschel, J. F. W. ..	1792	Cambridge, Senior Wrangler.
Malus, E. L.	1775	Military engineering: École polytechnique.
Nobili, L.	1784	Artillery Officer.
Ohm, G. S.	1787	Mathematics: University of Erlangen.
Quetelet, L. A. J. ..	1796	Mathematical education.
Sedgwick, A.	1785	Cambridge Wrangler: knew little or no geology until made Professor.
Talbot, W. H. F. ..	1800	Cambridge, 12th Wrangler.
Whewell, W.	1794	Cambridge, 2nd Wrangler.
Woodhouse, R. ..	1773	Cambridge, Senior Wrangler.

TABLE IV.—Men who attended general scientific courses, other than medical or mathematical, at a university.

Name	Year of Birth	Remarks
Buch, L. von . .	1774	Studied geology under Werner at Freiberg.
Chamisso, L. C. A. . .	1781	Royal College of France.
Cuvier, G. L. C. F. D.	1769	Course in philosophy, mathematics and natural science at Stuttgart.
Jameson, R	1773	Studied geology under Werner at Freiberg

TABLE V.—Men who began their scientific career through medicine or pharmacy.

Name	Year of Birth	Remarks
Beddoes, T.	1760	Medical education. London and Edinburgh.
Berzelius, J. J. . .	1779	Medical and chemical studies at Upsala.
Brande, W. T. . .	1788	Apprenticed to an apothecary with a view to medicine.
Brongniart, A. . .	1770	École de Medecin, but previously École des Mines.
Brown, R.	1789	Marischal College, Aberdeen, then medical studies at Edinburgh.
Candolle, A. P. de . .	1778	Medical studies, unwillingly undertaken.
Carlisle, A.	1768	Medical pupil: surgeon. Electrical researches incidental.
Cooper, A.	1768	Apprenticed to surgeon: also medical school at Edinburgh.
Daubeny, C. J. B. . .	1795	Medical studies at Edinburgh.
Davy, H.	1778	Apprenticed to a surgeon: then employed at Dr. Beddoes' Pneumatic Institution.
Dobereiner, J. F. . .	1780	Studied pharmacy, then set up chemical manufactory.
Dulong, P. L. . .	1785	Abandoned a medical career in favour of chemistry.

TABLE V.—(*cont.*)

Name	Year of Birth	Remarks
Dumas, J. B. A. ..	1800	Apprenticed to an apothecary: then moved to Geneva.
Dutrochet, R. J. H. ..	1776	Military marine: medical studies from 1802.
Gmelin, L.	1788	Medicine and chemistry; Göttingen, Tübingen, Vienna.
Henry, J.	1797	Studied for medicine.
Laënnec, R. T. H. ..	1781	Medical career.
Oersted, J. C. ..	1777	Pharmacist's apprentice.
Oken, L.	1779	Medical classes Wurzburg and Göttingen.
Poisson, S. D. ..	1781	Apprenticed to a surgeon, ultimately École polytechnique.
Savart, F.	1791	Medical education, then army.
Seebeck, T. J. ..	1770	Medical education.
Thénard, L. J. ..	1777	Studied pharmacy, then entered Vauquelin's laboratory.
Thompson, J. V. ..	1779	Apprentice in medicine and surgery: army surgeon.
Thomson, T.	1773	Medical studies at Edinburgh.
Wöhler, F.	1800	Medical studies at Marburg.
Wollaston, W. H. ..	1766	Medical degree, Cambridge.
Young, T.	1772	Studied for medical profession, London and Edinburgh.

TABLE VI.—Summary of Tables I–V.

Entry to a scientific profession	Number
Without higher education..	15
With higher education in non-scientific subjects	6
With higher mathematical education.	15
With scientific higher education other than medical	4
With pharmaceutical or medical education. ..	28
Total	68

It appears very clearly that the avenues that led to scientific eminence were :—

(i) The private pursuit of scientific studies by those who had no formal training.

(ii) Mathematics.

(iii) Medicine.

These three modes of education roughly correspond to the facilities available in the eighteenth century.

Self-education in science follows from the development of the taste for the subject and the availability of books that can afford a means of instruction for beginners. To understand the prevalence of self-education in science, we must therefore consider the means of popular scientific education in the eighteenth century.

The other two modes of entry follow the structure of the university courses of the time. Formal courses in mathematics were generally available, and this led to the study of its practical exemplification in mechanics, optics and astronomy. There were also formal courses in medicine, which became greatly developed in Scotland from the earlier half of the eighteenth century and in England in its closing years. These courses included instruction in chemistry and botany. The British universities also gave instruction in experimental philosophy, which covered physics and astronomy. These courses did not, however, lead to a degree or a profession, and, in fact, the attendance seems to have been small and desultory.

The same considerations seem to have applied on the continent, but here another factor emerges, namely the importance attached to military studies. The military tradition of Vauban in France, and the prestige of the wonderful succession of French mathematicians of the seventeenth and eighteenth centuries, led to a strong bias in favour of mathematical studies. It will be noted that the great scientific men who were receiving their education in the late eighteenth century were predominantly British or French: the majority of the great German scientists were born after 1800.

2. *The Popularization of Science in the Eighteenth Century.*

The tremendous import of the discoveries of the seventeenth century, especially of the unification of the cosmos by the Newtonian system, found its response in the wide popularization of science. The findings of science were conveyed to the interested public by means of encyclopædias, popularly written books and public lectures.

The first of the scientific encyclopædias written in English was John Harris's 'Lexicon Technicum,' first published in 1704. There followed throughout the eighteenth century a flood of similar and more extensive publications which were evidently widely read. Next in importance was Ephraim Chambers' 'Cyclopædia: or an Universal Dictionary of Art and Sciences' (1728), covering a wider range than Harris's work. The 'Encyclopædia Britannica or a Dictionary of Arts and Sciences, compiled by a society of gentlemen in Scotland' appeared in 1771: its special emphasis on scientific subjects is to be noted. There appeared numerous other English encyclopædic works on science, of which space forbids mention.

The same phenomenon was manifest on the continent. Thomas Corneille's 'Dictionnaire des arts et des sciences' appeared in 1694. In Germany, J. Hübner's 'Curieuses und reales Natur-Kunst-Berg-Gewerb- und Handlungs-Lexicon' appeared in 1712 and J. T. Jablonski, in 1721, published his 'Allgemeines Lexicon der Künste und Wissenschaften.' Zedler's 'Universal Lexicon' appeared in 64 volumes between 1732 and 1750. All these and others were eclipsed by the 'Encyclopédie ou dictionnaire raisonnée des sciences des arts et des mètiers par une société des gens de lettres, mis en ordre et publiée par MM. Diderot et d'Alembert,' 1751–72. This work was an admirable account of the sciences and arts. It was, moreover, an instrument of propaganda for science and its methods, and against the Church, the government and Christianity. Whatever its merits, its importance cannot be overestimated.

There were then, at the end of the eighteenth century, a number of encyclopædias dealing with special sciences, with sciences and arts, and with all knowledge, in a manner intended to inform and interest those without scientific knowledge or experience.

Numerous popular books on science appeared in the eighteenth century. In this country the most esteemed expositors of the Newtonian system were Colin Maclaurin * and James Ferguson †. The instrument-makers naturally found it to their interest to popularize scientific experiment. Thus, we find good clear works on the sciences or aspects thereof by Benjamin Martin the optician and George Adams, father and son, whose works cover a wide field in physics and astronomy and are far too numerous to record. Some popular works were addressed to "young gentlemen and ladies" and it is quite clear that science found a hearing in every type of society, from the nobleman who equipped a laboratory or observatory for the use of a man of science who enjoyed his patronage, to the apprentice seeking an education in his scanty leisure.

On the continent also many popular scientific works were published. In France the works of Voltaire and the Marquise du Châtelet were of importance in popularizing science. The former's 'Éléments de la philosophie de Newton' (1739), and the latter's translation of Newton's 'Principia' into French, published in 1756, were of importance. At a later period the influence of the works of Buffon can hardly be overestimated.

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* An account of Sir Isaac Newton's philosophical discoveries in four books. London, 1748.

† (*a*) 'Astronomy explained upon Sir Isaac Newton's principles and made easy to those who have not studied Mathematics.' James Ferguson. London, 1756.

(*b*) 'An easy introduction to astronomy for young gentlemen and ladies. James Ferguson. 2nd Edn.' 1769.

(*c*) 'Lectures on select subjects in Mechanics, Hydrostatics, Pneumatics and Optics; with the use of the Globes; the art of Dialling; and the calculation of the Mean Times of new and full Moons and Eclipses.' James Ferguson. 1760.

Scientific periodicals date from the seventeenth century; *e. g.*, 'Philosophical Transactions'; the 'Journal des Sçavans' and the 'Acta Eruditorum.' In the latter part of the eighteenth century three tendencies are notable, the publication of new general scientific journals, of specialized scientific journals and of popular periodicals dealing with scientific matters.

Between 1770 and 1800 we may note, in the British Isles, the 'Philosophical Magazine' (1798), 'Nicholson's Journal' (1797), 'Repertory of Arts and Manufactures' (1794), 'Medical and Philosophical Commentaries' (1773), 'Botanical Magazine' (1787), 'Memoirs of the Literary and Philosophical Society of Manchester' (1785); in Germany, the 'Allgemeines Journal de Chemie,' 'Archiv für physiologie' (Leipsic, 1798), 'Der Naturforscher' (Halle, 1774), 'Journal der physik' (1790); in Switzerland, numerous biological journals, 'Magasin für die Liebhaber der Entomologie' (1778), 'Archiv der Insectengeschichte' (1781), translated into English in 1795, 'Annalen der Botanik' (1781), and others. These works were chiefly intended for men of science, but throughout the century the numerous magazines, of which the 'Gentleman's Magazine' is the best known example, frequently published scientific news in a popular form.

An important means of popular education was the hearing of courses of lectures, specially prepared for general audiences. The first of these are said to be the lectures of J. T. Desaguliers *, between 1712 and 1742. He was followed by Stephen (Triboudet) Demainbray, LL.D., who lectured at various periods between 1740 and 1780, became tutor to George III in 1754, and was afterwards first Observer of the King's Observatory at Kew. The syllabuses of some of his courses are extant. They treat Mechanicks, Motion, Hydrostaticks, Pneumaticks, Magnetism, Opticks and Astronomy, and indeed cover much the same ground as the lectures of the universities in experimental philosophy, discussed in more detail in Section 4. He endeavoured "to render the Course entertaining and improving, being particularly calculated for such GENTLEMEN and LADIES as would chuse to be acquainted with the more rational and sublimer parts of Knowledge in the most expeditious and familiar Manner." Such lectures were always copiously illustrated by demonstrations and working-models. The lecturer depended on fees from his pupils and he had the strongest reasons to make his lectures valuable and pleasant. The conditions may be gathered from a pamphlet by James Ferguson † dated 1761.

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[*] The printed works of Desaguliers were of importance for popular scientific education. They include 'Physico-Mechanical Lectures, London,' 1717; 'A System of Experimental Philosophy,' London, 1719; 'A Course of Experimental Philosophy,' London, 1734.

[†] 'Analysis of a course of Lectures on Mechanics, Hydrostatics, Pneumatics, and Astronomy, read by James Ferguson, London MDCCLXI.'

> "Any gentleman, in or near *London*, may have this course of lectures read at his House, for Twelve Guineas, or any single Lecture for One Guinea, over and above the charge of carrying the Apparatus.
>
> Or if twenty persons subscribe a Guinea each, they may have the whole Course read, anywhere in *London*, or within Ten miles of it.—Greater distances require larger subscriptions.
>
> The Author teaches the Use of the Globes in a Month, attending one Hour every Day (Sunday excepted) for Two Guineas at Home . . . .

Some of these lectures, *e. g.*, those of James Arden [*] and James Dinwiddie [†], A.M., included Chemistry.

(3) *The Teaching of Mathematics in the Eighteenth Century.*

Of this subject, the less need be said because it was a regular and established feature of University courses, and a discipline that took rank with classical and philosophical studies. Mathematical Chairs in British Universities date back to the early seventeenth or even the sixteenth centuries. Nevertheless, the standard of mathematical attainment in the British Isles was, in the period under review, much inferior to that of France, possibly owing to the general adoption on the continent of the notation of Leibniz.

In France the study of mathematics after the Revolution was strongly encouraged. In 1794 were founded the École Normale Superieure and the École Centrale des Travaux Publics, which in the next year became the École Polytechnique. The École Normale Superieure had a magnificent staff. Mathematics were taught by Lagrange, Laplace and Monge, physics by Haüy, natural history by Daubenton and chemistry by Berthollet. It does not seem, however, in its earlier years to have fostered such remarkable men as did the École Polytechnique, which was chiefly devoted to producing engineers and artillery officers. Monge, Lagrange, Legendre, Prony and Hachette, later also Poisson and Fourier, represented mathematics : Fourcroy, Vauquelin, Berthollet and Chaptal were the chemists. The standards were high and the military discipline contrasted with the general idleness that prevailed in the English Universities. Arago, in his 'Autobiography' [‡], gives a lively account of his experiences therein.

(4) *The Teaching of Physics in the Eighteenth Century.*

The sciences which are today grouped as Physics were, in the eighteenth century, treated partly as Mathematics, partly as Chemistry, but chiefly under the heading of Experimental Philosophy. The teaching of experimental philosophy began long before the period we are discussing.

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[*] 'Analysis of Mr. Arden's Course of Lectures on Natural and Experimental Philosophy, by James Arden, 1782.'

[†] 'Syllabus of a course of Lectures on Experimental Philosophy, by James Dinwiddie, A.M., London 1790.'

[‡] 'The History of my youth, an autobiography of Francis Arago,' tr. Baden Powell, London, 1855.

Thus, at Oxford, John Keill was lecturing in 1700; according to his successor, J. T. Desaguliers, he was the first to teach natural philosophy "by experiments in a mathematical manner." Lectures of this kind were given at Oxford with only short intermissions throughout the eighteenth century. We may allow ourselves a brief survey of what was taught by Thomas Hornsby *, Reader in Experimental Philosophy from 1763 to 1810. His lectures are well spoken of and a syllabus is preserved. The headings are here transcribed: suffice it to say that it is a representative specimen of this type of instruction.

A SYLLABUS OF A COURSE OF LECTURES IN EXPERIMENTAL PHILOSOPHY.

Mechaniks.

Lect. 1 Of the Properties of Bodies.
Lect. 2 Of the Laws of Motion.
Lect. 3 Of the Attraction of Cohesion.
Lect. 4 Of Magnetism.
Lect. 5 Of Electricity.
Lect. 6 Of Gravity. Of the Mechanical Powers: Of the Lever.
Lect. 7 Of the Mechanical Powers, continued.
Lect. 8 Of compound Machines: Of Friction.
Lect. 9 Of accelerated and retarded Motion, and of Pendulums.
Lect. 10 Of Central Forces.
Lect. 11 Summary View of the Solar System.
Lect. 12 Of the Figures of the Earth, and the Motion of Projectiles.

Opticks.

Lect. 13 Of Catoptricks.
Lect. 14 Of Dioptricks; of Vision.
Lect. 15 Of several Circumstances respecting Vision; of Microscopes; of Telescopes.
Lect. 16 Of the theory of Light and Colours; and the Rainbow.
Lect. 17 [No title: optical demonstration in darkened room.]

Hydrostaticks.

Lect. 18 Of the Pressure of Fluids.
Lect. 19 Of the Pressure of Fluids, continued.
Lect. 20 Of the Sinking and Floating of Solid Bodies in Fluids; of Specific Gravity.
Lect. 21 Of the Motion of Fluids.

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* 'A syllabus of a course of Lectures in Experimental Philosophy.' No author. No date. One copy in the Museum of the History of Science is endorsed "Hornsby" in the handwriting of S. G. Rigaud.

PNEUMATICKS.

Lect. 22	Of the Air; of the State and Limits of the Atmosphere.
Lect. 23	Of Barometers, Thermometers, and Hygroscopes.
Lect. 24	Of the Structure of the Air Pump, with a variety of Experiments to prove the Pressure of the Air.
Lect. 25	Of the Elasticity of the Air; Structure of the Condensing Engine, etc.
Lect. 26	Miscellaneous Experiments upon Air.
Lect. 27	Different Sorts of Air.

It was illustrated by demonstrations. The apparatus apparently had to be provided by the Reader, though some colleges, *e. g.*, Oriel, had a considerable collection of "philosophical apparatus" presumably used for such lectures.

At Cambridge, Isaac Milner *, later President of Queen's College and Dean of Carlisle, read alternate courses of philosophical and chemical lectures. These continued from 1786 to 1792, and were attended with great interest. A German assistant, Hoffman, conduced to their success. Courses were also given by Antony Shepherd †, and these follow much the same lines as those of the non-academic lecturers, such as Ferguson, and of Hornsby. The lectures of Samuel Vince ‡ (1793) also follow this plan. All these lectures seem very antiquated in comparison with the active work in physics which characterized the first quarter of the nineteenth century, but they continued to be given with little change until, at least, its second decade. It will be noted that the study of heat was treated by the chemist rather than the physicist. Black's lectures 'On the effects of Heat and Mixture' give a much more adequate account of the phenomena of heat than these lectures on experimental philosophy, where thermometry is considered as a part of pneumatics Nearly all the great physicists educated during this period came to their subject, it would seem, through mathematical studies, and not through "experimental philosophy."

(5) *Medical Teaching as a source of Scientific Education.*

The eighteenth century saw the normal mode of entry to the medical profession change from private teaching or apprenticeship to public courses at the Medical School or University. The irregular modes of entry very often produced great men—but they doubtless also produced ignorant quacks. Thus, as Flexner points out, the two Hunters, Matthew Baillie, Bright, Addison and Hodgkin began as unknown youths, in the rôle of assistants in the dead-house or the out-patient departments.

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* 'The Life of Isaac Milner, D.D., F.R.S.,' Mary Milner, London and Cambridge, 1842.

† 'The Heads of a Course of lectures in Experimental Philosophy read at Christ College by A. Shepherd, D.D., F.R.S., and Plumian Professor of Astronomy and Experimental Philosophy, Cambridge N.D.'

‡ 'A plan of a Course of lectures on the principles of Natural Philosophy.' By the Rev. S. Vince, A.M., F.R.S., Cambridge, 1793.

Here they met with abundance of pathological and clinical material, and through that intimate acquaintance achieved their discoveries. Thereafter they became busy consultants, and research knew them no more. The medical student in a formal course listened to lectures, but had little opportunity of clinical or anatomical experience. Few schools had the opportunity to dissect half-a-dozen cadavers in the year, and since there were some 400 medical students in the largest courses, *e. g.*, that of Edinburgh, they simply looked on at demonstrations. Thus, John Bell wrote, as late as 1810.

> "In Dr. Monro's class, unless there be a fortunate succession of bloody murders, not three subjects are dissected in the year. On the remains of a subject fished up from the bottom of a tub of spirits, are demonstrated those delicate nerves which are to be avoided or divided in our operations, and these are demonstrated once at the distance of 100 feet!—nerves and arteries which the surgeon has to dissect at the peril of his patients' life."

Such a system obviously required the supplement of some kind of apprenticeship. None the less, the medical schools made admirable provision for teaching the theory and facts, as they were then known. In Great Britain, the most important schools in the eighteenth century were those of Edinburgh and Glasgow. The Edinburgh School dated from 1736. In England Guy's Hospital had a medical school of a somewhat informal character from 1723: the London Hospital Medical School, which arose from the private medical school of Sir William Blizard and Maclaurin in 1785, is said to be the first formal medical school in England. It was followed in 1790 by St. Bartholomew's medical school, under the guidance of Abernethy. The Meath Hospital, Dublin, formed a medical school as early as 1756. At Oxford, the Medical School dates from antiquity, but the chair of clinical medicine, founded in 1780, marks a virtual refounding of the School.

In France the Revolution had a salutary effect. The *Écoles de Santé*, were founded in 1794. In 1796 the medical school of Paris was reorganized with twelve professors. Its reputation rapidly increased, and in 1799 it had 500 students.

In Germany and Austria the conditions of medical education were very variable. At Wurzburg, Heidelberg, Göttingen, and Vienna, there was adequate provision for medical teaching, but in some minor universities medical degrees could be purchased after a minimum of examination. Medical teaching normally included Chemistry, but although this was often part of the medical course, its scientific importance is such as to warrant separate consideration.

(6) *The Teaching of Chemistry.*

It is not necessary to go back very far to find the effective origins of the formal teaching of Chemistry. The subject in the eighteenth century was normally treated as an ancillary, but not an unimportant, part of medical studies, and its cultivation runs parallel to these studies. Where medical education was chiefly by apprenticeship there was little or no formal teaching of chemistry.

The great source of instruction in these subjects was the University of Edinburgh, where Medicine, Chemistry, Materia Medica and Botany were subjects of study throughout the eighteenth century. Thus Dr. James Crawford exhibited experiments in chemistry from 1713. It may be noted that no practical classes in chemistry were in existence before 1800, but some students purchased apparatus and performed the experiments at home *. A considerable advance in the teaching of Chemistry was made by William Cullen, who was appointed lecturer at Glasgow in 1746 and later moved to Edinburgh. Lyon Playfair † tells us that he

> " saw that a science like Chemistry was not to be taught by mere lectures, but that there must be a free and unreserved communication between the teacher and the taught. He cultivated the personal acquaintance of his pupils and zealously aided them to overcome those difficulties we all experience in ascending the first steps of the ladder of knowledge. He taught professors of Chemistry to act as tutors as well as prelectors . . . . "

Cullen was succeeded at Glasgow in 1756 by Joseph Black, who in 1766 was appointed to the chair of Chemistry at Edinburgh. Black's influence on the teaching of Chemistry far exceeds that of any other eighteenth-century teacher.

Black is chiefly remembered for his work on magnesia, fixed air and the alkalies, and also for his researches on latent heat. These belong to the very beginning of his career. Thereafter he did little in the way of research: he is therefore stigmatized as indolent, but in fact he gave great attention to the teaching of the subject, and therein did a great work for science. Not only were his lectures the means of educating many chemists, but they were also the pattern of the chemical lectures given at Oxford and Cambridge in the last twenty years of the century.

There are some records of his powers as a lecturer. He was an admirable manipulator; thus Thomas Beddoes, who had been conducting a successful Chemistry course at Oxford, writes to him in 1788, in the following terms :—

> " What I find most difficult is to repeat some of those simple experiments, which in your hands are so striking and so instructive. How do you contrive to make that experiment which shows the burning of iron in dephlogisticated air ? " ‡

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* 'History of Scottish Medicine,' Comrie, Vol. I. p. 29.

† 'A Century of Chemistry in the University of Edinburgh being the Introductory Lecture to the Course of Chemistry in 1858,' by Lyon Playfair. Edinburgh, 1853.

‡ 'Life and Letters of Joseph Black, M.D.,' Sir Wm. Ramsay, p. 89.

Robison, who edited his lectures, tells us :—

> " That while he scorned the quackery of a showman, the simplicity, neatness, and elegance with which they were performed were truly admirable."

And Brougham also :—

> " I have seen him pour boiling water or boiling acid from one vessel to another, from a vessel that had no spout into a tube, holding it at such a distance as made the stream's diameter small, and so vertical that not a drop was spilt." . . . . " The long table on which the different processes had been carried on was as clean at the end of the lecture as it had been before the apparatus was planted upon it. Not a drop of liquid, not a grain of dust, remained."

That he took the preparation for his lectures very seriously appears from a letter written to the authorities in 1789, asking for a house to be built contiguous to the chemical laboratory.

> " The Professor has, it is true, only one hour of teaching, but he must spend several hours every day in his laboratory in preparing for the experiments and operations of the next lecture or finishing those already begun : and as these operations often last ten, twelve, or twenty-four hours, or some of them several days, he is under the necessity of looking into it frequently during the day and occasionally must be there early in the morning and late at night. Nor is this sort of labour confined to the session of the college : he must have recourse to his laboratory at all times to carry on his studies, and qualify himself the better for the discharge of his duty . . . . his office is much more laborious than theirs (*the professors' of medicine*) who have only an hour of teaching daily."

Lecture-demonstrations were, of course, of the greatest importance, since the students did no practical work, and the tradition of painstaking and skilled demonstration lasted far into the nineteenth century. Faraday, for example, regarded as one of his principal duties the perfect demonstration of the phenomena treated in his lectures.

The lectures of Joseph Black have fortunately been preserved * and can be read by anyone who wishes to taste one of the principal founts of scientific education in the late eighteenth century. It is to be noted that Black did not call his work ' Lectures on Chemistry ' but ' Lectures on the effect of Heat and Mixtures,' being much opposed, like Newton, to the precipitate formation of hypotheses. Black realized the imperfection of his account of these phenomena was such that it could not be regarded as forming a System of Chemistry. The style of the lectures is easy, and they are so phrased as to be comprehensible by and interesting to any intelligent person. Their arrangement seems to us to be somewhat unsystematic, with its long digressions following up some piece of work in which Black was peculiarly interested, but it is to be remembered that the chemistry of the time was informed by scarcely any theoretical structure and was in fact a natural history of bodies which might admit of many modes of arrangement.

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* ' Lectures on the Elements of Chemistry, delivered in the University of Edinburgh by the late Joseph Black, M.D. . . . . published from his manuscripts by John Robison, LL.D.,' Edinburgh, 1803.

Black's lectures, as edited by Robison, are in their latest form. We know that he adopted the views of Lavoisier before 1784: but his tendency was always to give the least possible weight to theories and the greatest to the correct description of phenomena.

A good idea of what Black taught can be gathered from the headings of Robison's work which are transcribed below. It has no table of contents, the compilation of which would doubtless have revealed to the editor the anomalies of his system of headings.

Part I.

Introductory.	Definition of Chemistry. Of Chemistry in General.

GENERAL DOCTRINES OF CHEMISTRY.

Part I. General Effects of Heat.

Introduction.	Of heat in general. Heat and Cold. Nature of Heat.
Sect. I.	Of Expansion. Of the Thermometer, Improvement of our knowledge of heat by means of the thermometer, (1) Extension of our ideas of heat, (2) Of the Distribution of heat, (3) Of the Celerity of the Communication of heat.
Sect. II.	Of fluidity.
Sect. III.	Of Vapour and Vaporization. Of Evaporation [*].
Sect. IV.	Of Ignition.
Sect. V.	Of Inflammation [†].

Part II.

General Effects of Mixture.

Introduction.	Of the theories of Chemical Mixture and Combination. Of Electric Attractions.

Part III. Chemical Apparatus.

THE CHEMICAL HISTORY OF BODIES.

Introduction.

Class I. Salts.

Of Salts in General.

Genus I. Alkaline Salts.

Species I	Vegetable Alkali.
Species II	Fossil Alkali.
Species III	Vegetable Alkali.

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[*] Sect. II and III contain Black's famous work on latent heat and the last-named deals with steam engines.

[†] This section deals with combustion.

Genus II. Acid Salts.

Vitriolic Acid.

Species III (*sic*)	Nitrous and Muriatic acids. Nitrous acid.
Species III	Muriatic acid.
Species IV	Acetous acid.
Species V	Acid of tartar.
Species VI	Sedative salt.

Genus III. Of the Compound or Neutral Salts.

Species I	Vitriolated tartar.
Species II	Glauber's Salt.
Species III	Nitre or Saltpetre *.
Species IV	Cubic nitre †.
Species V	Digestive Salt ‡.
Species VI	Common Salt.
Species VII	Vitriolic Ammoniac.
Species VIII	Nitrous Ammoniac.
Species IX	Sal Ammoniac.
Species X	Regenerated tartar.
Species XI	The Alkali Fossile Acetatum.
Species XII	Acetous Ammoniac.
Species XIII	Tartar.
Species XIV	Borax.

Synonymous Denominations of the Salts.

CHEMICAL HISTORY.

Class II. Earths.

Introductory.

Genus I. Alkaline Earths.

Species I	Calcareous earths.
Species II	Magnesia §. Calcareous Earths combined with acids. 1. *Gypsum*. 2. Fluor or Fluat of Lime 3. Phosphat of Lime. 4. Borat of Lime.
Species III	Barytes.
Species IV	Strontites.

Genus II. Argillaceous Earths.

Plastic Earths.

Genus III. Hard Stony Bodies.

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* Contains a considerable digression on oxygen, etc.

† Sodium nitrate.

‡ Potassium Chloride.

§ A long section treating of Black's work on the alkalies and fixed air, and subsequent discoveries and controversies.

GENUS IV. FUSIBLE STONES.

GENUS V. FLEXIBLE STONES.

Appendices on Precious stones, Gems, Porcelain, Quartz.

CLASS III. INFLAMMABLE SUBSTANCES.

Introductory. (On Combustion).

I. Inflammable Air *.
II. Phosphorus.
III. Sulphur.
IV. Charcoal. *Appendix.* Pyrophori.
V. Ardent Spirits.
VI. Oils. Aromatic Oils, Unctuous oils. Empyreumatic oils.
VII. Bitumens.

CLASS IV. METALLIC SUBSTANCES.

Introductory and General Account.

METALS.

Genus I Arsenic.
II Magnesium †.
III Iron ‡.
IV Mercury.
V Antimony.
VI Zinc, or Spelter.
VII Bismuth or Tinglass.
VIII Cobalt.
IX Niccolum.
X Lead.
XI Tin – Stannum.
XII Copper.
Silver and Gold.
XIII Silver.
XIV Gold.
XV Platina, or Platinum.

CLASS V. OF WATERS.

Introductory and General.

I. Experiments to discover the carbonic acid. II. Experiments to discover Alkaline Salts. III. Experiments to discover Sulphuric acid. IV. Experiments to discover the Muriatic Acid. V. Detection of Earths in Mineral Waters. VI. Detection of Sulphur.

This system of classification and, indeed, the greater part of the matter was adopted by those who set up similar courses elsewhere.

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* Treats of composition of water.

† Under this title Black describes manganese, and describes chlorine and the chlorates.

‡ A considerable section, dealing also with the metallurgy of iron and steel.

Scotland had many admirable chemists besides Black. Thomas Charles Hope taught at Glasgow from 1787 and in 1795 succeeded Black. Hope had some 500 students in chemistry and was the first to open a laboratory for practical classes. He is remembered for his discovery of strontia and for his work on the maximum density of water. Thomas Thomson, who studied under Black before 1799 became Professor of Chemistry at Glasgow in 1818. At the Andersonian College, Dr. Andrew Ure, well-known for his Dictionary, was professor from 1804.

At Oxford the first serious lectures in chemistry since the seventeenth century were given by Martin Wall, who was appointed Public Reader in Chemistry in 1782. He has left us a very full syllabus of his course *. In his preface he tells us that :—

> "The University during the last Course gave demonstrative Proofs of their sincere Zeal to raise a School of Medicine in this Place, by their Indulgence in Regard to the Degrees, and their liberal Assistance in the Improvement of the Laboratory."

The University did not, however, provide him with a salary, for he had to support himself by fees of two guineas a term, payable by his students. We learn from contemporary correspondence that he had an audience of fourteen or fifteen, which dropped away at the end of the course. The syllabus reveals that, like most unpractised lecturers, he had provided far more material than he could deliver. Thus a student wrote :—

> "It is unfortunate for us that the Doctor is tied down to a single hour, by which means his lectures are read with a rapidity that prevents much from being remembered, and anything from being taken down."

The general outline of the course resembles Black's, but the treatment of certain matters, *e. g.*, chemical apparatus and furnaces, seems to have been fuller. Moreover, the course concludes with a section on vegetable and animal chemistry.

Wall was succeeded by Thomas Beddoes, who seems to have been a lively lecturer. In the letter quoted in section 6, he tells Black that he has :—

> "the largest class in chemistry that has ever been seen at Oxford within the memory of man, in any department of knowledge, although my number cannot be put in competition with those in Edinburgh."

Neither his lectures nor even a syllabus of them survives. His popularity probably meant that he was a skilled and lively demonstrator, which is witnessed by his sending up a balloon on the 10th June, 1790, with a fuse of touch-paper arranged to set it alight after it had travelled about three miles, so as to discover whether the conflagration would give the appearance of an "igneous meteor." Beddoes found the conditions at Oxford unsatisfactory, for the fees from his class, his sole emolument, were scarcely enough to pay his expenses—for it seems that he had to find his own apparatus and materials. He was succeeded in 1784 by Robert Bourne, M.D., who has left a syllabus * of his course.

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* 'A syllabus of a course of Lectures in Chemistry. Read at the Museum, Oxford.' By Martin Wall, M.D., Public Reader in Chemistry. Oxford, 1782.

Bourne follows somewhat different lines, being more impressed with the importance of pneumatic chemistry. He starts with the general notion of attraction in natural philosophy and treats of Chemical attraction. He treats first of Light and Heat. He then gives a considerable section on aerial substances, "vital air, azotic air, inflammable air, alkaline air, fixed air, vitriolic acid air, muriatic acid air, oxygenated muriatic acid air, fluor acid air." He then treats saline substances; first the acids, then the alkalis and then the salts. He uses the new nomenclature. He then treats earthy substances from a more chemical point of view than does Wall. He continues with inflammable substances, sulphur, coal, alcohol, oils, resins, phosphorus, then goes on to the metals.

He adds at the end :—

> "For the sake of conciseness, the Reader in Chemistry has not specified the experiments, which will be shewn in his course of Lectures. But he thinks it right to observe, in a general manner, that as many experiments will be shown, and as many processes exhibited as the time will permit; in short that the Course will be practical rather than theoretical."

In 1803 John Kidd was appointed to the newly founded Aldrichian Professorship of Chemistry at Oxford. In 1808, he published a syllabus of a course of lectures which cover very much the same ground as those of Bourne. But the influence of the new way of thinking is manifest. Gases are no longer treated together as varieties of air, but are arranged according to their constituents. Light and Heat are still included in the course, but it seems that the discussion given to them was not so full.

Kidd scarcely kept pace with the rapid progress of chemistry. In 1818 he took exception to a statement of Brande in the supplement to the fourth and fifth editions of the 'Encyclopædia Britannica,' that "excepting in the Schools of London and Edinburgh, Chemistry, as a branch of education, is either neglected or, what is perhaps worse, superficially and imperfectly taught." Kidd rebuts this charge by printing the syllabus of his lectures, and incidentally gives his views on scientific education. He does not consider the purpose of such teaching to be primarily an introduction to a scientific career, but states :—

> "It is evident, then to those who reflect on the subject, that the whole tenor of an academical education, so far at least as intellectual endowments are concerned, regards the general improvement of its members, rather than their qualification for any particular profession: and hence the trite objection so often even now brought forward, that the Physical and Experimental Sciences are here neglected, can only proceed from want of candour or of information. For a candid and enlightened mind would readily allow, that though the discipline of

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* 'A Syllabus of a Course of Chemical Lectures read at the Museum Oxford in seventeen hundred ninety four.' (No author's name.)

† Syllabus of a Course of Lectures on Chemistry (by J. Kidd, M.A.). Oxford, 1808.

> Classical and Mathematical studies is well calculated to form the groundwork of excellence in the Physical and Experimental Sciences, the converse of this is by no means true; witness the deficiency, with respect to taste and reasoning, in the literary productions of individuals, whose fame in other points deservedly ranks high in the scientific and professional world.
>
> The Physical and Experimental Sciences then are not neglected in this place. They are not cultivated indeed to the same extent as in some other schools; but they are cultivated so far as is compatible with the views of a system of general education: and hence the object of the Lectures in the several branches of those sciences is, rather to present a liberal illustration of their principles and practical application, than to run into the minutiæ of a technical, or even a philosophical detail of facts. These branches of science, in this place at least, may be considered with reference to Divinity, Classics, and Mathematics, in the same light as the supernumerary war-horses of Homer's chariots: which were destined to assist, but not to regulate the progress of their nobler fellow coursers . . . .
>
> It has been the means, that is, of elevating to the title of Philosophers a host of individuals, whose talents were just equal to that species of inductive reasoning, the nature of which has been of late years so egregiously mistaken, and its importance so absurdly vaunted. That man, in truth, must be possessed of but ordinary abilities, who cannot draw a general conclusion from a number of analogous facts continually passing before his eyes; while, after all, it must be genius alone that can penetrate beyond the limits which apparently confine it, and connect at once the distant or hidden links in a chain of philosophic reasoning." (He argues that the powers of such a genius will be improved and exercised by a general education, though some are found with talents that supersede the necessity of any education.)

The Professorship of Chemistry at Cambridge was founded in 1713, but the first holder of it we may notice is that remarkable character Richard Watson, later Bishop of Llandaff. In his memoirs * he tells us with noble candour how he was appointed :—

> "In October 1764, I was made moderator for Christ's College. On the 19th of the following November, on the death of Dr. Hadley, I was unanimously elected by the Senate, assembled in full congregation, Professor of Chemistry. An eminent physician in London had expressed a wish to succeed Dr. Hadley, but on my signifying to him that it was my intention to read chemical lectures in the University, he declined the contest. At the time this honour was conferred upon me, I knew nothing at all of Chemistry, had never read a syllable on the subject; nor seen a single experiment in it; but I was tired with mathematics and natural philosophy, and the *vehementissima gloriae cupido* stimulated me to try my strength in a new pursuit, and the kindness of the University (it was always very kind to me) animated me to very extraordinary exertions. I sent immediately after my election for an operator to Paris; I buried myself as it were in my laboratory, at least as much as my other avocations would permit; and in fourteen months from my election, I read a course of chemical lectures to a very full audience, consisting of persons of all ages and degrees, in the University."

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* 'Anecdotes of the life of R. W. written by himself at different intervals, and revised in 1814.' Richard Watson. London, 1817.

We may gain some idea of Bishop Watson's lectures from his volumes of Chemical Essays *. They are pleasantly and lightly written accounts of interesting matters concerned with chemistry, but make little or no attempt to grapple with the unsolved problems of the subject. It is indeed such a work as a highly educated and intelligent man could compile in a year's reading, but it smells rather of the lamp than the laboratory. The experiments which he narrates as his own are chiefly concerned with manufacturing processes rather than with the advancement of pure science. He tells us † :—

> "Chemistry is cultivated abroad by persons of the first Rank, Fortune and Ability; they find in it a never failing source of honourable amusement for their private hours; and as public men, they consider its cultivation as one of the most certain means of bringing to perfection, the manufactures of their country ‡."

Watson was followed by a much greater man, Francis John Hyde Wollaston. We possess the syllabus of his lectures ‡. Their plan evidently derives from Black, but Wollaston uses the new nomenclature, and introduces a number of substances recently discovered.

A course of an entirely new type was initiated by William Farish, and is fully summarized in a printed syllabus §. This interesting course begins with practical mining and metallurgy. Under each metal he treats of practical applications, *e. g.*, tool-making under iron, engraving under copper. He treats of coal-mining; and of marble, slate, clay, flints, pottery, glass, optical glasses, vitriol, salt, nitre, gunpowder, sulphuric acid, alum, &c. He further treats of animal and vegetable substances, covering agriculture, charcoal-burning, woodworking, the industries connected with oil, soap, candles, whale-fishery, tannery, sugar-refining, textile industries, printing and dyeing. He then treats the construction of machines,—windmills, water-mills, steam-engines, various mechanical powers, lathes, cranes, jacks. He further treats water-works, *i. e.*, water-supply, pumps, canals, locks, docks, ship-construction.

If the lectures corresponded to the promise of the syllabus, they must have been of the first interest and are an eloquent witness of the effect of the industrial revolution. Something of the same practical character is shown in Davy's lectures ‖.

Davy begins, like Bourne, with a general discussion of the chemical powers. He then discusses first "Undecompounded substances, *i. e.*, elements, then binary compounds, then bodies composed of more than two simple substances. The treatment is recognizably that of the modern text-book and contrasts remarkably with that of Wall, twenty years earlier. He then treats the general phenomena of chemical action; goes on to the Chemistry of Imponderable substances, Heat or Caloric, Light, Electrical Influence, Galvanism; then treats of the Chemistry of the Arts, Agriculture, Tanning, Bleaching, Dyeing, Metallurgy, Glass, Porcelain, Food and Drink, and the Management of Heat and Light artificially produced."

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* 'Chemical Essays by R. Watson, D.D., F.R.S., and Regius Professor of Divinity in the University of Cambridge.' London, 1781.

† *Ibid.* Dedication, p. ii.

‡ 'Plan of a course of Chemical Lectures,' by Francis John Hyde Wollaston, M.A., F.R.S., Jacksonian Professor in the University of Cambridge. Cambridge, 1794.

§ 'A Plan of a Course of Lectures on Arts and Manufacturers most particularly such as relate to Chemistry.' By William Farish, M.A., Professor of Chemistry in the University of Cambridge, 1796.

‖ 'Syllabus of a course of Lectures on Chemistry delivered at the Royal Institution of Great Britain' (by Sir Humphry Davy). London, 1802.

(7) *Summary.*

Surveying the position of the teaching of the physical sciences at the end of the eighteenth century, we note first a general and living interest in science, both as a means of understanding the material world and of developing the practical arts which were already beginning to revolutionize human existence. That general interest had led to the means of satisfying it through scientific works—encyclopædias, books and periodicals—and through courses of private lectures.

Almost the only scientific professions were those of the medical man, and to a less extent the engineer, mechanical or military. We find, therefore, the medical schools teaching such departments of science as were ancillary to the making of physicians and surgeons, and the military academies giving instruction in mathematics and mechanics. The Universities, whether in England or abroad, if they taught science at all, did so in a somewhat perfunctory manner, with the object of giving those who were pursuing other studies the advantage of a superficial acquaintance with the scientific discoveries of the age. None the less, the desire for better instruction was already present by 1800, and gradually came to fruition throughout the succeeding century.

LXXI. *Determination of Meteor Radiants by Observation of Radio Echoes from Meteor Trails.*

By J. A. CLEGG, Physical Laboratories, University of Manchester *.

[Received March 23, 1948.]

ABSTRACT.

The rate of occurrence and the ranges of the radio echoes from meteor trails vary with time as the radiant moves across the heavens. This variation is explained and the ranges are estimated for any position of the radiant, on the assumption that the echoes are due to broadside reflection from meteor trails. Such theoretical estimates are shown to agree well with experimental results. The change in the rate and ranges is more marked for a highly beamed aerial, and it is shown that with a horizontally narrow beam directed at sufficiently low elevation the ranges attain a sharp maximum when the azimuth of the radiant differs from that of the beam by 90°. This forms the basis for a method of radiant determination.

For the known showers analysed so far the results obtained for the radiant coordinates are consistent with those given by visual observers, and it is shown how the method can be used for the determination of unknown radiant positions. The accuracy depends on the echo rate, the declination of the radiant and the width and elevation of the aerial beam. Using an aerial system whose half-amplitude beam width is ±10° it is possible, under the most favourable circumstances, to place the radiant within a circle of radius 1°, but this degree of precision cannot usually be attained, and probable errors of ±2° in right ascension and ±3° in declination are more common.

1. INTRODUCTION.

IT is now well established that the transient radio echoes observed on frequencies of the order of 70 Mc/s. are caused by meteor ionization in the atmosphere (see Lovell, 1948, for a review of this work). Pierce (1938) pointed out that the reflecting properties of the meteor trail should depend on its aspect with respect to the apparatus ; and subsequent experience (Hey and Stewart, 1946, 1947 ; Lovell, Banwell and Clegg, 1947) has confirmed that on radio-wave frequencies of this order the main radio echo is obtained only when the aerial beam is directed at right angles to the trail of the meteor. Hey and Stewart (1946 and

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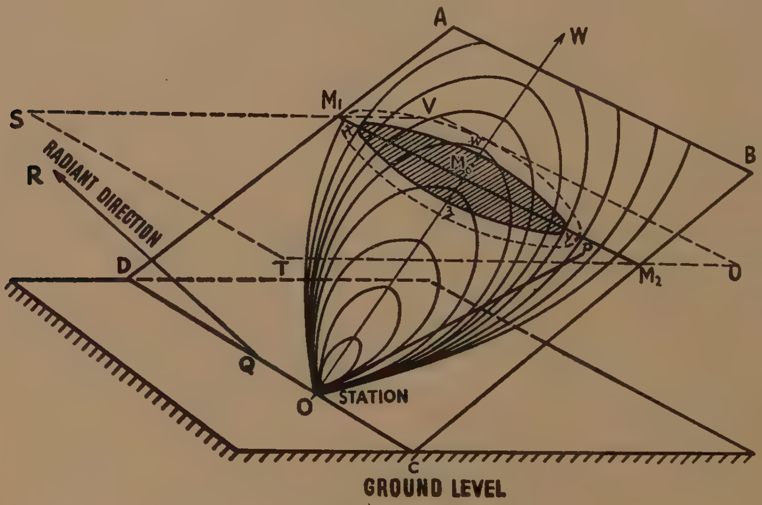
* Communicated by Prof. P. M. S. Blackett.

1947) used this aspect property to determine the radiant points of meteor streams from the diurnal variations in the hourly rates of the echoes observed on three stations. The present paper describes another method of radiant determination depending on the same property, but using only one station. By observing the variations in the ranges as well as the rates of the echoes the accuracy of measurement is increased by a factor of approximately five.

2. Radio Echoes from Meteors of a Single Stream.

It can be shown from diffraction theory that, provided the aerial beam is not too narrow and ionization varies uniformly over the first few Fresnel zones, reflection takes place effectively at a single point, which may be termed the reflecting point of the trail, and which is defined by the foot of the perpendicular from the station to the trail. In an ideal stream all meteors travel in parallel paths and due to effect of perspective appear to emanate from a point radiant. Fig. 1 represents such a stream viewed from a station at O. The radiant lies in the direction QR from the station, and the plane ABCD, which contains O, is perpendicular to QR and cuts the celestial sphere in a great circle of which the radiant is the pole. The path of all the meteors will be parallel to QR and will cross ABCD normally, so that the reflecting points will lie in this plane.

Fig. 1.



Effective collecting area of aerial system.

The majority of trails occur within a limited range of heights and this further restricts the reflecting points to a strip of the plane of limited depth. The echoes detected by the station will be from the portion of this strip lying within the aerial beam. We thus define a collecting area lying in the plane ABCD whose depth is limited by the depth Δh of the meteor zone, and whose lateral extent is determined by the width of the aerial beam. It will be shown that with a knowledge of the radiation pattern of the aerial it is possible to estimate the shape and size of this collecting area for any position of the radiant, and conversely by observation of the variations in range and echo rate to determine the radiant position.

3. ESTIMATION OF COLLECTING AREA.

The fundamental scattering formula, deduced and experimentally verified by Lovell (1947) and Lovell and Clegg (1948), may be written for the present purpose in the form

$$Z=\frac{kG(\phi,\psi)\,.\,\alpha}{R^{3/2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

Z=amplitude of the echo measured as a signal to noise ratio,

G=effective power gain of the aerial system in the direction of the reflecting point,

ϕ=azimuth of the reflecting point from the station,

ψ=elevation of the reflecting point from the station,

α=line density of electrons in the trail at the reflecting point,

R=range of the reflecting point from the station,

and k is a constant depending on the parameters of the apparatus. It will be convenient to refer to α as the effective electron density of the trail.

Equation (1) may be written

$$\frac{G(\phi,\psi)}{R^{3/2}}=s, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where

$$s=\frac{Z}{k\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If s remains constant, equation (2) defines a closed surface in space on which the overall sensitivity of the apparatus is constant. There will be a family of such surfaces corresponding to different values of s, those of lower value completely enveloping those of higher value. These surfaces will cut the great circle plane ABCD in a series of contours, as shown in fig. 1.

Visual observations show that the heights at which a meteor appears and disappears depend generally on its geocentric velocity and on its standard magnitude, the visible portion of the trail being at a greater mean height for a meteor of higher velocity and of greater depth of penetration for one of higher luminosity (Porter, 1944; Prentice, 1948). These results are in good agreement with contemporary theory, which predicts that the ionization and luminosity will vary as the meteor traverses the atmosphere, rising initially from zero to a maximum at some point on the trail and falling to zero again as the meteor evaporates, the intensity of ionization at the maximum point being proportional to the mass of the meteor, and the height of the maximum point depending on the velocity (Herlofson, 1948). For a single meteor stream, where the velocities can be assumed to be the same, all meteors of the same mass will produce identical trails. Let the horizontal surface STUV, in fig. 1, represent the height of maximum ionization for meteors of a certain mass group. For such meteors the effective electron density will be greatest when the trails cut the great circle plane in the line M_1M_2 and will fall off with the increasing distance from this line. Thus the amplitude of an echo will depend not only on the s contour on which the reflecting point lies, but also on the distance of this point from M_1M_2, and there will clearly be a closed curve such as $xwyz$ in the plane ABCD on which all trails of this class give echoes of the same amplitude. A family of such curves can be drawn for different values of the signal to noise ratio Z. If $xwyz$ corresponds to a value of Z equal to unity, it will represent the limit of visibility and the shaded area which it encloses will be the effective collecting area for meteors of this mass. For smaller meteors the corresponding area will be smaller, and for the smallest detectable mass group it will reduce to a point inside $xwyz$ lying on the axis of the beam OW. The collecting area for a larger mass group is shown as a dotted line in the figure, completely enveloping $xwyz$.

For a given mass group the horizontal width xy of the collecting area depends only on the conformation of the s contours on the great circle plane, and for the largest meteors tends to a limit defined by the width of the aerial beam. The depth wz increases indefinitely with the mass. The present method of radiant determination assumes that there are relatively few meteors of great mass * and that for the majority the depth of the collecting area is small in comparison with its mean height above the ground. In this case, if a surface such as STUV is drawn at the mean height of maximum ionization for the shower, all the echoes will originate at a point close to the line M_1M_2, being most numerous close to the point M_0 and occurring less frequently near to the edge of the beam.

4. SENSITIVITY CONTOURS ON THE HORIZONTAL SURFACE STUV.

To find the rate and ranges of the echoes for any position of the radiant it is convenient to represent the aerial beam by a set of sensitivity contours on the surface STUV at the mean height of maximum

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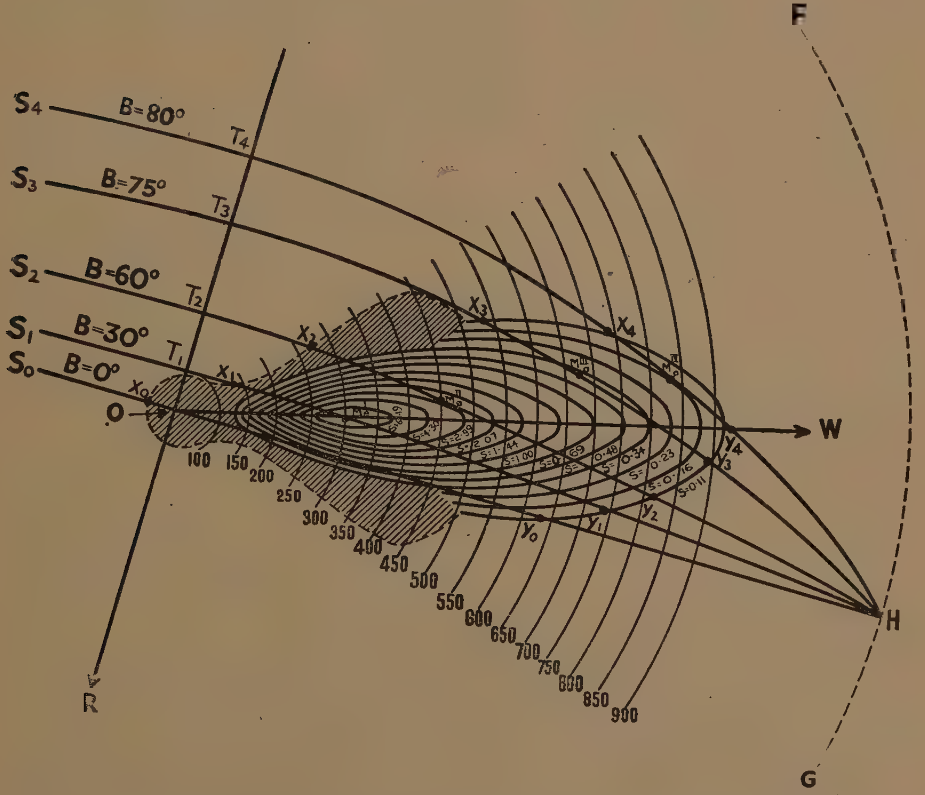
* The existence of a fairly well-defined upper mass limit is indicated by visual observations which show that for meteors brighter than first magnitude there is a rapid decrease in numbers with increasing luminosity, so that less than 5 per cent. of observed meteors exceed this magnitude (Cook, 1915; Olivier, 1925; Williams, 1939). According to Herlofson (1948), a meteor of first magnitude has a radius of 0·08 cms. and gives rise to a density of ionization of 10^{12} electrons per cm.

ionization as indicated in the map in fig. 2. Let this surface be at a height h above the ground. It can be shown that, allowing for earth curvature, the range R and the elevation ψ at any point on it are related by the equation

$$R^2+2\rho R\sin\psi-2\rho h+h^2=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where ρ is the radius of the earth. The surfaces of constant sensitivity will cut this horizontal surface in a series of contours defined by equations (2) and (4). The map in fig. 2 represents a polar projection of such a surface

Fig. 2.


F
S4
B=80°
T4
S3
B=75°
T3
S2
B=60°
T2
S1
B=30°
T1
S0
B=0°
X0
X1
X2
X3
X4
O
W
H
G
R
M
M II
M III
M IV
S=6·19
S=4·30
S=2·99
S=2·07
S=1·44
S=1·00
S=0·69
S=0·48
S=0·34
S=0·23
S=0·16
S=0·11
Y0
Y1
Y2
Y3
Y4
100
150
200
250
300
350
400
450
500
550
600
650
700
750
800
850
900

Map showing sensitivity contours on surface at height of 95 km. for aerial used at Jodrell Bank, with radiant in azimuthal direction OR.

at a height of 95 km., which may be taken as the mean height of occurrence of a typical shower of medium velocity (Prentice, 1948). The point O is vertically above the station and, in order to illustrate a specific case, the contours are drawn for the aerial at present in use at the Jodrell Bank Experimental Station (Lovell, Banwell and Clegg, 1947). This aerial has a half-amplitude beam width of ±8° horizontally and ±12° vertically. For the present purpose the beam is maintained at a constant elevation of 12·5°, but can be directed at any azimuth. The s values of the contours are marked, the contour $s=1$ corresponding to the sensitivity at a point in this plane, situated vertically above an aerial of unit gain. The shaded portions represent side lobes in which the conformation of the contours is not accurately known. Range circles are shown, giving distances from the station in kilometres. The dotted circle FHG indicates where the spherical 95-km. surface meets the horizon of the station. The radiant is in the azimuthal direction OR and the curved line S_1T_1H shows where the great circle plane (ABCD, in fig. 1) cuts the 95-km. surface when the elevation of the radiant is 30°. As the radiant elevation increases, the point T moves farther from O and corresponding lines are shown for elevations of 60°, 75°, and 80°. When the radiant elevation is 90° the line STH lies on the horizon circle FHG, and for zero elevation it becomes the straight line S_0OH, at right angles to OR.

Let the radiant be at the azimuth indicated in the figure and at an elevation of 75°. The reflecting points of the trails will then lie close to S_3T_3H, but visible echoes will occur only within the limits of the aerial beam. Consider the meteors whose mass is such that they give echoes of noise amplitude when their point of maximum ionization lies on the outermost contour, for which $s=0{\cdot}11$. The horizontal limits of the collecting area for meteors of this class are defined by the points x_3 and y_3 and, if the depth of the area (wz, in fig. 1) is not too great, these points will determine the maximum and minimum ranges at which such meteors can be detected. We have seen that there is a fairly well defined upper mass limit above which relatively few meteors are to be found *. If x_3y_3 defines the extent of the collecting area for this limiting mass, it can be seen from the map that the majority of echoes will occur between ranges of 530 km. and 890 km. At these ranges only the largest meteors will be seen, but at points between x_3 and y_3 meteors of smaller mass can be detected, and the most probable range of the echoes will be that of M_0'', where S_3T_3H touches the contour of highest value, and where the greatest number of meteors will give visible echoes.

It is evident from this figure that for a radiant at the same azimuth, but at elevation 60°, the ranges will be more scattered and the rate will tend to be higher, while for an elevation of 80° the range will be confined between narrower limits and the rate will be reduced.

5. VARIATION IN RANGE AND RATE OF THE ECHOES WITH TIME.

As the celestial sphere rotates, the elevation and azimuth of the radiant and consequently the position of the line STH in fig. 2 will change. A typical case of a radiant sweeping through a low narrow beam is illustrated in fig. 3. Let the aerial be directed at azimuth 90° and the radiant be about to cross the central meridian. Some time before transit it is in the azimuthal direction OR_1 and the line STH lies

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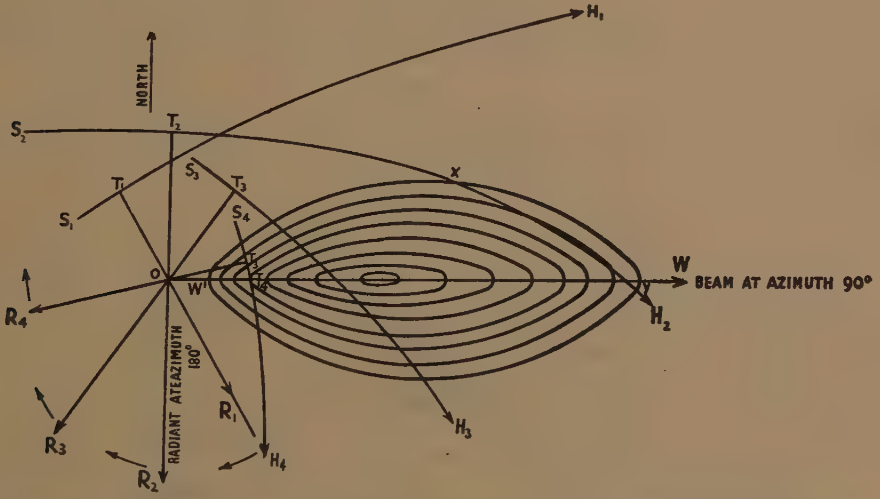
* See footnote, p. 580.

completely outside the beam so that no echoes will appear. As the azimuth approaches 180°, STH will cross the limiting contour and the rate will begin to rise, and at transit when the radiant is in the azimuthal direction OR_2 the collecting area for large meteors will be defined by xy. After transit the radiant passes through azimuths OR_3 and OR_4. This will cause a steady reduction in range and the echo rate will finally fall to zero as STH passes out of the beam at W′. The passage of the radiant through the beam is therefore characterized in this instance by a sudden appearance of echoes at long ranges followed by a slow fall in the mean range and a final decay of the echo rate.

If the polar diagram of the aerial and the mean height of occurrence of the meteors are known, it is possible to estimate the most probable ranges of the echoes for any radiant position by means of an aerial map such as that in fig. 2, and, hence, to plot the range as a function of time for a radiant at any right ascension and declination. In fig. 4 a set of theoretical range time curves is plotted for the aerial described in § 4 at azimuth 90°, assuming that the mean height of occurrence of the echoes is 95 km. These curves are drawn for five different values of radiant declination * and show the variation of maximum, minimum and most probable range with time after transit, assuming that meteors can be detected out to the contour $s=0{\cdot}11$. In each case the point B corresponds to the time at which the line STH first crosses this contour, and, therefore, at which echoes can first appear.

Fig. 3.



Passage of radiant through central meridian with aerial beam directed due east.

The curves show that there is a sudden rise in echo rate at a time close to that of transit, and the rapid increase of the maximum and most

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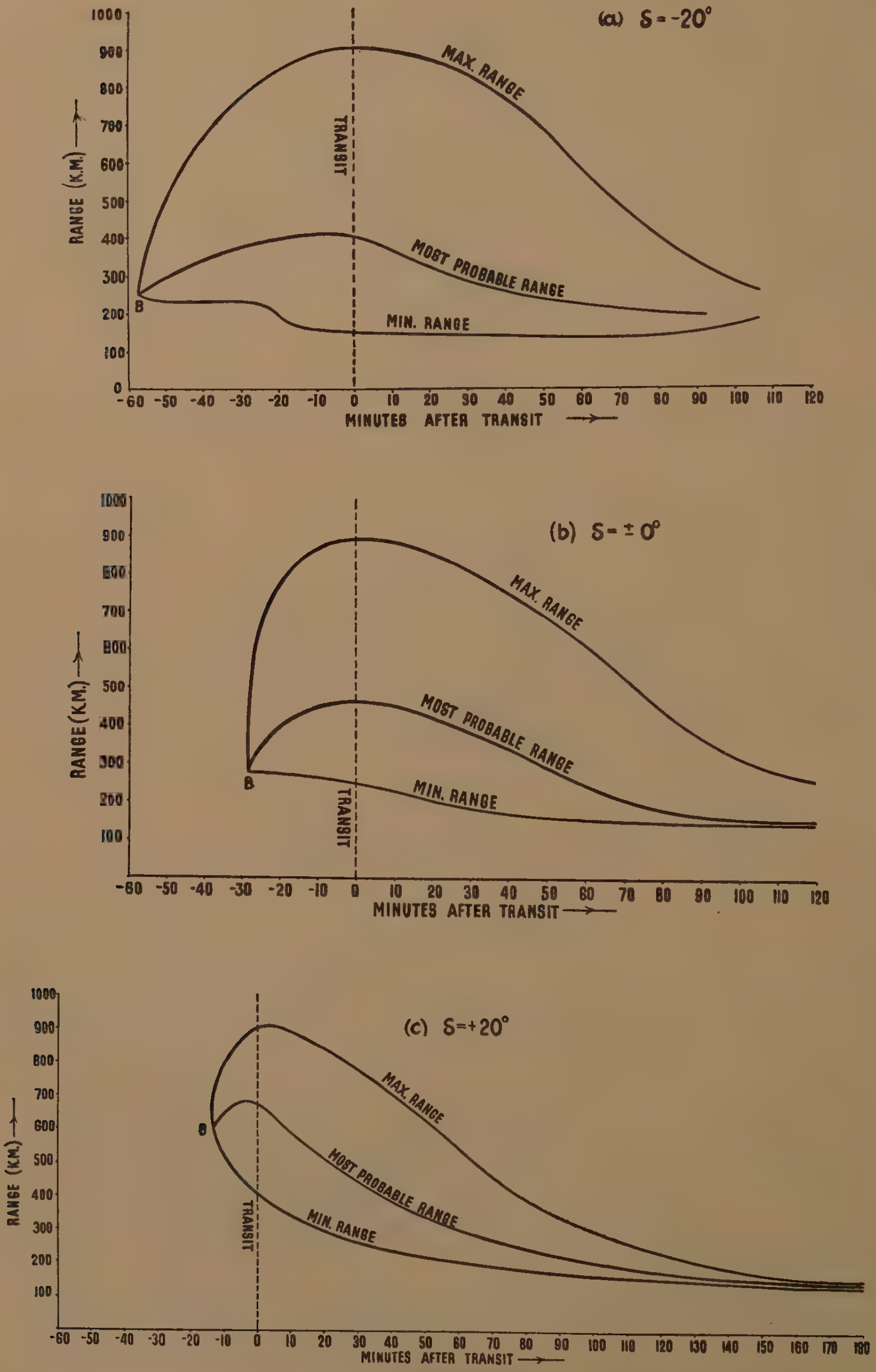
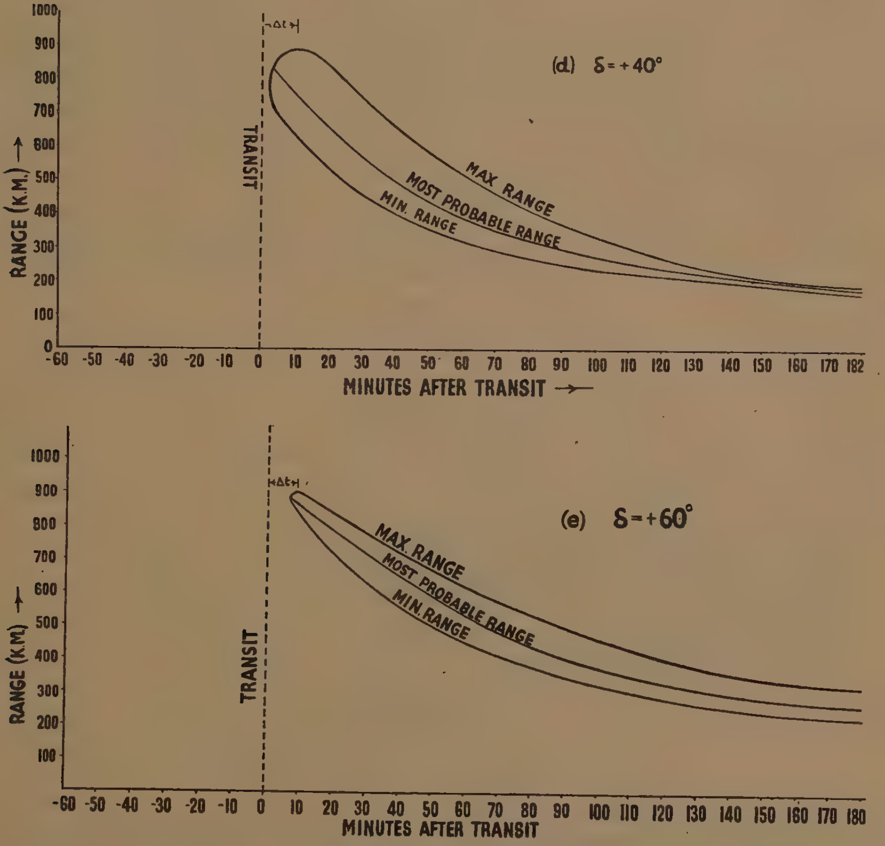
* The curves are drawn for a station at the latitude of Jodrell Bank (53° N.).

Fig. 4 (*a*–*c*).


(a) δ = -20°
RANGE (K.M.) →
1000
900
800
700
600
500
400
300
200
100
0
TRANSIT
MAX. RANGE
MOST PROBABLE RANGE
MIN. RANGE
B
-60 -50 -40 -30 -20 -10 0 10 20 30 40 50 60 70 80 90 100 110 120
MINUTES AFTER TRANSIT →
(b) δ = ± 0°
RANGE (K.M.) →
1000
900
800
700
600
500
400
300
200
100
MAX. RANGE
MOST PROBABLE RANGE
MIN. RANGE
TRANSIT
B
-60 -50 -40 -30 -20 -10 0 10 20 30 40 50 60 70 80 90 100 110 120
MINUTES AFTER TRANSIT →
(c) δ = +20°
RANGE (K.M.) →
1000
900
800
700
600
500
400
300
200
100
B
TRANSIT
MAX. RANGE
MOST PROBABLE RANGE
MIN. RANGE
-60 -50 -40 -30 -20 -10 0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180
MINUTES AFTER TRANSIT →

probable ranges to their highest value, followed by a steady fall. For radiants at low declination whose elevation at transit is small, the initial rise in echo rate and range commences earlier and is less rapid than for high declinations. Thus for declinations −20°, ±0°, and +20° the rate has already become appreciable before transit and the maximum ranges

Fig. 4 (*d–e*).



Variation of maximum, minimum and most probable range with time for observations taken on Jodrell Bank aerial directed at azimuth 90°.
Curves are computed from contour map of aerial assuming a mean height of occurrence of 90 km. Maximum and minimum ranges apply to meteors which would be on limit of visibility if their reflecting points lay on contours S=0·11.

occur at transit as indicated in figs. 4 (*a*), (*b*) and (*c*). For high declinations the line STH does not pass through contours of sufficiently high value at the time of transit to ensure an appreciable rate, and the range curves are similar to those shown in figs. 4 (*d*) and (*e*). For an aerial directed due west the variations occur in the opposite order and the time scale in fig. 4 is reversed.

Curves such as these can be constructed for any azimuth of the aerial and it is always found that maximum ranges occur when the axis of the beam is perpendicular to the horizontal direction of the radiant, provided that the radiant elevation is not too great. If a different limiting contour is chosen, or if the map is drawn for a different mean height of ionization, the curves are of the same general form, but the absolute values of the ranges are altered.

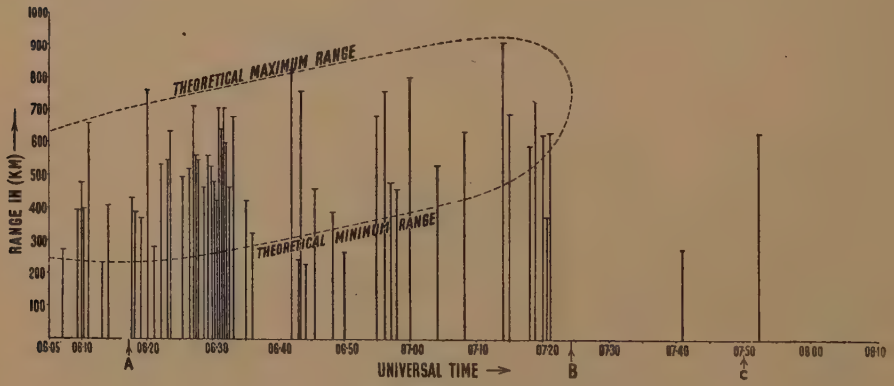
6. Experimental Test of the Predicted Range Time Distribution.

In order to test the predictions regarding the variation in range and rate of the echoes with time, measurements have been made on the following meteor streams for which the radiant positions are known from visual observations.

(*a*) 1947 *Quadrantid Shower.*

Visual observations on the Quadrantid shower have shown that the radiant probably consists of multiple centres. According to Prentice *, there was strong evidence that the 1947 shower had three subcentres at 232°+52°, 227°+48° and 218°+44°. For the period 1947, January 4, $06^{h}\ 05^{m}$ to $08^{h}\ 06^{m}$, the aerial was directed at azimuth 200°. The results are shown in the range-time diagram, fig. 5. The observed ranges of individual echoes are represented by lines parallel to the ordinate axis. During the observations the great circle plane swept through the beam from short to long range, and the arrows at A, B, and C show where the echo rate would be expected to fall suddenly for each of the three radiant positions, assuming that the echoes could be seen out to the contour $s=0{\cdot}11$. The results suggest that the centre at 227°+48° was highly active at this time. The theoretical range curves for a radiant in this

Fig. 5.



Quadrantid plots.
Period 1947, January 4, $06^{h}\,05^{m}$ to $08^{h}\,00^{m}$.
Aerial azimuth 200°.

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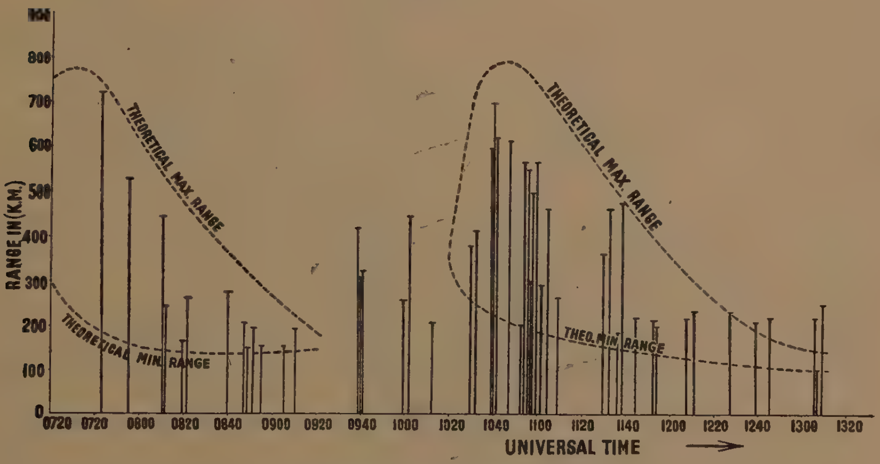
* Private communication.

position are shown as dotted lines, and it can be seen that they conform reasonably well with the observed range distribution.

(*a*) 1947 *η-Aquarid Shower.*

Fig. 6 shows the observations made during the η-Aquarid shower for the period 1947, May 7, $07^h\ 20^m$ to $13^h\ 17^m$ U.T., with the aerial directed due east. The majority of the echoes fall into two distinct groups with radiant positions separated in right ascension by about three hours. The first group conforms to the η-Aquarid radiant ($333° \pm 0°$), but the second is due to a new centre which transits at approximately $10^h\ 30^m$. The maximum ranges during the η-Aquarid shower were considerably less than those of the Quadrantids. This is to be expected since the elevation of the radiant at transit is only 37°, while that of the most active centre of the Quadrantids is 85°, and according to Herlofson (1947) the maximum ionization produced by a meteor is proportional to the sine of the elevation of the radiant. The theoretical range curves for the η-Aquarids have therefore been drawn for the same mean height of 95 km., but for the limiting contour of appropriately higher value. The activity of the shower during this period of observation was low, but the recorded ranges agree well with the theoretical values. Range curves are also shown for the second radiant which was found to lie in Pisces. They are drawn for a height of 95 km. and for the same limiting contour, but for a declination of +15°. The Piscid activity continued for some ten weeks and other active centres appeared in its vicinity and extended in both easterly and westerly directions (B.A.A. circular No. 282, and Clegg, Hughes and Lovell, 1947). The scattered echoes occurring between $09^h\ 35^m$ and $10^h\ 15^m$ are due to one of these minor radiants.

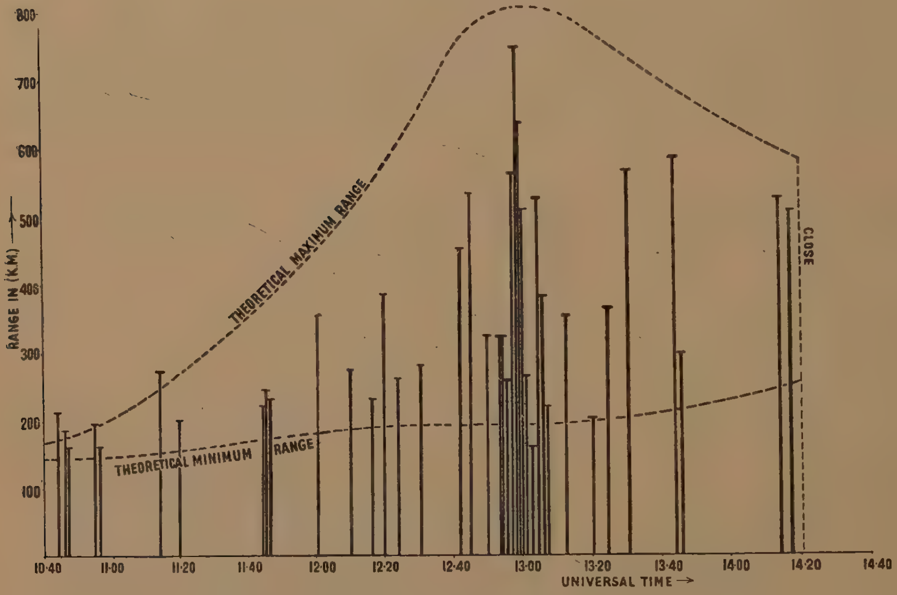
Fig. 6.



η-Aquarid and Piscid plots.
Period 1947, May 7, $07^h\,20^m$ to $13^h\,17^m$.
Aerial azimuth 30°.

At a later stage in this shower on May 29, the most active centre culminated at about $09^{\rm h}\,30^{\rm m}$. Fig. 7 shows the plots for the period $10^{\rm h}\,40^{\rm m}$ to $14^{\rm h}\,20^{\rm m}$ on an aerial azimuth of 340°. The ranges increased as the great circle plane swept into the beam, and the rate finally decayed as the radiant set. Again, the agreement between the theoretical and the observed ranges is reasonably good.

Fig. 7.



Piscid plots.
Period 1947, May 29, $10^{\rm h}\,40^{\rm m}$ to $19^{\rm h}\,30^{\rm m}$.
Aerial azimuth 340°.

7. Experimental Determination of Meteor Radiants.

It is evident from the measurements described in § 6 that for known radiant positions the predicted variations in range and rate of occurrence of the echoes with time are in good agreement with the experimental results. Hence the method can be applied to the determination of unknown radiants.

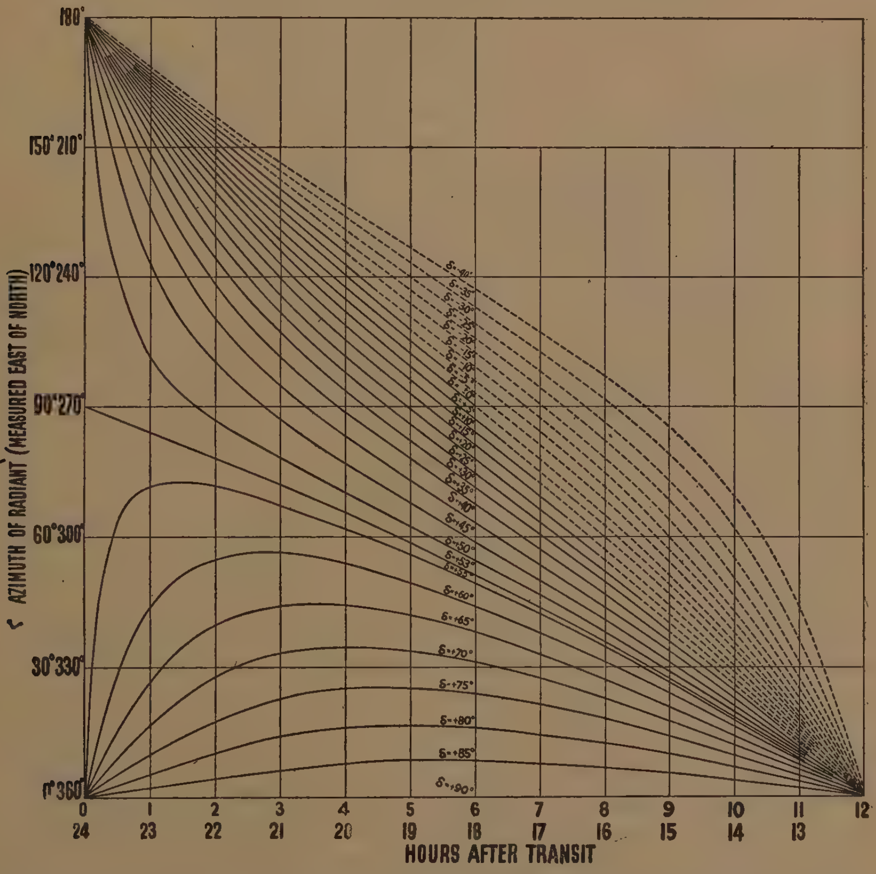
(*a*) *Determination of Right Ascension.*

Theoretical range-time curves such as those in figs. 4 (*a*), (*b*) and (*c*) show that for radiants at low elevation maximum ranges occur when the azimuth of the radiant differs from that of the aerial by 90°. The limiting elevation above which this no longer applies depends on the mean height of occurrence and the intensity of ionization of the trails, but comparison of experimental results with the theoretical curves shows that for the apparatus in use (Lovell, Banwell and Clegg, 1947) it is invariably greater than 60°. Thus for a radiant whose elevation never exceeds 60° (*i. e.* for which $\delta=+23°$) * the right ascension can be found by directing the aerial at azimuth 90°. Maximum ranges will then occur at the time of transit. If the declination of the radiant is greater than +23°, maximum ranges will generally occur after transit at some time $t_0+\Delta t_0$, as indicated in figs. 4 (*d*) and (*e*). The theoretical curves show that Δt_0 will never exceed 15^{m}, so that the consequent error in right ascension will not be greater than 4°.

(*b*) *Determination of Declination.*

It is possible to obtain some estimate of the declination from the range-time distribution of the echoes after transit, but for accurate determination

Fig. 8 (*a*).



Radiant azimuth plotted as function of time after transit for radiant declinations at 5° intervals between $\delta-40°$ and $\delta+90°$ and for latitude 53° N.

it is necessary to make use of the relationship between the declination and the rate of change of azimuth of the radiant at the times close to transit. This relationship is indicated in fig. 8, in which the azimuth and elevation of the radiant are plotted as functions of time. The curves are drawn for the approximate latitude of Jodrell Bank (53° N.) and for a number of declinations between $\delta-40^\circ$ and $\delta+90^\circ$.

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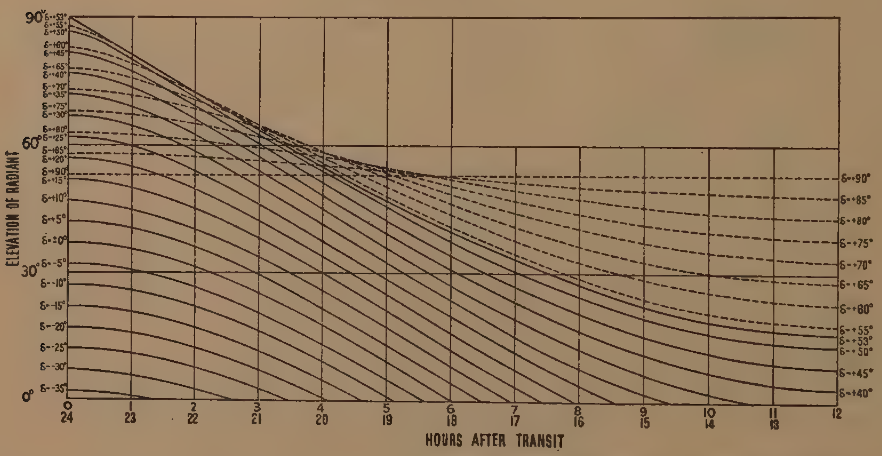
* See footnote, p. 583.

After observing the time t_0 of transit on azimuth 90°, the aerial is turned to some arbitrarily chosen azimuth $90^\circ+\theta$. If the radiant is at sufficiently low elevation, maximum ranges will now occur when its azimuth is $180^\circ+\theta$ at time t_θ. This gives the time $t_\theta-t_0$ required for the azimuth to change from 180° to $180^\circ+\theta$ and immediately determines the appropriate azimuth time curve in fig. 8 (*a*) and, hence, the declination.

If the elevation of the radiant is greater than 60°, there will again be an error, maximum ranges occurring on this new azimuth at some time $(t_\theta+\Delta t_\theta)$. The apparent time required for the azimuth to change by θ will then be $(t_\theta-t_0+\Delta t_\theta-\Delta t_0)$. If, however, θ does not exceed 20°, the elevation of the radiant will not change appreciably during this time and the errors Δt_0 and Δt_θ will be almost equal, so that the resultant error in the declination will be negligible.

Fig. 8 (*b*).



Radiant elevation plotted as function of time after transit for radiant declinations at 5° intervals between $\delta-35^\circ$ and $\delta+90^\circ$ and for latitude 53° N.

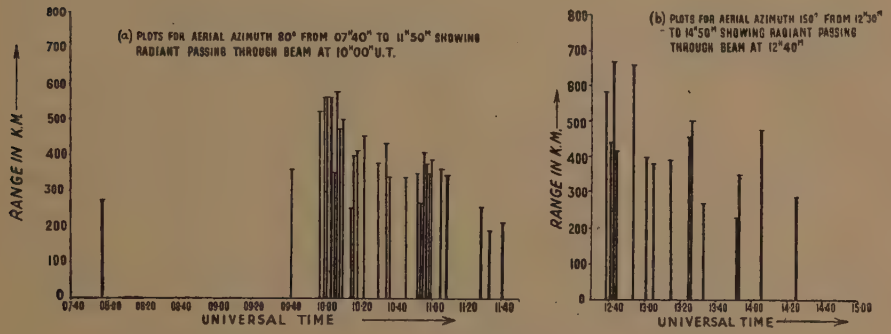
(*c*) *Method for Radiants at Higher Declinations.*

If the observations described in (*a*) and (*b*) indicate that the declination is greater than +23°, the right ascension can be found more accurately by choosing two aerial positions $90^\circ-\theta$ and $90^\circ+\theta$ such that the elevation of the radiant when at right angles to the beam is less than 60°. If maximum ranges occur on these azimuths at time $t_{-\theta}$ and $t_{+\theta}$ respectively, the time of transit will be $\frac{1}{2}(t_{-\theta}+t_{+\theta})$ and the radiant will have moved from azimuth $180^\circ-\theta$ to azimuth $180^\circ+\theta$ in time $(t_{+\theta}-t_{-\theta})$.

(*d*) *Alternative Methods.*

In practice, a set of observations is usually required to furnish data on other characteristics of a shower besides the coordinates of the radiant, and this may impose some restriction on the choice of aerial positions. Under these circumstances the method may be varied in detail to suit the particular set of conditions. For a circumpolar radiant, for example, it may often be more convenient to obtain the right ascension by observing the lower culmination. It is also possible to estimate the radiant position from observations on two azimuths not equally spaced about the east-west direction. An example is given in fig. 9, which shows the plots for azimuths 80° and 150° on 1947, June 10. It is clear that an isolated radiant passed through the beam at $10^{\mathrm{h}}\ 0^{\mathrm{m}}$ on the first azimuth. On turning to 150° long ranges occurred at $12^{\mathrm{h}}\ 40^{\mathrm{m}}$. From previous observations it was known that these two sets of echoes could be assumed to be from the same radiant, and, hence, the radiant azimuth changed from 170° to 240° in $2^{\mathrm{h}}\ 40^{\mathrm{m}}$. This places it on the curve $\delta+31°$ in fig. 8 (*a*) and gives the time of transit as $10^{\mathrm{h}}\ 18^{\mathrm{m}}$. This means that the elevation at $10^{\mathrm{h}}\ 0^{\mathrm{m}}$ was 67°, so that the radiant may not have been exactly at right angles to the beam at this time. The error, however, is unlikely to have been more than four minutes. Assuming that the times of maximum range were each subject to an error of measurement of ± 4 minutes, this places the time of transit at $10^{\mathrm{h}}\ 18^{\mathrm{m}}\ 6^{\mathrm{m}}$ and the declination at $28° \pm 2°$.

Fig. 9.



Plots for 1947, June 13, on aerial azimuths 80° and 150°.

8. Accuracy and Limitations of Radiant Determination.

For a given echo rate the accuracy with which the time of passage through the beam can be measured depends primarily on the rate of change of azimuth of the radiant during the period of observation. This is illustrated in figs. 4 (*a*), (*b*) and (*c*), where the rates of azimuthal rotation at the time of transit are respectively 15°, 19°, and 26° per hour. The increased sharpness of the maximum point for higher declinations is due almost entirely to the higher angular velocity with which the great circle plane is sweeping through the beam. Both the echo rate and the rate of azimuthal rotation are subject to diurnal variation and are maximum at transit, so that for high accuracy readings should be taken on azimuths as close to 90° or 270° as possible.

Consider the case when measurements are carried out on azimuths $90°-\theta$ and $90°+\theta$. The elevation and the rate of azimuthal rotation of the radiant will be the same at the times $t_{-\theta}$ and $t_{+\theta}$ of passage through these two positions, and the error Δt in the estimation of the times will be of the same order in both cases. The error in the time $\frac{1}{2}(t_{-\theta}+t_{+\theta})$ of transit will also be equal to Δt. The measurement of the time $(t_{+\theta}-t_{-\theta})$ required for a change in azimuth of 2θ will be subject to an error of $2\Delta t$, and the consequent error in the declination will depend on the value of θ chosen and on the particular curve in fig. 8 (*a*) on which the radiant lies.

Observations during the past 15 months with the apparatus at present in use (Lovell, Banwell and Clegg, 1947; Clegg, Hughes and Lovell, 1947) indicate that, with a steady rate of 30 echoes per hour and with an azimuthal velocity of 25° per hour, the time of passage through the beam can be measured to within ±4 minutes. For a slower moving radiant the accuracy will be less, and from the experimental results it seems reasonable to assume that the probable error varies inversely with the rate of azimuthal rotation. On the basis of these figures Table I. shows the probable error in the measurement of each of the coordinates for radiants at different declinations. It is assumed in every case that the optimum value has been chosen for θ.

Table I.

Probable errors for steady echo rate of 30 per hour.

Radiant declination	−30°	−20°	−10°	±0·0°	+10°	+20°	+30°	+40°	+50°	+60°	+70°	+80°	+90°
Probable error in R.A.	± 2°	± 2°	± 1·5°	± 1·5°	± 1°	± 1°	± 1°	± 1·5°	± 3°	± 3°	± 4°	± 5°	—
Probable error in δ	± 6°	± 6°	± 5°	± 4·5°	± 4°	± 3°	± 2°	± 1°	± 0·5°	± 2°	± 3°	± 5°	± 7°

It is evident that the method gives the most accurate values for the right ascension for radiant declinations of about +20°, but that it is most sensitive to changes in declination at +53° where transit occurs at the zenith. The limits of accuracy can be improved if readings are taken on more than two azimuths and the best mean values are found. For a radiant which remains active for a considerable period of time observations may be repeated on successive days, but it is important to remember that the coordinates may change considerably during the life of the shower (Prentice, 1948).

The method depends very critically on the echo rate and the figures quoted above apply for a steady rate of approximately 30 per hour. If the rate is significantly lower than this, or if the echoes occur irregularly in intermittent groups, the accuracy is markedly reduced.

Many radiants have multiple centres, several of which may be active at the same time, and may produce overlapping range curves. The ability of the aerial system to distinguish between two such centres depends principally on their elevations, the resolving power being greater for higher radiants. For a radiant at declination ±0° with an echo rate of 30 per hour two active centres must be separated by at least 20° in right ascension if they are to be distinguished. For uniformly diffuse radiants it is only possible to estimate the approximate coordinates of the eastern and western extremities of the active area by performing two experiments on successive days, one on azimuths close to 90° and the other on azimuths close to 270°.

The accuracy of the method can be increased by using an improved aerial system. Thus, by reducing the elevation of the beam, it would be possible to make measurements at times when the radiant elevation is greater than 60°, while a reduction in the horizontal beam width would give sharper definition and greater resolving power.

9. COMPARISON OF THE PRESENT METHOD WITH THAT OF HEY AND STEWART.

The method of radiant determination due to Hey and Stewart (1946, 1947) employed three stations situated 100 km. apart, but in its simplest form it requires simultaneous observations from two stations whose aerials are set on different bearings. The authors estimate that radiant positions can be determined with an accuracy of 10°, although this could be improved by using a narrower beam. It is clear, however, that with this method the probable error in the measurement must always be comparable with the beam width of the aerial employed, and that high precision can only be attained by building large aerial systems. In adopting a different approach to the problem, and in making use of the variation in the ranges of the echoes as well as the changes in echo rate, the present method is capable of greater accuracy, being subject to an error of only one-fifth of the aerial beam width.

ACKNOWLEDGEMENTS.

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LXXII. *Tables of Interpolation Coefficients for use in the Complex Plane.*

By P. M. WOODWARD, B.A.*

[Received Nov. 6, 1947.]

INTRODUCTION TO THE TABLES.

1. *Interpolation by the Method of Coupled Differences.*

IN an earlier paper ⁽¹⁾, the present author and others have described a new method of interpolation by differences for use in double-entry tables of functions satisfying Laplace's equation. The real and imaginary parts of an analytic function of a complex variable each satisfy this equation, and the method was in fact developed as an aid to subtabulation in the complex plane.

The object is to reduce the labour of ordinary bivariate interpolation by making use of symmetry with respect to the two variables. To this end, the interpolation is performed solely in terms of symmetrical or anti-symmetrical difference operators. If x and y denote the two arguments of a function u satisfying Laplace's equation, it has been shown ⁽¹⁾ that it is sufficient to tabulate “ coupled differences ”

$$\tfrac{1}{2}(\varDelta_x^2-\varDelta_y^2)u \quad \text{and} \quad \varDelta_x^2\varDelta_y^2u,$$

provided that sixth differences are negligible. The present tables give the coefficients by which these differences must be multiplied in performing an interpolation. The coefficients have been computed primarily to facilitate interpolation in those tables equipped with coupled differences which have already appeared in this magazine ⁽¹⁾, ⁽²⁾.

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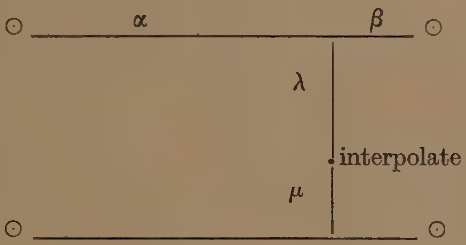
* Communicated by the Author.

2. *How to Use the Tables.*

Let the rectangular coordinates of the interpolate relative to the four nearest tabulated function values be as shown in the accompanying diagram, and let

$$\alpha+\beta=\lambda+\mu=1.$$



In performing an interpolation, each of the four function values and their associated differences are multiplied by the coefficients shown in the following scheme. The products so formed are added together to give the interpolated function value :—

(*a*) Coefficients of function values

$\beta\mu$	$\alpha\mu$
$\beta\lambda$	$\alpha\lambda$

(*b*) Coefficients of coupled second differences $\frac{1}{2}(\Delta_x^2-\Delta_y^2)$

$C_2(\beta, \mu)$	$C_2(\alpha, \mu)$
$C_2(\beta, \lambda)$	$C_2(\alpha, \lambda)$

(*c*) Coefficients of coupled (cross) fourth differences $\Delta_x^2\Delta_y^2$

$C_4(\beta, \mu)$	$C_4(\alpha, \mu)$
$C_4(\beta, \lambda)$	$C_4(\alpha, \lambda)$

The contribution from the four terms in (*b*) may be ignored if each coupled second difference is less than 4, and that from (*c*) if each fourth difference is less than 40, in units of the last decimal retained. Sixth differences may be ignored if less than 500.

3. *Interpolation in the Tables of Coefficients.*

It happens that the coefficients C_2 and C_4 themselves satisfy Laplace's equation. The method of interpolation described in section 2 may therefore be applied to the coefficients themselves.

4. *Description of the Tables.*

The coefficients C_2 and C_4 are defined as follows :—

$$C_2(p, q)=\tfrac{1}{6}pq(p^2-q^2),$$

$$C_4(p, q)=-\frac{pq}{360}(4+(p^2-3q^2)(3p^2-q^2)).$$

Table I., which gives C_2 for $p, q=0{\cdot}00\ (0{\cdot}05),\ 1{\cdot}00$, with coupled second differences, is exact to eight decimals. Table II. gives C_4 to seven decimals at the same intervals of p and q, also with coupled second differences. Fourth differences, which never exceed 60 in the seventh decimal place, are not tabulated.

5. *Acknowledgement.*

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6. *References.*

(1) P. M. and A. M. Woodward, "Four-figure Tables of the Airy Function in the Complex Plane," Phil. Mag. ser. 7, xxxvii. p. 236 (April, 1946).

(2) H. G. Hay, "Five-figure Table of the Function $\int_0^\infty e^{-zy}\mathrm{Ai}^2(y-j_1)dy$ in the Complex Plane," Phil. Mag. (to be published shortly).

TABLE I.—$C_2(p, q)$, with coupled second differences.

p q	0·00	0·05	0·10	0·15	0·20	0·25
	0·0					
0·00	0000000 0	0000000 0	0000000 0	0000000 0	0000000 0	0000000 0
0·05	0000000 0	0000000 625	0000625 1250	0002500 1875	0006250 2500	0012500 3125
0·10	0000000 0	−0000625 1250	0000000 2500	0003125 3750	0010000 5000	0021875 6250
0·15	0000000 0	−0002500 1875	−0003125 3750	0000000 5625	0008750 7500	0025000 9375
0·20	0000000 0	−0006250 2500	−0010000 5000	−0008750 7500	0000000 10000	0018750 12500
0·25	0000000 0	−0012500 3125	−0021875 6250	−0025000 9375	−0018750 12500	0000000 15625
0·30	0000000 0	−0021875 3750	−0040000 7500	−0050625 11250	−0050000 15000	−0034375 18750
0·35	0000000 0	−0035000 4375	−0065625 8750	−0087500 13125	−0096250 17500	−0087500 21875
0·40	0000000 0	−0052500 5000	−0100000 10000	−0137500 15000	−0160000 20000	−0162500 25000
0·45	0000000 0	−0075000 5625	−0144375 11250	−0202500 16875	−0243750 22500	−0262500 28125
0·50	0000000	−0103125 6250	−0200000 12500	−0284375 18750	−0350000 25000	−0390625 31250
0·55	0000000 0	−0137500 6875	−0268125 13750	−0385000 20625	−0481250 27500	−0550000 34375
0·60	0000000 0	−0178750 7500	−0350000 15000	−0506250 22500	−0640000 30000	−0743750 37500
0·65	0000000 0	−0227500 8125	−0446875 16250	−0650000 24375	−0828750 32500	−0975000 40625
0·70	0000000 0	−0284375 8750	−0560000 17500	−0818125 26250	−1050000 35000	−1246875 43750
0·75	0000000 0	−0350000 9375	−0690625 18750	−1012500 28125	−1306250 37500	−1562500 46875
0·80	0000000 0	−0425000 10000	−0840000 20000	−1235000 30000	−1600000 40000	−1925000 50000
0·85	0000000 0	−0510000 10625	−1009375 21250	−1487500 31875	−1933750 42500	−2337500 53125
0·90	0000000 0	−0605625 11250	−1200000 22500	−1771875 33750	−2310000 45000	−2803125 56250
0·95	0000000 0	−0712500 11875	−1413125 23750	−2090000 35625	−2731250 47500	−3325000 59375
1·00	0000000 0	−0831250 12500	−1650000 25000	−2443750 37500	−3200000 50000	−3906250 62500

TABLE I. (*cont.*)

q \ p	0·25	0·30	0·35	0·40	0·45	0·50
	0·0					
0·00	0000000 0	0000000 0	0000000 0	0000000 0	0000000 0	0000000 0
0·05	0012500 3125	0021875 3750	0035000 4375	0052500 5000	0075000 5625	0103125 6250
0·10	0021875 6250	0040000 7500	0065625 8750	0100000 10000	0144375 11250	0200000 12500
0·15	0025000 9375	0050625 11250	0087500 13125	0137500 15000	0202500 16875	0284375 18750
0·20	0018750 12500	0050000 15000	0096250 17500	0160000 20000	0243750 22500	0350000 25000
0·25	0000000 15625	0034375 18750	0087500 21875	0162500 25000	0262500 28125	0390625 31250
0·30	−0034375 18750	0000000 22500	0056875 26250	0140000 30000	0253125 33750	0400000 37500
0·35	−0087500 21875	−0056875 26250	0000000 30625	0087500 35000	0210000 39375	0371875 43750
0·40	−0162500 25000	−0140000 30000	−0087500 35000	0000000 40000	0127500 45000	0300000 50000
0·45	−0262500 28125	−0253125 33750	−0210000 39375	−0127500 45000	0000000 50625	0178125 56250
0·50	−0390625 31250	−0400000 37500	−0371875 43750	−0300000 50000	−0178125 56250	0000000 62500
0·55	−0550000 34375	−0584375 41250	−0577500 48125	−0522500 55000	−0412500 61875	−0240625 68750
0·60	−0743750 37500	−0810000 45000	−0831250 52500	−0800000 60000	−0708750 67500	−0550000 75000
0·65	−0975000 40625	−1080625 48750	−1137500 56875	−1137500 65000	−1072500 73125	−0934375 81250
0·70	−1246875 43750	−1400000 52500	−1500625 61250	−1540000 70000	−1509375 78750	−1400000 87500
0·75	−1562500 46875	−1771875 56250	−1925000 65625	−2012500 75000	−2025000 84375	−1953125 93750
0·80	−1925000 50000	−2200000 60000	−2415000 70000	−2560000 80000	−2625000 90000	−2600000 100000
0·85	−2337500 53125	−2688125 63750	−2975000 74375	−3187500 85000	−3315000 95625	−3346875 106250
0·90	−2803125 56250	−3240000 67500	−3609375 78750	−3900000 90000	−4100625 101250	−4200000 112500
0·95	−3325000 59375	−3859375 71250	−4322500 83125	−4702500 95000	−4987500 106875	−5165625 118750
1·00	−3906250 62500	−4550000 75000	−5118750 87500	−5600000 100000	−5981250 112500	−6250000 125000

Table I. (*cont.*)

q \ p	0·50	0·55	0·60	0·65	0·70	0·75
	0·0					
0·00	0000000 0	0000000 0	0000000 0	0000000 0	0000000 0	0000000 0
0·05	0103125 6250	0137500 6875	0178750 7500	0227500 8125	0284375 8750	0350000 9375
0·10	0200000 12500	0268125 13750	0350000 15000	0446875 16250	0560000 17500	0690625 18750
0·15	0284375 18750	0385000 20625	0506250 22500	0650000 24375	0818125 26250	1012500 28125
0·20	0350000 25000	0481250 27500	0640000 30000	0828750 32500	1050000 35000	1306250 37500
0·25	0390625 31250	0550000 34375	0743750 37500	0975000 40625	1246875 43750	1562500 46875
0·30	0400000 37500	0584375 41250	0810000 45000	1080625 48750	1400000 52500	1771875 56250
0·35	0371875 43750	0577500 48125	0831250 52500	1137500 56875	1500625 61250	1925000 65625
0·40	0300000 50000	0522500 55000	0800000 60000	1137500 65000	1540000 70000	2012500 75000
0·45	0178125 56250	0412500 61875	0708750 67500	1072500 73125	1509375 78750	2025000 84375
0·50	0000000 62500	0240625 68750	0550000 75000	0934375 81250	1400000 87500	1953125 93750
0·55	−0240625 68750	0000000 75625	0316250 82500	0715000 89375	1203125 96250	1787500 103125
0·60	−0550000 75000	−0316250 82500	0000000 90000	0406250 97500	0910000 105000	1518750 112500
0·65	−0934375 81250	−0715000 89375	−0406250 97500	0000000 105625	0511875 113750	1137500 121875
0·70	−1400000 87500	−1203125 96250	−0910000 105000	−0511875 113750	0000000 122500	0634375 131250
0·75	−1953125 93750	−1787500 103125	−1518750 112500	−1137500 121875	−0634375 131250	0000000 140625
0·80	−2600000 100000	−2475000 110000	−2240000 120000	−1885000 130000	−1400000 140000	−0775000 150000
0·85	−3346875 106250	−3272500 116875	−3081250 127500	−2762500 138125	−2305625 148750	−1700000 159375
0·90	−4200000 112500	−4186875 123750	−4050000 135000	−3778125 146250	−3360000 157500	−2784375 168750
0·95	−5165625 118750	−5225000 130625	−5153750 142500	−4940000 154375	−4571875 166250	−4037500 178125
1·00	−6250000 125000	−6393750 137500	−6400000 150000	−6256250 162500	−5950000 175000	−5468750 187500

TABLE I. (*cont.*)

$q \diagdown p$	0·75	0·80	0·85	0·90	0·95	1·00
	0·0					
0·00	0000000 0	0000000 0	0000000 0	0000000 0	0000000 0	0000000 0
0·05	0350000 9375	0425000 10000	0510000 10625	0605625 11250	0712500 11875	0831250 12500
0·10	0690625 18750	0840000 20000	1009375 21250	1200000 22500	1413125 23750	1650000 25000
0·15	1012500 28125	1235000 30000	1487500 31875	1771875 33750	2090000 35625	2443750 37500
0·20	1306250 37500	1600000 40000	1933750 42500	2310000 45000	2731250 47500	3200000 50000
0·25	1562500 46875	1925000 50000	2337500 53125	2803125 56250	3325000 59375	3906250 62500
0·30	1771875 56250	2200000 60000	2688125 63750	3240000 67500	3859375 71250	4550000 75000
0·35	1925000 65625	2415000 70000	2975000 74375	3609375 78750	4322500 83125	5118750 87500
0·40	2012500 75000	2560000 80000	3187500 85000	3900000 90000	4702500 95000	5600000 100000
0·45	2025000 84375	2625000 90000	3315000 95625	4100625 101250	4987500 106875	5981250 112500
0·50	1953125 93750	2600000 100000	3346875 106250	4200000 112500	5165625 118750	6250000 125000
0·55	1787500 103125	2475000 110000	3272500 116875	4186875 123750	5225000 130625	6393750 137500
0·60	1518750 112500	2240000 120000	3081250 127500	4050000 135000	5153750 142500	6400000 150000
0·65	1137500 121875	1885000 130000	2762500 138125	3778125 146250	4940000 154375	6256250 162500
0·70	0634375 131250	1400000 140000	2305625 148750	3360000 157500	4571875 166250	5950000 175000
0·75	0000000 140625	0775000 150000	1700000 159375	2784375 168750	4037500 178125	5468750 187500
0·80	−0775000 150000	0000000 160000	0935000 170000	2040000 180000	3325000 190000	4800000 200000
0·85	−1700000 159375	−0935000 170000	0000000 180625	1115625 191250	2422500 201875	3931250 212500
0·90	−2784375 168750	−2040000 180000	−1115625 191250	0000000 202500	1318125 213750	2850000 225000
0·95	−4037500 178125	−3325000 190000	−2422500 201875	−1318125 213750	0000000 225625	1543750 237500
1·00	−5468750 187500	−4800000 200000	−3931250 212500	−2850000 225000	−1543750 237500	0000000 250000

TABLE II.—$C_4(p, q)$, with coupled second differences.

q \ p	0·00	0·05	0·10	0·15	0·20	0·25
	−0·00					
0·00	00000	00000	00000	00000	00000	00000
	0	0	0	0	0	0
0·05	00000	00278	00556	00833	01112	01392
	0	0	0	−1	−1	−3
0·10	00000	00556	01111	01666	02223	02782
	0	0	0	−1	−3	−6
0·15	00000	00833	01666	02499	03331	04166
	0	1	1	0	−3	−6
0·20	00000	01112	02223	03331	04437	05544
	0	1	3	3	0	−5
0·25	00000	01392	02782	04166	05544	06917
	0	3	6	6	5	0
0·30	00000	01676	03346	05007	06655	08291
	0	6	10	13	13	9
0·35	00000	01965	03921	05861	07779	09674
	0	9	16	22	24	22
0·40	00000	02263	04512	06737	08928	11079
	0	14	25	35	40	41
0·45	00000	02574	05129	07648	10117	12525
	0	19	36	51	61	66
0·50	00000	02904	05782	08610	11367	14038
	0	26	50	71	88	98
0·55	00000	03259	06485	09643	12706	15649
	0	35	68	96	120	138
0·60	00000	03650	07255	10773	14165	17398
	0	45	88	127	160	187
0·65	00000	04085	08113	12030	15785	19334
	0	57	112	163	207	244
0·70	00000	04577	09084	13450	17613	21514
	0	71	140	204	263	312
0·75	00000	05141	10194	15075	19704	24007
	0	88	173	253	327	391
0·80	00000	05792	11478	16954	22123	26892
	0	107	210	309	400	481
0·85	00000	06550	12972	19143	24942	30259
	0	128	252	372	484	585
0·90	00000	07435	14719	21703	28246	34211
	0	152	300	443	578	701
0·95	00000	08472	16766	24708	32127	38865
	0	178	353	523	683	832
1·00	00000	09687	19167	28235	36693	44352
	0	208	413	612	800	977

TABLE II. (*cont.*)

p q	0·25	0·30	0·35	0·40	0·45	0·50
	−0·00					
0·00	00000 0	00000 0	00000 0	00000 0	00000 0	00000 0
0·05	01392 −3	01676 −6	01965 −9	02263 −14	02574 −19	02904 −26
0·10	02782 −6	03346 −10	03921 −16	04512 −25	05129 −36	05782 −50
0·15	04166 −6	05007 −13	05861 −22	06737 −35	07648 −51	08610 −71
0·20	05544 −5	06655 −13	07779 −24	08928 −40	10117 −61	11367 −88
0·25	06917 0	08291 −9	09674 −22	11079 −41	12525 −66	14038 −98
0·30	08291 9	09919 0	11547 −14	13190 −35	14869 −64	16612 −100
0·35	09674 22	11547 14	13407 0	15267 −22	17150 −53	19086 −93
0·40	11079 41	13190 35	15267 22	17323 0	19379 −32	21468 −75
0·45	12525 66	14869 64	17150 53	19379 32	21577 0	23777 −45
0·50	14038 98	16612 100	19086 93	21468 75	23777 45	26042 0
0·55	15649 138	18455 146	21116 144	23634 131	26022 103	28308 61
0·60	17398 187	20445 203	23291 208	25931 201	28371 177	30636 138
0·65	19334 244	22639 270	25676 285	28429 285	30899 268	33103 234
0·70	21514 312	25104 350	28346 376	31213 385	33697 377	35805 350
0·75	24007 391	27920 443	31392 482	34383 504	36873 507	38859 489
0·80	26892 481	31181 550	34921 604	38059 640	40558 656	42403 651
0·85	30259 585	34992 672	39055 744	42375 797	44901 829	46600 837
0·90	34211 701	39477 810	43934 903	47491 975	50074 1025	51635 1051
0·95	38865 832	44773 965	49718 1081	53583 1176	56275 1247	57723 1292
1·00	44352 977	51036 1138	56583 1280	60853 1400	63725 1496	65104 1563

TABLE II. (*cont.*)

q \ p	0·50	0·55	0·60	0·65	0·70	0·75
	−0·00					
0·00	00000 0	00000 0	00000 0	00000 0	00000 0	00000 0
0·05	02904 −26	03259 −35	03650 −45	04085 −57	04577 −71	05141 −88
0·10	05782 −50	06485 −68	07255 −88	08113 −112	09084 −140	10194 −173
0·15	08610 −71	09643 −96	10773 −127	12030 −163	13450 −204	15075 −253
0·20	11367 −88	12706 −120	14165 −160	15785 −207	17613 −263	19704 −327
0·25	14038 −98	15649 −138	17398 −187	19334 −244	21514 −312	24007 −391
0·30	16612 −100	18455 −146	20445 −203	22639 −270	25104 −350	27920 −443
0·35	19086 −93	21116 −144	23291 −208	25676 −285	28346 −376	31392 −482
0·40	21468 −75	23634 −131	25931 −201	28429 −285	31213 −385	34383 −504
0·45	23777 −45	26022 −103	28371 −177	30899 −268	33697 −377	36873 −507
0·50	26042 0	28308 −61	30636 −138	33103 −234	35805 −350	38859 −489
0·55	28308 61	30535 0	32764 −79	35074 −179	37565 −301	40358 −447
0·60	30636 138	32764 79	34816 0	36869 −102	39026 −228	41413 −380
0·65	33103 234	35074 179	36869 102	38564 0	40262 −128	42090 −285
0·70	35805 350	37565 301	39026 228	40262 128	41372 0	42485 −158
0·75	38859 489	40358 447	41413 380	42090 285	42485 158	42725 0
0·80	42403 651	43600 619	44181 560	44205 472	43759 350	42967 194
0·85	46600 837	47464 818	47512 770	46793 691	45386 577	43406 426
0·90	51635 1051	52148 1047	51616 1013	50075 945	47593 840	44274 696
0·95	57723 1292	57881 1306	56736 1289	54305 1235	50643 1143	45841 1010
1·00	65104 1563	64923 1599	63147 1601	59773 1564	54839 1488	48421 1368

TABLE II. (*cont.*)

q \ p	0·75	0·80	0·85	0·90	0·95	1·00
	−0·00					
0·00	00000 0	00000 0	00000 0	00000 0	00000 0	00000 0
0·05	05141 −88	05792 −107	06550 −128	07435 −152	08472 −178	09687 −208
0·10	10194 −173	11478 −210	12972 −252	14719 −300	16766 −353	19167 −413
0·15	15075 −253	16954 −309	19143 −372	21703 −443	24708 −523	28235 −612
0·20	19704 −327	22123 −400	24942 −484	28246 −578	32127 −683	36693 −800
0·25	24007 −391	26892 −481	30259 −585	34211 −701	38865 −832	44352 −977
0·30	27920 −443	31181 −550	34992 −672	39477 −810	44773 −965	51036 −1138
0·35	31392 −482	34921 −604	39055 −744	43934 −903	49718 −1081	56583 −1280
0·40	34383 −504	38059 −640	42375 −797	47491 −975	53583 −1176	60853 −1400
0·45	36873 −507	40558 −656	44901 −829	50074 −1025	56275 −1247	63725 −1496
0·50	38859 −489	42403 −651	46600 −837	51635 −1051	57723 −1292	65104 −1563
0·55	40358 −447	43600 −619	47464 −818	52148 −1047	57881 −1306	64923 −1599
0·60	41413 −380	44181 −560	47512 −770	51616 −1013	56736 −1289	63147 −1601
0·65	42090 −285	44205 −472	46793 −691	50075 −945	54305 −1235	59773 −1564
0·70	42485 −158	43759 −350	45386 −577	47593 −840	50643 −1143	54839 −1488
0·75	42725 0	42967 −194	43406 −426	44274 −696	45841 −1010	48421 −1368
0·80	42967 194	41984 0	41004 −234	40262 −510	40033 −832	40640 −1200
0·85	43406 426	41004 234	38372 0	35744 −279	33398 −606	31663 −983
0·90	44274 696	40262 510	35744 279	30951 0	26162 −330	21707 −713
0·95	45841 1010	40033 832	33398 606	26162 330	18601 0	11044 −386
1·00	48421 1368	40640 1200	31663 983	21707 713	11044 386	00000 0

LXXIII. *Variable Specific-heat Corrections for the Efficiency of the Basic Internal-combustion Thermodynamic Cycle and their Application to the Constant-volume, Constant-pressure, Diesel and Humphrey Pump Cycles.*

By R. M. Walker*.

[Received March 28, 1947.]

Summary.

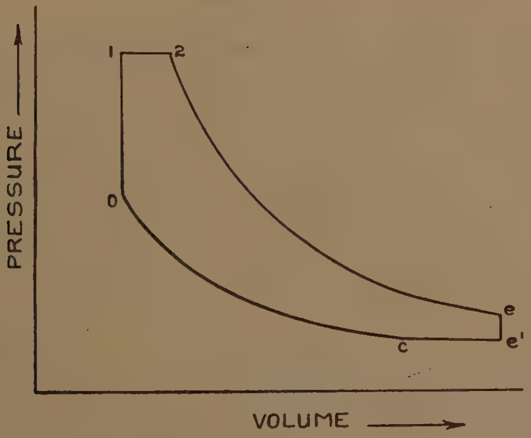
The following paper is offered as a contribution to the problem of facilitating variable specific-heat analysis as applied to internal-combustion motor problems in general.

The approach made to these problems is new, in so far as it involves the inclusion of a fifth ratio into the analysis of the basic internal-combustion engine cycle. This ratio, besides introducing a symmetry into the resulting expressions, also enables fairly convenient expressions to be obtained for the various thermodynamic quantities such as thermal efficiency, etc.

Introduction.

Previous work on this subject has been based on the use of four ratios (1) in the development of a formula for the thermal efficiency of the basic internal-combustion thermodynamic cycle shown in fig. 1. These four ratios are defined as follows :—

Fig. 1.



The explosion ratio :

$$\alpha = \frac{P_1}{P_0} = \frac{T_1}{T_0}.$$

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* Communicated by Professor W. J. Walker.

The cut-off ratio :

$$\rho=\frac{V_2}{V_1}=\frac{T_2}{T_1}.$$

The extended expansion ratio :

$$\beta=\frac{V_{e'}}{V_c}=\frac{T_{e'}}{T_c}.$$

The compression ratio :

$$r=\frac{V_c}{V_0}.$$

The introduction of a fifth ratio, namely :—

The exhaust-pressure ratio :

$$\sigma=\frac{P_e}{P_{e'}}=\frac{T_e}{T_{e'}},$$

into the expression for thermal efficiency is found to be of value from four points of view :—

(1) It introduces a symmetry into the expressions and thereby simplifies the analysis of the effect of variable specific heat.

(2) It leads to much simplified expressions for the efficiency of the Constant-Volume and Constant-Pressure Cycles under variable specific-heat conditions.

(3) It is fundamental in the determination of T_c, the initial compression temperature.

(4) It is also fundamental in the determination of the gravimetric ratio $\dfrac{\text{Fuel}}{\text{Fuel+Air+Retained Exhaust Gases}}$.

(Points (3) and (4) will not be discussed in this paper.)

The derivation of an efficiency expression under variable specific-heat conditions depends on the solution of an exponential equation, an exact solution of which does not seem possible and earlier workers (2), (3), (4) have based their analysis on approximations such as $e^x=1+x$ and $e^{-x}=1-x$, which become completely inadequate when x is at all large. In the following analysis similar relationships are used, but the equality signs are replaced by inequality signs, as for example,

$$e^x>1+x. \qquad \ldots \ldots \ldots \ldots \quad (1)$$

This technique, combined with the method of Newton for solving equations, is valuable in showing the order of the approximation.

A slide-rule method (5) of solving this particular type of equation was used in numerical calculations and in the drawing up of tables.

The analysis is developed on the assumption that the specific heats of the gases vary linearly with the temperature, the relationships being given by

$$K_v=D+ST=224+0{\cdot}0672T \text{ ft.-lbs./lb./° C.}$$

and

$$K_p=E+ST=320+0{\cdot}0672T \text{ ft.-lbs./lb./° C.}$$

The thermal efficiency of the cycle of fig. 1 is given by

$$\eta_t = 1 - \frac{\left[D(T_e - T_{e'}) + \frac{S}{2}(T_e^2 - T_{e'}^2)\right] + \left[E(T_{e'} - T_c) + \frac{S}{2}(T_{e'}^2 - T_c^2)\right]}{\left[D(T_1 - T_0) + \frac{S}{2}(T_1^2 - T_0^2)\right] + \left[E(T_2 - T_1) + \frac{S}{2}(T_2^2 - T_1^2)\right]}.$$

Let $\frac{E}{D} = m = 1{\cdot}43$ and $\frac{S}{D} = \lambda = 0{\cdot}0003$.

Then
$$\eta_t = 1 - \frac{\beta(\sigma-1) + m(\beta-1) + \frac{\lambda T_c}{2}(\beta^2\sigma^2 - 1)}{(\alpha-1) + m\alpha(\rho-1) + \frac{\lambda T_0}{2}(\alpha^2\rho^2 - 1)} \cdot \left(\frac{T_c}{T_0}\right). \quad . \quad . \quad (2)$$

The relationship between T_c and T_0 is obtained from

$$P_0 V_0^m e^{\lambda T_0} = P_c V_c^m e^{\lambda T_c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2\,a)$$

and
$$\frac{P_0 V_0}{T_0} = \frac{P_c V_c}{T_c},$$

which give
$$T_0 e^{\lambda T_0} = r^{m-1} T_c e^{\lambda T_c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also, the relationship between four of the ratios is obtained from (2 *a*) and

$$P_e V_e^m e^{\lambda T_e} = P_2 V_2^m e^{\lambda T_2},$$

which give
$$\sigma\beta^m e^{\lambda T_c(\sigma\beta-1)} = \alpha\rho^m e^{\lambda T_0(\alpha\rho-1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The formulæ (2), (3) and (4) are taken as the fundamental relationships under variable specific-heat conditions. Expressions (2) and (4), in particular, illustrate the symmetry obtained by the introduction of the new ratio.

The formulæ which are derived from (2), (3) and (4) in the following analysis show accurately the effect of variable specific heat and should be of value in enabling those interested to decide whether or not, in any given case, variable specific heat is the most important factor in the reduction of efficiency, or whether other factors such as dissociation are, at least, equally important.

2. *The Formula for* T_0 *derived from Equation* (2).

In the following, two expressions are derived, lying fairly closely to one another, between which T_0 must lie.

Equation (3) may be written :

$$\log T_0 + \lambda T_0 - [\log T_c + (m-1)\log r + \lambda T_c] = 0.$$

Let
$$F(T) = \log T + \lambda T - [\log T_c + (m-1)\log r + \lambda T_c]. \quad . \quad . \quad . \quad (5)$$

The problem is, therefore, to find the root of the equation $F(T)=0$. Adopting Newton's method of approximation :

$$F'(T)=\frac{1}{T}+\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

As a first approximation to T_0,

$$T'=r^{m-1}T_c>T_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

For a second approximation,

$$T''=r^{m-1}T_c-\frac{\lambda T_c(r^{m-1}-1)}{\dfrac{1}{T_c r^{m-1}}+\lambda}$$

$$=r^{m-1}T_c\left[\frac{1+\lambda T_c}{1+\lambda T_c r^{m-1}}\right]<T_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

since the graph of $F(T)$ against T is convex upwards.

Also, by equation (3),

$$T_0=r^{m-1}T_c e^{\lambda(T_c-T_0)}.$$

$$\therefore \qquad T_0<\frac{r^{m-1}T_c}{1+\lambda(T_0-T_c)}<\frac{r^{m-1}T_c}{1+\lambda T_c\left[\dfrac{r^{m-1}(1+\lambda T_c)}{1+\lambda T_c r^{m-1}}-1\right]},$$

$$\therefore \qquad T_0<r^{m\quad 1}T_c\left[\frac{1+\lambda T_c r^{m-1}}{1+\lambda T_c(2r^{m-1}-1)}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\therefore \quad r^{m-1}T_c\left[\frac{1+\lambda T_c}{1+\lambda T_c r^{m-1}}\right]<T_0<r^{m-1}T_c\left[\frac{1+\lambda T_c r^{m-1}}{1+\lambda T_c(2r^{-m1}-1)}\right]. \quad . \quad . \quad (10)$$

The difference between the two terms in square brackets is

$$\frac{\lambda^2 T_c^2(r^{m-1}-1)^2}{[1+\lambda T_c r^{m-1}][1+\lambda T_c(2r^{m-1}-1)]},$$

which in an extreme case of $r=20$ has the value 0·033, representing only 3·3 per cent. of $r^{m-1}T_c$.

It is now suggested that the geometric mean of the two approximations in (10) should give an accurate value for T_0, namely,

$$T_0=T_c r^{m-1}\sqrt{\frac{1+\lambda T_c}{1+\lambda T_c(2r^{m-1}-1)}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This is confirmed by the slide-rule method for determining T_0. The error involved by using this geometric mean is about 0·5 per cent. in the extreme case of $r=20$, so that the error is negligible for all practical cases. It should further be noted that the error involved in using either of the approximations to T_0 in (10) is less than 2 per cent. for the extreme case of $r=20$.

3. *Two Inequalities derived from Equation* (4).

The first of these inequalities is of value when considering the Constant-Volume and Diesel-Engine cycles, the second when considering the Constant-Pressure and Humphrey-Pump cycles.

Equation (4) gives

$$\sigma=\frac{\alpha\rho^m}{\beta^m}e^{\lambda T_0(\alpha\varrho-1)-\lambda T_c(\sigma\beta-1)},$$

$$\therefore \qquad \sigma>\frac{\alpha\rho^m}{\beta^m}[1+\lambda T_0(\alpha\rho-1)-\lambda T_c\sigma\beta+\lambda T_c], \text{ by (1)},$$

$$\therefore \qquad \sigma\left[1+\frac{\alpha\rho^m}{\beta^{m-1}}\lambda T_c\right]>\frac{\alpha\rho^m}{\beta^m}[1+\lambda T_0(\alpha\rho-1)+\lambda T_c],$$

$$\therefore \qquad \sigma>\frac{\alpha\rho^m}{\beta^m}\left[\frac{1+\lambda T_0(\alpha\rho-1)+\lambda T_c}{1+\dfrac{\alpha\rho^m}{\beta^{m-1}}\lambda T_c}\right],$$

$$\therefore \qquad (\alpha\rho-1)\lambda T_0-(\sigma\beta-1)\lambda T_c<\frac{(\alpha\rho-1)\lambda T_0-\left(\dfrac{\alpha\rho^m}{\beta^{m-1}}-1\right)\lambda T_c}{1+\lambda T_c\dfrac{\alpha\rho^m}{\beta^{m-1}}}$$

$$=\left[\frac{1+\lambda T_0(\alpha\rho-1)+\lambda T_c}{1+\lambda T_c\dfrac{\alpha\rho^m}{\beta^{m-1}}}\right]-1,$$

$$\therefore \qquad \frac{\alpha\rho^m}{\beta^m}e^{\left[\frac{1+\lambda T_0(\alpha\varrho-1)+\lambda T_c}{1+\lambda T_c\frac{\alpha\varrho^m}{\beta^{m-1}}}\right]-1}>\sigma>\frac{\alpha\rho^m}{\beta^m}\left[\frac{1+\lambda T_0(\alpha\rho-1)+\lambda T_c}{1+\lambda T_c\dfrac{\alpha\rho^m}{\beta^{m-1}}}\right]. \qquad (12)$$

Similarly, it can be shown that

$$\frac{\alpha^{1/m}\rho}{\sigma^{1/m}}e^{\left[\frac{1+\frac{\lambda T_0}{m}(\alpha\varrho-1)+\frac{\lambda T_c}{m}}{1+\frac{\lambda T_c}{m}\alpha^{1/m}\varrho\sigma^{m-1/m}}\right]-1}>\beta>\frac{\alpha^{1/m}\rho}{\sigma^{1/m}}\left[\frac{1+\dfrac{\lambda T_0}{m}(\alpha\rho-1)+\dfrac{\lambda T_c}{m}}{1+\dfrac{\lambda T_c}{m}\alpha^{1/m}\rho\sigma^{m-1/m}}\right]. \quad . \quad (13)$$

The values of σ and β are closer to the right-hand side than the left-hand side value of these inequalities. The right-hand side of (12) and (13) may also be found by Newton's method as second approximations to σ and β respectively. When the quantity in the square bracket is close to unity, the difference between the two sides of the inequality is small and the right-hand side may be used as a good approximation to σ in (12) and β in (13).

4. *A Formula for the Efficiency of the Basic Cycle.*

$$\eta_t=1-\frac{\beta(\sigma-1)+m(\beta-1)+\dfrac{\lambda T_c}{2}(\beta^2\sigma^2-1)}{(\alpha-1)+m\alpha(\rho-1)+\dfrac{\lambda T_0}{2}(\alpha^2\rho^2-1)}\cdot\frac{T_c}{T_0}$$

$$=1-\frac{\beta(\sigma-1)+m(\beta-1)}{(\alpha-1)+m\alpha(\rho-1)}\left[\frac{(1+\lambda BT_c)T_cr^{m-1}}{(1+\lambda AT_0)T_0}\right]\frac{1}{r^{m-1}},$$

where

$$\left.\begin{aligned}A&=\frac{\alpha^2\rho^2-1}{2[(\alpha-1)+m\alpha(\rho-1)]},\\ B&=\frac{\beta^2\sigma^2-1}{2[\beta(\sigma-1)+m(\beta-1)]}.\end{aligned}\right\}\quad .\ .\ .\ .\ .\ (14)$$

Let

$$\left.\begin{aligned}\delta&=\frac{\beta(\sigma-1)+m(\beta-1)}{(\alpha-1)+m\alpha(\rho-1)}\\ K&=\left[\frac{(1+\lambda BT_c)T_cr^{m-1}}{(1+\lambda AT_0)T_0}\right].\end{aligned}\right\}\quad .\ .\ .\ .\ .\ (15)$$

and

Then

$$\eta_t=1-\frac{K\delta}{r^{m-1}}.\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (16)$$

The efficiency expression (16) has been cast in this form because of its similarity to the Air-Standard formula for the Constant-volume and Constant-Pressure Cycle. For practical purposes it is better not to examine the effects of variable specific heat on the efficiency of this cycle, but rather to do so for the four most important cycles in practice, as will be done hereafter.

5. *The Constant-Volume Cycle.*

For this cycle $\beta=1$ and $\rho=1$.

$$\therefore\qquad A=\frac{\alpha+1}{2},\quad B=\frac{\sigma+1}{2},\quad \delta=\frac{\sigma-1}{\alpha-1}$$

and

$$\eta_t=1-\left(\frac{\sigma-1}{\alpha-1}\right)\frac{K}{r^{m-1}}.$$

Under constant specific-heat conditions $K=1$ and $\sigma=\alpha$, so that $\eta_t=1-\dfrac{1}{r^{m-1}}$, which is the well-known air (C.S.H.) standard expression.

On making calculations for K under variable specific-heat conditions for different values of r ranging from 6 to 20 and for different values of α ranging from 2 to 15, it was found that the values of K were all close to 1, the lowest value being 0·96, the highest 1·03. A constant value of 400° C. abs. was used for T_c.

We therefore have the important result that a good approximation to the true efficiency is the formula

$$\eta_t=1-\left(\frac{\sigma-1}{\alpha-1}\right)\frac{1}{r^{m-1}},\quad .\ .\ .\ .\ .\ .\ .\ .\ (17)$$

the maximum error being in the neighbourhood of 5 per cent only.

σ is always greater than α under variable specific-heat conditions, so that the efficiency will be less than the air-standard efficiency. If σ, α and r can be readily determined, then formula (17) is a useful one.

It is also possible, using the approximations from sections 2 and 3, to derive an expression for the correction factor by which the air-standard formula must be multiplied to give the true efficiency.

From equation (12),

$$\sigma > \alpha\left[\frac{1+\lambda T_0(\alpha-1)+\lambda T_c}{1+\lambda T_c\alpha}\right],$$

$$\therefore \qquad \frac{\sigma-1}{\alpha-1} > \frac{1+\lambda T_0\alpha}{1+\lambda T_c\alpha}.$$

For all practical conditions K is less than unity. As a good approximation, therefore, we put K=1 and

$$\frac{\sigma-1}{\alpha-1} = \frac{1+\lambda T_0\alpha}{1+\lambda T_c\alpha},$$

$$\therefore \quad \eta_t = 1-\left(\frac{\sigma-1}{\alpha-1}\right)\frac{1}{r^{m-1}}, \text{ by (17)},$$

$$= 1-\left(\frac{1+\lambda T_0\alpha}{1+\lambda T_c\alpha}\right)\frac{1}{r^{m-1}}$$

$$= \frac{(r^{m-1}-1)+\alpha\lambda(r^{m-1}T_c-T_0)}{r^{m-1}(1+\alpha\lambda T_c)}$$

$$= \left(1-\frac{1}{r^{m-1}}\right)\left[\frac{1+\dfrac{\lambda T_c r^{m-1}}{1+\lambda T_c r^{m-1}}\alpha\lambda T_c}{1+\alpha\lambda T_c}\right] \text{ by second approximation (8),}$$

thus giving, as the correction factor to the Air-Standard Efficiency $\left(1-\frac{1}{r^{m-1}}\right)$, the term in square brackets.

A previous correction factor ⁽⁴⁾, deduced by Professor W. J. Walker, is $\left[1-\left(\frac{\alpha+1}{2}\right)\lambda T_c\right]$. This implies that the efficiency decreases more rapidly with α than the foregoing analysis in the paper indicates.

6. *The Constant-Pressure Cycle.*

In this cycle $\alpha=1$ and $\sigma=1$, so that $A=\frac{\rho+1}{2m}$, $B=\frac{\beta+1}{2m}$, $\delta=\frac{\beta-1}{\rho-1}$ and $\eta_t=1-\left(\frac{\beta-1}{\rho-1}\right)\frac{K}{r^{m-1}}$. Under constant specific-heat conditions as with the constant-volume cycle, $\eta_t=1-\frac{1}{r^{m-1}}$. If calculations are made for K for a range of values of r from 6 to 20 and of ρ from 1·5 to 12, it is again found that K lies within this range of values very close to 1. The highest value of K is 1·06, the lowest 0·97. We may, therefore, use the formula

$$\eta_t=1-\left(\frac{\beta-1}{\rho-1}\right)\frac{1}{r^{m-1}}$$

as a good approximation to the efficiency under Variable Specific-Heat conditions.

If the approximations from sections 2 and 3 are developed as in the case of the Constant-Volume Cycle, the following Correction Factor may be derived :—

$$\left[\frac{1+\dfrac{\lambda T_c r^{m-1}}{1+\lambda T_c r^{m-1}}\,\dfrac{\rho\lambda T_c}{m}}{1+\dfrac{\rho\lambda T_c}{m}}\right].$$

7. *The Diesel-Engine Cycle.*

Here $\alpha=1$ and $\beta=1$.

$$A=\frac{\rho+1}{2m},\quad B=\frac{\sigma+1}{2}\quad\text{and}\quad \delta=\frac{\sigma-1}{m(\rho-1)},$$

so that

$$\eta_t=1-\frac{K(\sigma-1)}{m(\rho-1)}\cdot\frac{1}{r^{m-1}}.$$

Under constant specific-heat conditions $K=1$ and $\sigma=\rho^m$, so that

$$\eta_t=1-\frac{\rho^m-1}{m(\rho-1)}\frac{1}{r^{m-1}}.$$

In this cycle, under variable specific-heat conditions, K does not approximate to unity, so that it is not possible to simplify the efficiency formula. One interesting point is that the values of σ approximate fairly closely to ρ^m even under variable specific-heat conditions, so that the first approximation (12), namely,

$$\sigma=\rho^m\left[\frac{1+\lambda T_0(\rho-1)+\lambda T_c}{1+\lambda T_c\rho^m}\right],$$

is close enough for all practical purposes. K is, however, always greater than 1, so that for this cycle the reduction in efficiency under variable specific-heat conditions is mainly contained in the factor K.

8. *The Humphrey-Pump Cycle.*

Here $\rho=1$ and $\sigma=1$.

Hence

$$A=\frac{\alpha+1}{2},\quad B=\frac{\beta+1}{2m},$$

$$\delta=\frac{m(\beta-1)}{\alpha-1}\quad\text{and}\quad\eta_t=1-\frac{m(\beta-1)}{\alpha-1}\,\frac{K}{r^{m-1}}.$$

Under constant specific-heat conditions K=1 and $\beta=\alpha^{1/m}$ so that

$$\eta_t=1-\frac{m(\alpha^{1/m}-1)}{\alpha-1}\cdot\frac{1}{r^{m-1}}.$$

Under variable specific-heat conditions K is always less than 1, but β is much greater than $\alpha^{1/m}$ and for large values of r and α may be greater than $3\alpha^{1/m}$. Thus the approximation (12) cannot be used in this case and the best method of calculating β is the slide-rule method.

9. *Method of Calculating the Variables.*

The procedure followed in compiling tables from which the results of sections 5 to 8 were derived is as follows :

First, from given values of r and an average value of T_c, namely $T_c=400°$ C. Absolute, T_0 was calculated from formula (11).

Secondly, equation (4) was written in a form suitable for solution by the slide-rule method referred to above. Thus the formula

$$\begin{aligned}&\log_{10}\sigma+(0{\cdot}4343\lambda T_c\beta)\sigma\\&\qquad=[\log_{10}\alpha+m\log_{10}\rho-m\log_{10}\beta]+0{\cdot}4343[\lambda T_0(\alpha\rho-1)+\lambda T_c]\end{aligned}$$

was used for the determination of σ in the Constant-Volume and Diesel-Engine cycles, while for the determination of β in the Constant-Pressure and Humphrey-Pump cycles the formula

$$\begin{aligned}&\log_{10}\beta+\left(\frac{0{\cdot}4343}{m}\lambda T_c\sigma\right)\beta\\&\qquad=\left[\frac{1}{m}\log_{10}\alpha+\log_{10}\rho-\frac{1}{m}\log_{10}\sigma\right]+\frac{0{\cdot}4343}{m}[\lambda T_0(\alpha\rho-1)+\lambda T_c]\end{aligned}$$

was used.

Both equations are of the form $\log_{10}x+nx=k$, which can be solved with a suitable slide-rule, such as the Faber, which has a $\log_{10}$ scale along the lower edge of the rule.

References.

(1) W. J. Walker, "A Standard Cycle for all types of internal-combustion motor," Proc. Inst. C. E. (1926). Selected Paper No. 33 ; see formula (7), p. 4.

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(3) S. Lees, "Variable Specific-Heat Effects," Engineering (1st January, 1915).

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(5) R. M. Walker, "The Use of the Slide-Rule in the Solution of Equations," Phil. Mag. (in the Press).

LXXIV. *Representation of Psychophysical Data by Sigmoid and Linear Logarithmic Graphs.*

By Frank Allen, Ph.D., LL.D., F.R.S.C., Professor Emeritus in Physics, University of Manitoba, Winnipeg, Canada *.

[Received February 2, 1947.]

It has been urged as a reproach to modern physics that it consists of a mass of *ad hoc* hypotheses and " laws," each being valid for the limited range of data for which it was designed and for no others. The whole vast subject still awaits a synthesizing mind, as the discoveries of Copernicus, Kepler, and Galileo remained without coherence until Newton unified them in his laws of motion and gravitation, and in his system of the world.

Much the same criticism can be urged against experimental psychology, since it also has *ad hoc* hypotheses and " laws," sometimes mathematical ones, which apply only to restricted groups of measurements. Some of these laws, which take the form of graphical representation, are derived from measurements of the responses to stimulation of many kinds. Among the various types employed, two general methods of graphical treatment have been found to apply to numerous groups of phenomena, which are known as sigmoid and linear logarithmic curves. Sometimes it has been made to appear that either one of these forms, especially the former, is to be preferred, through assumed superior accuracy, to the exclusion of the other.

The purpose of this short discussion is to show that these curves are not antagonistic or mutually exclusive modes of representation, but that they are supplementary to each other, and rest ultimately upon the same foundation. Each form has its own peculiar advantages and gives its own necessary information on the responses of the organism to stimulation.

The sigmoid curve, so-called from its double curvature like the old English letter ʃ, which mathematicians employ as the symbol of integration, is similar to the saturation curve in physics. A familiar example is the curve representing the magnetization of iron by the field of an electric current. The first application of weak currents produces very little magnetization, which is denoted by the initial gradual upward slope of the curve. When a certain strength is obtained, the increase of magnetization becomes very great for small increments of current

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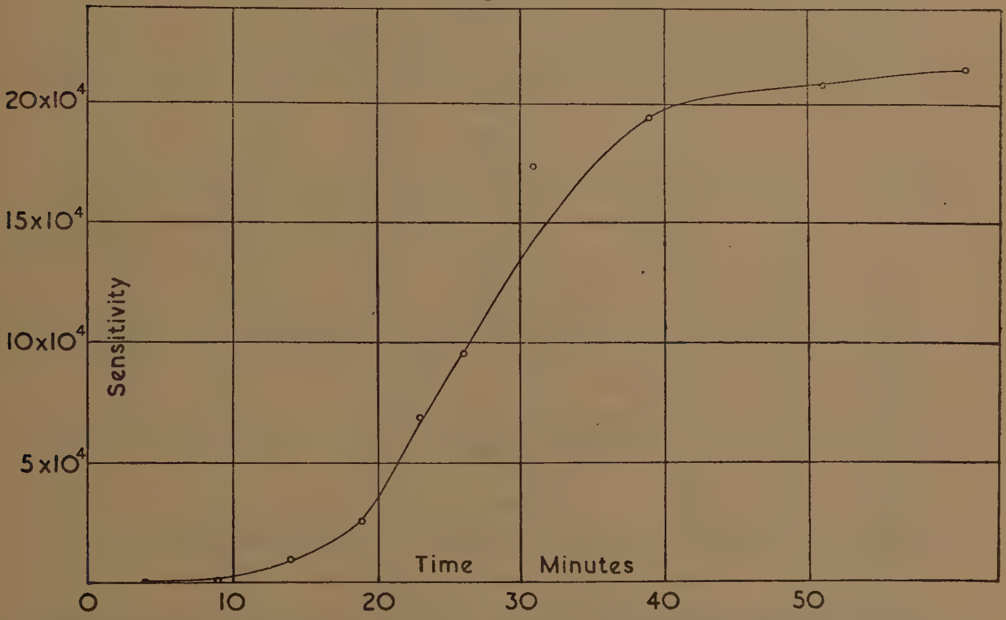
[*] Communicated by the Author.

strength. This condition is indicated by the steep central part of the curve. But when the iron approaches full magnetization, or magnetic saturation, there is very little increase in its amount with large increments of current. This condition is represented by the horizontal flattening of the upper part of the curve.

When a stimulus is applied to a sense organ, or other responding mechanism, such as muscles and glands of the human or animal body, an appropriate response is evoked, the magnitude of which can be measured and related to the strength of the stimulus. The sigmoid or saturation curve, plotted from the measurements, shows that when gently stimulated above the threshold value the response of the organ is small, which is represented by the slight upward slope of the lowest part of the graph. At a certain value of the stimulus the responses quite suddenly become relatively great for small increments of stimulation, which accounts for the steep central part of the curve. At a certain strength of the stimulus the increment of the response diminishes in magnitude, for no organ can have its responses augmented indefinitely by increasing stimulation. This condition is therefore represented by the flattening of the uppermost part of the graph. The sigmoid or saturation curve, though a very general type, has therefore a definite value in representing the responses to stimulation. A curve of this kind is shown in fig. 1.

Fig. 1.



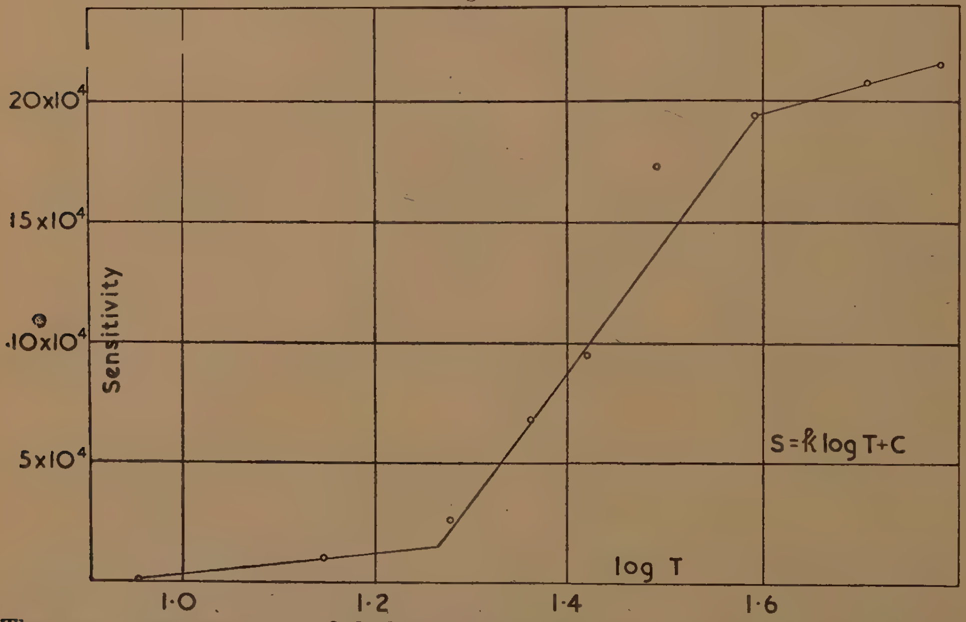
Measurements of darkness adaptation with time in minutes by Nagel. Helmholtz: 'Physiological Optics,' Eng. trans. Southall, ii. p. 320. One measurement is clearly wrong.

Equally, or even more valuable, is the linear logarithmic curve. The logarithmic representation of the responses to stimulation generally conforms to the linear equation :

$$R = \pm k \log S + C,$$

where R represents the response characteristic of the organ stimulated, S the magnitude of the stimulus, and k and C are constants, the values of which are determined by the nature of the organic mechanism, the type of stimulus and the mode of experimentation. While numerical values of the constants can readily be obtained from the graphs, the individual anatomical and physiological factors involved in them are separately indeterminate. When the graph consists of two or more intersecting straight lines, as is usually the case, the same equation holds for each of them with suitable changes in the numerical values of the constants. The positive or negative sign is determined by the relationship of R and S. If R increases with increasing S the positive sign is necessary; if R decreases as S increases the sign becomes negative. It is indicated by the positive or negative slope of the graph.

Fig. 2.



The same measurements of darkness adaptation, as in fig. 1, are plotted with logarithms of time as abscissæ. They are represented by the equation, $S = k \log T + C$.

The logarithmic law is far more significant than appears on first consideration. It is also remarkable in applying to wide ranges of measurements in the senses of vision, hearing, touch, taste, temperature and pain, the post-contraction of muscles and the secretion of the salivary glands. No doubt it applies to all responses of the organism which follow stimulation. An example is given in fig. 2. The underlying reason for the logarithmic law is clear. All responding mechanisms are composed of numerous, not to say innumerable, anatomical elements. The retina, the most complex of all, has 130 million rods, 3·3 million cones and half a million nerves. The minute central rod-free area has 30,000 cones. The ear, the vestibular organs and semi-circular canals have great numbers of receptors. In touch, the receptors number 500,000 and in pain over 3 millions ; in smell and taste there are multitudes of end-organs. The muscles are composed of great numbers of fibrils, and the glands of secreting cells. All are connected with the nervous system. The response of each nerve to stimulation is by a volley of discharges in an impulse, the number varying with the intensity of stimulation. When a changing stimulus is applied to any organ, therefore, it will elicit responses in varying numbers of receptors, nerves, muscle-fibres or glandular cells. The smallest number of responding elements involved in liminal stimulation, and the greatest number in maximal, are impossible to determine. It is this complex anatomical structure underlying physiological activity which renders possible the elegant and precise gradation of responses. Fewer and larger units would fail in this regard. In just lifting one pound, a certain number of muscle fibres are contracted ; if the next larger number stimulated into action would just lift two pounds, it would follow that two pounds weight of muscular force would be put forth to lift all weights between one and two pounds. All finely graduated differences would accordingly become impossible to obtain. These conditions may be expressed in mathematical form. In making measurements, such as lifting weights, it is customary to note the weight, W, lifted, and the difference, $d\mathrm{W}$, whether an increase or a decrease, which can just be detected. The ratio, $\frac{d\mathrm{W}}{\mathrm{W}}$, may then be computed. But what is fundamental to this measurement, however, is not the weight lifted, but the number of muscle-fibres, N, required by their contraction to lift the weight, and the additional number, $d\mathrm{N}$, necessary to be brought into action to give the just noticeable increase. The ratio, $\frac{d\mathrm{N}}{\mathrm{N}}$, where both factors vary from one weight lifted to another should, in reality, be determined in successive measurements. This, however, is impossible. Still more fundamental is the number of nerves involved, or the number of neural discharges in the impulses in those nerves, or both factors combined. Incidentally it may be remarked that groups of muscle-fibres, and doubtless other structures, work in relays as fatigue sets in. If light, or sound, or any other stimulus of intensity, S, is employed, the just noticeable increment is $d\mathrm{S}$, or primarily the corresponding receptors or nerves as before.

If the just perceptible changes in the response are denoted by $d\mathrm{R}$, then $d\mathrm{R}$ is proportional to $\frac{d\mathrm{S}}{\mathrm{S}}$ or $\frac{d\mathrm{N}}{\mathrm{N}}$, that is, $d\mathrm{R}=k\,\frac{ds}{\mathrm{S}}$; or $k'\,\frac{d\mathrm{N}}{\mathrm{N}}$.

The magnitudes of dR, dS and dN are not infinitesimal, since they are the measure of the Weber increment which, however, in most cases is very small. Though consequently lacking in mathematical rigor, which may not apply to the underlying structures, the equations may be integrated :

$$\int dR = k\int\frac{dS}{S} \quad \text{or} \quad k'\int\frac{dN}{N},$$

from which

$$R = k \log S + C \quad \text{or} \quad k' \log N + C'.$$

Since it is impossible to count or otherwise evaluate the number of responding anatomical elements in any stimulation, the best that can be done is to measure the external stimuli, the brightness of the light, the loudness of the sound, the lifted weight, etc., and their just noticeable differences, to which the numbers of excited elements are proportional. Such measurements therefore follow a logarithmic law, as experiment invariably shows to be the case. The logarithmic law of responses to stimulation is accordingly founded upon the mathematical fact that $\int\frac{dN}{N} = \log N$, and this rests ultimately upon the fact that the human organism consists of multitudes of anatomical units of which varying but limited numbers are excited by corresponding magnitudes of the stimulus. Since this condition is inherent both in anatomy and mathematics it is impossible for the logarithmic law of response to fail.

So general a law of response must rest upon some condition common to all organs, and this common factor is multitudinousness of units responding in definite groups, numerically proportioned to the strength of the stimulus on the lower physical side and to perception on the higher psychical plane. The logarithmic law is therefore the connecting medium between the anatomical, physiological and perceptual levels of organic response.

The integral, $\int\frac{dN}{N}$, is essentially a definite one, the limits of which are unknown, since there seems to be no way by which the number of responding elements can be ascertained. The lower limit is complicated by the fact that a certain liminal number must respond before consciousness is affected in a perceptible degree.

Depression and enhancement of excitability of the responding elements also occur, so that the liminal response varies slightly from time to time in magnitude. Consciousness again is not aware of minute gradations of increasing and decreasing stimulation ; that is, small changes in the number of responding elements, less than the equivalent Weber fraction, though they occur, are unperceived. The upper limit of stimulation implies that the theoretical maximal intensity is reached when all the responding elements are stimulated, which probably never occurs. Between minimum and maximum stimulation, a sensation, or other response, occurs whatever number of responding elements are stimulated, and the Weber increment or decrement is measured from that point; just as in sound the musical scale may start with any frequency of vibration the intervals being computed with reference to that basic number. Though the external stimulus alone can be observed and measured, the fact should never be lost sight of that fundamentally in all responses definite but unknown numbers of anatomical units are excited into action. In order to obtain comparable measurements for either form of graphical representation, sigmoid or logarithmic, they should all be taken continuously in one period of experimentation. Averages of measurements taken on different days, in different conditions of the nervous system, may be quite misleading.

The sigmoid curve is more difficult to express in mathematical terms. But many instances of its application to biological measurements, and theoretical discussions of its character, have been collected from the researches of various investigators by D'Arcy W. Thompson (1). One expression for its shape is $y=\int e^{-x^2}dx$. A function of this form would lead one to suspect a logarithmic substitution for an expression of the derivative.

When a population, or its equivalent, grows by geometrical progression, it can be represented by the equation

$$\frac{dy}{dt}=by \quad \text{or} \quad \frac{dy}{y}=b\,dt,$$

which gives a logarithmic curve. But when any factor of retardation is introduced the equation becomes more complicated.

In the growth of bacteria or of population, for example, the sigmoid curve is used to express the results graphically. The rate of growth, coupled with a retarding factor, can be expressed by the differential equation

$$\frac{dy}{dt}=by(a-y),$$

that is, the rate of increase is proportional to the number of organisms present, y, and to the concentration of the food supply, $(a-y)$, which diminishes as y increases. Integration of this equation yields the sigmoid curve

$$t=\frac{1}{ab}\log\frac{y}{a-y}.$$

In this case the sigmoid curve has therefore an ultimate logarithmic basis.

An empirical equation has also been proposed for the sigmoid curve of growth by Backman, as quoted by Thompson, with the form

$$\log \mathrm{H}=k_0+k_1\log \mathrm{T}-k_2\log^2 \mathrm{T},$$

where H is the growth and T the time. This equation is made up, according to Backman, of a constant velocity, an acceleration varying with the time and a retardation varying with the square of the time.

By analogy with this interpretation, the equation for the sigmoid curve, when it represents the responses to stimulation of any organ, may be considered to be made up of a steady increase in the number of responding anatomical units, a process of facilitation which represents the acceleration, and a process of inhibition which corresponds to the retardation, all varying at different rates.

The sigmoid curve, therefore, rests ultimately upon logarithmic functions. So that in representing measurements of responses to stimulation by either logarithmic or sigmoid curves, logarithms are employed, in the former case explicitly and in the latter implicitly. Fundamentally both types of curves rest upon the anatomical structure of multitudes of units of which varying numbers are excited into activity by increasing intensities of stimulations.

The sigmoid curves and equations employed by Thompson refer to populations increasing in geometrical progression by the addition of new units. But while in human organic responses no new units are coming into existence, increasing numbers of those already present are coming into action, also in geometrical progression, which amounts to the same thing mathematically. Any preference between the two graphs is therefore a question whether it is desirable to employ logarithms explicitly or implicitly. In every case it is the same organism responding with its multitudinous units, and out of that condition the logarithmic function, in some form, must invariably arise.

The reason for the occurrence of the saturation curve in magnetism is that the magnetization of iron is due to the gradual forced alignment of innumerable natural atomic magnets from a state of initial disorder.

The sigmoid curve which enters into so many diverse natural phenomena represents, as Thompson expresses it, a certain common pattern among nature's mechanisms, in which multitudes of small units of some kind figure prominently. It also expresses the philosophical principle of continuity. For in any response the units employed with small intensities of stimulation are still used, with additions to them, by large intensities.

In those parts of physics where multitudes of atoms and molecules are involved, such as radioactivity ⁽²⁾ and the kinetic theory of gases ⁽³⁾, the logarithmic function naturally emerges. In the theory of absorption of alpha particles passing through vast numbers of atoms the logarithmic function appears. Such also is the case in the kinetic theory of gases, especially in that portion, for example, dealing with statistical mechanics.

In statistical thermodynamics, also, there is the Boltzmann equation for the entropy of a body :—

$$\text{Entropy} = k \log \text{D},$$

where k is the Boltzmann constant and D is a quantitative measure of the atomistic disorder of the body in question, which is partly that of heat motion and partly that derived from different kinds of atoms or molecules being mixed at random ⁽⁴⁾.

A basic characteristic of the multiplicity of small units, atoms and molecules, in the activity of inorganic matter, is thus carried to the organic realm, where the significant units are larger and fewer in number in the form of cells. In human and animal bodies, particularly in muscle-fibres, glands, sense organs, nerves and the billion-celled brain, that is, in the complete receptor and effector mechanisms, the responding elements are exceedingly numerous. Living and non-living matter therefore exhibit many phenomena characterized alike by actions involving groups of innumerable small units, and one feature common to them all is the logarithmic uniformity of behaviour. From the mathematical and physical standpoint, the logarithmic law is an important link connecting certain uniform activities of all matter, organic and inorganic alike, where the operations of numerous discrete particles and minute structures are concerned.

References.

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(3) E. H. Kennard, 'Kinetic Theory of Gases,' Chap. IX. (1938) (McGraw-Hill). *See also*: Thompson, Inman and Brownfield, 'On the Cutaneous Nerve Areas of the Forearm and Hand,' p. 218 (1934) (University of California Press).
(4) E. Schrödinger, 'What is Life ? ' p. 73 (1946) (Cambridge University Press).

LXXV. *The Diffusion of Decomposition Products through Plastic Materials.*

By A. G. Ward *.

[Received June 21, 1947.]

1. Introduction.

Certain plastics, particularly those based on nitrocellulose, show slow decomposition, and resulting evolution of gases if stored continuously at elevated temperatures. In thin sections the diffusion process is sufficiently rapid so that high concentrations of the decomposition products are not built up. In thicker sections, however, the concentration may increase to a point where the pressure in a minute cavity becomes enough to split the material †. It is of interest to obtain the relation between concentration, time and position for certain simple cases, and the following calculations show how this may be done. Given the rate of formation of a product, its diffusion constant, its solubility and the appropriate initial and boundary conditions, the problem can be solved exactly for certain geometrical shapes. The calculation rests on the assumption of Fick's law of diffusion.

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* Communicated by the Author.

† This explanation of the cause of crack formation, together with a considerable amount of experimental work on this problem, have been communicated to me by Dr. R. J. Rosser of the Ministry of Supply.

The interest in the mathematical solutions is increased by the fact that the equations concerned are precisely those for dielectric heating, in which heat is being generated at a constant rate per unit volume and lost from the boundaries of the material. Particular solutions required for practical problems in this field may require the introduction of different boundary conditions, but the general methods given still apply.

Further instances where the equations may find useful application are biological systems of a consistency to arrest convection currents, in which substances are being formed throughout the bulk of the material and being removed at the boundary.

2. FUNDAMENTAL EQUATIONS.

Let Q be the mass of a particular gas produced in grams, in one second from one gram of the plastic. Then the mass of gas produced per ml. of plastic is $Q\sigma$, where σ is the plastic density in gms./ml. Then we may write

$$Q = A\, e^{-B/RT}, \qquad \text{(2.1)}$$

where B is the activation energy for the rate determining reaction by which the gas is produced, R the gas constant, A a constant, T the absolute temperature.

Let C be the concentration of the gas in gms./ml. of plastic. Then we write

$$C = k\, p, \qquad \text{(2.2)}$$

where k is the solubility and p is the partial pressure in mm. of mercury. Equation (2.2) is Henry's Law and is probably obeyed within reasonable limits for gas solubility in plastics (1).

It is not possible to predict the general form of the dependence of solubility on temperature. It must be remembered, however, that it is a function of temperature, and requires determination for each pair of substances, plastic and gas.

The equation for diffusion may be written

$$P = -D \frac{\partial C}{\partial x}, \qquad \text{(2.3)}$$

P is the quantity of gas in gms. passing through 1 sp. cm. of surface per sec. normal to the gradient of concentration $\partial C/\partial x$ and D is the diffusion constant. The units for D are $\text{cm.}^2\ \text{sec.}^{-1}$, as can be seen from equation (2.3). Diffusion is generally an activated process so we may write

$$D = D_0\, e^{-B'/RT}, \qquad \text{(2.4)}$$

where B' is the activation energy and D_0 a constant.

By consideration of an element of volume, it is now possible to derive the fundamental diffusion equation, under isothermal conditions. This derivation follows closely that for normal diffusion, only modified by the addition of the gas generated, leading to the equation

$$\frac{\partial C}{\partial t}=D\nabla^2 C+Q\sigma. \qquad (2.5)$$

It can be seen that for $Q=0$ this reduces to the ordinary diffusion equation. If we write $Q'=Q\sigma$, *i. e.* Q'=the mass of gas generated per ml. of plastic per second, then

$$\frac{\partial C}{\partial t}=D\nabla^2 C+Q'. \qquad (2.6)$$

The coordinate system used can then be selected according to the symmetry properties of the particular problem, in order to simplify (2.6) as far as possible.

We may now distinguish two types of solution of (2.6). The first is the stationary state, for which $\partial C/\partial t=0$. This is presumably reached in shapes and sizes which never exhibit cracking at a given temperature, and in which the maximum concentration is less than the critical value for crack formation. The second type of solution is the approach to the stationary state, which may be interrupted by the onset of cracking.

3. THE STATIONARY STATES.

We shall assume, with good justification, that the boundary condition is always given by equation (2.2) Henry's Law. The two instances of practical importance are (i) as for nitrogen, p is constant, and hence C at boundary is constant, (ii) as for carbon dioxide where p, and hence C, are effectively zero at the boundary throughout.

(a) *Parallel plate, effectively infinite in y, z directions, 2a thick in x direction* (fig. 1).

Equation (2.6) becomes, for the steady state,

$$D\frac{d^2C}{dx^2}=-Q'. \qquad (3.1)$$

Integrating and applying boundary conditions (i) above we obtain

$$C=C_0+\frac{Q'}{D}\left(ax-\frac{x^2}{2}\right). \qquad (3.2)$$

Using boundary conditions (ii) gives

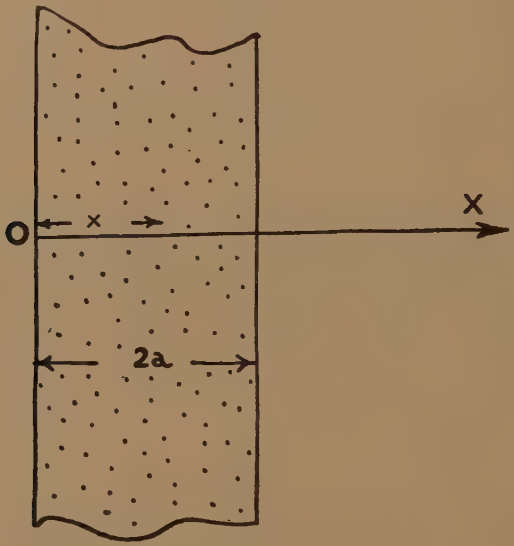
$$C=\frac{Q'}{D}\left(ax-\frac{x^2}{2}\right). \qquad (3.3)$$

Thus (3.2) and (3.3) differ only by the constant C_0, the concentration at the boundary in equilibrium with the surrounding gas. Transferring the origin of coordinates to the centre of the plate gives for (3.3)

$$C=\frac{Q'}{2D}(a^2-x^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.4)$$

The maximum concentration is $Q'a^2/2D$ at the centre of the plate, increasing, therefore, as the square of the thickness.

Fig. 1.



(b) *Parallel plate with concentration* C_0 *on one side and* 0 *on the other.*

Applying the boundary conditions $C=0$ at $x=0$ and $C=C_0$ at $x=2a$ to the equation obtained by integrating (3.1) we obtain

$$C=\frac{Q'}{D}\left(ax-\frac{x^2}{2}\right)+\frac{C_0 \,.\, x}{2a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.5)$$

From this equation and (2.3)

$$P=-Q'(a-x)-\frac{DC_0}{2a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.6)$$

Hence at $x=0$

$$P=-aQ'-\frac{DC_0}{2a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.7)$$

This gives the quantity of gas passing into the vacuum on one side of the plate. The condition corresponds to that in most of the experimental arrangements for measuring permeability, at temperatures at which the amount of gas evolved by decomposition is becoming an important fraction of the amount passing through the plate. To this must be added allowances based on equation (3.3) in conjunction with (2.3) for gases other than the one whose permeability constant is being measured. These will each contribute a term

$$\mathrm{P}_r = -\mathrm{Q}_r'(a). \qquad \text{(3.8)}$$

(c) *Effect of temperature on the solution for* $\mathrm{C}_{\text{max.}}$ *in* 3(*a*).

Combining equations (2.1) and (2.4) with the value of $\mathrm{C}_{\text{max.}}$ gives

$$\mathrm{C}_{\text{max.}} = \frac{\mathrm{A}a}{2\mathrm{D}_0} e^{(\mathrm{B}'-\mathrm{B})/\mathrm{RT}}. \qquad \text{(3.9)}$$

Now (2.2) gives $\mathrm{C}=kp$, with k some function of temperature, which can be determined from the solubility-temperature curve. Where these data are available, the change of $p_{\text{max.}}$ (equilibrium) with temperature can readily be obtained from diffusion and gas evolution measurements.

(d) *Infinite cylinder, radius a, zero concentration on the surface.*

It is evident that cylindrical coordinates will be the most convenient, since from symmetry

$$\frac{\partial \mathrm{C}}{\partial \phi} = \frac{\partial \mathrm{C}}{\partial z} = 0.$$

Equation (2.6) then becomes for the steady state

$$\mathrm{D}\left\{\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\mathrm{C}}{d\rho}\right)\right\} = -\mathrm{Q}'. \qquad \text{(3.10)}$$

Integration and insertion of the boundary conditions gives

$$\mathrm{C} = \frac{\mathrm{Q}'}{4\mathrm{D}}(a^2-\rho^2), \qquad \text{(3.11)}$$

$$\mathrm{C}_{\text{max.}} = \frac{\mathrm{Q}'a^2}{4\mathrm{D}}. \qquad \text{(3.12)}$$

Again $\mathrm{C}_{\text{max.}}$ increases as the square of the relevant linear dimension. The temperature dependence can be introduced as before.

(e) *Infinite tube, radii* a_1, a_2, *zero concentration inside and outside surfaces.*

Integration and substitution of the boundary conditions gives

$$\frac{d\mathrm{C}}{d\rho} = -\frac{\mathrm{Q}'\rho}{2\mathrm{D}} + \frac{\mathrm{Q}'(a_1^2-a_2^2)}{4\mathrm{D}\rho\log(a_1/a_2)}, \qquad \text{(3.13)}$$

$$\mathrm{C} = -\frac{\mathrm{Q}'\rho^2}{4\mathrm{D}} + \frac{\mathrm{Q}'(a_1^2-a_2^2)}{4\mathrm{D}\log(a_1/a_2)}\log\rho + \frac{\mathrm{Q}'}{4\mathrm{D}}\frac{(a_1^2\log a_2 - a_2^2\log a_1)}{\log(a_2/a_1)}. \qquad \text{(3.14)}$$

$\mathrm{C}_{\text{max.}}$ occurs for $d\mathrm{C}/d\rho=0$, *i. e.*

$$\rho^2 = \frac{a_1^2-a_2^2}{\log(a_1^2/a_2^2)}. \qquad \text{(3.15)}$$

Hence $C_{max.}$ does not depend in any simple way on wall thickness, but can be readily computed for given dimensions.

By substitution of boundary conditions $C=0$ for $a=a_1$, $C=C_0$ for $a=a_2$, we can easily derive solutions for a tube with gas at a pressure p inside and zero outside.

(f) *The cube of side 2a and the cylinder of length equal to the diameter.*

Neither of these shapes give easy solutions. It is proposed, both for the equilibrium and for the transient case, to substitute for both the sphere of equal volume.

(i) Cube. If a_1 is the radius of the equivalent sphere, then

$$a_1=1{\cdot}24a. \qquad (3.16)$$

The maximum and minimum distances from the centre of the cube to its surface are 1·73a and a.

(ii) Cylinder. Radius a, length $2a$. Then if a_1 is the radius of the equivalent sphere

$$a_1=1{\cdot}14a. \qquad (3.17)$$

The maximum and minimum distances from the centre to the surface are 1·14a and a.

It can therefore be seen that the sphere of equal volume constitutes a fair approximation to the cylinder and rather less good an approximation to the cube. It must, however, be remembered that only conditions near the centre are important as far as the concentrations causing cracking are concerned, and also that the error in estimating times of cracking renders greater accuracy unnecessary.

(g) *The sphere of radius a_1.*

Equation (2.6) gives, in the steady state, with boundary conditions of spherical symmetry :

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dC}{dr}\right)=-\frac{Q'}{D}. \qquad (3.18)$$

Integrating, with the condition that $C=0$ for $r=a_1$

$$C=\frac{Q'}{6D}(a_1^{\,2}-r^2), \qquad (3.19)$$

$$C_{max.}=\frac{Q'a_1^{\,2}}{6D}. \qquad (3.20)$$

As before, the temperature dependence can be introduced into these equations.

4. THE APPROACH TO EQUILIBRIUM.

The stationary state solutions are only applicable for plastics of dimensions small enough to give no cracking at the given temperature, or at the limiting size for which cracking commences just as equilibrium is being reached. This latter condition would be very useful in giving the criterion as regards concentration for cracking to occur.

It is now proposed to derive expressions for the concentration as a function of time and position for the shapes already considered. It will be assumed that initially $C=C_1$ everywhere inside the plastic. Any other distribution can easily be dealt with if it is given in an analytical form, and even if only given as a series of figures for concentration-distance these can be used for a numerical Fourier analysis (see Whittaker and Robinson, Calculus of Observations).

(a) *Parallel plate effectively infinite in y, z directions 2a thick in x direction* (fig. 1).

The equation from (2.6) is

$$D\frac{\partial^2 C}{\partial x^2}+Q'=\frac{\partial C}{\partial t}. \qquad (4.1)$$

The mode of solution is :

Take $C=(\text{equilibrium distribution})+\Theta(x, t)$,

i. e.
$$C=C_0+\frac{Q'}{D}\left(ax-\frac{x^2}{2}\right)+\Theta(x, t).$$

Then

$$D\frac{\partial^2 \Theta}{\partial x^2}=\frac{\partial \Theta}{\partial t}. \qquad (4.2)$$

Now equation (4.2) is identical formally with the diffusion or heat conduction equations and can be solved by familiar methods (*cf.* Barrer, Diffusion in and through Solids).

We write $$\Theta(x, t)=\phi(x)\,F(t).$$

Substitute in (4.2) and divide by Θ, then

$$\frac{1}{\phi}\frac{\partial^2\phi}{\partial x^2}=\frac{1}{DF}\frac{\partial F}{\partial t}=-k^2,$$

where k is a constant.

This gives $$F=F_0\,e^{-Dk^2t}$$

$$\phi=M_1\cos kx+M_2\sin kx,$$

where F_0, M_1, M_2 are constants. Hence the complete solution is

$$C=C_0+\frac{Q'}{D}\left(ax-\frac{x^2}{2}\right)+\sum_k e^{-Dk^2t}\,(M_1k\cos kx+M_2k\sin kx).$$

At all time after $t=0$ we require $C=C_0$ at the boundaries. Hence

$$M_1 k=0,$$

and

$$k\,.\,2a=n\pi, \qquad k=\frac{n\pi}{2a}.$$

The solution is therefore

$$C=C_0+\frac{Q'}{D}\left(ax-\frac{x^2}{2}\right)+\sum_{n=1}^{\infty} e^{\frac{-Dn^2\pi^2}{4a^2}t}\,.\,M_n \sin\frac{n\pi x}{2a}. \quad . \quad . \quad (4.3)$$

This has taken full account of the conditions at the exposed surface. It is now necessary to fit the initial distribution. At $t=0$ we have

$$C=C_0+\frac{Q'}{D}\left(ax-\frac{x^2}{2}\right)+\sum_{n=1}^{\infty} M_n \sin\frac{n\pi x}{2a}.$$

We may now take C as any desired $f(x)$. In fact, we will use $C=C_1$, then

$$\sum_{n=1}^{\infty} M_n \sin\frac{n\pi x}{2a}=C_1-C_0-\frac{Q'}{D}\left(ax-\frac{x^2}{2}\right).$$

Writing

$$\frac{\pi x}{2a}=z \text{ so that } x=\frac{2az}{\pi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.4)$$

we have

$$\sum_{n=1}^{\infty} M_n \sin nz=C_1-C_0-\frac{2a^2Q'}{\pi D}\left(z-\frac{z^2}{\pi}\right). \quad . \quad . \quad . \quad (4.5)$$

Write

$$C_2=C_1-C_0 \text{ and } \frac{2a^2Q'}{\pi D}=b, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.6)$$

we have

$$\sum_{n=1}^{\infty} M_n \sin nz=C_2-bz+\frac{bz^2}{\pi}=f(z).$$

M_n are the coefficients of a half Fourier series and are given by

$$M_n=\frac{2}{\pi}\int_0^{\pi} f(z)\sin nz\,dz,$$

$$=\frac{2}{\pi}\int_0^{\pi}\left(C_2-bz+\frac{bz^2}{\pi}\right)\sin nz\,dz.$$

Hence M_n vanishes for n even and for n odd

$$M_n=\frac{2}{\pi}\left(\frac{2C_2}{n}-\frac{4b}{\pi n^3}\right)$$

$$=\frac{4(C_1-C_0)}{\pi n}-\frac{16a^2Q'}{\pi^3 D n^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.8)$$

Hence the complete solution is

$$C=C_0+\frac{Q'}{D}\left(ax-\frac{x^2}{2}\right)+\sum_{n=1,3,5}^{\infty}\left\{\frac{4(C_1-C_0)}{\pi n}-\frac{16a^2Q'}{\pi^3 D n^3}\right\}e^{\frac{-Dn^2\pi^2}{4a^2}t}\,.\,\sin\frac{n\pi x}{2a}. \ (4.9)$$

It is evident from the form of the solution that after a comparatively short period of time only the term for which $n=1$ will give an appreciable contribution, so that for all reasonably long intervals

$$\mathrm{C}=\mathrm{C}_0+\frac{\mathrm{Q}'}{\mathrm{D}}\left(ax-\frac{x^2}{2}\right)+\left\{\frac{4(\mathrm{C}_1-\mathrm{C}_0)}{\pi}-\frac{16a^2\mathrm{Q}'}{\pi^3\mathrm{D}}\right\}e^{\frac{-\mathrm{D}\pi^2t}{4a^2}}\sin\frac{\pi x}{2a}, \quad . \quad (4.10)$$

where $\mathrm{C}_1=\mathrm{C}_0=0$ the solution simplifies to

$$\mathrm{C}=\frac{\mathrm{Q}'}{\mathrm{D}}\left(ax-\frac{x^2}{2}\right)-\sum_{n=1,3,5}^{\infty}\frac{16a^2\mathrm{Q}'}{\pi^2\mathrm{D}n^3}e^{\frac{-\mathrm{D}n^2\pi^2}{4a^2}t}\sin\frac{n\pi x}{2a}, \quad . \quad (4.11)$$

or for large t

$$\mathrm{C}=\frac{\mathrm{Q}'}{\mathrm{D}}\left(ax-\frac{x^2}{2}\right)-\frac{16a^2\mathrm{Q}'}{\pi^3\mathrm{D}}e^{\frac{-\mathrm{D}\pi^2t}{4a^2}}\sin\frac{\pi x}{2a}. \quad . \quad . \quad . \quad (4.12)$$

As before, the dependence of Q′ and D on T may be introduced.

(b) *Parallel plate with concentration* C_0 *on one side* 0 *on the other.*

This may easily be obtained by the method in (*a*), using the equilibrium concentration already derived.

(c) *Infinite cylinder with concentration* C_0 *on boundary.*

The equation from (2.6) is

$$\frac{\mathrm{D}}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\mathrm{C}}{\partial\rho}\right)+\mathrm{Q}'=\frac{\partial\mathrm{C}}{\partial t}.$$

As before, we use a summation of the equilibrium solution (3.11) and the transient

$$\mathrm{C}=\mathrm{C}_0+\frac{\mathrm{Q}'}{4\mathrm{D}}(a^2-\rho^2)+\Theta(\rho,t).$$

Then we obtain

$$\frac{\mathrm{D}}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Theta}{\partial\rho}\right)=\frac{\partial\Theta}{\partial t}.$$

Writing

$$\frac{\partial\Theta}{\partial t}=\phi(\rho)\,.\,\mathrm{F}(t),$$

$$\frac{1}{\phi\rho}\cdot\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right)=\frac{1}{\mathrm{DF}}\frac{\partial\mathrm{F}}{\partial t}=-k^2.$$

From this

$$\mathrm{F}=\mathrm{F}_0\,e^{-\mathrm{D}k^2t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.13)$$

$$\frac{\partial^2\phi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\phi}{\partial\rho}+k^2\phi=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.14)$$

This is Bessel's equation of order zero.

The general solution includes terms of the form $\mathrm{J}_0(k\rho)$, Bessel Functions of the first kind, and $\mathrm{Y}_0(k\rho)$, Bessel Functions of the second kind, the latter are omitted from the solution since they give infinity at $\rho=0$.

Individual solutions are therefore

$$\phi = M\, e^{-Dk^2t}\, J_0(k\rho). \tag{4.15}$$

The solution must $=0$ at $\rho=a$ since $C=C_0$ for all time after $t=0$. Therefore

$$J_0(ka)=0. \tag{4.16}$$

This (*cf.* Carslaw 'The Conduction of Heat') gives an infinite number of positive roots, k_1, k_2, etc. The values are $k_1a=2{\cdot}405$; $k_2a=5{\cdot}520$; $k_3a=8{\cdot}654$; $k_4a=11{\cdot}791$.

The complete solution must therefore be of the form

$$C=C_0+\frac{Q'}{4D}(a^2-\rho^2)+\sum_{r=1}^{\infty} e^{-Dk_r^2t} M_r \,.\, J_0(k_r\rho) \tag{4.17}$$

At $t=0$

$$C_1-C_0-\frac{Q'}{4D}(a^2-\rho^2)=\sum_{r=1}^{\infty} M_r \,.\, J_0(k_r\rho), \tag{4.18}$$

i. e.

$$f(\rho)=\sum_{r=1}^{\infty} M_r\, J_0(k_r\rho).$$

Using the standard integrals

$$\int_0^a \rho\{J_0(k_r\rho)\}^2\, d\rho=\frac{a^2}{2}\{J'_0(k_ra)\}^2,$$

$$\int_0^a \rho\, J_0(k_r\rho)\, J_0(k_s\rho)\, d\rho=0 \quad r\neq s,$$

$$M=\frac{2}{a^2}\,\frac{\int_0^a \rho(f(\rho))\, J_0(k_r\rho)\, d\rho}{\{J'_0(k_ra)\}^2}. \tag{4.19}$$

We require to evaluate therefore

$$I_1=\int_0^a \rho\, J_0(k_r\rho)\, d\rho \quad \text{and} \quad I_2=\int_0^a \rho^3\, J_0(k_a\rho)\, d\rho.$$

For I_1. Bessel's equation of zero order may be written

$$\frac{d}{d\rho}\left(\rho\frac{dJ_0}{d\rho}\right)=-k_r^2\rho J_0.$$

Hence

$$\int_0^a \rho\, J_0\, d\rho=-\frac{1}{k_r^2}\left\{\rho\frac{\partial J_0}{\partial\rho}\right\}_0^a=-\frac{a}{k_r}J'_0(k_ra). \tag{4.20}$$

We use

$$J'_0(z)=\frac{d}{dz}\{J_0(z)\}.$$

For I_2. From Bessel's equation

$$\rho^3\frac{d}{d\rho}\left(\rho\frac{dJ_0}{d\rho}\right)=-k_r^2\,\rho^3\,.\,J_0,$$

$$\int_0^a \rho^3\, J_0\, d\rho=-\frac{1}{k_r^2}\int_0^a \rho\frac{d}{d\rho}\left(\rho\frac{dJ_0}{d\rho}\right)d\rho.$$

Integrating by parts

$$\begin{aligned} I_2 &= -\frac{1}{k_r^2}\left\{\left[\rho^3 \frac{dJ_0}{d\rho}\right]_0^a - 2\int_0^a \rho^2 \frac{dJ_0}{d\rho}\, d\rho\right. \\ &= -\frac{1}{k_r^2}\left\{k_r\, a^3\, J_0'(k_r a) - \Big[\, 2\rho^2\, J_0\Big]_0^a + 4\int_0^a \rho\, J_0\, d\rho\right\} \\ &= -\frac{1}{k_r^2}\left\{k_r\, a^3\, J'_0(k_r a) - \frac{4a}{k_r}\, J'_0(k_r a)\right\} \\ &= \frac{a\, J'_0(k_r a)}{k_r^3}\{4 - a^2 k_r^2\}. \qquad \ldots\ldots\ldots \quad (4.21) \end{aligned}$$

Making use of the values for I_1 and I_2 we have

$$\begin{aligned} \int_0^a \rho f(\rho)\, J_0(k_r\rho) d\rho &= \int_0^a \left\{\left(C_1 - C_0 - \frac{Q' a^2}{4D}\right)\rho + \frac{Q'}{4D}\rho^3\right\} J_0(k_r\rho) d\rho \\ &= \left(C_1 - C_0 - \frac{Q' a^2}{4D}\right)\left\{-\frac{a}{k_r}\, J_0(k_r a)\right\} + \frac{Q'}{4D}\,\frac{a\, J'_0(k_r a)}{k_r^3}\{4 - a^2 k_r^2\} \\ &= \frac{a}{k_r}\, J'_0(k_r a)\left\{C_0 - C_1 + \frac{Q'}{4D}\left(a^2 - a^2 + \frac{4}{k_r^2}\right)\right\} \\ &= \frac{a}{k_r}\, J_0(k_r a)\left\{\{C_0 - C_1\} + \frac{Q'}{Dk_r^2}\right\}. \end{aligned}$$

From (4.19)

$$M_r = \frac{2}{a\, k_r\, J'_0(k_r a)}\left\{\{C_0 - C_1\} + \frac{Q'}{Dk_r^2}\right\}. \qquad \ldots \quad (4.22)$$

So (4.17) becomes

$$C = C_0 + \frac{Q'}{4D}(a^2 - \rho^2) + \frac{2}{a}\sum_{r=1}^{\infty} \frac{e^{-Dk_r^2 t}.\ (C_0 - C_1 + Q'/Dk_r^2)}{k_r\, J'_0(k_r a)}\,.\, J_0(k_r\rho). \quad . \quad (4.23)$$

Usually the first four terms of the series are enough (*i. e.* $k_1 a = 2{\cdot}405$; $k_1 a = 5{\cdot}520$; $k_3 a = 8{\cdot}654$; $k_4 a = 11{\cdot}791$) and as t increases these can be omitted as (starting with k_4) the contribution of each becomes negligible owing to the operation of the exponential term.

As before, the temperature dependence of Q' and D can easily be introduced.

(d) *Infinite tube, radii a_1, a_2, with concentration C_0 on boundary.*

The differential equation is as for (*c*)

$$\frac{D}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial C}{\partial\rho}\right) + Q' = \frac{\partial C}{\partial t}.$$

Using the summation of equilibrium solution (3.19) and transient

$$C = C_0 - \frac{Q'\rho^2}{4D} + \frac{Q'}{4D}\,\frac{(a_1^2 - a_2^2)}{\log(a_1/a_2)}\log\rho + \frac{Q'}{4D}\left(\frac{a_1^2\log a_2 - a_2^2\log a_1}{\log(a_2/a_1)}\right) + \Theta(\rho, t).$$

$$\ldots\ldots\ldots \quad (4.24)$$

Then

$$\frac{D}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Theta}{\partial\rho}\right)=\frac{\partial\Theta}{\partial t}.$$

A solution $\Theta(\rho, t)=\phi(x)e^{-Dk^2t}$, where

$$\frac{\partial^2\phi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\phi}{\partial\rho}+k^2\phi=0.$$

This is again Bessel's equation for zero order. However, whereas for the solid cylinder solutions of the second kind were omitted owing to their giving ∞ at $\rho=0$, this is no longer so since the solution is only to hold between $a^2\geqslant\rho\geqslant a_1$.

Suitable forms of the solution can be obtained (Carslaw, 'The Conduction of Heat,' p. 127) and then the initial conditions fitted by an expansion as before.

(e) *The sphere equivalent to cube or short cylinder.*

The initial concentration is taken as C_1 throughout, and the concentration at the boundary $C=C_0$. Then equation (2.6) becomes

$$D\left(\frac{\partial^2 C}{\partial r^2}+\frac{2}{r}\frac{\partial C}{\partial r}\right)+Q'=\frac{\partial C}{\partial t}. \quad \ldots \ldots \quad (4.25)$$

Applying the same methods of solution as in the previous instances

$$C=C_0+\frac{Q'}{6D}(a_1{}^2-r^2)+\frac{2}{\pi r}\sum_{n=1}^{\infty}e^{-Dn^2\pi^2t/a_1{}^2}(-1)^{n+1}$$
$$\times\left\{\frac{a_1}{n}(C_1-C_0)+\frac{Q'a_1{}^3}{D\pi^2n^3}\right\}\sin\left(\frac{n\pi r}{a_1}\right). \quad \ldots \ldots \quad (4.26)$$

For appreciable times this will simplify to give

$$C=C_0+\frac{Q'}{6D}(a_1{}^2-r^2)+\frac{2}{\pi r}e^{-D\pi^2t/a_1{}^2}\left\{a_1(C_1-C_0)+\frac{Q'a_1{}^3}{D\pi^2}\right\}\sin\left(\frac{\pi r}{a_1}\right). \quad . \quad . \quad (4.27)$$

As usual, the factors covering temperature dependence may be introduced.

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LXXVI. *On the Forced Electro-Magnetic Oscillations in Spherical Resonators.*

By O. E. H. Rydbeck, Sc.D., Chalmers Institute of Technology,
Gothenburg, Sweden*.

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§ 1. *Introduction.*

The electro-magnetic resonators have attracted much interest in recent years in connection with the development of the technique of radio microwaves. In many cases the theoretical methods used are fairly old. Debye, for example, already in 1908 studied the diffraction of a plane electro-magnetic wave by a sphere and introduced what is now often called a radial Hertzian vector.

The fundamental papers on cavity resonators are written by Hansen (1), Condon (2) and Borgnis (3). A contemporary discussion of the spherical cavity case by Berg (4), as well as an important paper by Richtmyer (2) on the dielectric resonators, should be mentioned in this connection.

In most cases only the resonant modes are studied and little attention is paid to their coupling to the exciting source, the presence of which generally modifies the boundary relations. In the present paper the electro-magnetic oscillations in spherical systems are studied as functions of the exciting source, and the solutions obtained show that degree of excitation of the various modes and orders. Transverse electric (TE) waves and transverse magnetic (TM) waves are studied separately, and the equivalent network for a resonator with two current loops is shown.

§ 2. *TE-Waves.*

(a) *General considerations.*

In this case the magnetic field lines are contained in the meridian planes and the electric oscillations are generated by a small current loop of area S_1 and strength S_1I_1, where I_1 is the loop current, as shown in fig. 1.

We have with a time factor $e^{-j\omega t}$ in practical units

$$H_r=\left(k^2+\frac{\partial^2}{\partial r^2}\right)\Pi ; \quad H_\theta=\frac{1}{r}\frac{\partial^2}{\partial r\,\partial\theta}(\Pi) ; \quad H_\phi=0,$$

and
$$E_r=E_\theta=0 ; \quad E_\phi=jzk\,.\,\frac{1}{r}\frac{\partial}{\partial\theta}(\Pi),$$

where z is the intrinsic impedance of the medium, $z=\sqrt{\frac{\mu}{\epsilon}}\cdot\frac{1}{\sqrt{1+j60\sigma\lambda_0(\epsilon_0/\epsilon)}}$

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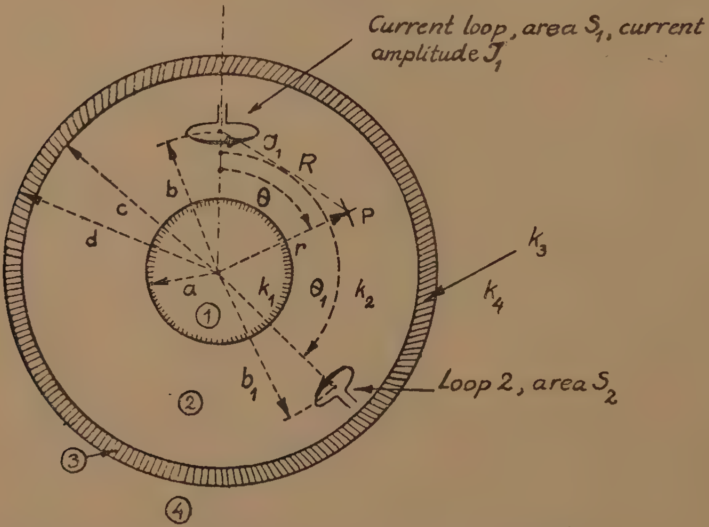
* Communicated by the Author.

σ is the conductivity in mhos/meter, and ϵ_0 is the dielectric constant of free space and λ_0 the corresponding wave-length in metres. The intrinsic impedance of free space $z_0=\sqrt{\frac{\mu_0}{\epsilon_0}}\approx 120\pi$ ohms, k is the complex wave number $k=\frac{2\pi}{\lambda}$, and $k^2=\omega^2\epsilon\mu\{1+j60\sigma\lambda_0(\epsilon_0/\epsilon)\}$.

In the media 1, 2, 3, and 4 we denote k by k_1, k_2, k_3, k_4. For copper $\sigma\approx 5,8\cdot 10^7$ mhos/meter, and therefore for a resonator with a copper shell $\frac{k_2}{k_3}\approx e^{-j\pi/2}\sqrt{f}\,.\,10^{-9}\,.\sqrt{\frac{\epsilon_2\mu_2}{\epsilon_0\mu_3}}$, where f is the wave-frequency.

Fig. 1.



In connection with studies regarding the propagation of radio waves it has been shown by the present author that [(6)]

$$\Pi(r,\theta)=\mathrm{A}\,.\,\frac{j}{k_2{}^2b^2}\sum_{n=0}^{\infty}(n+1/2)\,\zeta_n^{(1)}(k_2b)\,\zeta_n^{(2)}(k_2r)\left\{1+\mathrm{R}_1\,.\,\frac{\zeta_n^{(2)}(k_2a)\,\zeta_n^{(1)}(k_2r)}{\zeta_n^{(1)}(k_2a)\,\zeta_n^{(2)}(k_2r)}\right\}$$
$$\times\frac{1+\mathrm{R}_4\,.\,\dfrac{\zeta_n^{(1)}(k_2c)\,\zeta_n^{(2)}(k_2b)}{\zeta_n^{(2)}(k_2c)\,\zeta_n^{(1)}(k_2b)}}{1-\mathrm{R}_1\mathrm{R}_4\dfrac{\zeta_n^{(1)}(k_2c)\,\zeta_n^{(2)}(k_2a)}{\zeta_n^{(2)}(k_2c)\,\zeta_n^{(1)}(k_2a)}}\,.\,\mathrm{P}_n(\cos\theta),\qquad(b>r>a)\quad.\quad(1\,a)$$

and

$$\Pi(r,\theta)=\mathrm{A}\,.\,\frac{j}{k_2{}^2b^2}\sum_{n=0}^{\infty}(n+1/2)\,\zeta_n^{(1)}(k_2r)\,\zeta_n^{(2)}(k_2b)\left\{1+\mathrm{R}_1\,.\,\frac{\zeta_n^{(2)}(k_2a)\,\zeta_n^{(1)}(k_2r)}{\zeta_n^{(1)}(k_2a)\,\zeta_n^{(2)}(k_2r)}\right\}$$
$$\times\frac{1+\mathrm{R}_4\,.\,\dfrac{\zeta_n^{(1)}(k_2c)\,\zeta_n^{(2)}(k_2r)}{\zeta_n^{(2)}(k_2c)\,\zeta_n^{(1)}(k_2r)}}{1-\mathrm{R}_1\mathrm{R}_4\dfrac{\zeta_n^{(1)}(k_2c)\,\zeta_n^{(2)}(k_2a)}{\zeta_n^{(2)}(k_2c)\,\zeta_n^{(1)}(k_2a)}}\,.\,\mathrm{P}_n(\cos\theta),\qquad(r>b>a)\quad.\quad(1\,b)$$

where

$$\mathrm{A}=\mathrm{I}_1\mathrm{S}_1\,.\,\frac{k_2}{4\pi},\quad \zeta_n^{(1)}(x)=\left(\frac{\pi x}{2}\right)^{1/2}\mathrm{H}^{(1)}_{n+1/2}(x),\quad\text{and}\quad \zeta_n^{(2)}(x)=\left(\frac{\pi x}{2}\right)^{1/2}\mathrm{H}^{(2)}_{n+1/2}(x).$$

We further have

$$\mathrm{R}_1=\frac{-\beta(k_2a)+\gamma(k_1a)k_1/k_2}{\alpha(k_2a)-\gamma(k_1a)k_1/k_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where

$$\alpha(x)=\frac{\zeta_n^{(1)\prime}(x)}{\zeta_n^{(1)}(x)},\quad \beta(x)=\frac{\zeta_n^{(2)\prime}(x)}{\zeta_n^{(2)}(x)},\quad \gamma(x)=\frac{\psi_n'(x)}{\psi_n(x)},\quad\text{and}\quad \psi_n(x)=\left(\frac{\pi x}{2}\right)^{1/2}\mathrm{I}_{n+1/2}(x).$$

It has been shown elsewhere (7) that R_1 can be expanded

$$\mathrm{R}_1=\mathrm{R}_{21}+\mathrm{T}_{21}\mathrm{T}_{12}\sum_{m=0}^{\infty}(\mathrm{R}_{12})^m\,.\,\{\eta(k_1a)\}^{m+1}, \quad . \quad . \quad . \quad (3)$$

where $\eta(x)=\zeta_n^{(1)}(x)/\zeta_n^{(2)}(x)$,

$$\mathrm{R}_{21}=\frac{-\beta(k_2a)+\beta(k_1a)k_1/k_2}{\alpha(k_2a)-\beta(k_1a)k_1/k_2},\quad \mathrm{T}_{21}=1+\mathrm{R}_{21},$$

and

$$\mathrm{R}_{12}=\frac{-\alpha(k_2a)-\alpha(k_1a)k_1/k_2}{\alpha(k_2a)-\beta(k_1a)k_1/k_2},\quad \mathrm{T}_{12}=1+\mathrm{R}_{12}.$$

R_{21} is the spherical reflexion coefficient for a wave approaching the boundary 1, 2 in medium 2. T_{21} is the corresponding transmission coefficient, etc.

R_1 therefore contains all reflected and refracted waves in the central sphere. Naturally for a metallic sphere $\mathrm{R}_1\approx\mathrm{R}_{21}$.

One further finds

$$\mathrm{R}_4=-\frac{\alpha(k_3c)-\alpha(k_2c)k_2/k_3+\{\beta(k_3c)-\alpha(k_2c)k_2/k_3\}\,\mathrm{R}_{34}\,\eta(k_3d)/\eta(k_3c)}{\alpha(k_3c)-\beta(k_2c)k_2/k_3+\{\beta(k_3c)-\beta(k_2c)k_2/k_3\}\,\mathrm{R}_{34}\,\eta(k_3d)/\eta(k_3c)}, \quad (4)$$

where

$$\mathrm{R}_{34}=\frac{-\alpha(k_4d)+\alpha(k_3d)k_3/k_4}{\alpha(k_4d)-\beta(k_3d)k_3/k_4}.$$

One easily finds that R_4 permits the expansion

$$\mathrm{R}_4=\mathrm{R}_{23}+\frac{(1+\mathrm{R}_{23})(1+\mathrm{R}_{32})}{1-\mathrm{R}_{32}\mathrm{R}_{34}\,\eta(k_3d)/\eta(k_3c)}$$

$$=\mathrm{R}_{23}+\mathrm{T}_{23}\mathrm{T}_{32}\{1+\mathrm{R}_{32}\mathrm{R}_{34}\,\eta(k_3d)/\eta(k_3c)+(\qquad)^2+\ldots\},$$

with $\mathrm{R}_{23}=\dfrac{-\alpha(k_3c)+\alpha(k_2c)k_2/k_3}{\alpha(k_3c)-\beta(k_2c)k_2/k_3}$, and $\mathrm{R}_{32}=\dfrac{-\beta(k_3c)+\beta(k_2c)k_2/k_3}{\alpha(k_3c)-\beta(k_2c)k_2/k_3}$.

Thus R_4 contains the entire complex of waves reflected from the resonator shell. For a metallic shell $\mathrm{R}_4\approx\mathrm{R}_{23}$. It should be remarked at this point that it is possible to find an expression for R_4 even in the more general case, which is of interest in problems of radio wave propagation only, that the reflecting shell is radially inhomogeneous (8).

(b) *Resonance properties and Q-values.*

Introducing $\phi_{1_n}=j\frac{\mathrm{A}}{k_2{}^2b^2}(n+1/2)\{1+\mathrm{R}_4\eta(k_2c)/\eta(k_2b)\}\,.\,\zeta_n^{(1)}(k_2b)$,

$$\phi_{2_n}=\zeta_n^{(2)}(k_2r)\{1+\mathrm{R}_1\eta(k_2r)/\eta(k_2a)\},$$

$$\phi_{3_n}=1-\mathrm{R}_1\mathrm{R}_4\eta(k_2c)/\eta(k_2a),$$

$$\phi_{1_n}^{(1)}=j\frac{\mathrm{A}}{k_2{}^2b^2}(n+1/2)\{1+\mathrm{R}_1\eta(k_2b)/\eta(k_2a)\}\,.\,\zeta_n^{(2)}(k_2b),$$

and $\phi_{2_n}^{(1)}=\zeta_n^{(1)}(k_2r)\{1+\mathrm{R}_4\eta(k_2c)/\eta(k_2r)\}$,

we have

$$\Pi(r,\theta)=\sum_{n=0}^{\infty}\frac{\phi_{1n}\phi_{2n}}{\phi_{3_n}}\,.\,\mathrm{P}_n(\cos\theta)=\sum_{n=0}^{\infty}a_n\mathrm{P}_n(\cos\theta),\qquad(b>r>a)$$

$$\Pi(r,\theta)=\sum_{n=0}^{\infty}\frac{\phi_{1n}^{(1)}\phi_{2n}^{(1)}}{\phi_{3_n}}\,.\,\mathrm{P}_n(\cos\theta)=\sum_{n=0}^{\infty}a_n^{(1)}\mathrm{P}_n(\cos\theta).\qquad(r>b>a)$$

We observe that the zeros of ϕ_{3_n} yield the natural frequencies of the resonators. In the simple case $k_1=k_2$, $\mathrm{R}_1\zeta_n^{(2)}(k_2a)/\zeta_n^{(1)}(k_2a)=1$, and therefore, for the natural frequencies in this case,

$$\mathrm{R}_4\zeta_n^{(1)}(k_2c)/\zeta_n^{(2)}(k_2c)=1.\quad\ldots\ldots\quad(5)$$

When $\mathrm{R}_4=\mathrm{R}_{23}$ (good metallic shell, good dielectric reflector) it is easy to show that (5) can be written

$$\frac{\psi_n(k_2c)}{\psi_n'(k_2c)}=\frac{k_2}{k_3}\cdot\frac{\zeta_n^{(1)}(k_3c)}{\zeta_n^{(1)\prime}(k_3c)}.\quad\ldots\ldots\quad(5\,a)$$

This is the classical boundary relation (boundary 2–3) of the free oscillations. It is satisfied by complex ω-values ("natural oscillation constants") only, so that $\omega=\Omega_{nm}=\omega_{nm}-j\beta_{nm}$, where β_{nm} is the *m*th order attenuation per second of the *n*th mode.

For a metallic shell and small reflection losses $|k_3|\gg|k_2|$, we write $\Omega_{nm}=\omega_{nm}^0+\Delta\Omega_{nm}$, where ω_{nm}^0 is the no-loss natural frequency obtained from $\psi_n(k_2c)=0$, *i. e.* $\mathrm{I}_{n+1/2}(k_2c)=0$. Therefore,

$$\Delta\Omega_{nm}=\Delta\omega_{nm}-j\beta_{nm}\approx-j\frac{\omega_{nm}}{k_2c}\cdot\frac{k_2}{k_3}\approx\frac{\omega_{nm}}{k_2c}\cdot\left|\frac{k_2}{k_3}\right|\cdot e^{-j3\pi/4}.$$

Thus

$$\mathrm{Q}_{nm}=\frac{\omega_{nm}}{2\beta_{nm}}=\frac{k_2c}{\sqrt{2}}\cdot\left|\frac{k_3}{k_2}\right|,\quad\ldots\ldots\quad(6)$$

a familiar result.

If we instead consider a dielectric sphere we assume that it is possible to make $k_2\gg k_3$ so that the reflection losses become fairly small. Since the no-loss zeros are this time determined by $\psi_n'(k_2c)=0$, *i. e.* $\left[\frac{d}{dz}\{\sqrt{z}\,\mathrm{I}_{n+1/2}(z)\}\right]_{z=k_2c}=0$, and as, further, $\psi_n''(x)=-\psi_n(x)\left\{1-\frac{n(n+1)}{x^2}\right\}$, one immediately finds

$$\mathrm{Q}_{nm}\approx\frac{k_2c}{2}\cdot\frac{k_2}{k_3}\cdot\left\{1-\frac{n(n+1)}{(k_2c)^2}\right\}\cdot|\zeta_n^{(1)}(k_3c)|^2,\quad\ldots\quad(6\,a)$$

if the dielectric has no losses. If $k_3c \gg 1$, then $|\zeta_n^{(1)}(k_3c)|^2 \approx 1$ and the Q-values of (6) and (6 a) have the same magnitude. If, however, as is the case for the lowest order zeros, $k_3c < 1$, then $|\zeta_n^{(1)}(k_3c)|^2 > 1$. In this case the increase in the Q-value is due to the poor radiation from the dielectric sphere when $k_3c \ll 1$.

* * * * * *

With the notations $R_4R_1 = e^{-2\tau}$, and $\eta(k_2c)/\eta(k_2a) = e^{j\{2\phi+m2\pi\}}$, we have

$$\phi_{3n} = 1 - e^{-2\tau+j2\phi}, \quad \ldots\ldots\ldots\ldots \quad (7)$$

where ϕ is real when k_2 is real. When the reflexion losses are moderate τ is practically constant in the neighbourhood of the steady state resonance frequencies. The resonance value of ϕ_{3n} thus becomes

$$\phi_{3n\ \mathrm{res}} = 1 - e^{-2\mathrm{Re}\{\tau\}} = 1 - |R_1R_4|. \quad \ldots\ldots \quad (8)$$

At the transmission band limits we have approximately

$$2\phi \approx 2\mathrm{I}m\{\tau\} \pm 2\phi_b, \quad \text{and} \quad \sin\phi_b \approx \sinh\{\mathrm{Re}(\tau)\}. \quad \ldots \quad (9)$$

Thus there is no resonance effect to speak of when $\sinh\{\mathrm{Re}(\tau)\} > 1$, *i. e.* when $|R_1R_4| < (\sqrt{2}-1)^2$. In the simple spherical case, $k_1 = k_2$, we similarly obtain $|R_4| < (\sqrt{2}-1)^2 \approx 0{,}17$.

When $k_2a > 1$, then

$$2\phi_b \approx 2\Delta k_2 \,.\, (c-a), \quad \alpha(k_2a) \approx \alpha(k_2c) \approx -\beta(k_2a) \approx -\beta(k_2c) \approx j,$$

so that τ varies fairly slowly through the transmission band. The minimum Q-value thus approximately becomes

$$Q_{nm\ \mathrm{min}} \approx \frac{2(c-a)}{\lambda_2} \approx m, \quad \ldots\ldots\ldots \quad (10)$$

where λ_2 is the wave-length in medium 2.

(1) $|k_3| \gg k_2$, $|k_1| \gg k_2$, *i. e.* a cavity resonator. In the metallic case we then have

$$R_1R_4 \approx R_{21}R_{23} \approx 1 - \frac{k_2}{k_1} \cdot \frac{2}{\zeta_n^{(1)}(k_2a)\,\zeta_n^{(2)}(k_2a)} - \frac{k_2}{k_3} \cdot \frac{2}{\zeta_n^{(1)}(k_2c)\,\zeta_n^{(2)}(k_2c)},$$

i. e.

$$\tau \approx \frac{k_2}{k_1} \cdot \frac{1}{\zeta_n^{(1)}(k_2a)\,\zeta_n^{(2)}(k_2a)} + \frac{k_2}{k_3} \cdot \frac{1}{\zeta_n^{(1)}(k_2c)\,\zeta_n^{(2)}(k_2c)}, \quad \ldots \quad (11)$$

when $x > 1$, $\zeta_n^{(1)}(x)\,\zeta_n^{(2)}(x) \approx 1$. However, when $x \ll 1$, $\zeta_n^{(1)}(x)\,\zeta_n^{(2)}(x) \gg 1$. When, therefore, $k_2a > 1$ the qualities of the inner and outer shells are of equal importance irrespective of the ratio c/a, as one naturally expects. When $k_2a \ll 1$ the central sphere or inner shell cannot absorb much energy from the cavity field and, therefore, only affects the Q-value and, as will be shown, the resonance frequencies slightly.

Introducing $\phi_n(x) = \sqrt{\frac{\pi x}{2}}\, N_{n+1/2}(x)$, we have $$2\phi+m2\pi=2\arctan\left\{\frac{\phi_n(k_2c)}{\psi_n(k_2c)}\right\}-2\arctan\left\{\frac{\phi_n(k_2a)}{\psi_n(k_2a)}\right\}.$$

Thus, when the frequency changes are relatively small,

$$\Delta\phi\approx\frac{\Delta\omega_{nm}}{\omega_{nm}}\left\{\frac{k_2c}{|\zeta_n^{(1)}(k_2c)|^2}-\frac{k_2a}{|\zeta_n^{(1)}(k_2a)|^2}\right\}. \quad . \quad . \quad . \quad (12)$$

It is readily seen that the central sphere hardly affects the resonance conditions when $k_2a\ll 1$.

From (9), (11) and (12) we finally obtain for metallic shells

$$Q_{nm}\approx\frac{1}{\sqrt{2}}\cdot\frac{\dfrac{k_2c}{|\zeta_n^{(1)}(k_2c)|^2}-\dfrac{k_2a}{|\zeta_n^{(1)}(k_2a)|^2}}{\left|\dfrac{k_2}{k_3}\right|\cdot\dfrac{1}{|\zeta_n^{(1)}(k_2c)|^2}+\left|\dfrac{k_2}{k_1}\right|\cdot\dfrac{1}{|\zeta_n^{(1)}(k_2a)|^2}}. \quad . \quad . \quad (13)$$

This reduces to (6) when $k_2a\ll 1$.

(2) $k_2\gg k_3$, $k_2\gg k_1$, *i. e.* a dielectric resonator. We now have $|R_1|=1$, and $$R_1=\exp[2j\{\arctan[\psi_n'(k_2a)\psi_n(k_2a)+\phi_n'(k_2a)\phi_n(k_2a)$$ $$+|\zeta_n^{(1)}(k_2a)|^2 . k_1\psi_n'(k_1a)/k_2\psi_n(k_1a)]\}].$$

When $k_1\ll k_2$, a convenient approximation is

$$R_1\,\zeta_n^{(2)}(k_2a)/\zeta_n^{(1)}(k_2a)\approx-\zeta_n^{(2)\prime}(k_2a)/\zeta_n^{(1)}(k_2a).$$

Further

$$R_4\frac{\zeta_n^{(1)}(k_2c)}{\zeta_n^{(2)}(k_2c)}\approx-\frac{\zeta_n^{(1)\prime}(k_2c)}{\zeta_n^{(2)\prime}(k_2c)}\cdot\overbrace{\left\{1+2j\alpha(k_3c)\frac{k_3}{k_2}\cdot\frac{1}{|\zeta_n^{(1)\prime}(k_2c)|^2}\right\}}^{\eta_4}.$$

With $$\frac{\zeta_n^{(1)\prime}(k_2c)\,\zeta_n^{(2)\prime}(k_2a)}{\zeta_n^{(2)\prime}(k_2c)\,\zeta_n^{(1)\prime}(k_2a)}=e^{j(2\phi_1+m2\pi)},\quad\text{and}\quad\eta_4=e^{-2\tau_1},$$

the transmission band limits are determined by $2\phi_1=2Im(\tau_1)\pm 2\phi_{b_1}$, with $\sin\phi_{b_1}\approx\sinh\{Re(\tau)\}$. Since

$$\tau_1\approx-j\alpha(k_3c)\cdot\frac{k_3}{k_2}\cdot\frac{1}{|\zeta_n^{(1)\prime}(k_2c)|^2}, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and

$$\Delta\phi_1\approx\frac{\Delta\omega_{nm}}{\omega_{nm}}\left[\frac{k_2c}{|\zeta_n^{(1)\prime}(k_2c)|^2}\cdot\left\{1-\frac{n(n+1)}{(k_2c)^2}\right\}-\frac{k_2a}{|\zeta_n^{(1)\prime}(k_2a)|^2}\cdot\left\{1-\frac{n(n+1)}{(k_2a)^2}\right\}\right], \quad . \quad . \quad . \quad (15)$$

when the frequency changes are small, we finally obtain

$$Q_{nm}\approx\frac{1}{2}\cdot\frac{k_2}{k_3}\cdot k_2\left[c\left\{1-\frac{n(n+1)}{(k_2c)^2}\right\}-a\frac{|\zeta_n^{(1)\prime}(k_2c)|^2}{|\zeta_n^{(1)\prime}(k_2a)|^2}\cdot\left\{1-\frac{n(n+1)}{k({}_2a)^2}\right\}\right]. \quad (16)$$

This reduces to (6 *a*) when $k_2a\ll 1$.

* * * * * *

Denoting by E_ϕ^{nm} the E_ϕ-component of the nmth mode natural oscillation and assuming, for the sake of simplicity, that $k_1=k_2$, we may write

$$E_\phi^{nm}=jz_2k_2\alpha_{nm}\cdot\frac{\psi_n(k_2r)}{r}\cdot P_n^1(\cos\theta)\,;$$

$$H_\theta^{nm}=k_2\alpha_{nm}\cdot\frac{\psi_n'(k_2r)}{r}\cdot P_n^1(\cos\theta)\,;\qquad\text{(medium 2)}$$

and

$$E_\phi^{nm}=jz_3k_3\alpha_{nm}\cdot\frac{\zeta_n^{(1)}(k_3r)}{r}\cdot P_n^1(\cos\theta)\cdot\beta_{nm}\,;$$

$$H_\theta^{nm}=k_3\alpha_{nm}\cdot\frac{\zeta_n^{(1)\prime}(k_3r)}{r}\cdot P_n^1(\cos\theta)\cdot\beta_{nm}.\qquad\text{(medium 3)}$$

The radiation flow into 3 from medium 2 thus becomes

$$P^{nm}=\frac{1}{2}\iint Re\{E_\phi^{nm}\,H_\theta^{nm*}\}dS$$

$$=z_2k_2|\alpha_{nm}|^2\cdot\frac{2\pi n(n+1)}{2n+1}\cdot|\beta_{nm}|^2\cdot Re\{jk_3^*\,\zeta_n^{(1)}(k_3c)\,\zeta_n^{(1)}(k_3c)^*\}.\qquad(17a)$$

It is to be observed that for the natural frequencies $\phi_{1_n}^{(1)}\,\phi_{2_n}^{(1)}$ is identical to $\phi_{1_n}\,\phi_{2_n}$ and the energy stored in the dielectric medium 2 thus becomes

$$N^{nm}=\frac{1}{2}\iiint\epsilon_2|E_\phi^{nm}|^2dV=\epsilon_2(z_2k_2)^2|\alpha_{nm}|^2\frac{2\pi n(n+1)}{2n+1}\int_0^c\psi_n^2(k_2r)dr.\qquad(17b)$$

For Q_{nm} we therefore obtain

$$Q_{nm}=\omega\frac{N^{nm}}{P^{nm}}=\frac{\displaystyle\int_0^{k_2c}\psi_n^2(x)\,dx}{Re\underbrace{\left\{j\psi_n(k_2c)\left(\frac{k_3}{k_2}\right)^*\zeta_n^{(1)\prime}(\overset{*}{k_3}c)\,\overset{*}{\beta}_{nm}\right\}}_{\rho}}\quad .\ \ .\ \ .\qquad(17c)$$

One further has

$$\int_0^z\psi_n^2(x)dx=\frac{z}{2}\left[\left\{1-\frac{n(n+1)}{z^2}\right\}\{\psi_n(z)\}^2+\{\psi_n'(z)\}^2-\frac{1}{z}\psi_n(z)\,\psi_n'(z)\right],$$

where the last term always is insignificant when the reflection losses are very small. For the two characteristic cases

(1) $|k_3|\ll k_2$, a cavity resonator, $\psi_n(k_2c)\approx 0$, and $\rho=\dfrac{k_2}{k_3}\{\psi_n'(k_2c)\}^2$,

(2) $k_3\ll k_2$, a dielectric resonator,

$$\psi_n'(k_2c)\approx 0,\quad\text{and}\quad Re\{\rho\}=\frac{k_3}{k_2}\cdot\{\psi_n(k_2c)\}^2\cdot\frac{1}{|\zeta_n^{(1)}(k_3c)|^2},$$

we thus obtain the Q_{nm}-values of (6) and (6a) respectively. β_{nm} has been considered so small that $\psi_n(k_2c)$ in (2) is approximately real.

It should be stressed, however, that the when reflection losses are no longer very small, only a direct examination of ϕ_{3_n}, as has already been shown, yields a correct Q_{nm}-value.

(c) *The impedance of the coupling loops.*

In order to calculate the impedance of the loop we find it convenient to separate the " primary " and " secondary " fields. We have

$$\Pi(r,\theta)=\mathrm{A}\left\{\frac{r}{b}\cdot\frac{e^{jk_2\mathrm{R}}}{k_2\mathrm{R}}\right.$$
$$+\frac{j}{k_2{}^2b^2}\sum_{n=0}^{\infty}(n+1/2)\,\zeta_n^{(1)}(k_2b)\,\zeta_n^{(1)}(k_2r)\left\{\frac{\mathrm{R}_1}{\eta(k_2a)}-1\right\}\mathrm{P}_n(\cos\theta)$$
$$+\frac{j}{k_2{}^2b^2}\sum_{n=0}^{\infty}(n+1/2)\,\zeta_n^{(1)}(k_2b)\,\zeta_n^{(1)}(k_2r)\left\{\frac{1}{\eta(k_2r)}+\frac{\mathrm{R}_1}{\eta(k_2a)}\right\}$$
$$\times\left\{\frac{1}{\eta(k_2b)}+\frac{\mathrm{R}_1}{\eta(k_2a)}\right\}\cdot\frac{\mathrm{R}_4\eta(k_2c)}{\phi_{3n}}\cdot\mathrm{P}_n(\cos\theta). \quad . \quad . \quad . \quad . \quad (18)$$

We denote the impedance of the loop in free space by z_1, the impedance of the loop with respect to the central sphere (medium 1) by z_{11_a} and the resonator impedance by z_{11ac}. Applying a voltage V_1 to the loop, we obtain $\mathrm{V}_1=\mathrm{Z}_1\mathrm{I}_1+\frac{d\Lambda}{dt}$, where Λ is the flux through the loop. Therefore, from (17),

$$\mathrm{Z}_{11ac}=120\sqrt{\frac{\epsilon_0}{\epsilon_2}}\cdot\frac{\mathrm{S}_1{}^2}{b^4}\sum_{n=1}^{\infty}n(n+1/2)(n+1)\{\zeta_n^{(1)}(k_2b)\}^2\left\{\frac{\mathrm{R}_1}{\eta(k_2a)}-1\right\},\ \text{ohms}, \quad (19)$$

and

$$\mathrm{Z}_{11ac}=120\sqrt{\frac{\epsilon_0}{\epsilon_2}}\cdot\frac{\mathrm{S}_1{}^2}{b^4}\sum_{n=1}^{\infty}n(n+1/2)(n+1)\Delta_b{}^2\frac{\mathrm{R}_4\eta(k_2c)}{\phi_{3n}},\ \text{ohms}, \quad . \quad . \quad . \quad (20)$$

where $\Delta_b=\frac{1}{2}\{\zeta_n^{(2)}(k_2b)+\zeta_n^{(1)}(k_2b)\,\mathrm{R}_1/\eta(k_2a)\}$,

ϵ_0 is the dielectric constant of free space, and μ_2 is assumed equal to μ_0.

We have seen that the central sphere hardly disturbs the field when $k_2a\ll 1$. Similarly another very small loop, area S_2, coordinates b_1, θ_1 (and axis parallel to b_1) does not change the picture. The mutual cavity impedance between the two loops apparently becomes

$$\mathrm{Z}_{12ac}=120\sqrt{\frac{\epsilon_0}{\epsilon_2}}\cdot\frac{\mathrm{S}_1\mathrm{S}_2}{b^2b_1{}^2}\cdot\sum_{n=1}^{\infty}n(n+1/2)(n+1)\Delta_b\Delta_{b_1}\cdot\frac{\mathrm{R}_4\eta(k_2c)}{\phi_{3n}}.\mathrm{P}\ (\cos\theta),\ \text{ohms}. \quad . \quad . \quad . \quad (21)$$

Equations (20) and (21) are conveniently written

$$\mathrm{Z}_{11ac}=\sum_{n=1}^{\infty}\Delta\mathrm{Z}_{11ac}^{(n)},\quad \mathrm{Z}_{12ac}=\sum_{n=1}^{\infty}\Delta\mathrm{Z}_{12ac}^{(n)}. \quad . \quad . \quad . \quad . \quad (22)$$

Near a resonance $\Delta\mathrm{Z}_{ac}^{(n)}$ gets the distinguishing notation $\Delta\mathrm{Z}_{ac}^{(nm)}$. Further

$$\Delta\mathrm{Z}_{12ac}^{(n)}=\mathrm{P}_n(\cos\theta)\sqrt{\Delta\mathrm{Z}_{11ac}^{(n)}\,\Delta\mathrm{Z}_{22ac}^{(n)}}. \quad . \quad . \quad . \quad . \quad (22\,a)$$

The power delivered by loop 1 at resonance (when loop 2 is open) in the simple case $k_1=k_2$ becomes

$$\mathrm{P}^{(nm)}=\mathrm{I}_1{}^2\,.\,30\,.\,\frac{\mathrm{S}_1{}^2}{b^4}\,.\,n(n+1/2)(n+1)\frac{\psi_n{}^2(k_2b)}{\mathrm{Re}\{\tau\}}. \quad . \quad . \quad (23)$$

For the steady state

$$\alpha_{nm}=j\mathrm{A}\,.\,(2n+1)\,.\,\psi_n(k_2b)\,.\,\frac{1}{k_4{}^2b^2\,\mathrm{Re}\{\tau\}}.$$

Relation (16 *a*) therefore yields

$$\mathrm{P}^{(nm)}=\mathrm{I}_1{}^2\,.\,30\,.\,\frac{\mathrm{S}_1{}^2}{b^4}\,.\,n(n+1/2)(n+1)\,\psi_n{}^2(k_2b)\,\mathrm{Re}\{\rho\}/\mathrm{Re}^2\{\tau\}.$$

However, $$\frac{1}{|\zeta_n^{(1)}(x)|^2}=\{\psi_n'(x)\}^2,\quad \text{when } \psi_n(x)=0,$$

and $$\frac{1}{|\zeta_n^{(1)\prime}(x)|^2}=\{\psi_n(x)\}^2,\quad \text{when } \psi_n'(x)=0.$$

For the two specific cases (cavity resonator and dielectric resonator) the result of (17 *a*) thus is identical with (23).

In the resonance band we write for small losses $\phi_{3_n}=2\{\tau-j\phi\}$ and therefore, for the cavity case for example,

$$\Delta\mathrm{Z}_{11ac}^{(nm)}\approx 120\sqrt{\frac{\epsilon_0}{\epsilon_2}}\cdot\frac{\mathrm{S}_1{}^2}{b^4}\,.\,n(n+1/2)(n+1)$$

$$\times\,\psi_n{}^2(k_2b)\cdot\frac{1}{\dfrac{k_2c}{|\zeta_n^{(1)}(k_2c)|^2}-\dfrac{k_2a}{|\zeta_n^{(1)}(k_2a)|^2}}\cdot\frac{\mathrm{Q}_{nm}}{1-j\dfrac{\Delta\omega}{\Delta\omega_{nm}}},\ (\text{ohms}).\qquad(24)$$

From the corresponding expression for the impedance of a low loss parallel resonance circuit,

$$\mathrm{Z}\approx\mathrm{L}\omega_0\,.\,\frac{\mathrm{Q}_0}{1-j\dfrac{\Delta\omega}{\Delta\omega_0}}=\frac{\mathrm{W}}{1-j\dfrac{\Delta\omega}{\Delta\omega_0}},\quad \mathrm{L}^{(nm)}\ \text{and}\ \mathrm{R}^{(nm)}\ \text{or}\ \mathrm{W}^{(nm)}$$

for such a circuit equivalence are immediately obtained from the above expression for $\Delta\mathrm{Z}_{11ac}^{(nm)}$.

If loop 2 is closed by an external impedance $\mathrm{Z}e_2$ the cavity or resonator impedance seen from loop 1 becomes (free space impedance of the loops neglected and index *ac* omitted)

$$\Delta\mathrm{Z}_1^{(nm)}=\frac{\Delta\mathrm{Z}_{12}^{(\mathrm{S})}+\chi^2\,\mathrm{Z}e_2}{1+\mathrm{Z}e_2/\Delta\mathrm{Z}_{22}^{(nm)}},\quad .\ .\ .\ .\ .\ .\ .\qquad(25)$$

where $\Delta\mathrm{Z}_{12}^{(\mathrm{S})}=\Delta\mathrm{Z}_{11}^{(nm)}\{1-\mathrm{P}_n{}^2(\cos\theta)\}=$ the short circuit impedance, and $\chi=\mathrm{S}_1\,\Delta_b/\mathrm{S}_2\,\Delta_{b_1}$, a non-uniformity or transformation coefficient. For $\theta_1=\pm\pi$*, and

$$\Delta\mathrm{Z}_1^{(nm)}=\chi^2\,\mathrm{Z}e_2\,\Delta\mathrm{Z}_{11}^{(nm)}/(\chi^2\,\mathrm{Z}e_2+\Delta\mathrm{Z}_{11}^{(nm)}),$$

---

* This case roughly corresponds to a transmission line $n\lambda$ long.

with the equivalent circuit shown by fig. 2. From this the decrease in Q_{nm}-value due to the secondary impedance Ze_2 is easily calculated.

Fig. 2.

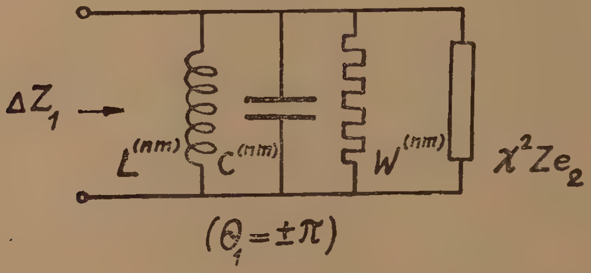
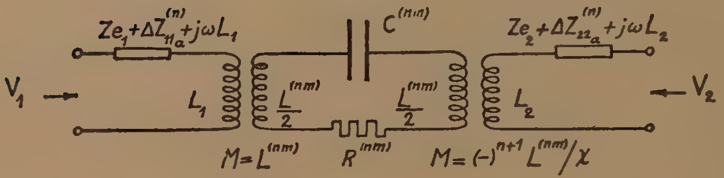

ΔZ_1
$L^{(nm)}$
$C^{(nm)}$
$W^{(nm)}$
$\chi^2 Ze_2$
$(\theta_1 = \pm\pi)$

Fig. 3.


$Ze_1+\Delta Z^{(n)}_{11a}+j\omega L_1$
$C^{(nm)}$
$Ze_2+\Delta Z^{(n)}_{22a}+j\omega L_2$
V_1
L_1
$\frac{L^{(nm)}}{2}$
$\frac{L^{(nm)}}{2}$
L_2
V_2
$M=L^{(nm)}$
$R^{(nm)}$
$M=(-)^{n+1}L^{(nm)}/\chi$

Finally, the complete equivalent network of the high-Q resonator near the mth resonance of the nth mode is shown by fig. 3 (note: time factor $e^{j\omega t}$, $Z^{(n)}_{12a}$ neglected) for the characteristic case $\theta_1 = \pm\pi$.

§ 3. *TM-Waves.*

(a) *General considerations.*

The current loop (magnetic dipole) is now replaced by a current element (electric dipole) $I_1 L_0$-meter-amperes strong. We then have

$$E_r = \left(k^2 + \frac{\partial^2}{\partial r^2}\right)\Pi^1; \quad E_\theta = \frac{1}{r}\cdot\frac{\partial^2}{\partial r \partial \theta}(\Pi^1); \quad E_\phi = 0,$$

and

$$H_r = H_\theta = 0; \quad H_\phi = -j\frac{k}{z}\cdot\frac{1}{r}\frac{\partial}{\partial\theta}(\Pi^1).$$

For Π^1 one can use the relations (1 *a*) and (1 *b*) for Π if A is replaced by $A^1 = I_1 L_0 \cdot \frac{z_2}{4\pi}$, $\frac{k_2}{k_1}$ by $\frac{k_1}{k_2}$, $\frac{k_2}{k_3}$ by $\frac{k_3}{k_2}$, etc. Instead of R_1 we thus obtain

$$R_1^1 = \frac{-\beta(k_2 a) + \gamma(k_1 a)k_2/k_1}{\alpha(k_2 a) - \gamma(k_1 a)k_2/k_1}.$$

(b) *Q-values.*

Not much can be said in this connection since the treatment of $\phi^1_{3_n}$ is practically the same as that of ϕ_{3_n}. We therefore limit ourselves to the case of very small losses.

(1) $|k_3| \gg k_2$, $|k_1| \gg k_2$, a cavity resonator. We have in the metallic case

$$\mathrm{R}_1\mathrm{R}_4\frac{\zeta_n^{(1)}(k_2c)\,\zeta_n^{(2)}(k_2a)}{\zeta_n^{(2)}(k_2c)\,\zeta_n^{(1)}(k_2a)}$$

$$\approx\frac{\zeta_n^{(1)\prime}(k_2c)\,\zeta_n^{(2)\prime}(k_2a)}{\zeta_n^{(2)\prime}(k_2c)\,\zeta_n^{(1)\prime}(k_2a)}\left[1-2\left\{\frac{k_2}{k_3}\cdot\frac{1}{|\zeta_n^{(1)\prime}(k_2c)|^2}+\frac{k_2}{k_1}\cdot\frac{1}{|\zeta_n^{(1)\prime}(k_2a)|^2}\right\}\right].$$

i. e.

$$\tau_1{}^1\approx\frac{k_2}{k_1}\cdot\frac{1}{|\zeta_n^{(1)\prime}(k_2a)|^2}+\frac{k_2}{k_3}\cdot\frac{1}{|\zeta_n^{(1)\prime}(k_2c)|^2}. \quad . \quad . \quad . \quad . \quad (28)$$

Since $\varDelta\phi_1{}^1$ is expressed by (15), we finally obtain for metallic shells

$$\mathrm{Q}_{nm}\approx\frac{1}{\sqrt{2}}\cdot\frac{\dfrac{k_2c}{|\zeta_n^{(1)\prime}(k_2c)|^2}\cdot\left\{1-\dfrac{n(n+1)}{(k_2c)^2}\right\}-\dfrac{k_2a}{|\zeta_n^{(1)\prime}k_2a)|^2}\cdot\left\{1-\dfrac{n(n+1)}{(k_2a)^2}\right\}}{\left|\dfrac{k_2}{k_3}\right|\cdot\dfrac{1}{|\zeta_n^{(1)\prime}(k_2c)|^2}+\left|\dfrac{k_2}{k_1}\right|\cdot\dfrac{1}{|\zeta_n^{(1)\prime}(k_2a)|^2}}. \quad . \quad . \quad . \quad (29)$$

This resembles the corresponding result for the TE-waves (13). For such high order zeros that $k_2a \gg 1$ relation (13) yields

$$\mathrm{Q}_{nm}\approx\frac{k_2(c-a)}{\sqrt{2}}\cdot\frac{1}{\left|\dfrac{k_2}{k_3}\right|+\left|\dfrac{k_2}{k_1}\right|},$$

and relation (29)

$$\mathrm{Q}_{nm}\approx\frac{k_2(c-a)}{\sqrt{2}}\cdot\frac{1}{\left|\dfrac{k_2}{k_3}\right|+\left|\dfrac{k_2}{k_1}\right|}\cdot\left\{1+\frac{n(n+1)}{k_2{}^2ac}\right\}.$$

The present author has shown in an entirely different way (in connection with studies of the propagation of radio waves)$^{(9)}$ that the attenuation per second for the travelling wave is

$$\frac{\omega}{k_2(c-a)\sqrt{2}}\left\{\left|\frac{k_2}{k_3}\right|+\left|\frac{k_2}{k_1}\right|\right\}$$

for the TM-waves and

$$\frac{\omega}{k_2(c-a)\sqrt{2}}\left\{\left|\frac{k_2}{k_3}\right|+\left|\frac{k_2}{k_1}\right|\right\}\cdot\frac{1}{1+\dfrac{n(n+1)}{k_2{}^2ac}}$$

for the TE-waves, when

$$c-a \ll a \quad \text{and} \quad n(n+1)<k_2{}^2a^2.$$

This is the "plane" wave attenuation with $\dfrac{n(n+1)}{k_2{}^2ca}\approx\sin^2\xi$, where ξ is the angle of incidence. By the relation $\mathrm{Q}=\dfrac{\omega}{2\beta}$, it is seen that these attenuation values yield the same Q-value as do relations (13) and (29) for the "plane" case.

(2) $k_2 \gg k_3$, $k_2 \gg k_1$, a dielectric resonator. We now have

$$R_4{}^1 \approx 1 - \frac{k_3}{k_2} \cdot \frac{\zeta_n^{(1)}(k_3c)}{\zeta_n^{(1)\prime}(k_3c)} \cdot \frac{2j}{|\zeta_n^{(1)}(k_2c)|^2}$$

and

$$R_1 \frac{\zeta_n^{(2)}(k_2a)}{\zeta_n^{(1)}(k_2a)} \approx \frac{\zeta_n^{(2)}(k_2a)}{\zeta_n^{(1)}(k_2a)}. \qquad (30)$$

Thus

$$Re\{\tau^1\} \approx -\frac{k_3}{k_2} \cdot \frac{1}{|\zeta_n^{(1)}(k_2c)|^2} \cdot \frac{1}{|\zeta_n^{(1)\prime}(k_3c)|^2}.$$

By (12), which expresses $\Delta\phi$ in this case, we therefore obtain

$$Q_{nm} \approx \frac{1}{2} \cdot \frac{k_2}{k_3} \cdot k_2 \left\{ c - a \frac{|\zeta_n^{(1)}(k_2c)|^2}{|\zeta_n^{(1)}(k_2a)|^2} \right\} \cdot |\zeta_n^{(1)\prime}(k_3c)|^2, \qquad (31)$$

which also resembles the corresponding Q_{nm}-expression for the TE-waves (16). When $k_3c \ll 1$ the radiation from the dielectric sphere is considerably reduced with a corresponding increase in Q_{nm}.

(c) *The impedance of the current elements.*

The self- and mutual impedances are easily obtained from the following relation :

$$E_r^{(nm)} = \frac{n(n+1)}{r^2} \cdot \Pi^{1(nm)}.$$

It is practically useful only when the nature of the coupling and excitation of the current element is known.

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* Due to the war situation this list of references cannot be regarded as complete as far as later publications are concrned.

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LXXVII. *Irany's Temperature Function of Viscosity.*

By Balbhadra Prasad *.

[Received November 14, 1945.]

Irany⁽¹⁾ developed a function of viscosity linear in temperature. He worked out the function graphically. He revised the function and gave the value of the function (θ) for various values of viscosity in a subsequent paper⁽²⁾. When Irany's function of viscosity⁽¹⁾ was plotted against temperature in cases of gallium and tin by Srinivasan and the author⁽³⁾ a curved line was obtained instead of a straight one. Even with the revised values⁽²⁾ of "θ" (which were not available at the time the paper was communicated), there is no improvement. This lead to a doubt in the universal applicability of the Irany's function.

It has been definitely established⁽³⁾ that the variation of viscosity of unassociated liquids (including liquids whose association does not vary with temperature) is accurately represented by simple Andrade's⁽⁴⁾ equation, $\eta=Ae^{b/T}$. If a certain function of viscosity is linear in temperature, then it should not give conditions which are not compatible with Andrade's equation. Irany's function (θ) can be represented as $\theta=a+bT$, where "a" and "b" are constants for the liquid concerned. Let us consider two liquids which have the same viscosity (hence the same value of θ), one at "T_1" and the other at "T_2." It follows that $a_1+b_1T_1=a_2+b_2T_2$. If a set of temperatures T_1 and T_2 are selected for the two liquids at which they have the same viscosity, then T_1 plotted against T_2 should give a straight line. Further, Andrade's equation may be written in the form $\log\eta=\alpha+\beta/T$. At the two temperatures at which two liquids have the same viscosity, $\alpha_1+\beta_1/T_1=\alpha_2+\beta_2/T_2$ or $\beta_1T_2=\beta_2T_1+(\alpha_2-\alpha_1)T_1T_2$. Since T_1T_2 cannot be constant, as both "T_1" and "T_2" will simultaneously increase for lower viscosity and decrease for higher viscosity, it is necessary that α_1 is equal to α_2, in order that T_2 plotted against T_1 may give a straight line. In other words, no function like Irany's function will be applicable to a number of liquids whose "α" value differ appreciably and which obey Andrade's simple equation, in spite of Irany's claim ⁽⁵⁾ to the contrary.

Irany has given the formula

$$\theta=\theta_1+\frac{1\cdot69}{1+\dfrac{1\cdot38}{\log\eta}}$$

for his function "θ." The "θ" values thus calculated do not agree with the graphical values below 0·5 centipoise. If a liquid obeys both Irany's equation $\theta=a+bT$ and Andrade's equation $\log\eta=\alpha+\beta/T$, the function can be obtained by eliminating "T" from the two equations, and it can be expressed in a much simpler way: $\theta = a + \beta b / \log \eta - \alpha$ or $\theta = a + B / \log \eta - \alpha$. The whole of Irany's function from 5000 centipoises down to 0·8 centipoise is represented by

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* Communicated by the Author.

$$\theta = 1{\cdot}630 - \frac{2{\cdot}087}{\log \eta + 1{\cdot}282}$$

to an accuracy of ±0·005 in almost all cases. This means that in this region Irany's function corresponds to a liquid whose "α" is about $\bar{2}{\cdot}7$. It may be noted that from 6 centipoises to 0·6 centipoise the function is expressed to an accuracy of ±0·002 by the equation

$$\theta = 1{\cdot}72 - \frac{2{\cdot}41}{\log \eta + 1{\cdot}400},$$

corresponding to $\alpha = \bar{2}{\cdot}600$. From 0·6 to 0·2, "$\theta$" can be represented by

$$\theta = 3{\cdot}02 - \frac{6{\cdot}47}{\log \eta + 2{\cdot}16}.$$

This region corresponds to a liquid whose $\alpha = \bar{3}{\cdot}84$. In order to represent the "θ" value of 0·1 centipoise also a much different value of "α" is needed.

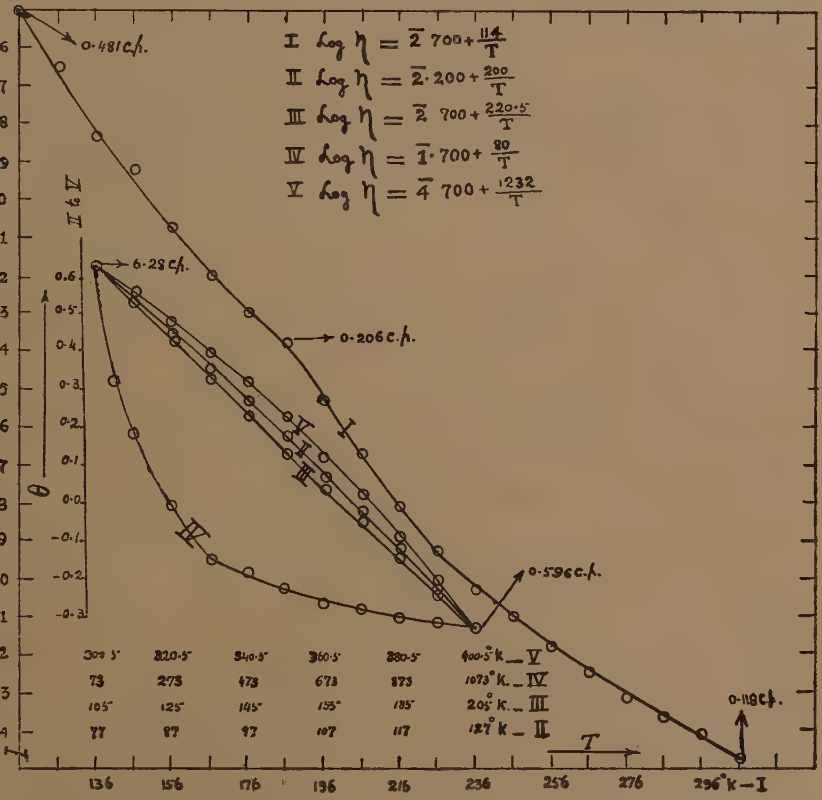
Irany's function (θ) has been calculated for a number of hypothetical liquids obeying Andrade's equation, whose "α" values are different for a number of temperatures, by interpolation from figures given in his paper (2). In fig. 1 "θ" has been plotted against absolute temperature in case of all these liquids. Liquid I represents viscosity from 0·481 to 0·118 centipoise. In this region "θ" changes rapidly with viscosity. The line obtained in the case of this liquid is not even a regular curve. This is due to the fact that a number of liquids having different values of "α" have been used by Irany in constructing the function (θ) corresponding to this region of viscosity. Liquids II to V have been deliberately made to cover the same region, in order to show the effect of change of "α" unhindered by some other factors. Different temperature scales have been chosen for the different liquids so as to make the points corresponding to 6·28 and 0·596 c.p. coincide in case of all the liquids in order to see the departure from linearity in a clear way. Liquids II ($\alpha = \bar{2}{\cdot}200$) and III ($\alpha = \bar{2}{\cdot}700$) may be compared. Both of them are found to have slight curvature on careful analysis, but straight lines can be made of either of them in a rough way. It is remarkable that when "α" is decreased the departure from linearity in θ—T plot is not so pronounced as when it is increased. This point is seen more clearly by comparing the curves of liquids IV and V. The curvature in case of liquid IV ($\alpha = \bar{1}{\cdot}700$) is far more pronounced than in the case of liquid V ($\alpha = \bar{4}{\cdot}700$) though the "$\alpha$" value of liquid IV differs by 1·1, whereas that of liquid V differs by 1·9 from that of the hypothetical liquid ($\alpha = \bar{2}{\cdot}600$) corresponding to Irany's function in this region. The cause of the failure of Irany's function in the case of gallium ($\alpha=1{\cdot}618$) and tin ($\alpha=\bar{1}{\cdot}638$) now becomes clear.

It is possible to develop any number of functions of Irany's type. One of these functions (θ') is given by

$$\theta'=6-\frac{1{\cdot}8}{\log\eta+1{\cdot}3}.$$

This function (which also corresponds to $\alpha=\bar{2}{\cdot}7$), when plotted against temperature, exhibits same peculiarities as Irany's function, as may be

Fig. 1.



seen in fig. 2. A regular curve is obtained in case of liquid VI, which covers the same region as liquid "I" but has a different value of "α" because the function θ' corresponds to one hypothetical liquid in all regions of viscosity. The behaviour of liquid I in fig. 1 may be examined to serve as a contrast. Functions corresponding to other values of "α" also give good results. For instance, the function

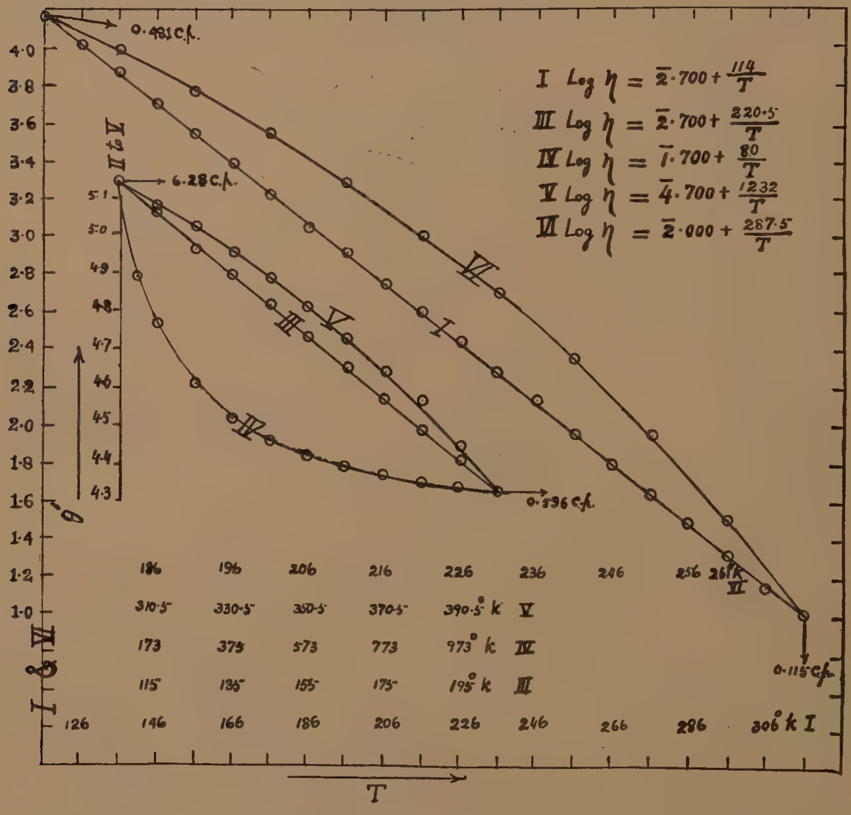
$$\theta''\left(\theta''=5-\frac{3{\cdot}5}{\log\eta+1{\cdot}7}\right),$$

corresponding to $\alpha=\bar{2}{\cdot}3$, works well and suffers from the same defects as the other two functions, θ and θ'.

The apparent success of Irany's function is due to two reasons :—(1) a large number of organic liquids have their " α " values between $\bar{2}\cdot0$ and $2\cdot7$, and (2) the range of few of them is fairly wide to bring out the defects of Irany's function.

Throughout the whole paper viscosity has been expressed in centipoises to keep in a line with Irany. The whole argument advanced will remain intact if viscosity is expressed in poises, only the corresponding values of " α " shall have to be decreased in every case by two.

Fig. 2.



Summary.

1. It has been shown that Irany's function (θ) of viscosity, which is supposed to be linear in temperature, is not compatible with Andrade's simple equation, $\log \eta = \alpha + \beta/T$. The function (θ) has only a limited range inasmuch as it is applicable only to liquids which have their " α " values round about $\bar{2}\cdot7$. The curvature in θ—T plot is not so pronounced in cases of liquids having the values of " α " smaller than $\bar{2}\cdot7$ as it is in the cases of liquids having their " α " values larger than $\bar{2}\cdot7$. The apparent linearity " θ " in " t " in a large number of cases

is due to the facts that some liquids have their " α " round about $\bar{2}\cdot 7$ and many others do not cover a sufficiently large range of viscosity to show the failure of the function.

2. It has also been shown how a number of functions like Irany's can be developed. All these functions are linear in temperature to the same extent as Irany's and suffer from the same drawbacks.

3. It is impossible to develop any function of viscosity which is linear in temperature over a wide range of viscosity for all liquids.

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Mayurbhanj Chemical Laboratory,
Ravenshaw College, Cuttack,
India.

LXXVIII. *An Alternative Definition of Reciprocal Differences.*

By Herbert E. Salzer *†.

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Thiele's definition of his reciprocal differences $\rho_n(x_1, x_2 \ldots x_{n+1})$, given by

$$\rho_n(x_1, x_2 \ldots x_{n+1}) = \frac{x_1 - x_{n+1}}{\rho_{n-1}(x_1 \ldots x_n) - \rho_{n-1}(x_2 \ldots x_{n+1})} + \rho_{n-2}(x_2 \ldots x_n), \quad \ldots \quad (A')$$

where $\rho_{-1}=0$, $\rho_0(x)=f(x)$, leads to his continued fraction interpolation formula, which terminates (*i. e.* for some n, $\rho_n(x\, x_1 \ldots x_n)=$constant) whenever $f(x)$ is a rational function, namely,

$$f(x) = f(x_1) + \frac{x - x_1}{\rho_1(x_1 x_2)} + \frac{x - x_2}{\rho_2(x_1 x_2 x_3) - f(x_1)} + \frac{x - x_3}{\rho_3(x_1 x_2 x_3 x_4) - \rho_1(x_1 x_2)} \cdots$$
$$+ \frac{x - x_{n-1}}{\rho_{n-1}(x_1 \ldots x_n) - \rho_{n-3}(x_1 \ldots x_{n-2})}$$
$$+ \frac{x - x_n}{\rho_n(x\, x_1 \ldots x_n) - \rho_{n-2}(x_1 x_2 \ldots x_{n-1})}. \quad \ldots \quad (B')$$

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* Communicated by the Author.

† Mathematical Tables Project, National Bureau of Standards.

Now Thiele's definition of the reciprocal difference $\rho_n(x_1 \dots x_{n+1})$, although it possesses the advantage of symmetry in the x_i's, is roundabout in its generation according to (A′) and in its application according to (B′). Also, in a sense, Thiele's reciprocal difference is not the continued fraction analogue of the divided difference, because the latter is the coefficient of $(x-x_1)\dots(x-x_n)$ in Newton's general divided difference expansion, and its analogue in a continued fraction ought to be a coefficient that is written as a single term which occurs in either the numerator or denominator of the nth partial quotient.

One can characterize Thiele's expansion by noting that it is a continued fraction of the form

$$\phi(x)=a_0+\frac{x-x_1}{a_1}+\frac{x-x_2}{a_2}+\dots \text{etc.}$$

where the a_n are determined so that $\phi(x_i)=f(x_i)$; in other words, $a_n=\rho_n(x_1x_2\dots x_{n+1})-\rho_{n-2}(x_1x_2\dots x_{n-1})$. It seems more natural to denote this a_n as the reciprocal difference, and to avoid confusion with Thiele's notation, a " capital ρ " or " P " will denote this new reciprocal difference, which, of course, will not be a symmetric function of the arguments. Here Thiele's $\rho_n(x_1\dots x_{n+1})-\rho_{n-2}(x_1\dots x_{n-1})$ is denoted by $\mathrm{P}_n(x_{n+1}\,x_n\dots x_1)$, which can be shown to satisfy the recursion relation

$$\mathrm{P}_n(x_{n+1}\,x_n\dots x_1)=\frac{x_{n+1}-x_n}{\mathrm{P}_{n-1}(x_{n+1}x_{n-1}\dots x_1)-\mathrm{P}_{n-1}(x_nx_{n-1}\dots x_1)}. \quad . \quad . \quad \text{(A)}$$

(Notice the fixed order of the x_i's, which is essential since $\mathrm{P}_n(x_{n+1}\dots x_1)$ is not symmetric.)

For $n=1$, (A) is obvious if $\mathrm{P}_0(x_1)\equiv f(x_1)$. For $n>1$, the right member of (A), from the definition of P_n, is equal to

$$\frac{x_{n+1}-x_n}{\rho_{n-1}(x_1\dots x_{n-1}x_{n+1})-\rho_{n-3}(x_1\dots x_{n-2})-\rho_{n-1}(x_1\dots x_{n-1}x_n)+\rho_{n-3}(x_1\dots x_{n-2})},$$

which, by the symmetry of $\rho_{n-1}(x_1\dots x_{n-1}\,x_{n+1})$, is equal to

$$\frac{x_{n+1}-x_n}{\rho_{n-1}(x_{n+1}\,x_1\dots x_{n-1})-\rho_{n-1}(x_1\dots x_{n-1}x_n)},$$ or, making use of (A′),

$\rho_n(x_{n+1}\,x_1\dots x_{n-1}\,x_n)-\rho_{n-2}(x_1\dots x_{n-1})$, which, by the symmetry of ρ_n, is equal to $\rho_n(x_1\dots x_{n-1}\,x_n\,x_{n+1})-\rho_{n-2}(x_1\dots x_{n-1})$, or $\mathrm{P}_n(x_{n+1}x_n\dots x_1)$, which is the left member of (A), *q.e.d.*

With this definition of $\mathrm{P}_n(x_{n+1}\dots x_1)$, Thiele's interpolation formula can be written as

$$f(x)=f(x_1)+\frac{x-x_1}{\mathrm{P}_1(x_2x_1)}+\frac{x-x_2}{\mathrm{P}_2(x_3x_2x_1)}+\frac{x-x_3}{\mathrm{P}_3(x_4x_3x_2x_1)}+\dots+\frac{x-x_{n-1}}{\mathrm{P}_{n-1}(x_n\dots x_1)}$$
$$+\frac{x-x_n}{\mathrm{P}_n(xx_n\dots x_1)}. \quad . \quad . \quad . \quad . \quad \text{(B)}$$

On comparison of (A′) and (B′) with (A) and (B), the saving of numerical work in using the latter form is quite obvious.

It is worth noting that, instead of starting with Thiele's definition of reciprocal differences and his interpolation formula (*i. e.* (A′) and (B′)), establishing the symmetry of his $\rho_n(x_1 \ldots x_{n+1})$ in all the x_i's and then proving that one could convert to the simpler forms (A) and (B), it is much easier to begin by defining $P_n(x_{n+1}x_n \ldots x_1)$ according to (A), starting with $P_0(x) \equiv f(x)$. From (A) one obtains the expansion (B) by an immediate induction. Then the rational function obtained by truncating the right member of (B) by omitting the final partial quotient $\dfrac{x-x_n}{P_n(xx_n \ldots x_1)}$, through the use of (A) in an obvious induction argument, can be shown to equal $f(x_i)$ for $x=x_i$, $i=1, 2, \ldots, n$.

To demonstrate by a numerical example the advantage of this new definition of reciprocal differences, consider the function $f(x)=\dfrac{x+1}{x^2+4}$ and the computation of its reciprocal differences according to both Thiele and this newer scheme: *

x	$f(x)$
$x_1=0$	1/4
$x_2=1$	2/5
$x_3=2$	3/8
$x_4=3$	4/13
$x_5=4$	1/4
$x_6=5$	6/29

Thiele's reciprocal differences.

First Order:

$$\rho_1(x_1x_2) = \frac{x_1-x_2}{f(x_1)-f(x_2)} = \frac{-1}{\frac{1}{4}-\frac{2}{5}} = \frac{20}{3}.$$

$$\rho_1(x_2x_3) = \frac{x_2-x_3}{f(x_2)-f(x_3)} = \frac{-1}{\frac{2}{5}-\frac{3}{8}} = -40.$$

$$\rho_1(x_3x_4) = \frac{x_3-x_4}{f(x_3)-f(x_4)} = \frac{-1}{\frac{3}{8}-\frac{4}{13}} = \frac{-104}{7}.$$

New definition of reciprocal differences.

First Order:

$$P_1(x_2x_1) = \frac{x_2-x_1}{f(x_2)-f(x_1)} = \frac{1}{\frac{2}{5}-\frac{1}{4}} = \frac{20}{3}.$$

$$P_1(x_3x_1) = \frac{x_3-x_1}{f(x_3)-f(x_1)} = \frac{2}{\frac{3}{8}-\frac{1}{4}} = 16.$$

$$P_1(x_4x_1) = \frac{x_4-x_1}{f(x_4)-f(x_1)} = \frac{3}{\frac{4}{13}-\frac{1}{4}} = 52.$$

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* It is only fair to point out that this newer scheme, unlike Thiele's, is not adaptable to interpolation for x within varying subintervals of the range; but such cases rarely occur in the use of reciprocal differences.

Thiele's reciprocal differences.

First Order (*cont.*):

$$\rho_1(x_4x_5) = \frac{x_4-x_5}{f(x_4)-f(x_5)} = \frac{-1}{\frac{4}{13}-\frac{1}{4}} = \frac{-52}{3}.$$

$$\rho_1(x_5x_6) = \frac{x_5-x_6}{f(x_5)-f(x_6)} = \frac{-1}{\frac{1}{4}-\frac{6}{29}} = \frac{-116}{5}.$$

Second Order:

$$\rho_2(x_1x_2x_3) = \frac{x_1-x_3}{\rho_1(x_1x_2)-\rho_1(x_2x_3)} + f(x_2) = \frac{-2}{\frac{20}{3}+40} + \frac{2}{5} = \frac{-3}{70} + \frac{2}{5} = \frac{5}{14}.$$

$$\rho_2(x_2x_3x_4) = \frac{x_2-x_4}{\rho_1(x_2x_3)-\rho_1(x_3x_4)} + f(x_3) = \frac{-2}{-40+\frac{104}{7}} + \frac{3}{8} = \frac{7}{88} + \frac{3}{8} = \frac{5}{11}.$$

$$\rho_2(x_3x_4x_5) = \frac{x_3-x_5}{\rho_1(x_3x_4)-\rho_1(x_4x_5)} + f(x_4) = \frac{-2}{\frac{-104}{7}+\frac{52}{3}} + \frac{4}{13} = \frac{-21}{26} + \frac{4}{13} = \frac{-1}{2}.$$

$$\rho_2(x_4x_5x_6) = \frac{x_4-x_6}{\rho_1(x_4x_5)-\rho_1(x_5x_6)} + f(x_5) = \frac{-2}{\frac{-52}{3}+\frac{116}{5}} + \frac{1}{4} = \frac{-15}{44} + \frac{1}{4} = \frac{-1}{11}.$$

New definition of reciprocal differences.

First Order (*cont.*):

$$P_1(x_5x_1) = \frac{x_5-x_1}{f(x_5)-f(x_1)} = \frac{4}{\frac{1}{4}-\frac{1}{4}} = \infty.$$

$$P_1(x_6x_1) = \frac{x_6-x_1}{f(x_6)-f(x_1)} = \frac{5}{\frac{6}{29}-\frac{1}{4}} = -116.$$

Second Order:

$$P_2(x_3x_2x_1) = \frac{x_3-x_2}{P_1(x_3x_1)-P_1(x_2x_1)} = \frac{1}{16-\frac{20}{3}} = \frac{3}{28}.$$

$$P_2(x_4x_2x_1) = \frac{x_4-x_2}{P_1(x_4x_1)-P_1(x_2x_1)} = \frac{2}{52-\frac{20}{3}} = \frac{3}{68}.$$

$$P_2(x_5x_2x_1) = \frac{x_5-x_2}{P_1(x_5x_1)-P_1(x_2x_1)} = \frac{3}{\infty-\frac{20}{3}} = 0.$$

$$P_2(x_6x_2x_1) = \frac{x_6-x_2}{P_1(x_6x_1)-P_1(x_2x_1)} = \frac{4}{-116-\frac{20}{3}} = \frac{-3}{92}.$$

Thiel's reciprocal differences.

Third Order:

$$\rho_3(x_1x_2x_3x_4)$$
$$=\frac{x_1-x_4}{\rho_2(x_1x_2x_3)-\rho_2(x_2x_3x_4)}$$
$$+\rho_1(x_2x_3)=\frac{-3}{\frac{5}{14}-\frac{5}{11}}-40$$
$$=\frac{154}{5}-40=\frac{-46}{5}.$$

$$\rho_3(x_2x_3x_4x_5)$$
$$=\frac{x_2-x_5}{\rho_2(x_2x_3x_4)-\rho_2(x_3x_4x_5)}$$
$$+\rho_1(x_3x_4)=\frac{-3}{\frac{5}{11}+\frac{1}{2}}-\frac{104}{7}$$
$$=-\frac{22}{7}-\frac{104}{7}=-18.$$

$$\rho_3(x_3x_4x_5x_6)$$
$$=\frac{x_3-x_6}{\rho_2(x_3x_4x_5)-\rho_2(x_4x_5x_6)}$$
$$+\rho_1(x_4x_5)=\frac{-3}{\frac{-1}{2}+\frac{1}{11}}-\frac{52}{3}$$
$$=-10.$$

New definitions of reciprocal differences.

Third Order:

$$P_3(x_4x_3x_2x_1)$$
$$=\frac{x_4-x_3}{P_2(x_4x_2x_1)-P_2(x_3x_2x_1)}$$
$$=\frac{1}{\frac{3}{68}-\frac{3}{28}}=\frac{-238}{15}.$$

$$P_3(x_5x_3x_2x_1)$$
$$=\frac{x_5-x_3}{P_2(x_5x_2x_1)-P_2(x_3x_2x_1)}$$
$$=\frac{2}{0-\frac{3}{28}}=\frac{-56}{3}.$$

$$P_3(x_6x_3x_2x_1)$$
$$=\frac{x_6-x_3}{P_2(x_6x_2x_1)-P_2(x_3x_2x_1)}$$
$$=\frac{3}{\frac{-3}{92}-\frac{3}{28}}=\frac{-322}{15}.$$

Fourth Order:

$$\rho_4(x_1x_2x_3x_4x_5)$$
$$=\frac{x_1-x_5}{\rho_3(x_1x_2x_3x_4)-\rho_3(x_2x_3x_4x_5)}$$
$$+\rho_2(x_2x_3x_4)=\frac{-4}{\frac{-46}{5}+18}+\frac{5}{11}$$
$$=\frac{-5}{11}+\frac{5}{11}=0.$$

Fourth Order:

$$P_4(x_5x_4x_3x_2x_1)$$
$$=\frac{x_5-x_4}{P_3(x_5x_3x_2x_1)-P_3(x_4x_3x_2x_1)}$$
$$=\frac{1}{-\frac{56}{3}+\frac{238}{15}}=\frac{-5}{14}.$$

Thiele's reciprocal differences.

Fourth Order (*cont.*)

$$\rho_4(x_2x_3x_4x_5x_6)=\frac{x_2-x_6}{\rho_3(x_2x_3x_4x_5)-\rho_3(x_3x_4x_5x_6)}+\rho_2(x_3x_4x_5)=\frac{-4}{-18+10}-\frac{1}{2}=\frac{1}{2}-\frac{1}{2}=0.$$

Then substituting in Thiele's interpolation formula (B′):

$$f(x)=\frac{1}{4}+\frac{x}{\frac{20}{3}}+\frac{x-1}{\frac{5}{14}-\frac{1}{4}}+\frac{x-2}{\frac{-46}{5}-\frac{20}{3}}+\frac{x-3}{0-\frac{5}{14}}.$$

which becomes

$$f(x)=\frac{1}{4}+\frac{x}{\frac{20}{3}}+\frac{x-1}{\frac{3}{28}}+\frac{x-2}{\frac{-238}{15}}+\frac{x-3}{\frac{-5}{14}}.$$

New definition of reciprocal differences.

Fourth Order (*cont*).

$$P_4(x_6x_4x_3x_2x_1)=\frac{x_6-x_4}{P_3(x_6x_3x_2x_1)-P_3(x_4x_3x_2x_1)}=\frac{2}{-\frac{322}{15}+\frac{238}{15}}=\frac{-5}{14}.$$

Substituting in (B), one obtains at once:

$$\rho(x)=\frac{1}{4}+\frac{x}{\frac{20}{3}}+\frac{x-1}{\frac{3}{28}}+\frac{x-2}{\frac{-238}{15}}+\frac{x-3}{\frac{-5}{14}}.$$

As an illustration of the value of this simpler and more direct approach to the interpolation of rational functions, consider the question of showing that the reciprocal difference (new definition) of a certain order of any rational function, is a constant. Let $\frac{A_m(x)}{B_n(x)}$ denote a particular $P_i(xx_i\ldots x_1)$ which, of course, may be $f(x)$ itself. Here $A_m(x)$ and $B_n(x)$ denote polynomials of degrees m and n respectively. Then

$$P_{i+1}(xx_{i+1}x_i\ldots x_1)=\frac{x-x_{i+1}}{P_i(xx_i\ldots x_1)-P_i(x_{i+1}x_i\ldots x_1)}=\frac{x-x_{i+1}}{\frac{A_m(x)}{B_n(x)}-\frac{A_m(x_{i+1})}{B_n(x_{i+1})}}$$

$$=\frac{B_n(x)B_n(x_{i+1})(x-x_{i+1})}{A_m(x)B_n(x_{i+1})-A_m(x_{i+1})B_n(x)}.$$

The denominator of the last expression contains $x-x_{i+1}$ as a factor, so after the cancellation it will be of degree $m-1$ or $n-1$, whichever is larger. Thus if $\frac{A_m(x)}{B_n(x)}$ is written as the quotient of two polynomials of equal degree M, where M is the larger of m or n (by adding terms with 0 coefficients if necessary), taking the reciprocal differences will successively yield rational expressions of degrees M over M$-$1 and then M$-$1 over M$-$1. Hence no more than 2M steps will yield a constant. Now a similar argument for Thiele's ρ_n works only if we make use of the fact that

$$\rho_n(x_1 \dots x_n\; x_{n+1}) = \rho_n(x_{n+1}\; x_1 \dots x_n).$$

That last relation is a consequence of the symmetry of his $\rho_n(x_1 \dots x_{n+1})$ in all the x_i's. But to prove the symmetry requires a considerable amount of exertion. Also, a direct proof that the Thiele reciprocal difference of a certain order of a rational function is constant, may be had by troublesome transformations of the explicit determinant form of ρ_n, which is rather involved.

In the completely confluent case, where all the x_i's approach x_1, it follows immediately from (A) that

$$\underbrace{P_n(x_1x_1 \dots x_1)}_{n+1\text{ times}} = \lim_{x\to x_1} \frac{x-x_1}{\underbrace{P_{n-1}(xx_1 \dots x_1)}_{n\text{ times}} - \underbrace{P_{n-1}(x_1x_1 \dots x_1)}_{n\text{ times}}}$$

$$= \left.\frac{1}{\frac{\partial}{\partial x}\underbrace{P_{n-1}(xx_1 \dots x_1)}_{n\text{ times}}}\right|_{x=x_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\bar{A})$$

The use of $(\bar{A})$ as a scheme to obtain $\underbrace{P_n(x_1x_1 \dots x_1)}_{n+1\text{ times}}$ for successive values of n, requires the knowledge of $\underbrace{P_{n-1}(xx_1 \dots x_1)}_{n\text{ times}}$. But from (A), any

$$\underbrace{P_i(xx_1 \dots x_1)}_{i+1\text{ times}} = \frac{x-x_1}{\underbrace{P_{i-1}(xx_1 \dots x_1)}_{i\text{ times}} - \underbrace{P_{i-1}(x_1x_1 \dots x_1)}_{i\text{ times}}}.$$

Thus to apply $(\bar{A})$, assuming that everything is known up to P_{i-1}, one finds alternately $\underbrace{P_i(x_1 \dots x_1)}_{i+1\text{ times}}$ from $(\bar{A})$ and $\underbrace{P_i(xx_1 \dots x_1)}_{i+1\text{ times}}$ from (A), then again $\underbrace{P_{i+1}(x_1 \dots x_1)}_{i+2\text{ times}}$ and $\underbrace{P_{i+1}(xx_1 \dots x_1)}_{i+2\text{ times}}$, etc. In the confluent form of (B), which is Thiele's continued fraction expansion analogous to Taylor's theorem, given by

$$f(x) = f(x_1) + \frac{x-x_1}{P_1(x_1x_1)} + \frac{x-x_1}{P_2(x_1x_1x_1)} + \frac{x-x_1}{P_3(x_1x_1x_1x_1)} + \dots$$

$$\dots + \frac{x-x_1}{\underbrace{P_{n-1}(x_1 \dots x_1)}_{n\text{ times}}} + \frac{x-x_1}{\underbrace{P_n(xx_1 \dots x_1)}_{n+1\text{ times}}}, \quad . \quad . \quad . \quad (\bar{B})$$

it is necessary to go as far as $P_{n-1}(\underbrace{x_1 \dots x_1}_{n\text{ times}})$ and $P_{n-1}(\underbrace{xx_1 \dots x_1}_{n\text{ times}})$, and finally to obtain $P_n(\underbrace{xx_1 \dots x_1}_{n+1\text{ times}})$. Omission of the last partial quotient to obtain a rational approximation to $f(x)$, makes the calculation of $P_{n-1}(\underbrace{xx_1 \dots x_1}_{n\text{ times}})$ and $P_n(\underbrace{xx_1 \dots x_1}_{n+1\text{ times}})$ both unnecessary.

In practice $(\bar{\text{A}})$ and $(\bar{\text{B}})$ are somewhat simpler to apply than Thiele's reciprocal derivatives $r_nf(x)$ which are generated by the recursion formula

$$r_nf(x)=r_{n-2}f(x)+nrr_{n-1}f(x), \quad \dots\dots \quad (\bar{\text{A}}')$$

where the operator " r " denotes the reciprocal of the ordinary derivative. In $(\bar{\text{B}})$, $P_n(\underbrace{x_1 \dots x_1}_{n+1\text{ times}})$ is identical with Thiele's $nrr_{n-1}f(x_1)$, but the former is obtained from $(\bar{\text{A}})$ in a more direct manner than the latter is obtained from $(\bar{\text{A}}')$,

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LXXIX. *Resolution of Devices Actuated by Random Events.*

By P. G. Klemens, B.Sc.,
Commonwealth Research Student, University of Sydney *.

[Received July 27, 1947.]

1. *Introduction.*

If a device dealing with a series of events can only deal with each event when there is a minimum interval between successive events, it is said that this device has a resolving time. This resolving time is connected with the time that this device requires to return to the equilibrium position after it has been actuated by the event.

Such devices can be pulse recorders, scaling circuits and other triggered electronic circuits, or detecting devices such as Geiger Counters.

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* Communicated by Prof. V. A. Bailey.

The simple statistical theory * usually employed in such problems is based on the hypothesis that there exists an interval τ, which must elapse after an event—whether recorded or not—before the following event can be recorded.

The probability that n events occur in time t is given by a Poisson distribution

$$P_n = \frac{(Nt)^n}{n!\,e^{Nt}},$$

and hence the probability of finding an interval between two successive events to be between t_1 and t_2 is given by

$$P_{1,2} = e^{-Nt_1} - e^{-Nt_2} \qquad t_2 > t_1,$$

where N is the number of events per unit time.

Hence the fraction of events following with an interval between 0 and τ is given by

$$P = 1 - e^{-N\tau}, \quad \ldots\ldots\ldots \quad (1)$$

and this equals the fraction of events which fail to actuate, *i. e.* the fraction of events lost.

If N′ is the number of events "recorded,"

$$N' = N(1-P) = Ne^{-N\tau} \quad \ldots\ldots\ldots \quad (2)$$

and $dN'/dN = 0$ if $N\tau = 1$. Thus the number of events recorded increases with N in a non-linear way, till it reaches a maximum, and then it actually decreases with increasing N. This is known as "jamming." The maximum value of N′ is $1/\tau e$. Plotting log N/N′ against N, a straight line is obtained and τ is determined from the slope.

2. *Generalization of the Theory.*

This theory is not always adequate. It is accordingly proposed to investigate here those devices, whose behaviour may be described as follows :—

There exists an interval τ_1 which must elapse (without any further event arriving) after an event has occurred and been recorded, before a following event can be recorded ; but if an event has been lost, a time τ_2 must elapse (without any further events occurring) before an event can again be recorded. The time τ_2 may be larger or smaller than τ_1, so that, if an event is lost, the ensuing recovery time is either lengthened or shortened. If τ_2 equals τ_1, the device behaves as if there was a single resolving time τ as in the previous section.

Case 1 : Suppose, firstly, that $\tau_1 < \tau_2$. Let P be the probability that a single event is lost. Then an individual event will be lost if (*a*) it follows with an interval less than τ_1, or (*b*) it follows with an interval between τ_1 and τ_2, the previous event having been lost.

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* Volz, *Zeit. f. Physik*, xciii. p. 539 (1935). Strong, 'Procedures in Experimental Physics,' p. 298, Prentice-Hall (1943).

These two cases are mutually exclusive, hence the probability that a single event is lost is the sum of two probabilities :

$$P=(1-e^{-N\tau_1})+P(e^{-N\tau_1}-e^{-N\tau_2}), \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

and thus

$$P=\frac{1-e^{-N\tau_1}}{1-e^{-N\tau_1}+e^{-N\tau_2}}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

In the special case when $\tau_1=\tau_2$ this becomes the original expression (1) for P. The number of events recorded, N′, is now given by

$$N'=N\frac{e^{-N\tau_2}}{1-e^{-N\tau_1}+e^{-N\tau_2}}. \quad . \; . \; . \; . \; . \; . \; . \quad (5)$$

To obtain the maximum value of N′ we differentiate with respect to N and equate to zero, obtaining

$$\frac{dN'}{dN}\propto E\equiv[N(\tau_2-\tau_1)-1]e^{-N\tau_1}-(\tau_2N-1)+e^{-N\tau_2}=0, \quad . \; . \quad (6)$$

which is a transcendental equation in N.

To obtain τ_1 and τ_2 from the observed relation between N and N′ we proceed as follows :—

From (5), $$\log\frac{N'}{N}=-N\tau_2-\log(1-e^{-N\tau_1}+e^{-N\tau_2}), \quad . \; . \; . \; . \; . \quad (7)$$

$$\frac{d}{dN}\log\frac{N'}{N}=-\tau_2-\frac{\tau_1e^{-N\tau_1}-\tau_2e^{-N\tau_2}}{1-e^{-N\tau_1}+e^{-N\tau_2}}. \quad . \; . \; . \; . \; . \quad (8)$$

When $N\rightarrow 0$, $$e^{-N\tau_1}\rightarrow 1, \qquad e^{-N\tau_2}\rightarrow 1,$$

$$\frac{d}{dN}\log\frac{N'}{N}\rightarrow-\tau_2-\tau_1+\tau_2=-\tau_1,$$

hence τ_1 can be determined from the initial slope of the curve relating log N/N′ to N.

This slope will then increase with N, till for large values it will be τ_2. In practice, however, it may not be possible to measure N′ at a sufficiently high value, so that the following relation can be used to find τ_2 by trial or by successive approximation as shown below :

$$\tau_2=-\frac{1}{N}\log\frac{N'}{N}-\frac{1}{N}\log(1-e^{-N\tau_1}+e^{-N\tau_2}). \quad . \; . \; . \; . \quad (7\,a)$$

For a given pair of values N′ and N and given τ_1, a trial value of τ_2 is substituted into the right side and a better value for τ_2 is found. This process can be repeated. The approximation converges * because the

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* For if we consider a sequence of numbers defined by $z_{n+1}=F(z_n)$ and we let $z_{n+1}\rightarrow\zeta$ as $n\rightarrow\infty$ so that

$$z_{n+1}+\epsilon_{n+1}=\zeta=z_n+\epsilon_n,$$

then, for sufficiently small ϵ_n,

$$\epsilon_{n+1}\doteqdot\epsilon_nF'(\zeta), \qquad \text{since } F(\zeta)=\zeta.$$

In order that $|\epsilon_{n+1}|<|\epsilon_n|$ it is necessary that $|F'(\zeta)|<1$, differential of the right-hand side with respect to τ_2 is

$$-\frac{1}{N}\frac{Ne^{-N\tau_2}}{1-e^{-N\tau_1}+e^{-N\tau_2}},$$

whose modulus is always less than unity.

For given values of τ_1 and τ_2, the maximum value of N′ can be obtained by substituting trial values for N into the expression

$$E\equiv[N(\tau_2-\tau_1)-1]e^{-N\tau_1}-(\tau_2 N-1)+e^{-N\tau_2}, \quad . \quad . \quad (6\,a)$$

until a value is obtained for which E=0. Here sufficient accuracy is usually obtained without iteration. Since dE/dN can be shown to be always negative, E is positive if the trial value for N is taken too low and *vice versa*. Given N_{max}, the corresponding N′ is found.

Case 2: Similar considerations hold in the case $\tau_1>\tau_2$, when the resolving time is shortened if the previous event has been lost. An event is then lost if (*a*) it follows with an interval less than τ_2, or (*b*) it follows with an interval between τ_2 and τ_1, the previous event not having been lost.

This leads to the equation

$$P=(1-e^{-N\tau_2})+(1-P)(e^{-N\tau_2}-e^{-N\tau_1}), \quad . \quad . \quad . \quad . \quad (9)$$

whence

$$P=\frac{1-e^{-N\tau_1}}{1-e^{-N\tau_1}+e^{-N\tau_2}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

This is the same result as in the previous case (4), except that $\tau_1>\tau_2$. All above conclusions and formulæ hold with appropriate modifications, save that dE/dN can be either positive or negative.

3. *Application of the Present Theory.*

The hypothesis of lengthened resolving time was applied to a mechanical pulse recorder of the Australian P.M.G. (Telephone Exchange) type.

Random pulses were obtained from a Geiger Counter and a 4·2-mc. Ra source in a lead container. By varying the distance between source and counter, N was obtained from the inverse square law, making use of the fact that for large distances N′=N. Counts were taken over sufficiently long times to eliminate statistical fluctuations.

With the dependence of N′ on N determined, it was found that the simple theory was satisfactory only for values of N up to 400 per min., giving a resolving time $\tau=1{\cdot}9\times10^{-4}$ min. For higher values of N, N′ was lower than (2) would predict, and also N_{max} was found to be at 1240 instead of 5300 per min. as indicated by $N\tau=1$.

Applying the present theory (case 1) with $\tau_1=1{\cdot}9\times10^{-4}$ min. and $\tau_2=10{\cdot}2\times10^{-4}$ min., the following values were obtained :—

N	300	700	1000	1300
N′ calc.	278	560	625	720
N′ obs.	286	590	684	708
N′ calc. by simple theory	284	615	582	1020

It can be seen that formula (5) gives, in general, a better fit than does formula (2).

The improvement is even more striking when the maximum is determined. The present theory yields a maximum value N′=720, corresponding to N=1300, which makes E=0·00±0·01. This compares well with an observed maximum value N′=710, when N=1240. On the other hand, the simple theory applied to the same data gives N′=1900 as maximum, when N=5300.

Thus the present theory (case 1) gives a much better representation of counting losses in this particular mechanical recorder.

The measurements referred to were carried out by Mr. G. K. White, in connection with work on Geiger Counters in progress at the Department of Physics, University of Sydney.

Acknowledgment.

The writer wishes to express his thanks to Professor V. A. Bailey and to Dr. R. E. B. Makinson for making valuable suggestions.

LXXX—*Notices respecting New Books.*

Note Sur un Problème de Sondage. By J. Bouzitat. [Pp. 57+6 folding charts.] (Office National d'Études et de Recherches Aéronautiques, Paris, 1947.) 300 francs.

M. Bouzitat has taken the problem of sampling when a random sample from a finite population has been examined, and the properties of the population are to be deduced from the sample. With the aid of methods much less refined than those used by Wald in the United States for the same study, he obtains a number of results agreeing with his ; the French work was evidently carried out independently of the American, and in ignorance of the fact that the ground had already been covered.

The most useful result which the author reaches is the following : if the population contains A members of category 1, B of category 2, and so on (total N) and if, of n members examined, a are of category 1, b of category 2 and so on, then, there being ν categories in all, we may take $\frac{a+1}{n+\nu}$ as an estimate of $\frac{A+1}{N+\nu}$. Thus, if an urn might contain balls of 5 colours, and 3 drawings have all given white, the probability of all N being white is one-half. J. H. A.

Introduction to the Theory of Equations. By LOIS WILFRED GRIFFITHS. (Second Edition.) [Pp. 278.] (London: Chapman & Hall, 1947.) Price 18*s.* 0*d.*

THE preface states: "In this book students are guided slowly through the proofs of the important theorems in the elementary theory of algebraic equations. . . . . Development from the particular to the general is an outstanding feature."

On the whole, these aims are faithfully, at times tediously, carried out, but it is disappointing to find, in a book of this size and at this price, that some of the important theorems are not proved at all. For instance, Descartes's rule of signs is stated and applied in discussing some numerical equations, the reader being referred to the "long but not difficult" proofs given in other text-books.

It seems likely that the reader who required to be guided so slowly would be bewildered by Chapter I on Binomial Equations. The quadratic formula is not established nor even stated, but is applied to obtain the complex cube roots of unity, the number i being nowhere defined. In the next paragraph appears the statement: "By the quadratic formula the roots of $z^2+1=0$ are i and $-i$." Although geometric representation of complex numbers is considered, there is no mention of the representation of i itself.

The other chapters treat cubics and quartics, Sturm's theorem (well done), determinants, and symmetric functions of the roots. The fundamental theorem of algebra is stated without proof. A good feature is the presence of so many exercises for the student.

The paper is admirable and the printing good; but it is a pity that in so many places formulæ are broken and made to occupy more than one line.

S. L. GREEN.

Mechanical Admittances and their Applications to Oscillation Problems. By Professor W. J. DUNCAN. (His Majesty's Stationery Office, R. & M. No. 2000, 1947.) Price 22*s.* 6*d.*

IN this monograph the concept of admittance, widely used in electrical engineering, is applied to solve mechanical problems which give rise to differential equations analogous to those that occur in certain electrical problems. The mechanical admittance is defined, in the first instance, as the ratio of the amplitude of the displacement of a point in the oscillating system to that of the (simple harmonic) force which causes the displacement. This concept is then generalized, admittances and cross-admittances being, in general, complex numbers. Thus is developed a powerful and elegant method of treating oscillation problems.

The first three chapters introduce the method which is then used in the discussion of continuous deformable bodies (such as shafts, beams and cantilevers) and gyrostatic, aerodynamic and engine-airscrew systems. One chapter deals with the experimental determination of admittances. A valuable feature of the book is the inclusion of tables of admittances for various systems. Some numerical examples are included.

A knowledge of only the elementary functions is required of the reader. The matrix notation is introduced in some paragraphs, but these may be omitted without serious loss. The argument throughout is admirably clear and there is no doubt that the book will be invaluable to researchers in a field in which the author is so distinguished a worker. S. L. GREEN.

Cables and Transmissions, No. 2. (King's Crown Press: London, Geoffrey Cumberlege, 1948.)

THIS is a French publication, somewhat akin to the British Post Office Engineers' Journal, but sponsored by L'Administration des Postes, Télégraphs et Téléphones, in co-operation with four French firms manufacturing telecommunication equipments.

The paper is published at three-monthly intervals and the present issue contains both theoretical and practical articles (in the French language, of course), dealing mainly with cable telegraphy and telephony.

The two theoretical articles deal with the synthesis of filter units (dipoles) for given frequency characteristics and the testing of co-axial cables for uniformity by means of square wave pulses and the reflected wave-form as observed in a cathode ray oscillograph. Both articles are interesting and advance their subjects. A good knowledge of those branches of modern applied mathematics applicable to these subjects is pre-supposed.

The four descriptive articles relate to the jointing of co-axial cables; experimental measurements on recently installed 12-channel carrier cables; the possible methods of providing for the very rapid starting of Diesel-electric generators in the event of failure of the normal supply to repeater stations; and the laying of a loaded telegraph and telephone cable along the electrified railway line between Nimes and Sète (Cette). These articles are of interest to British Telecommunications engineers as some of the methods are novel.

The production of the paper is in characteristic French style; excellent type, clear diagrams, no apparent misprints and—what to many will be a real relief—no advertisements. C. L. F.

Radon: Its Technique and Use. By W. A. JENNINGS and S. RUSS. [Pp. x+222.] (Published for the Middlesex Hospital Press by John Murray, Albemarle Street, London, W. 1.) Price 18*s.* net.

THE radioactive gas radon, the "emanation" of radium, is widely used in medical work as a source of radiation, which, in a suitable container, can be inserted into the body of a patient and left there. Its half-life is only a few days, so that in some circumstances, it may even be left permanently, and will within a reasonable time decay to a negligible activity. Radon is continuously being formed from radium, and is "milked" off at suitable intervals, after which it must soon be used and then becomes inactive and valueless.

The authors, both of whom have experience as hospital physicists, describe for the benefit of their fellow-workers the techniques required in handling and measuring this gas, from the separation from its parent radium to its sealing up in "seeds." They keep rigidly to their subject, and do not discuss other elements like cobalt, carbon or iodine whose radioactive isotopes seem likely to become extensively used in medicine. It is of interest to note (not by way of criticism, but as throwing light on the conditions and needs of hospital work) that in hospital practice, the main instrument is still the gold-leaf electroscope, and that little use is apparently made of scalers and counters; not much attention appears to be given either to methods of measurement which distinguish between gamma rays of different energies.

The book contains a number of tables and graphs which must be of very great value to practitioners who have to deal regularly with radon, and to them it will be well worth its quite moderate price. J. H. A.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

Zirconium Hydride: We are informed that Messrs Murex Ltd., of Rainham, Essex, are now producing Zr. Hyd. in powder form.

LXXXI. *Frequency Analysis of Modulated Pulses.*

By S. H. Moss, B.A., B.Sc.*

[Received December 3, 1946.]

List of Symbols.

n	integral parameter, defining order of pulse in time.
a_n	amplitude of nth pulse.
t_n	epoch of nth pulse.
m	integral parameter, defining one of denumerable set of frequency components (except in section 2.2.2 and section 4).
ω_m	mth frequency component.
$f_0(t)$	individual pulse waveform.
$\delta(t)$	unit improper pulse.
$\mathrm{F}(t)$	recurrent set of pulses of type $f_0(t)$, amplitude and phase modulated.
$\mathrm{E}(t)$	recurrent set of pulses of type $\delta(t)$, with same modulation as $\mathrm{F}(t)$.
$\triangle(t)$	recurrent set of pulses of type $\delta(t)$, unmodulated.
$\mathrm{G}(\omega_m)$	coefficient of $e^{j\omega_m t}$ in $\mathrm{F}(t)$, (for denumerable modulations).
$g(\omega)$	coefficient of $e^{j\omega t}.\,d\omega/2\pi$ in $\mathrm{F}(t)$, (for continuous modulations).
$\Gamma(\omega_m)$	modulation factor of $\mathrm{G}(\omega_m)$.
$\gamma(\omega)$	modulation factor of $g(\omega)$.
$g_0(\omega)$	pulse shape factor of $\mathrm{G}(\omega)$ or of $g(w)$, (for pulse $f_0(t)$).
$f(t)$	modulation waveform, or signal waveform.
$\bar{f}(t)$	signal waveform.
r	reference time variable.
s	scanning time variable.
t	physical time variable.
$s(t)$	scanning waveform (for weighting signal or modulation function $f(t)$).
$\hat{f}(t)$	equivalent point-scanned signal or modulation function.
$\sigma(\omega)$	scanning spectral factor, imposed on $f(t)$ giving $\hat{f}(t)$.
m	(section 2.2.2) reference gradient of auxiliary scanning sawtooth waveform, for time-modulated scanning points s_n.
k	reference upper excursion limit of auxiliary sawtooth waveform.
Ω	recurrence (radian) frequency of unmodulated pulses.
T	recurrence period.
ρ	depth of amplitude modulation.
ϵ	depth of time modulation of physical pulse (relative to reference pulse).
η	depth of time modulation of scanning pulse (relative to reference pulse).
ζ	depth of time modulation of physical and scanning pulses relative to each other.
θ	phase of reference vector
ϕ	phase of scanning vector (except in section 4).

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* Communicated by the Author.

ψ	phase of physical vector.
p	integral parameter, defining order of carrier frequency component $e^{jp\Omega t}$.
q	integral parameter defining one of the combination tones of a denumerable set of modulation tones.
μ_q	qth intermodulation frequency component (for denumerable modulation).
μ	typical intermodulation frequency component (for continuous modulations except in section 4).
$\Gamma_q(p\,\Omega, \mu_q)$	coefficient of $e^{j(p\Omega+\mu_q)t}$ in discrete sideband about pth carrier.
$\gamma_\mu(p\,\Omega,\mu)$	coefficient of $e^{j(p\Omega+\mu)t}.\ d\mu/2\pi$ in continuous sideband about pth carrier.
$E_p(t)$	modulated pth carrier component of $E(t)$.
$C_q(x)$, $D_q(x)$	denumerable sequences of functions related only to $f(t)$ (equation (56)).
$d_\mu(x)$	continuous sequence of function related only to $f(t)$ (equation (56′)).
$J_q(x)$	Bessel function of 1st kind of order q.
$h(t;\ u_n, v_n)$	rectangular pulse of unit amplitude, between $t=u_n$ and $t=v_n$.
$k(t;\ u_n, v_n)$	pulse with edges of general shape and epochs $t=u_n$ and $t=v_n$.
$f_1(t)$	growth (or decay) waveform of pulse edge $\left(=\int_{-\infty}^{t} f_0(t)\,dt\right)$.
$Z(t)$	recurrent set of width modulated pulses of type $h(t;\ u_n, v_n)$.
a	constant time shift of physical pulses at $t=t_n$.
b	constant time shift of scanning pulses at $t=s_n$.
c	constant relative time shift of physical and scanning pulses.
F_r, $\Omega/2\pi$	recurrence frequency $(=1/T)$.
ϕ	depth of phase modulation (section 4).
f_d, $\omega_d/2\pi$	depth of frequency modulation.
f_μ, $\mu/2\pi$	modulation frequency (section 4).
m	(section 4) ratio of modulating to recurrence frequency, $=\dfrac{\mu}{\Omega}=\dfrac{f_\mu}{F_r}$.

SUMMARY.

The recent development of decimetric and centimetric valves for radiolocation purposes has stimulated interest in the possibility of extending pulse techniques to communication applications. In the design of pulse communication systems a knowledge of the carrier and sideband frequency structure of modulated recurrent pulses of different types is necessary to complete the information presented by a direct inspection of the time waveforms themselves. Not only is any relative distortion between the fundamental modulating tones of interest but also the degree of intermodulation between them when the pulses are demodulated.

In this paper, only the abstract analytical problem is considered, although the physical parallel to the various mathematical situations described will be recognized by those versed in the art. The analysis is developed in a general way for pulses of arbitrary shape modulated in amplitude and time. Introducing a spectral shape-factor for each shape of pulse, and making use of Heaviside's and Dirac's improper impulse function, $\delta(t)$, it is possible to concentrate the development on the essential features of each type of modulation. The virtue of using indefinitely narrow pulses is that time modulation (T.M.) can be studied

by deforming the time base (*i. e.* transforming the time variable), and amplitude modulation by a simple weighting function, since the shape distortion produced by these imaginary processes then becomes immaterial.

The relation between the signal on the one hand, and the modulation values applied to the pulses on the other, requires careful definition; and, in particular, where T.M. exists, the distinction between periodic and synchronous point scanning is made. In general, the modulating waveform is scanned (*i. e.* each modulating value is recorded) over a finite time interval, but in most practical cases this is analytically reducible to a point scanning operation, often by the device of a spectral scanning-factor for the modulating function (analogous to the two dimensional aperture-factor of television scanning theory).

General expressions are obtained for various types of modulated pulses, and, in the final section, some particular forms for single-tone and double-tone modulations are given.

(1) INTRODUCTION.

THE sets of pulses which are the subject of the following analysis are combinations of specified functions, f_0, of the time t, each of the form

$$y=k\,.\,f_0(t\,;\,\lambda_1,\lambda_2,\ldots,\lambda_N), \quad \ldots\ldots \quad (1)$$

where y is some physical quantity such as current or voltage, whose dimensions and scale, covered by the constant k, do not concern us, and where λ_1, λ_2, . . . , λ_N, are a *finite* number of parameters, such "amplitude," "epoch," "width," etc., which fully define the position and shape of the pulse. A set of pulses is said to be modulated when one or more of these constants differ in value from one pulse to another. These modulation values will generally be derived from some given signal function, $f(t)$. The case when the parameters are independently modulated by different signal functions will not be considered.

Another type of pulse modulation, not considered here, has a denumerable or continuous infinity of modulation parameters. This occurs for example when the resultant waveform, $g(t)$, differs from the signal waveform, $f(t)$, only by a "switching function" factor, $S(t)$,

$$g(t)=f(t)\,.\,S(t), \quad \ldots\ldots \quad (2)$$

where $S(t)$ is, *e. g.* unity within some set of time intervals (a_n, b_n), and zero elsewhere; or more generally when, for example,

$$g(t)=f(u)\,.\,S(u), \quad \ldots\ldots \quad (3)$$

where u is a transformed time variable which enables $g(t)$ to represent a set of pulses whose individual shapes are continuously deformable not only in amplitude but in duration and time of occurrence.

The following sections deal first with pulses having at most two modulation parameters, for a general signal function, and finally, with a few special cases for single and double tone modulations.

In a practical system, various time delays will occur between a signal waveform and the pulses which it eventually modulates at various points; but in the analysis, corresponding intervals on the signal and pulse waveforms will be treated as contemporary, a constant relative time shift between the two waveforms being considered irrelevant. Thus, there is no inconsistancy when, for example, a pulse appears to be modulated by a value of the signal occurring subsequently.

(2) AMPLITUDE AND TIME MODULATED PULSES.

(2.1) *Pulses of Arbitrary Shape.*

Section (2) is concerned only with sets of recurrent pulses of some fixed but arbitrary transient shape. The individual pulse waveforms only differ from each other in their relative amplitudes and in the times of their occurrence. If $f_0(t)$ denotes the individual pulse waveform measured at time t from some reference instant, which may be called the "epoch" of the pulse, and at some reference amplitude, then a set of recurrent pulses, $F(t)$, may be written

$$F(t)=\sum_{n=-\infty}^{\infty} a_n \,.\, f_0(t-t_n), \quad . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

where a_n is the relative amplitude and t_n the epoch of the nth pulse.

If $F(t)$ can be resolved into a denumerable set of discrete frequency components, say,

$$F(t)=\sum_{m=-\infty}^{\infty} G(\omega_m) \,.\, e^{j\omega_m t}, \quad . \; . \; . \; . \; . \; . \; . \quad (5)$$

then the factor for a typical component, $e^{j\omega_m t}$, is (as is formally evident),

$$G(\omega_m)=\lim_{c\to\infty} \frac{1}{2c}\int_{-c}^{c} F(t) \,.\, e^{-j\omega_m t} \,.\, dt, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

$$=\lim_{c\to\infty} \frac{1}{2c}\left\{\sum_n a_n e^{-j\omega_m t_n}\int_{-c}^{c} f_0(t-t_n) \,.\, e^{-j\omega_m(t-t_n)} \,.\, dt\right\}, \quad . \; . \quad (7)$$

i. e.

$$G(\omega_m)=g_0(\omega_m) \,.\, \Gamma(\omega_m), \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

where

$$\Gamma(\omega_m)=\lim_{c\to\infty} \frac{1}{2c}\left\{\sum_{(|t_n|<c)} a_n e^{-j\omega_m t_n}\right\}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

and

$$g_0(\omega)=\int_{-\infty}^{\infty} f_0(t) \,.\, e^{-j\omega t} \,.\, dt. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (10)$$

Thus each spectral term, $G(\omega_m)$, can be expressed as the product of two factors. One of these factors, $\Gamma(\omega_m)$, which may be called the "modulation factor," is independent of the reference pulse waveform, $f_0(t)$, but depends only upon the amplitudes and epochs, a_n, t_n, of the recurrent pulses. The other factor, $g_0(\omega_m)$, which may be called the "pulse shape factor," depends only upon the spectral density of the reference transient, $f_0(t)$, at the frequency $\omega=\omega_m$ in question, since by inverting the Fourier relation (10),

$$f_0(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} g_0(\omega)\,.\,e^{j\omega t}\,.\,d\omega. \qquad (11)$$

Thus, in resolving the recurrent pulse waveform into its frequency components it is sufficient to consider pulses of some standard shape, and to correct each term subsequently by the appropriate pulse shape factor ratio. A suitable standard pulse would be one for which $g_0(\omega)=1$ at all frequencies, but this does not correspond to any attainable pulse waveform. For any finite frequency range, however, this result is approached by any sequence of pulse waveforms, $\{\delta_s t()\}$, whose durations become indefinitely small, and whose amplitudes at the same time become large in such a way that their areas remain finite, approaching unity (*i. e.* unit zero'th moment) whilst all the higher temporal moments about the pulse tend to zero, *i. e.*

$$\lim_{s\to\infty}\int_{-\infty}^{\infty} t^r\,.\,\delta_s(t)\,.\,dt \doteq \begin{cases} 1, & (r=0), \\ 0, & (r\geqslant 1). \end{cases} \qquad (12)$$

It is convenient to write as if such a sequence possessed a limiting function $\delta(t)$, (see fig. 3(0), (*a*)). Then, when the same amplitude and time modulation is applied to such a standard pulse, the set of recurrent pulses becomes formally, from equations (4), (5) and (8),

$$\mathrm{E}(t)=\sum_n a_n\,.\,\delta(t-t_n), \qquad (13)$$

$$\mathrm{E}(t)=\sum_m \Gamma(\omega_m)\,.\,e^{j\omega_m t}. \qquad (14)$$

Although it must be remembered that $\delta(t)$ is not a proper function and that both the Fourier integral representation

$$\delta(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}.\,d\omega, \qquad (15)$$

and the summed array of Fourier terms (14) for the recurrent pulse, are divergent, these will be freely used for calculating the required coefficients $\Gamma(\omega_m)$. Thus the analysis is only formal, but the same results may be obtained by the repeated use of limiting processes, with convergent expansions and integrals for proper narrow pulses $\delta_s(t)$. $\delta(t)$ is to be regarded as an operator, or incomplete symbol, defined by its auxiliary usage in the above context. Physically, narrow pulses $\delta_s(t)$, whose durations are much less than any of the relevant time-constants of a given system, will produce terms $\Gamma_s(\omega_m)\,.\,e^{j\omega_m t}$ indistinguishable from the terms $\Gamma(\omega_m)\,.\,e^{j\omega_m t}$ for any frequency ω_m within the significant range of the system.

The existence of a pulse shape factor, for an arbitrary pulse waveform, now becomes physically evident. For we may suppose this pulse waveform, $f_0(t)$, to be obtained by applying the ideal improper pulse, $\delta(t)$, to a linear network of frequency characteristic $g_0(\omega)$, by (11) and (15). This same network will transform the recurrent improper pulses, $\mathrm{E}(t)$, into the recurrent proper pulse waveform, $\mathrm{F}(t)$; consequently, the amplitude of the $e^{j\omega_m t}$ component, $\Gamma(\omega_m)$, is multiplied by the factor $g_0(\omega_m)$ imposed by this pulse shaping network. (The network may not be physically realizable unless an additional constant delay is imposed between $\delta(t)$ and $f_0(t)$).

The result corresponding to (8) remains true when the recurrent proper and improper pulse waveforms (4) and (13) resolve into continuous instead of discrete frequency distributions (as for transient modulations); then

$$\mathrm{F}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)\,.\,e^{j\omega t}\,.\,d\omega, \quad \dots\dots \quad (5')$$

and

$$\mathrm{E}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} \gamma(\omega)\,.\,e^{j\omega t}\,.\,d\omega\,; \quad \dots\dots \quad (14')$$

where

$$\gamma(\omega)=\sum_{n=-\infty}^{\infty} a_n\,.\,e^{-j\omega_m t_n}\,; \quad \dots\dots \quad (9')$$

and

$$g(\omega)=g_0(\omega)\,.\,\gamma(\omega). \quad \dots\dots \quad (8')$$

(2.2) *Scanning of Signal or Modulation Waveforms.*

Consider a recurrent pulse waveform in which only one pulse parameter is being modulated (*e. g.* amplitude, a_n, or epoch, t_n, or both, if their deviations are proportional). Then the modulation value, f_n, which is proportional to the deviation of this parameter from its reference or unmodulated value for the nth pulse, is itself generally derived from a continuous waveform in time, $f(t)$, which may be called the modulation function. This does not necessarily coincide with the applied signal waveform, $\bar{f}(t)$, but is usually effectively related to it by a specified linear operation, such as

$$\left.\begin{aligned} & f(t)=a\,.\,\bar{f}(t)+b, \\ \text{or}\quad & f(t)=a\,.\,\frac{d}{dt}\{\bar{f}(t)\}+b, \\ \text{or}\quad & f(t)=a\,.\int_{-\infty}^{t}\bar{f}(t)dt+b, \end{aligned}\right\} \quad \dots\dots \quad (16)$$

according to the type of modulation in question. Such a transformation affects the relative values of the frequency components in the signal waveform in a known way, but does not introduce any intermodulation products, *i. e.* linear combinations of multiple harmonic frequencies

(In engineering language, it produces "linear distortion," but no "non-linear distortion," at this stage. Since the transformation is deliberate, it does not imply distortion in the system as a whole). The distinction between the signal and modulation waveforms will not always be made below, and will be taken as understood. In an actual system, the modulation waveform may be only virtual; or the processes to be described may in fact be carried out on either waveform.

The process of obtaining the modulation values, f_n, from the waveform $f(t)$, which may be described as a scanning process, is called "*point-scanning*" if it selects the values of $f(t)$ at well-defined instants, $t=s_n$, so that

$$f_n=f(s_n)\,; \qquad (17)$$

but if f_n is derived from all the values of $f(t)$ over an extended continuous period of time, the process is called "*interval-scanning*." In many types of interval scanning processes, f_n is proportional to some form of average, or integrated value, of $f(t)$ over the interval. Some frequently occurring types are given below.

(2.2.1) *Interval Scanning, Type I.*

In this type, in place of the ideal scanning points $t=s_n$, a scanning-function $s(u)$ (of epoch $u=0$) is used to obtain weighted average values of $f(t)$ around well-defined epochs, $t=s_n$,

$$f_n=\int_{-\infty}^{\infty} f(u)\,.\,s(u-s_n)\,.\,du, \qquad (18)$$

the scanning function $s(u)$ applying most of its weight in the neighbourhood of $u=0$. It follows from the "folding" theorem (of Fourier integral theory) that

$$f_n=\hat{f}(s_n), \qquad (19)$$

where

$$\hat{f}(t)=\int_{-\infty}^{\infty} f(u)\,.\,s(u-t)\,.\,du \qquad (20)$$

is the resultant of the functions $f(t)$ and $s(-t)$ (1); *i.e.* if $\sigma(\omega)$ is the spectral distribution function for the (reverse) scanning-function $s(-t)$,

$$s(-t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} \sigma(\omega)\,.\,e^{j\omega t}\,.\,d\omega, \qquad (21)$$

then $\hat{f}(t)$ is derived from $f(t)$ by multiplying each of its frequency components by the corresponding value of the "scanning-factor" $\sigma(\omega)$; so that if

$$f(t)=\sum_q b_q e^{j\lambda_q t}, \qquad \text{then } \hat{f}(t)=\sum_q \{b_q\sigma(\lambda_q)\}\,.\,e^{j\lambda_q t}\,; \qquad (22)$$

or if

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} b(\lambda)\,.\,e^{j\lambda t}\,.\,d\lambda, \text{ then } \hat{f}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} \{b(\lambda)\sigma(\lambda)\}\,.\,e^{j\lambda t}\,.\,d\lambda. \qquad (23)$$

The significance of equations (19) and (22), or (23), is that Type I. interval-scanning of a given modulation (or signal) waveform $f(t)$ is equivalent to point-scanning of a modified waveform $\hat{f}(t)$. This modification consists of imposing a known "linear distortion" upon the waveform (such as might be obtained by passing it through a linear network) and does not introduce combination tones.

As the scanning-function, $s(u)$, approaches the limiting form of the improper function $\delta(u)$, the scanning-factor, $\sigma(\omega)$, approaches unity, and the interval-scanning process becomes indistinguishable from point scanning.

(2.2.2) *Interval-Scanning, Type II.*

In this case, a sequence of instants $t=s_n$ is determined by the integration of the signal waveform $\bar{f}(t)$ between successive instants (s_{n-1}, s_n), or in intervals (r_n, s_n) where the instants $t=r_n$ form some pre-assigned sequence, in such a way that

$$\int_{s_{n-1}}^{s_n} \{m+\bar{f}(t)\}\,.\,dt=k, \quad \dots \dots \quad (24)$$

or

$$\int_{r_n}^{s_n} \{m+\bar{f}(t)\}\,.\,dt=k, \quad \dots \dots \quad (25)$$

respectively, where m and k are usually constants. Each of these relations reduces to the form

$$s_n+f(s_n)=u_n, \quad \dots \dots \quad (26)$$

where the instant $t=u_n$ is well-defined and independent of the values of the modulation waveform in the scanning-interval, and where

$$f(t)=\frac{1}{m}\int_0^t \bar{f}(t)dt \quad \dots \dots \quad (27)$$

(when m is constant). Thus, Type II. interval-scanning of a given signal waveform is equivalent to a point-scanning process upon the signal waveform modified by integration.

The equivalence to a point scanning process still holds but without modifiyng the effective signal waveform, for the alternative modulation scanning process defined by

$$\int_{n-1}^{s_n} m\,dt=k-f(s_n), \quad \dots \dots \quad (24')$$

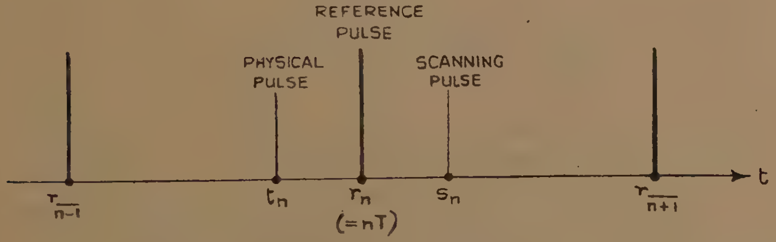
or

$$\int_{r_n}^{s_n} m\,dt=k-f(s_n). \quad \dots \dots \quad (25')$$

(2.3) *Narrow Pulse Modulation.*

As shown by the discussion in sections 1 and 2, for the frequency analysis of modulated pulses of arbitrary, but fixed shape, it is generally sufficient to consider the case of indefinitely narrow pulses whose modulation-values have been obtained by point-scanning a given modulation waveform. Sufficiently narrow pulses of simple shape effectively offer only two parameters for modulation, namely, time of occurrence and "content" (*i. e.* time-integral). In the reference or unmodulated condition, the pulses are conventionally of constant content and occur at points separated by constant intervals, both constants being arbitrarily pre-assigned for the system. If the pulses are made to occur at instants differing from the reference points, but their contents are not varied they are said to be (purely) *time-modulated*; whilst if the contents are made to differ from the standard value but the pulses still occur at the reference points, they are said to be (purely) *amplitude-modulated* (by analogy with the case of proper pulses of fixed shape) although this description is not strictly applicable. More generally, the time and content may be simultaneously modulated, but the only case considered here is when they are proportionately varied, *i. e.* when there is effectively only a single modulation parameter for each pulse whose value is obtained at one corresponding scanning-point. Thus, it is useful to think of a virtual *reference-pulse* and a virtual *scanning-pulse* associated with, but in general distinct from, each *physical pulse* (see fig. 1).

Fig. 1.



Pulse epoch diagram.

If each scanning pulse is coincident with its reference pulse, so that the modulation numbers are proportional to the values of the modulation function at strictly periodically recurring instants, the modulation is said to be *periodic*; whilst if each scanning pulse coincides with its physical pulse, so that the modulation-numbers are proportional to the values of the modulation function at the very instants when the physical pulses occur, the modulation is said to be *synchronous*. If, as for synchronous modulation, the scanning pulses are still synchronously time-modulated, but the time displacements of the physical and scanning pulses are merely proportional instead of being equal, the modulation may be said to be *sub-synchronous*. This includes periodic and synchronous modulations as special cases.

If t denotes the time variable, T the recurrence period, and $\frac{\Omega}{2\pi}=\frac{1}{T}$ the recurrence frequency of the system, then the reference pulses occur at instants $t=r_n$, say,

$$r_n=nT, \quad (n=\ldots, -1, 0, 1, 2, \ldots). \qquad (28)$$

whilst the modulated physical pulses occur at the instants $t=t_n$,

$$t_n=r_n-\epsilon\,.\,f(s_n), \quad \ldots \quad (29)$$

with relative "amplitudes"

$$a_n=1+\rho\,.\,f(s_n), \quad \ldots \quad (30)$$

where f_n is the nth value of the modulation parameter, and is here equal to the value of the bounded modulation-function $f(t)$ at the scanning-point $t=s_n$,

$$f_n=f(s_n). \quad \ldots \quad (17)$$

ϵ and ρ are factors put into evidence to indicate relative depths of modulation.

For periodic modulation ($s_n=r_n$):

$$t_n=r_n-\epsilon\,.\,f(r_n), \quad \ldots \quad (31)$$

whilst for synchronous modulation ($s_n=t_n$):

$$t_n=r_n-\epsilon\,.\,f(t_n). \quad \ldots \quad (32)$$

More generally, for sub-synchronous modulation:

$$s_n=r_n+\eta\,.\,f(s_n), \quad \ldots \quad (33)$$

where $-\eta$ is the depth of synchronous modulation of the scanning pulse, so that from (29) and (33):

$$\frac{(t_n-r_n)}{-\epsilon}=\frac{(s_n-r_n)}{\eta}=\frac{(t_n-s_n)}{-\zeta}=f(s_n), \quad \ldots \quad (34)$$

where

$$\zeta=(\epsilon+\eta), \quad \ldots \quad (35)$$

(the sign conventions for ϵ, η and ζ being chosen for subsequent convenience). Although the instants $t=r_n$, $t=s_n$, $t=t_n$ are actually values of the same time-variable, t (fig. 2), it is useful to consider them alternatively as corresponding discrete values of three different continuous time-variables, r, s and t, related always by the condition

$$t=r-\epsilon\,.\,f(s), \quad \ldots \quad (36)$$

and also, for sub-synchronous modulation, by the further condition

$$s=r+\eta\,.\,f(s), \quad \ldots \quad (37)$$

or by the equivalent condition

$$t=s-\zeta\,.\,f(s), \quad \ldots \quad (38)$$

the discrete values being selected at the points $r=r_n=n\mathrm{T}$. Thus, r, s and t, are to be considered as simultaneous measures of the time on three different time scales, "reference-time," "scanning-time" and "physical-time," respectively. These scales are identical in the absence of modulation, $r=s=t$; but the presence of modulation distorts these time-bases relative to each other, so that the same pulses which still occur periodically at the points $r=r_n=n\mathrm{T}$ in reference-time occur also at the new points $s=s_n$ measured in scanning-time, and also at the new points $t=t_n$ measured in physical-time.

This point of view leads to the vector representation of the time-modulation by the three simultaneous vectors ; $e^{j\theta}$, $e^{j\phi}$, $e^{j\psi}$, where

$$\theta = \Omega r, \qquad \phi = \Omega s, \qquad \psi = \Omega t, \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

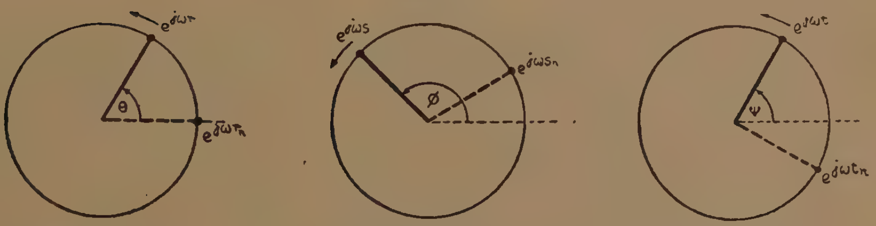
each vector rotating uniformly in its *own* time (fig. 2, (*a*), (*b*) and (*c*).) The pulse occurs whenever the r-vector crosses its positive real axis.

When the modulation is sub-synchronous, then, by scanning-time s, the reference and physical vectors are each phase-modulated (in the conventional continuous-wave sense) by the modulation-function, $f(s)$, with respect to the uniformly rotating scanning-vector as carrier-vector, but in general to different depths of modulation :

$$\theta = \Omega s - \eta\Omega \, . \, f(s), \qquad \frac{d\theta}{ds} = \Omega - \eta\Omega \, . \, f'(s) \, ; \quad . \quad . \quad . \quad (40)$$

$$\psi = \Omega s - \zeta\Omega \, . \, f(s), \qquad \frac{d\psi}{ds} = \Omega - \zeta\Omega \, . \, f'(s). \quad . \quad . \quad . \quad (41)$$

Fig. 2.



(a) REFERENCE-VECTOR. (b) SCANNING-VECTOR. (c) PHYSICAL-VECTOR.

Vector representation of time-modulation of pulses.

Thus, if the signal-function $\bar{f}(t)$ is proportional to the modulation-function $f(t)$, the pulses are said to be "*phase-modulated*," whilst if $\bar{f}(t)$ is proportional to $f'(t)$, they are said to be "*frequency-modulated*," but these are both only special cases of time-modulation.

In particular, for synchronous modulation ($s=t$, $\zeta=0$), the reference vector $e^{j\Omega r}$ is time-modulated in physical t-time. Conventional C.W. phase or frequency modulation is synchronous and is represented by the vector $e^{j\Omega r}$ in a more direct sense.

For periodic modulation ($s=r$, $\eta=0$) on the other hand, the physical-vector $e^{j\Omega t}$ is time-modulated in reference r-time. This implies some " non-linear " distortion of the time-modulation of the reference-vector $e^{j\Omega r}$ as reckoned in physical t-time.

In the following analysis of the modulated pulse waveform

$$\mathrm{E}(t) = \sum_{n} a_n \, . \, \delta(t - t_n), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

two important properties of the δ-function are used ; firstly, that it acts as a selection operator, *i. e.* for an arbitrary function $f(t)$,

$$f(t) \, . \, \delta(t - t_n) = f(t_n) \, . \, \delta(t - t_n) \, ; \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

and secondly, that upon a change of variable, say, from t to r, it transforms into a similar improper pulse, but with a change of content, given by

$$\delta(t-t_n)=\frac{dr}{dt}\,.\,\delta(r-r_n) \tag{43}$$

(as may be verified by integrating each side with respect to t). Particular use is made of the unmodulated recurrent pulse waveform $\triangle(r)$ of period $\mathrm{T}=\dfrac{2\pi}{\Omega}$.

$$\triangle(r)=\sum_{n=-\infty}^{\infty}\delta(r-r_n)=\frac{1}{\mathrm{T}}\sum_{p=-\infty}^{\infty}e^{jp\Omega r}, \tag{44}$$

representing unit improper pulses at all the points $r_n=n\mathrm{T}$.

(2.3.1) *Sub-Synchronous Amplitude and Time-Modulated Pulses.*

This system has been defined by

$$\mathrm{E}(t)=\sum_n a_n\,.\,\delta(t-t_n)\,; \tag{13}$$

where
$$\begin{cases} a_n=1+\rho\,.\,f(s_n)\,; & (30)\\ t_n=r_n-\epsilon\,.\,f(s_n)\,; & (29)\end{cases}$$

and
$$\begin{cases} t=r-\epsilon\,.\,f(s), & (36)\\ s=r+\eta\,.\,f(s), & (37)\\ t=s-\zeta\,.\,f(s). & (38)\end{cases}$$

Thus, the modulated pulse waveform may be written

$$\mathrm{E}(t)=\{1+\rho\,.\,f(s)\}\,.\,\sum_n\delta(t-t_n)=\{1+\rho\,.\,f(s)\}\,.\,\frac{dr}{dt}\,.\,\triangle(r)\,; \tag{45}$$

i. e.
$$\mathrm{E}(t)=\sum_{p=-\infty}^{\infty}\mathrm{E}_p(t), \tag{46}$$

where
$$\mathrm{E}_p(t)=\frac{1}{\mathrm{T}}\,.\,\{1+\rho\,.\,f(s)\}\,.\,\frac{dr}{dt}\,.\,e^{jp\Omega r}, \tag{47}$$

$$=\frac{1}{\mathrm{T}}\,.\,\{1+\rho\,.\,f(s)\}\,.\left[\frac{1-\eta\,.\,f'(s)}{1-\zeta\,.\,f'(s)}\right].\,e^{jp\Omega r}, \tag{48}$$

$$=\sum_q \Gamma_q(p\Omega,\mu_q)\,.\,e^{j(p\Omega+\mu_q)t}, \tag{49}$$

say, when the modulation-function $f(t)$ is resolvable into a denumerable set of frequency components and $e^{j\mu_q t}$ represents one of its typical combination tones. Then

$$\Gamma_q(p\Omega,\mu_q)=\lim_{c\to\infty}\frac{1}{2c}\int_{-c}^{c}\mathrm{E}_p(t)\,.\,e^{-j(p\Omega+\mu_q)t}\,.\,dt \tag{50}$$

$$=\lim_{c\to\infty}\frac{1}{2c\mathrm{T}}\int_{-c}^{c}\{1+\rho\,.\,f(s)\}\,.\,e^{j\{p\Omega(r-t)-\mu_q t\}}\,.\,dr \tag{51}$$

$$=\lim_{c\to\infty}\frac{1}{2c\mathrm{T}}\int_{-c}^{c}\{1+\rho\,.\,f(s)\}\,.\,\{1-\eta\,.\,f'(s)\}\,.\,e^{j\{(\epsilon.p\Omega+\zeta.\mu_q)f(s)-\mu_q s\}}\,.\,ds. \tag{52}$$

But the functions $C_q(x)$ and $D_q(x)$, where

$$C_q(x) = \lim_{c\to\infty} \frac{1}{2c} \int_{-c}^{c} e^{j\{xf(s)-\mu_q s\}} \,.\, ds \quad . \quad . \quad . \quad . \quad . \quad . \qquad (53)$$

and

$$\left\{\begin{array}{ll} D_q(x) = \left\{\dfrac{C_q(x)}{jx}\right\}, & (\mu_q \neq 0), \quad . \quad . \quad . \quad . \quad . \quad . \qquad (54) \\ D_0(x) = \left\{\dfrac{C_0(x)-1}{jx}\right\}, & (\mu_q = 0), \quad . \quad . \quad . \quad . \quad . \quad . \qquad (55) \end{array}\right.$$

are known, since these occur as spectral coefficients in the expansions

$$e^{jxf(s)} = \sum_q C_q(x) \,.\, e^{j\mu_q s} = 1 + jx \,.\, \sum_q D_q(x) \,.\, e^{j\mu_q s}. \quad . \quad . \quad . \qquad (56)$$

Moreover, making $x \to 0$, it is seen that the spectral terms for $f(s)$ itself are given by

$$f(s) = \sum_q C_q'(0) \,.\, e^{j\mu_q s} = \sum_q D_q(0) \,.\, e^{j\mu_q s}, \quad . \quad . \quad . \quad . \qquad (57)$$

the coefficients $D_q(0)$ being zero for those tones which are absent from this original waveform. Thus, by successive differentiations with respect to s and x respectively,

$$\{1-\eta \,.\, f'(s)\} \,.\, e^{jxf(s)} = 1 + j \sum_q (x - \eta\mu_q) D_q(x) \,.\, e^{j\mu_q s}, \quad . \quad . \qquad (58)$$

$$\{1+\rho \,.\, f(s)\} \,.\, \{1-\eta \,.\, f'(s)\} \,.\, e^{jxf(s)}$$
$$= 1 + \sum_q \Big[(x-\eta\mu_q)\{jD_q(x) + \rho D_q'(x)\} + \rho D_q(x) \Big] e^{j\mu_q s}, \quad . \quad . \qquad (59)$$

and so, putting $x = (p\Omega \,.\, \epsilon + \mu_q \,.\, \zeta)$, we have

$$E(t) = \sum_p \sum_q \Gamma_q(p\Omega, \mu_q) \,.\, e^{j(p\Omega+\mu_q)t}, \quad . \quad . \quad . \quad . \quad . \qquad (60)$$

where (for $\mu_q \neq 0$)

$$\left.\begin{array}{l} \Gamma_q(p\Omega, \mu_q) = \dfrac{1}{T} \Big[\{\rho + j\epsilon \,.\, (p\Omega+\mu_q)\} D_q(p\Omega \,.\, \epsilon + \mu_q \,.\, \zeta) \\ \qquad\qquad + \rho\epsilon \,.\, (p\Omega+\mu_q) \,.\, D_q'(p\Omega \,.\, \epsilon + \mu_q \,.\, \zeta) \Big], \\ \text{and (for } \mu_q = 0) \\ \Gamma_0(p\Omega, 0) = \dfrac{1}{T} \Big[1 + \{\rho + j\epsilon \,.\, p\Omega\} \,.\, D_0(p\Omega\epsilon) + \rho\epsilon \,.\, p\Omega \,.\, D_0'(p\Omega\epsilon) \Big]. \end{array}\right\} \quad (61)$$

The coefficient $\Gamma_q(p\Omega, \mu_q)$ appears as the sum of three terms, multiples of ρ, ϵ, and $\rho\epsilon$, which respectively exhibit the effects of amplitude-modulating the pulses, of time-modulating them, and of the interaction of these modulations.

If the modulation-function $f(t)$ resolves into a continuous instead of a discrete spectrum then in place of equations (49) "*et seq.*", we have (say)

$$e^{jxf(s)} = 1 + \frac{jx}{2\pi} \int_{-\infty}^{\infty} d_\mu(x) \,.\, e^{j\mu s} \,.\, d\mu, \quad . \quad . \quad . \quad . \quad . \qquad (56')$$

$$f(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d_{\mu}(0)\,.\,e^{j\mu s}\,.\,d\mu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (57')$$

the modulated pulse waveform resolving into

$$\mathrm{E}(t)=\sum_{p}\frac{1}{2\pi}\int_{-\infty}^{\infty} \gamma_{\mu}(p\Omega,\mu)\,.\,e^{j(p\Omega+\mu)t}\,.\,d\mu, \quad . \quad . \quad . \quad . \qquad (60')$$

where

$$\gamma_{\mu}(p\Omega,\mu)=\frac{1}{\mathrm{T}}[\{\rho+j\epsilon(p\Omega+\mu)\}\,.\,d_{\mu}(p\Omega\,.\,\epsilon+\mu\zeta)$$
$$+\rho\epsilon(p\Omega+\mu)\,.\,d_{\mu}'(p\Omega\epsilon+\mu\zeta)\,; \quad . \quad . \qquad (61')$$

i.e. each of the spectral lines for the recurrent unmodulated pulses broadens, on modulation, into a continuous band.

The more general results for the case when the amplitude and time modulation values are independently obtained by different scanning pulses (acting possibly upon different modulation functions, or themselves undergoing more complex motions) can be similarly derived, but the formula (61) covers all the special cases now to be considered.

In all cases, the modulated pulse waveform $\mathrm{E}(t)$, ((60) or (60′)), can be resolved into separate modulated carrier components of different orders, $\mathrm{E}_p(t)$, each being further resolvable into a discrete or continuous sideband group. Equation (48) shows that all the modulated carrier components (including the zero'th carrier) possess the same envelope

$$\frac{1}{\mathrm{T}}\,.\,\{1+\rho\,.\,f(s)\}\,.\left\{\frac{1-\eta\,.\,f'(s)}{1-\zeta\,.\,f'(s)}\right\}; \quad . \quad . \quad . \quad . \quad . \quad . \qquad (62)$$

whilst, since

$$r=t+\epsilon\,.\,f(s), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (36)$$

each carrier, $e^{j(p\Omega)r}$, is phase-modulated by $f(s)$ to a depth $(p\Omega\,.\,\epsilon)$ proportional to its order p (the carrier of zero order being phaseless). Thus, the sideband groups of time-modulated pulses will spread with increasing carrier order and must eventually overlap. If the depth of modulation is sufficiently small, and if the modulation frequencies are sufficiently low in relation to the recurrence frequency, then it may be possible to isolate physically any one of the real modulated carrier groups by a simple filter network.

Modulations giving continuous sideband structures will not receive any further consideration.

(2.3.2) *Time-Modulated Pulses.*

In the absence of amplitude modulation ($\rho=0$), the (sub-synchronously) modulated pulse waveform may be resolved into

$$\mathrm{E}(t)=\frac{1}{\mathrm{T}}\,.\,\frac{dr}{dt}\,.\,\Sigma e^{jp\Omega r}=\frac{1}{\mathrm{T}}\,.\left\{\frac{1-\eta\,.\,f'(s)}{1-\zeta\,.\,f'(s)}\right\}.\,\sum_{p} e^{jp\Omega r}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (63)$$

$$=\frac{1}{\mathrm{T}}[1+j\epsilon\,.\,\sum_{q}(p\Omega+\mu_q)\,.\,\mathrm{D}_q(p\Omega\,.\,\epsilon+\mu_q\,.\,\zeta)\,.\,e^{j\mu_q t}]e^{jp\Omega t} \quad . \quad . \quad . \qquad (64)$$

(2.3.2.1) When the pulses are *synchronously modulated* ($\zeta=0, \eta=-\epsilon$; $s=t$)

$$t=r-\epsilon . f(t) ; \qquad (65)$$

$$\mathrm{E}(t)=\frac{1}{\mathrm{T}}\sum_{p}[1+j\epsilon . \sum_{q}(p\Omega+\mu_q) . \mathrm{D}_q(p\Omega\epsilon) . e^{j\mu_q t}]e^{jp\Omega t}, \qquad (66)$$

$$=\mathrm{E}_0(t)+\frac{1}{\mathrm{T}}\sum_{p\neq 0}\sum_{q}\left(\frac{p\Omega+\mu_q}{p\Omega}\right) . \mathrm{C}_q(p\Omega\epsilon) . e^{j(p\Omega+\mu_q)t}. \qquad (67)$$

The zero-order terms, $\mathrm{E}_0(t)$, are simply

$$\mathrm{E}_0(t)=\frac{1}{\mathrm{T}} . \frac{dr}{dt}=\frac{1}{\mathrm{T}} . \{1+\epsilon . f'(t)\}, \qquad (68)$$

$$=\frac{1}{\mathrm{T}}\left[1+j\epsilon . \sum_{q}\mu_q\mathrm{D}_q(0) . e^{j\mu_q t}\right], \qquad (69)$$

so that this waveform (*i. e.* the envelope waveform of the modulated carrier component of any order) contains no intermodulation products of the modulation-waveform, and is a faithful reproduction of the signal-waveform when the (narrow) pulses are frequency-modulated, (*cf.* equation (57)). Since the phase of the pth carrier component, $\mathrm{E}_p(t)$, is

$$\theta_p=(p\Omega) . r=(p\Omega) . t+(p\Omega\epsilon) . f(t), \qquad (70)$$

each carrier of non-zero order is faithfully phase or frequency modulated when the pulses are respectively phase or frequency modulated.

(2.3.2.2) When the pulses are *periodically modulated* ($\eta=0$, $\zeta=\epsilon$; $s=r$),

$$t=r-\epsilon . f(r) ; \qquad (71)$$

$$\mathrm{E}(t)=\frac{1}{\mathrm{T}} . \sum_{p}\left[1+j\epsilon . \sum_{q}(p\Omega+\mu_q) . \mathrm{D}_q(\overline{p\Omega+\mu_q} . \epsilon) . e^{j\mu_q t}\right]e^{jp\Omega t}, \qquad (72)$$

$$=\frac{1}{\mathrm{T}} . \sum_{p}\sum_{q}\mathrm{C}_q(\overline{p\Omega+\mu_q} . \epsilon) . e^{j(p\Omega+\mu_q)t}; \qquad (73)$$

and the zero-order terms ($p=0$) are

$$\mathrm{E}_0(t)=\frac{1}{\mathrm{T}} . \frac{dr}{dt}=\frac{1}{\mathrm{T}} . \left[1-\epsilon . f'(r)\right]^{-1}_{(r-\epsilon . f(r)=t)}, \qquad (74)$$

$$=\frac{1}{\mathrm{T}} . \left[1+j\epsilon . \sum_{q}\mu_q\mathrm{D}_q(\mu_q\epsilon) . e^{j\mu_q t}\right], \qquad (75)$$

so that intermodulation products of the modulation waveform appear here, in general; (*cf.* equations (57) and (69)). The result may be interpreted as a time-scale inversion of the relation (71) (see section (2.3.3.1)).

(2.3.3) *Amplitude Modulated Pulses.*

When the physical pulses are not time-modulated ($\epsilon=0$, $\eta=\zeta$; $t=r$), and when their amplitude-modulation values are recorded by a synchronously time-modulated scanning pulse, according to the relation

$$s=r+\eta . f(s), \qquad (37)$$

the modulated pulse waveform may be written

$$E(t)=\frac{1}{T}\cdot\sum_{p}\left[1+\rho\cdot\sum_{q}D_q(\mu_q\eta)\cdot e^{j\mu_q t}\right]e^{jp\Omega t}, \quad \ldots \quad (76)$$

and the zero-order terms ($p=0$) are

$$E_0(t)=\frac{1}{T}\cdot\{1+\rho\cdot f(s)\}=\frac{1}{T}\cdot\left[1+\rho\cdot\sum_{q}D_q(\mu_q\eta)\cdot e^{j\mu_q t}\right]. \quad \ldots \quad (77)$$

which, in general, include intermodulation products of the modulation function $f(t)$; (*cf.* (75)).

Note that the sideband structures about all the component carrier frequencies, $p\Omega$, are identical. This result is true of any periodically recurring (narrow) pulse which is only " amplitude "-modulated, however the modulation-values are derived.

(2.3.3.1) The results (75) and (77) may be alternatively expressed as an *inversion formula* for the relation (37), (when $r=t$),

$$r=s-\eta\cdot f(s), \quad \ldots \quad (37)$$

into the form

$$s=r+\eta\cdot g(r), \quad \ldots \quad (78)$$

(when $f(s)$ and $g(r)$ are bounded functions with denumerable frequency resolutions), namely,

$$f(s)=g(r)=\sum_{q}D_q(\mu_q\eta)\cdot e^{j\mu_q r}, \quad \ldots \quad (79)$$

where

$$e^{jxf(s)}\equiv 1+jx\cdot\sum_{q}D_q(x)\cdot e^{j\mu_q s}. \quad \ldots \quad (56)$$

The classical example of this formula is Bessel's inversion of the relation between the mean-anomaly, r, and eccentric-anomaly, s, of an ellipse of eccentricity η (2) :

$$r=s-\eta\cdot\sin s, \quad \ldots \quad (80)$$

$$s=r+\eta\cdot\sum_{q\neq 0}\left\{\frac{J_q(q\eta)}{j(q\eta)}\right\}\cdot e^{jqr}. \quad \ldots \quad (81)$$

(2.3.3.2) In the special case of *periodic-scanning* ($\eta=0$; $r=s=t$) the periodically recurrent pulse waveform reduces simply to

$$E(t)=\frac{1}{T}\cdot\sum_{p}\{1+\rho\cdot f(t)\}\cdot e^{jp\Omega t}=\frac{1}{T}\cdot\sum_{p}\left[1+\rho\cdot\sum_{q}D_q(0)\cdot e^{j\mu_q t}\right]e^{jp\Omega t}, \quad \ldots \quad (82)$$

the zero-order terms ($p=0$) faithfully reproducing the modulation-waveform $f(t)$.

(2.3.4) *Simultaneously Amplitude and Time Modulated Pulses.*

(2.3.4.1) In the special case when the pulses are synchronously time-modulated and at the same time synchronously amplitude modulated ($\zeta=0$, $\eta=-\epsilon$; $s=t$), according to

$$t=r-\epsilon\cdot f(t), \quad \ldots \quad (65)$$

the modulated pulse waveform resolves into

$$\mathrm{E}(t)=\frac{1}{\mathrm{T}}\sum_{p}\Big[1+\sum_{q}\{(\rho+j\epsilon\,.\,\overline{p\Omega+\mu_q})\,.\,\mathrm{D}_q(p\Omega\epsilon)$$
$$+\rho\epsilon\,.\,(p\Omega+\mu_q)\,.\,\mathrm{D}_q'(p\Omega\epsilon)\}e^{j\mu_q t}\Big]e^{jp\Omega t}, \quad . \quad (83)$$

and the zero-order terms ($p=0$) are

$$\mathrm{E}_0(t)=\frac{1}{\mathrm{T}}\Big[1+\sum_{q}\{(\rho+j\epsilon\,.\,\mu_q)\,.\,\mathrm{D}_q(0)+\rho\epsilon\,.\,\mu_q\,.\,\mathrm{D}_q'(0)\}e^{j\mu_q t}\Big]. \quad . \quad (84)$$

(2.3.4.2) If the pulses are *periodically* time-modulated, and at the same time periodically amplitude modulated ($\eta=0$, $\zeta=\epsilon$; $s=r$), according to

$$t=r-\epsilon\,.\,f(r), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

the modulated pulse waveform resolves into

$$\mathrm{E}(t)=\frac{1}{\mathrm{T}}\sum_{p}\Big[1+\sum_{q}\{(\rho+j\epsilon\,.\,\overline{p\Omega+\mu_q})\,.\,\mathrm{D}_q(\overline{p\Omega+\mu_q}\,.\,\epsilon)$$
$$+\rho\epsilon\,.\,(p\Omega+\mu_q)\,.\,\mathrm{D}_q'(\overline{p\Omega+\mu_q}\,.\,\epsilon)\}e^{j\mu_q t}\Big]e^{jp\Omega t}, \quad . \quad (85)$$

and the zero-order terms ($p=0$) are

$$\mathrm{E}_0(t)=\frac{1}{\mathrm{T}}\Big[1+\sum_{q}\{(\rho+j\epsilon\,.\,\mu_q)\,.\,\mathrm{D}_q(\mu_q\epsilon)+\rho\epsilon\,.\,\mu_q\,.\,\mathrm{D}_q'(\mu_q\epsilon)\}e^{j\mu_q t}\Big]. \quad (86)$$

(3) WIDTH MODULATED PULSES.

A "rectangular" pulse being one which is of constant amplitude between two instants $t=u$, $t=v$ and zero elsewhere, a sequence of rectangular pulses is said to be width-modulated when the pulses are of equal amplitudes, recur periodically on the average, and their durations (v_n-u_n), undergo variations in proportion to successive modulation-values derived from a modulation-function, $f(t)$. As in the case of narrow time-modulated pulses, $f(t)$ is not necessarily proportional to the signal-function $\bar{f}(t)$ but is generally related to it by some well-defined linear operation. In general, both edges $t=u_n$, $t=v_n$, will be time-modulated, but not more than one modulation-function is assumed in the following.

The rectangular pulse $h(t\,;\,u_n,\,v_n)$ of unit amplitude,

$$h(t\,;\;u_n,\,v_n)=\begin{Bmatrix}1,\ (u_n<t<v_n)\,; \quad \text{or } -1,\ (v_n<t<u_n)\,; \\ 0,\ \text{elsewhere}\,; \end{Bmatrix} \quad (87)$$

may be formally regarded as the time-integral of the difference of two improper pulses of unit contents at $t=u_n$, $t=v_n$,

$$h(t\,;\;u_n,\,v_n)=\int_{-\infty}^{t}\{\delta(t-u_n)-\delta(t-v_n)\}\,.\,dt\,; \quad . \quad . \quad . \quad . \quad . \quad (88)$$

so that the analysis of the waveform of recurrent width-modulated rectangular pulses of unit amplitude

$$\mathrm{Z}(t)=\sum_{n=-\infty}^{\infty}h(t\,;\;u_n,\,v_n)=\sum_{n}\int_{-\infty}^{t}\{\delta(t-u_n)-\delta(t-v_n)\}\,.\,dt, \quad . \quad (89)$$

into its frequency components, may be directly derived from the two corresponding cases of indefinitely narrow time-modulated pulses. (Physically (89) means that the modulated rectangular pulses may be obtained by passing the two sets of narrow pulses through a linear integrating network).

Although the rectangular pulse, having infinitely steep edges, is an ideal waveform, the analysis of any practical case, when the leading and trailing " edges " spread in time, immediately follows from the ideal case, provided that all these edges are of identical waveform with epochs at the instants $t=u_n$, $t=v_n$, *i. e.* that the nth pulse is of the form

$$k(t\,;\,u_n,\,v_n)=\int_{-\infty}^{t}\{f_0(t-u_n)-f_0(t-v_n)\}\,.\,dt, \quad . \quad . \quad . \quad (90)$$

$f_1(t)=\int_{-\infty}^{t}f_0(t)dt$ being the reference " growth " waveform of each pulse (and $-f_1(t)$ its reference " decay " waveform). The frequency components will only differ from the corresponding ideal case $Z(t)$ in that each is multiplied by its " pulse-shape " factor which depends only upon the growth waveform. As in the case of time-modulated pulses, these factors may be conceived as being imposed by passing the modulated rectangular pulses through a linear pulse shaping network.

Also, as before, it is considered sufficient to discuss only point-scanned modulations.

(3.1) *Single Edge Modulation.*

When one edge of the recurrent pulse, say $t=v_n$, remains unmodulated, then there is little loss of generality in supposing that this always coincides with the reference position ($t=r_n=n\mathrm{T}$,) of the modulated edge ($t=u_n=t_n$), *i. e.* that the unmodulated width of the pulse is zero. For if, in fact, the pulses are of non-zero unmodulated width, c, so that $v_n=r_n\pm c$, they differ from the supposed case only by a strictly periodic waveform of (unmodulated) rectangular pulses, which makes only fixed contributions to the carrier frequency components, $e^{jp\Omega t}$, of the modulated waveform. The sign-convention is such that the modulated edge, at $t=t_n$, is always positive-going ; so that the sign of $\pm c$ for the unmodulated edge would depend upon whether the modulated edge would be leading (and the pulse positive) or trailing (and the pulse negative).

If, as before, $\mathrm{E}(t)$ denotes the waveform of time-modulated unit improper pulses at $t=t_n$, whilst $\triangle(t)$ denotes the corresponding unmodulated waveform, then the recurrent width-modulated rectangular pulse waveform may be written as

$$\mathrm{Z}(t)=\int_{-\infty}^{t}\{\mathrm{E}(t)-\triangle(t)\}\,.\,dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

(3.1.1) When the time-modulation is *sub-synchronous, i. e.* (from (63)),

$$\mathrm{E}(t)=\frac{1}{\mathrm{T}}\,.\,\sum_{p} e^{jp\Omega r}.\frac{dr}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63')$$

where

$$r=t+\epsilon\,.\,f(s), \qquad \text{(36')}$$

$$s=t+\zeta\,.\,f(s), \qquad \text{(38')}$$

then
$$Z(t)=\frac{\epsilon}{T}\cdot f(s)+\frac{1}{T}\sum_{p\neq 0}\cdot\left\{\frac{e^{jp\Omega.\epsilon f(s)}-1}{jp\Omega}\right\}\cdot e^{jp\Omega t}. \qquad (92)$$

In expanded form, (from (64)),

$$E(t)=\triangle(t)+\frac{j\epsilon}{T}\cdot\sum_p\sum_q(p\Omega+\mu_q)\,.\,D_q(p\Omega\,.\,\epsilon+\mu_q\,.\,\zeta_q).\,e^{j(p\Omega+\mu_q)t}\,; \qquad (93)$$

$$Z(t)=\frac{\epsilon}{T}\cdot\sum_p\sum_q D_q(p\Omega\,.\,\epsilon+\mu_q\,.\,\zeta)\,.\,e^{j(p\Omega+\mu_q)t}\,; \qquad (94)$$

the constant component of $Z(t)$ being $\left(\frac{\epsilon}{T}\right)\cdot D_0(0)$, since $D_0(0)$ is the constant component of $f(t)$.

(3.1.2) When the pulses are *synchronously* modulated ($\zeta=0\,;\ s=t$),

$$t=r-\epsilon\,.\,f(t), \qquad (65)$$

$$Z(t)=\frac{\epsilon}{T}\cdot\sum_p\sum_q D_q(p\Omega\epsilon)\,.\,e^{j(p\Omega+\mu_q)_t}\,; \qquad (95)$$

the zero-order terms ($p=0$) being simply

$$Z_0(t)\doteq\frac{\epsilon}{T}\cdot\sum_q D_q(0)\,.\,e^{j\mu_q t}\doteq\frac{\epsilon}{T}\cdot f(t), \qquad (96)$$

i. e. a faithful reproduction of the modulation-waveform, and hence also of the signal-waveform, $\tilde{f}(t)$, if the pulse edge is phase-modulated (but introducing a " linear-distortion " if it is frequency-modulated).

(3.1.3) When the pulses are *periodically* modulated ($\zeta=\epsilon\,;\ s=r$),

$$t=r-\epsilon\,.\,f(r), \qquad (71)$$

$$Z(t)=\frac{\epsilon}{T}\cdot\sum_p\sum_q D_q(\overline{p\Omega+\mu_q}\,.\,\epsilon)\,.\,e^{j(p\Omega+\mu_q)t}\,; \qquad (97)$$

the zero-order terms ($p=0$) being

$$Z^0(t)=\frac{\epsilon}{T}\cdot\sum_q D_q(\mu_q\epsilon)\,.\,e^{j\mu_q t}, \qquad (97)$$

so that, in general, intermodulation tones are introduced (*cf.* (57) (96)).

(3.2) *Double Edge Modulation.*

There is a large number of essentially different ways in which the two edges of a rectangular pulse of constant amplitude can be time-modulated by a single modulation-function, and these ways fall into two main groups, according as the two edges are modulated jointly or individually, *i. e.* according as whether the modulating waveform is effectively scanned once or twice. For double-edge width-modulation there would generally be a significant loss of generality in taking the unmodulated width to be zero.

(3.2.1) Before considering these cases, the formula (64) for sub-synchronously time-modulated unit improper pulses, will be generalized by shifting the reference points for the physical pulses $t=t_n$, and scanning pulses $t=s_n$, by fixed times a and b respectively, from the points $t=r_n=n\mathrm{T}$;

$$t_n=(r_n-a)-\epsilon\,.\,f(s_n), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (99)$$

$$s_n=(r_n+b)+\eta\,.\,f(s_n), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

so that,

$$t_n=(s_n-c)-\zeta\,.\,f(s_n), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

where

$$c=(a+b)\,; \qquad \zeta=(\epsilon+\eta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

The Fourier coefficients of the new pulse waveform, $\overline{\mathrm{E}}(t)$,

$$\overline{\mathrm{E}}(t)=\sum_n \delta(t-t_n)=\sum_p \sum_q \overline{\Gamma}_q(p\Omega,\mu_q)\,.\,e^{j(p\Omega+\mu_q)t}, \quad . \quad . \quad . \quad . \quad (103)$$

are still given by the equation (51), for $\Gamma_q(p\Omega,\mu_q)$, but now become (instead of (52)),

$$\overline{\Gamma}_q(p\Omega,\mu_q)=e^{j(p\Omega.a+\mu_q.c)}\,.\,\Gamma_q(p\Omega,\mu_q). \quad . \quad . \quad . \quad . \quad (104)$$

Hence, instead of (64) we have

$$\overline{\mathrm{E}}(t)=\frac{1}{\mathrm{T}}\,.\,\sum_p\left[1+j\epsilon\sum_q(p\Omega+\mu_q)\,.\,\mathrm{D}_q(p\Omega\,.\,\epsilon+\mu_q\,.\,\zeta)\,.\,e^{j\mu_q(t+c)}\right]e^{jp\Omega(t+a)}. \quad (105)$$

In particular, when the scanning and physical pulses have the same reference points, $c=0$ and the pulses will be said to be "*locally scanned.*"

(3.2.2) *Edges Individually Modulated.*

If the two edges of the rectangular pulses are independently (sub-synchronously) time-modulated, the waveform may be represented by

$$\mathrm{Z}(t)=\int_{-\infty}^{t}\{\overline{\mathrm{E}}_1(t)-\overline{\mathrm{E}}_2(t)\}\,.\,dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

where $\overline{\mathrm{E}}_1(t)$ and $\overline{\mathrm{E}}_2(t)$ are each given by (105) with suffixes 1 and 2 respectively on the symbols ϵ_i, ζ_i; a_i, c_i. Hence

$$\mathrm{Z}(t)=\frac{1}{\mathrm{T}}\,.\,\sum_p\left[\left[\frac{e^{jp\Omega a_i}}{jp\Omega}\right]_2^1+\sum_q\left[\epsilon_i e^{j(p\Omega a_i+\mu_q c_i)}\mathrm{D}_q(p\Omega\epsilon_i+\mu_q\,.\,\zeta_i)\right]_2^1 e^{j\mu_q t}\right]e^{jp\Omega t},$$
$$. \quad . \quad . \quad (107)$$

the constant term ($p=0$, $\mu_q=0$) being interpreted as

$$\frac{1}{\mathrm{T}}\{(a_1-a_2)+(\epsilon_1-\epsilon_2)\mathrm{D}_0(0)\}.$$

(3.2.2.1) In particular, when the edges have the same depths of modulation for their physical and scanning points ($\epsilon_1=\epsilon_2$, $\zeta_1=\zeta_2$) and when the interval between the physical and scanning reference points is the same for the two edges ($c_1=c_2$), (as, for example, when each edge is locally scanned, $c_1=0=c_2$) ; then

$$Z(t)=\frac{1}{T}\cdot\sum_p\left(e^{jp\Omega a_1}-e^{jp\Omega a_2}\right)\cdot\left[\frac{1}{jp\Omega}+\epsilon\,.\sum_q D_q\,(p\Omega\,.\,\epsilon+\mu_q\,.\,\zeta)\,.\,e^{j\mu_q(t+c)}\right]e^{jp\Omega t}, \quad . \;.\;.\; (108)$$

and all the terms of zero order ($p=0$) disappear, except the constant term $\left(\frac{a_1-a_2}{T}\right)$.

This is the case when the rectangular pulse may be considered as being essentially time-modulated, the variation in width arising as an incidental feature (and in such a way as to cancel the low frequency components) ; for although the two edges have the same depths of modulation, their modulation-values are selected by different scanning points.

(3.2.2.2) If the two edges are each locally scanned ($c_1=c_2=0$), and *symmetrically* in antiphase ($\epsilon_1=-\epsilon_2=\epsilon$; $\zeta_1=-\zeta_2=\zeta$), about the points $r_n=nT$ (say) so that $a_1=-a_2=a$, then

$$Z(t)=\frac{1}{T}\cdot\sum_p\Bigg[\left\{\frac{2\sin(p\Omega a)}{p\Omega}\right\}$$
$$+\epsilon\,.\sum_q\{e^{jp\Omega a}\,.\,D_q(p\Omega\epsilon+\mu_q\zeta)+e^{-jp\Omega a}\,.\,D_q(-\overline{p\Omega\epsilon+\mu_q\zeta})\}e^{j\mu_q t}\Bigg]e^{jp\Omega t}, \quad . \;.\; (109)$$

the constant term being $\frac{2}{T}\{a+\epsilon\,.\,D_0(0)\}$.

(3.2.3) *Edges Jointly Modulated.*

When there is only one scanning pulse for the two edges, $b_1=b_2=b$; $\eta_1=\eta_2=\eta$, the waveform (107) becomes

$$Z(t)=\frac{1}{T}\cdot\sum_p\Bigg[\left[\frac{e^{jp\Omega a_i}}{jp\Omega}\right]_2^1$$
$$+\sum_q\left[\epsilon_i\,.\,e^{j(p\Omega+\mu_q)a_i}\,.\,D_q(\overline{p\Omega+\mu_q}\,.\,\epsilon_i+\mu_q\eta)\right]_2^1 e^{j\mu_q(t+b)}\Bigg]e^{jp\Omega t}, \quad . \; (110)$$

and here there is little loss of generality in assuming that $b=0$ and that $a_1=-a_2=a$.

(3.2.3.1) When the two edges have the same depth (and sub-synchronous type) of modulation, $\epsilon_1=\epsilon_2=\epsilon$, they now necessarily have constant separation since there is only one modulation parameter, and this case has already been considered under time-modulation of pulses of fixed shape. The waveform (110) becomes with $b=0$; and $a_1=-a_2=a$),

$$Z(t)=\frac{1}{T}\sum_p\Bigg[\left(\frac{e^{jp\Omega a}-e^{-jp\Omega a}}{jp\Omega}\right)$$
$$+\epsilon\,.\sum_q\{(e^{j(p\Omega+\mu_q)a}-e^{-j(p\Omega+\mu_q)a})\,.\,D_q(p\Omega\,.\,\epsilon+\mu_q\,.\,\xi)\}e^{j\mu_q t}\Bigg]e^{jp\Omega t}, \quad (111)$$

which only differs from the time-modulated improper pulse waveform

$$\mathrm{E}(t)=\frac{1}{\mathrm{T}}\sum_{q}\left[1+j\epsilon\,.\sum_{q}\{(p\Omega+\mu_q)\,.\,\mathrm{D}_q(p\Omega\,.\,\epsilon+\mu_q\,.\,\zeta)\}e^{j\mu_q t}\right]e^{jp\Omega t}, \quad (64)$$

in that each frequency component $e^{j\omega t}$ is further multiplied by the shape-factor $\left\{\frac{2\sin(\omega a)}{\omega}\right\}$ appropriate to the shape and (symmetric) reference position of the rectangular pulse.

The zero-order terms ($p=0$) do not cancel out in this case, for although the contributions from the two edges are still identical, they now occur relatively displaced in time by the duration of the pulse $(a_1-a_2)=2a$.

(3.2.3.2) For the *symmetric width-modulation* case, the time-displacements of the edges are now exactly equal and opposite, $\epsilon_1=-\epsilon_2=\epsilon$. If, also, the scanning-pulse is unmodulated ($\eta=0$), *i. e.* the modulation is *periodic*, the waveform becomes

$$\mathrm{Z}(t)=\frac{1}{\mathrm{T}}\,.\sum_{p}\left[\left\{\frac{2\sin(p\Omega a)}{p\Omega}\right\}+\epsilon\,.\sum_{q}\{e^{j(p\Omega+\mu_q)a}.\,\mathrm{D}_q(\overline{p\Omega+\mu_q}\,.\,\epsilon)\right.$$
$$\left.+e^{-j(p\Omega+\mu_q)a}.\,\mathrm{D}_q(-\overline{p\Omega+\mu_q}\,.\,\epsilon)\}e^{j\mu_q t}\right]e^{jp\Omega t}. \quad . \quad . \quad (112)$$

(4) SINGLE AND DOUBLE TONE MODULATIONS.

(4.1) When the modulation function is a *single tone*,

$$\epsilon\,.\,f(t)=\epsilon\,.\,\sin\mu t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113)$$

then the expression (56),

$$e^{jxf(s)}=\sum_{q}\mathrm{C}_q(x)\,.\,e^{j\mu_q s}=1+jx\,.\sum_{q}\mathrm{D}_q(x)\,.\,e^{j\mu_q s} \quad . \quad . \quad . \quad . \quad . \quad (56)$$

becomes

$$e^{jx\sin\mu s}=\sum_{=-\infty}^{\infty}\mathrm{J}_q(x)\,.\,e^{jq\mu s}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (114)$$

(J_q being the Bessel function of order q), the combination angular frequencies μ_q for $f(t)$ being simply the multiple harmonic frequencies $q\mu$, ($q=\ldots,-1,0,1,2,\ldots$), and the functions $\mathrm{D}_q(x)$ are

$$\left.\begin{aligned}\mathrm{D}_q(x)&=\left\{\frac{\mathrm{J}_q(x)}{jx}\right\}, \quad (q\neq 0)\,;\\ \mathrm{D}_0(x)&=\left\{\frac{\mathrm{J}_0(x)-1}{jx}\right\}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (115)$$

Since

$$\mathrm{J}_{-q}(x)=(-1)^q\,.\,\mathrm{J}_q(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (116)$$

and, for small values of x, and for q$\geqslant 0$,

$$\mathrm{J}_q(x)=\frac{(\frac{1}{2}x)^q}{q\,!}\left\{1-\frac{(\frac{1}{2}x)^2}{(q+1)}\right\}; \text{ (approx.)}, \quad . \quad . \quad . \quad (117)$$

it follows that

$$\left.\begin{aligned} D_q(0)&=0, \quad (q\neq\pm 1)\,; \\ D_1(0)&=\frac{1}{2j}\,; \\ D_{-1}(0)&=-\frac{1}{2j}\,; \end{aligned}\right\}, \quad \dots \dots \quad (118)$$

in accordance with the expression (57),

$$f(s)=\sum_q D_q(0)\,.\,e^{j\mu_q s}, \quad \dots \dots \quad (57)$$

since

$$\sin \mu s=\frac{1}{2j}\,(e^{j\mu s}-e^{-j\mu s}). \quad \dots \dots \quad (119)$$

(4.2) For the *double-tone* modulation-function, say,

$$\epsilon\,.\,f(t)=\epsilon(\alpha_1 \sin \mu_1 t+\alpha_2 \sin \mu_2 t)=(\epsilon_1 \sin \mu_1 t+\epsilon_2 \sin \mu_2 t), \quad . \; . \quad (120$$

the expression (56) becomes

$$e^{jx(\alpha_1 \sin \mu_1 s+\alpha_2 \sin \mu_2 s)}=\sum_{q_1=-\infty}^{\infty}\;\sum_{q_2=-\infty}^{\infty} J_{q1}(\alpha_1 x)J_{q2}(\alpha_2 x)\,.\,e^{j(q_1\mu_1+q_2\mu_2)s}, \quad . \; . \quad (121)$$

there being a doubly-infinite array of combination tones for $f(t)$, of which a typical (angular) frequency is $\mu_q=(\mu_1 q_1+\mu_2 q_2)$, the symbol q denoting the integer-pair (q_1, q_2). Then, if $q=0$ refers to the pair (0, 0), for which $\mu_q=0$,

$$\left.\begin{aligned} D_q(x)&=\left\{\frac{J_{q1}(\alpha_1 x)J_{q2}(\alpha_2 x)}{jx}\right\}, \quad (q\neq 0)\,; \\ D_0(x)&=\left\{\frac{J_0(\alpha_1 x)J_0(\alpha_2 x)-1}{jx}\right\}. \end{aligned}\right\} \quad . \; . \; . \quad (122)$$

In particular,

$$\left.\begin{aligned} &D_q(0)=0\,; \quad \text{unless } q=(\pm 1, 0), \text{ or } (0, \pm 1)\,; \\ &D_{(\pm 1,\,0)}(0)=\pm\left(\frac{\alpha_1}{2j}\right), \text{ (respectively)}\,; \\ &D_{(0,\,\pm 1)}(0)=\pm\left(\frac{\alpha_2}{2j}\right), \text{ (respectively)}\,; \end{aligned}\right\}, \quad . \quad (123)$$

in accordance with the formula (57), since

$$(\alpha_1 \sin \mu_1 s+\alpha_2 \sin \mu_2 s)=\frac{\alpha_1}{2j}\,(e^{j\mu_1 s}-e^{-j\mu_1 s})+\frac{\alpha_2}{2j}\,(e^{j\mu_2 s}-e^{-j\mu_2 s}). \quad . \quad (124)$$

(4.3) The frequency analysis for the pure tone modulation function (113), of frequency $f_\mu=\mu/2\pi$, is summarized in real form, in Table I., but only for five types of physical pulse-modulation (phase-modulation, Ph.M., frequency modulation, F.M., and amplitude-modulation, A.M., of indefinitely narrow unit-content pulses; and asymmetric and symmetric width-modulations, W.M., of ideal rectangular pulses of unit amplitude), and for two types of scanning (periodic and synchronous point-scanning),

TABLE I.

Single Tone Modulation.

TYPE	PERIODIC SCAN	SYNCHRONOUS SCAN
Ph. M.	$\frac{1}{T}\left[1+2\sum_{q=1}^{\infty}J_q(qm\emptyset).\cos(q\mu t)\right]$ $+\frac{2}{T}\sum_{p=1}^{\infty}\sum_{q=-\infty}^{\infty}J_q(\overline{p+qm}.\emptyset).\cos(\overline{p\Omega+q\mu}.t)$	$\left[\frac{1}{T}+f_\mu\emptyset.\cos(\mu t)\right]$ $+\frac{2}{T}\sum_{p=1}^{\infty}\sum_{q=-\infty}^{\infty}\left(1+\frac{qm}{p}\right).J_q(p\emptyset)\cos(\overline{p\Omega+q\mu}.t)$
F. M.	$F_r\left[1+2\sum_{q=1}^{\infty}J_q\left(q\cdot\frac{f_d}{F_r}\right).\cos(q\mu t)\right]$ $+2F_r\sum_{p=1}^{\infty}\sum_{q=-\infty}^{\infty}J_q\left(\overline{\frac{p}{m}+q\cdot}\frac{f_d}{F_r}\right).\cos(\overline{p\Omega+q\mu}.t)$	$\left[F_r+f_d.\cos(\mu t)\right]$ $+2F_r\sum_{p=1}^{\infty}\sum_{q=-\infty}^{\infty}\left(1+\frac{qm}{p}\right).J_q\left(\frac{p}{m}\cdot\frac{f_d}{F_r}\right).\cos(\overline{p\Omega+q\mu}.t)$
A. M.	$\frac{1}{T}\left[1+\rho\sin(\mu t)\right]$ $+\frac{1}{T}\sum_{p=1}^{\infty}\left[2\cos(p\Omega t)+\rho\left\{\sin(\overline{p\Omega+\mu}.t)-\sin(\overline{p\Omega-\mu}.t)\right\}\right]$	$\frac{1}{T}\left[1+2\rho\sum_{q=1}^{\infty}\left\{\frac{J_q(q\mu\eta)}{(q\mu\eta)}\right\}\sin(q\mu t)\right]$ $+\frac{2}{T}\sum_{p=1}^{\infty}\left[\cos(p\Omega t)+\rho\sum_{q=1}^{\infty}\left\{\frac{J_q(q\mu m)}{(q\mu\eta)}\right\}.\left\{\begin{matrix}\sin(\overline{p\Omega+q\mu}.t)\\ \sin(\overline{p\Omega-q\mu}.t)\end{matrix}\right\}\right]$
ASYMMETRIC W. M.	$\frac{\emptyset}{\pi}\left[\sum_{q=1}^{\infty}\left\{\frac{J_q(qm\emptyset)}{(qm\emptyset)}\right\}\cdot\sin(q\mu t)\right]$ $+\frac{1}{\pi}\sum_{p=1}^{\infty}\left[\left\{\sum_{q=-\infty}^{\infty}\left(\frac{J_q(\overline{p+qm}.\emptyset)}{(p+qm)\cdot}\right)\sin(\overline{p\Omega+q\mu}.t)\right\}-\frac{\sin(p\Omega t)}{p}\right]$	$\frac{\emptyset}{2\pi}\cdot\sin(\mu t)$ $+\frac{1}{\pi}\sum_{p=1}^{\infty}\frac{1}{p}\left[\left\{\sum_{q=-\infty}^{\infty}J_q(p\emptyset)\sin(\overline{p\Omega+q\mu}.t)\right\}-\sin(p\Omega t)\right]$
SYMMETRIC W. M.	$\frac{2\emptyset}{\pi}\left[\sum_{q=1}^{\infty}\left\{\frac{J_q(qm\emptyset)}{(qm\emptyset)}\right\}\cdot\sin(q\mu t)\cdot\sin^2\left(q\cdot\frac{\pi}{2}\right)\right]$ $+\frac{2}{\pi}\sum_{p=1}^{\infty}\left[\sum_{q=-\infty}^{\infty}\left\{\frac{J_q(\overline{p+qm}.\emptyset)}{(p+qm)}\right\}\cdot\left\{\begin{matrix}\cos(p\Omega a).\sin(\overline{p\Omega+q\mu}.t).\sin^2\left(q\cdot\frac{\pi}{2}\right)\\ +\\ \sin(p\Omega a).\cos(\overline{p\Omega+q\mu}.t).\cos^2\left(q\cdot\frac{\pi}{2}\right)\end{matrix}\right\}\right]$	$\frac{\emptyset}{\pi}\cdot\sin(\mu t)$ $+\frac{2}{\pi}\sum_{p=1}^{\infty}\left[\sum_{q=-\infty}^{\infty}\left\{\frac{J_q(p\emptyset)}{p}\right\}\cdot\left\{\begin{matrix}\cos(p\Omega a).\sin(\overline{p\Omega+q\mu}.t).\sin^2\left(q\cdot\frac{\pi}{2}\right)\\ +\\ \sin(p\Omega a).\cos(\overline{p\Omega+q\mu}.t).\cos^2\left(q\cdot\frac{\pi}{2}\right)\end{matrix}\right\}\right]$

selected from the previous sections. In all cases the recurrence-frequency is $F_r = \frac{\Omega}{2\pi} = \frac{1}{T}$.

All the physically time-modulated epochs are assumed to have the same maximum time-durations $\pm\epsilon$; but changes of notation are made so that the depth of modulation is expressed as a phase-deviation $\pm\phi_d$ (or simply $\pm\phi$), relative to a recurrence-period, or as a frequency-deviation $\pm f_d = \pm\,\omega_d/2\pi$, where

$$\phi_d = \Omega\epsilon = 2\pi\left(\frac{\epsilon}{T}\right) = \frac{\omega_d}{\mu} = \frac{f_d}{f_\mu}; \quad \ldots \ldots \quad (125)$$

also, the modulation frequency, f_μ, sometimes appears as (mF_r), where

$$m = \frac{\mu}{\Omega} = \frac{f_\mu}{F_r}, \quad \ldots \ldots \ldots \ldots \quad (126)$$

so that the time-deviation, $\pm\epsilon$, also represents a phase-deviation $\pm(m\phi_d)$ relative to a modulation-period

$$(m\phi_d) = \mu\epsilon = 2\pi\left(\frac{\epsilon}{f_\mu^{-1}}\right) = \frac{\omega_d}{\Omega} = \frac{f_d}{F_r}. \quad \ldots \ldots \quad (127)$$

Apart from this change of notation, the results for Ph.M. and F.M. are identical, the signal-function being proportional to $(\phi_d \,.\, \sin\mu t)$ for the Ph.M. (and W.M.) pulses, and to $(f_d \,.\, \cos\mu t)$ for the F.M. pulses.

Both types of A.M. pulses are periodically recurrent, the synchronous type having a (virtual) synchronously time-modulated scanning pulse of maximum time-deviations $\pm\eta$.

All the W.M. cases have phase-modulated edges. The asymmetric W.M. pulses have one unmodulated edge, and are taken as having zero unmodulated width. The symmetric W.M. pulses are cases of formula (109), *i. e.* the two edges are individually and locally scanned, and are of finite unmodulated width, $2a$, (since there would be a significant loss of generality in taking this to be zero); and the D.C. (but not A.C.) contributions of the background of unmodulated pulses, namely $2a/T$, are omitted. (The factors, $\sin^2(q\,.\,\pi/2)$ and $\cos^2(q\,.\,\pi/2)$, which appear in expansions in the symmetric W.M. cases, simply serve to select the odd terms, or even terms, respectively, from these expansions).

In all cases, the terms of zero order ($p=0$) are displayed separately; the remaining carriers ($p=1, 2, 3, \ldots$) with their upper and lower sideband structures, being lumped together.

(4.4) The corresponding cases of double tone modulations are summarized in Table II. The signal function is taken to be proportional to

$$(\phi_1 \sin\mu_1 t + \phi_2 \sin\mu_2 t)$$

for the Ph.M. and W.M. cases, and to $(f_{d_1} \cos\mu_1 t + f_{d_2} \cos\mu_2 t)$ for the F.M. cases. For the synchronous A.M. case, the time-deviations η_1, η_2, of the (virtual) scanning pulse, for the two frequency components are taken to be proportional to their depths of A.M., ρ_1, ρ_2, for the physical pulse, so that ρ/η may be written for either ρ_1/η_1, or ρ_2/η_2.

TABLE II.

Double Tone Modulation

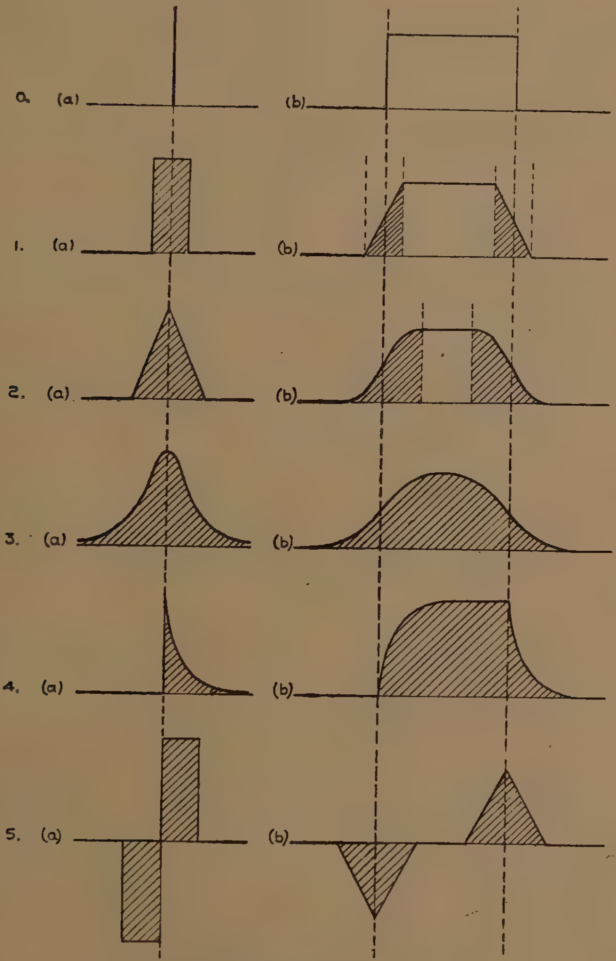
TYPE	PERIODIC SCAN	SYNCHRONOUS SCAN
Ph. M.	$\frac{1}{T}\left[1+2\sum_{q_1=0}^{\infty}{}'\sum_{q_2=0}^{\infty}{}' J_{q_1}\left(\overline{q_1m_1+q_2m_2}\cdot\varnothing_1\right)\cdot J_{q_2}\left(\overline{q_1m_1+q_2m_2}\cdot\varnothing_2\right)\cdot\cos\left(\overline{q_1\mu_1+q_2\mu_2}\cdot t\right)\right]$ $+\frac{2}{T}\sum_{p=1}^{\infty}\sum_{q_1=-\infty}^{\infty}\sum_{q_2=-\infty}^{\infty}J_{q_1}\left(\overline{p+q_1m_1+q_2m_2}\cdot\varnothing_1\right)\cdot J_{q_2}\left(\overline{p+q_1m_1+q_2m_2}\cdot\varnothing_2\right)\cdot\cos\left(\overline{p\Omega+q_1\mu_1+q_2\mu_2}\cdot t\right)$	$\left[\frac{1}{T}+f_{\mu_1}\varnothing_1\cdot\cos(\mu_1 t)+f_{\mu_2}\varnothing_2\cdot\cos(\mu_2 t)\right]$ $+\frac{2}{T}\sum_{p=1}^{\infty}\sum_{q_1=-\infty}^{\infty}\sum_{q_2=-\infty}^{\infty}\left(1+\frac{q_1m_1+q_2m_2}{p}\right)\cdot J_{q_1}(p\varnothing_1)\cdot J_{q_2}(p\varnothing_2)\cdot\cos\left(\overline{p\Omega+q_1\mu_1+q_2\mu_2}\cdot t\right)$
F. M.	$F_r\left[1+2\sum_{q_1=0}^{\infty}{}'\sum_{q_2=0}^{\infty}{}' J_{q_1}\left(\frac{\overline{q_2m_2}}{m_1}+q_1\cdot\frac{f_{d_1}}{F_r}\right)\cdot J_{q_2}\left(\frac{\overline{q_1m_1}}{m_2}+q_2\cdot\frac{f_{d_2}}{F_r}\right)\cdot\cos\left(\overline{q_1\mu_1+q_2\mu_2}\cdot t\right)\right]$ $+2F_r\sum_{p=1}^{\infty}\sum_{q_1=-\infty}^{\infty}\sum_{q_2=-\infty}^{\infty}J_{q_1}\left(\frac{\overline{p+q_2m_2}}{m_1}+q_1\cdot\frac{f_{d_1}}{F_r}\right)\cdot J_{q_2}\left(\frac{\overline{p+q_1m_1}}{m_2}+q_2\cdot\frac{f_{d_2}}{F_r}\right)\cdot\cos\left(\overline{p\Omega+q_1\mu_1+q_2\mu_2}\cdot t\right)$	$\left[F_r+f_{d_1}\cdot\cos(\mu_1 t)+f_{d_2}\cdot\cos(\mu_2 t)\right]$ $+2F_r\sum_{p=1}^{\infty}\sum_{q_1=-\infty}^{\infty}\sum_{q_2=-\infty}^{\infty}\left(1+\frac{q_1m_1+q_2m_2}{p}\right)\cdot J_{q_1}\left(\frac{p}{m_1}\cdot\frac{f_{d_1}}{F_r}\right)\cdot J_{q_2}\left(\frac{p}{m_2}\cdot\frac{f_{d_2}}{F_r}\right)\cdot\cos\left(\overline{p\Omega+q_1\mu_1+q_2\mu_2}\cdot t\right)$
A.M.	$\frac{1}{T}\left[1+\rho_1\cdot\mathrm{SIN}(\mu_1 t)+\rho_2\cdot\mathrm{SIN}(\mu_2 t)\right]$ $+\frac{1}{T}\sum_{p=1}^{\infty}\left[2\cos(p\Omega t)+\rho_1\left\{\mathrm{SIN}\left(\overline{p\Omega+\mu_1}\cdot t\right)-\mathrm{SIN}\left(\overline{p\Omega-\mu_1}\cdot t\right)\right\}+\rho_2\left\{\mathrm{SIN}\left(\overline{p\Omega+\mu_2}\cdot t\right)-\mathrm{SIN}\left(\overline{p\Omega-\mu_2}\cdot t\right)\right\}\right]$	$\frac{1}{T}\left[1+2\left(\frac{\rho}{\eta}\right)\sum_{q_1=0}^{\infty}{}'\sum_{q_2=0}^{\infty}{}'\left\{\frac{J_{q_1}\left(\overline{q_1\mu_1+q_2\mu_2}\cdot\eta_1\right)\cdot J_{q_2}\left(\overline{q_1\mu_1+q_2\mu_2}\cdot\eta_2\right)}{(q_1\mu_1+q_2\mu_2)}\right\}\cdot\mathrm{SIN}\left(\overline{q_1\mu_1+q_2\mu_2}\cdot t\right)\right]$ $+\frac{2}{T}\sum_{p=1}^{\infty}\left[\cos(p\Omega t)+\left(\frac{\rho}{\eta}\right)\sum_{q_1=0}^{\infty}{}'\sum_{q_2=0}^{\infty}{}'\left\{\frac{J_{q_1}\left(\overline{q_1\mu_1+q_2\mu_2}\cdot\eta_1\right)\cdot J_{q_2}\left(\overline{q_1\mu_1+q_2\mu_2}\cdot\eta_2\right)}{(q_1\mu_1+q_2\mu_2)}\right\}\cdot\left\{\begin{matrix}\mathrm{SIN}\left(\overline{p\Omega+q_1\mu_1+q_2\mu_2}\cdot t\right)\\ -\\ \mathrm{SIN}\left(\overline{p\Omega-q_1\mu_1-q_2\mu_2}\cdot t\right)\end{matrix}\right\}\right]$
ASYMMETRIC W. M.	$\frac{1}{\pi}\sum_{q_1=0}^{\infty}{}'\sum_{q_2=0}^{\infty}{}'\left\{\frac{J_{q_1}\left(\overline{q_1m_1+q_2m_2}\cdot\varnothing_1\right)\cdot J_{q_2}\left(\overline{q_1m_1+q_2m_2}\cdot\varnothing_2\right)}{(q_1m_1+q_2m_2)}\right\}\cdot\mathrm{SIN}\left(\overline{q_1\mu_1+q_2\mu_2}\cdot t\right)$ $+\frac{1}{\pi}\sum_{p=1}^{\infty}\left[\left\{\sum_{q_1=-\infty}^{\infty}\sum_{q_2=-\infty}^{\infty}\left(\frac{J_{q_1}\left(\overline{p+q_1m_1+q_2m_2}\cdot\varnothing_1\right)\cdot J_{q_2}\left(\overline{p+q_1m_1+q_2m_2}\cdot\varnothing_2\right)}{(p+q_1m_1+q_2m_2)}\right)\cdot\mathrm{SIN}\left(\overline{p+q_1\mu_1+q_2\mu_2}\cdot t\right)\right\}-\frac{\mathrm{SIN}(p\Omega t)}{p}\right]$	$\frac{1}{2\pi}\left[\varnothing_1\cdot\mathrm{SIN}(\mu_1 t)+\varnothing_2\cdot\mathrm{SIN}(\mu_2 t)\right]$ $+\frac{1}{\pi}\sum_{p=1}^{\infty}\frac{1}{p}\left[\left\{\sum_{q_1=-\infty}^{\infty}\sum_{q_2=-\infty}^{\infty}J_{q_1}(p\varnothing_1)\cdot J_{q_2}(p\varnothing_2)\cdot\mathrm{SIN}\left(\overline{p\Omega+q_1\mu_1+q_2\mu_2}\cdot t\right)\right\}-\mathrm{SIN}(p\Omega t)\right]$
SYMMETRIC W. M.	$\frac{2}{\pi}\sum_{q_1=0}^{\infty}{}'\sum_{q_2=0}^{\infty}{}'\left\{\frac{J_{q_1}\left(\overline{q_1m_1+q_2m_2}\cdot\varnothing_1\right)\cdot J_{q_2}\left(\overline{q_1m_1+q_2m_2}\cdot\varnothing_2\right)}{(q_1m_1+q_2m_2)}\right\}\cdot\mathrm{SIN}\left(\overline{q_1\mu_1+q_2\mu_2}\cdot t\right)\cdot\mathrm{SIN}^2\left(\overline{q_1+q_2}\cdot\frac{\pi}{2}\right)$ $+\frac{2}{\pi}\sum_{p=1}^{\infty}\left[\sum_{q_1=-\infty}^{\infty}\sum_{q_2=-\infty}^{\infty}\left\{\frac{J_{q_1}\left(\overline{p+q_1m_1+q_2m_2}\cdot\varnothing_1\right)J_{q_2}\left(\overline{p+q_1m_1+q_2m_2}\cdot\varnothing_2\right)}{(p+q_1m_1+q_2m_2)}\right\}\left\{\begin{matrix}\mathrm{SIN}(p\Omega a)\cdot\cos\left(\overline{p\Omega+q_1\mu_1+q_2\mu_2}\cdot t\right)\cdot\cos^2\left(\overline{q_1+q_2}\cdot\frac{\pi}{2}\right)\\ +\\ \cos(p\Omega a)\cdot\mathrm{SIN}\left(\overline{p\Omega+q_1\mu_1+q_2\mu_2}\cdot t\right)\mathrm{SIN}^2\left(\overline{q_1+q_2}\cdot\frac{\pi}{2}\right)\end{matrix}\right\}\right]$	$\frac{1}{\pi}\left[\varnothing_1\cdot\mathrm{SIN}(\mu_1 t)+\varnothing_2\cdot\mathrm{SIN}(\mu_2 t)\right]$ $+\frac{2}{\pi}\sum_{p=1}^{\infty}\left[\sum_{q_1=-\infty}^{\infty}\sum_{q_2=-\infty}^{\infty}\left\{\frac{J_{q_1}(p\varnothing_1)\cdot J_{q_2}(p\varnothing_2)}{p}\right\}\left\{\begin{matrix}\mathrm{SIN}(p\Omega a)\cdot\cos\left(\overline{p\Omega+q_1\mu_1+q_2\mu_2}\cdot t\right)\cdot\cos^2\left(\overline{q_1+q_2}\cdot\frac{\pi}{2}\right)\\ +\\ \cos(p\Omega a)\cdot\mathrm{SIN}\left(\overline{p\Omega+q_1\mu_1+q_2\mu_2}\cdot t\right)\cdot\mathrm{SIN}^2\left(\overline{q_1+q_2}\cdot\frac{\pi}{2}\right)\end{matrix}\right\}\right]$

No further explanation of this table is required except that the symbol $\overset{\infty}{\underset{q_1=0}{\Sigma'}}\ \overset{\infty}{\underset{q_2=0}{\Sigma'}}$ denotes that the term corresponding to the integer-pair (q_1, q_2) $=(0, 0)$ is to be omitted from the summation.

(5) PULSE SHAPE FACTORS.

(5.1) The results so far obtained apply only to the ideal pulse waveforms, of unit-content improper-pulses (for amplitude and time-modulated pulses of fixed shape), and of unit-amplitude rectangular pulses (for

Fig. 3.



Reference waveforms.

(*a*) PH.M., F.M. and A.M. pulse waveforms., (*b*) W.M. pulse waveforms.

(0) Ideal narrow pulse and rectangular waveforms.
(1) Rectangular and trapezoidal waveforms.
(2) (Symmetric) triangular pulse and confluent parabolic edge waveforms.
(3) Gaussian pulse and gaussian-integral edge waveforms.
(4) Exponential pulse and exponential edge waveforms.
(5) Rectangular doublet pulse and (symmetric) triangular edge waveforms.

TABLE III.

Pulse Shape Factors.

REFERENCE PULSE WAVEFORM FOR Ph.M., F.M., & A.M. PULSES. $f_0(t)$:	REFERENCE GROWTH WAVEFORM FOR W.M. PULSES. $f_1(t) = \int_{-\infty}^{t} f_0(t)\,dt$:	PULSE SHAPE FACTOR. $g_0(\omega)$:
1. RECTANGULAR. $\left\{\begin{array}{ll} \mathcal{R} & ,(-\tau < t < \tau) \\ 0 & ,(\text{ELSEWHERE}) \end{array}\right\}.$ WHERE $A_0 = 2(\mathcal{R}\tau)$	1. TRAPEZOIDAL $\left\{\begin{array}{ll} 0 & ,(t < -\tau) \\ \mathcal{R}(t+\tau) & ,(-\tau < t < \tau) \\ A_0 & ,(t > \tau) \end{array}\right\}$	1. $A_0 \cdot \left\{\frac{\text{SIN}(\omega\tau)}{(\omega\tau)}\right\}$
2. TRIANGULAR (SYMMETRIC). $\left\{\begin{array}{ll} \mathcal{R}(1+t/2\tau) & ,(-2\tau < t < 0) \\ \mathcal{R}(1-t/2\tau) & ,(0 < t < 2\tau) \\ 0 & ,(\text{ELSEWHERE}) \end{array}\right\}.$ WHERE $A_0 = 2(\mathcal{R}\tau)$.	2. CONFLUENT PARABOLIC. $\left\{\begin{array}{ll} 0 & ,(t < -2\tau) \\ \left\{\frac{\mathcal{R}(t+2\tau)^2}{4\tau}\right\} & ,(-2\tau < t < 0) \\ A_0 - \left\{\frac{\mathcal{R}(t-2\tau)^2}{4\tau}\right\} & ,(0 < t < 2\tau) \\ A_0 & ,(t > 2\tau) \end{array}\right\}$	2. $A_0 \cdot \left\{\frac{\text{SIN}(\omega\tau)}{(\omega\tau)}\right\}^2$
3. GAUSSIAN. $\mathcal{R}e^{-\frac{1}{2}(t/\tau)^2}$, (ALL t), WHERE $A_0 = \sqrt{2\pi}(\mathcal{R}\tau)$.	3. GAUSSIAN INTEGRAL. $A_0 \cdot G(t/\tau) = \frac{A_0}{\sqrt{2\pi}} \cdot \int_{-\infty}^{(t/\tau)} e^{-\frac{1}{2}x^2}\,dx$ (ALL t).	3. $A_0 \cdot e^{-\frac{1}{2}(\omega\tau)^2}$
4. EXPONENTIAL. $\left\{\begin{array}{ll} 0 & ,(t < 0) \\ \mathcal{R}e^{-t/\tau} & ,(t > 0) \end{array}\right\}$ WHERE $A_0 = (\mathcal{R}\tau)$	4. EXPONENTIAL. $\left\{\begin{array}{ll} 0 & ,(t < 0) \\ A_0(1-e^{-t/\tau}) & ,(t > 0) \end{array}\right\}$	4. $\frac{A_0}{(1 + j.\omega\tau)}$
5. RECTANGULAR DOUBLET. $\left\{\begin{array}{ll} -\mathcal{R}, & (-2\tau < t < 0) \\ \mathcal{R}, & (0 < t < 2\tau) \\ 0, & (\text{ELSEWHERE}) \end{array}\right\}.$ WHERE $A_1 = 4(\mathcal{R}\tau^2)$	5. COMPLEMENTARY TRIANGULAR (SYMMETRIC) $\left\{\begin{array}{ll} -2\mathcal{R}\tau(1+t/2\tau), & (2\tau < t < 0) \\ -2\mathcal{R}\tau(1-t/2\tau), & (0 < t < 2\tau) \\ 0 & ,(\text{ELSEWHERE}) \end{array}\right\}$	5. $-j\omega . A_1 . \left\{\frac{\text{SIN}(\omega\tau)}{(\omega\tau)}\right\}^2$

width-modulated pulses of fixed growth and decay waveforms). To pass from these ideal results to the general case, it is only necessary to multiply each frequency component, $e^{j\omega t}$, by the further shape-factor, $g_0(\omega)$,

$$g_0(\omega)=\int_{-\infty}^{\infty} f_0(t) \,.\, e^{-j\omega t} \,.\, dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

where $f_0(t)$ is the actual reference waveform for the pulses of fixed shape, or where $\pm\int_{-\infty}^{t} f_0(t)dt$ is the actual growth, or decay, waveform for the width-modulated pulses.

If $g_0(\omega)$ is real (as for symmetric reference pulses), then this factor may be applied directly to the real form of the Fourier expansion, but, more generally, the individual terms in the real form of the expansion must be modified in both magnitude and phase, in the usual way, by $g_0(\omega)$.

A few examples (see fig. 3) are listed in Table III., in terms of the content A_0 of the reference-pulse $f_0(t)$, or of the virtual final amplitude A_0, of the growth-waveform $f_1(t)=\int_{-\infty}^{t} f_0(t)dt$ of the W.M. pulses ; and in terms of some time-parameter, τ, for these reference waveforms. When A_0 is zero (as for a "doublet-type" pulse, $f_0(t)$), the "edges" of the W.M. pulses, which have the form $\pm f_1(t)$, may be regarded as separate pulses, possibly overlapping, and the content of one of these pulses, A_1 (or, first-moment for the doublet pulse) may be used as an alternative parameter.

(5.2) Note that the last column of Table III. will also serve to give scanning-factors, $\sigma(\omega)$, for Type I interval-scanning, when the (reverse) scanning-functions, $s(-t)$, are given in the first column.

ACKNOWLEDGMENT.

The author is indebted to Siemens Brothers & Co. Ltd., for permission to publish this paper.

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(2) A. Gray, C. B. Mathews and T. M. MacRobert, 'A Treatise on Bessel Functions and their Applications to Physics' (Macmillan, 1931).

LXXXII. *On the Relation between certain Distributions and their corresponding Patterson Distributions.*

By Dorothy Wrinch, M.A. (Camb.), D.Sc. (Oxon. and Lond.), Department of Physics, Smith College *.

[Received September 20, 1947.]

This communication discusses two series of finite distributions in one-dimensional space and the corresponding Patterson distributions. It constitutes part of a general investigation into the relation between distributions and their Pattersons which is being undertaken as a necessary preliminary to the interpretation of Patterson and Patterson-Harker maps of megamolecular crystals, computed from intensity data obtained from X-ray studies of such crystals †.

It has recently been shown ‡ that Fourier transforms provide a convenient method for deriving the Patterson of a given distribution and for studying the relation between the morphologies of distributions so related §. For if a distribution $\rho(x)$ of total weight w, in " atomic " space s, has the transform T(X), in S, the space reciprocal to s, then

$$\mathrm{T(X)}=\int_{-\infty}^{\infty} e^{-2\pi i \mathrm{X}x}\frac{\rho(x)dx}{w},$$

$$\frac{\rho(x)}{w}=\int_{-\infty}^{\infty} e^{2\pi i x\mathrm{X}}\,\mathrm{T(X)}d\mathrm{X},$$

where

$$w=\int_{-\infty}^{\infty} \rho(x)dx.$$

Since the corresponding Patterson has the transform $|\mathrm{T(X)}|^2$, it follows that

$$\frac{\rho_p(x)}{w_p}=\int_{-\infty}^{\infty} e^{2\pi i x\mathrm{X}}|\mathrm{T(X)}|^2d\mathrm{X},$$

where w_p, the total weight of $\rho_p(x)$, is w^2.

We first discuss a homologous series of distributions $\rho(x, 0)$, $\rho(x, 1)$, $\rho(x, 2)$, . . . $\rho(x, n)$, . . . which have the interesting property that the Patterson of any member of the series is also a member of the series, the first member being its own Patterson. The second series discussed is obtained by interchanging " atomic " space and reciprocal space.

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* Communicated by the Author.

† A. D. Booth and D. Wrinch, J. Chem. Phys. xiv (1946).

‡ D. Wrinch, 'Nature,' pp. 157, 226 (1946).

§ D. Wrinch, "Fourier Transforms and Structure Factors" (American Society for X-ray and Electron Diffraction, Monograph No. 2, 1946).

Let

$$F_n(X, A) = (1-|X|/A)^n, \quad 0$$

according as X^2 is less than or greater than A^2, A being a positive quantity. Further let

$$f_n(x, a) = \tfrac{1}{2}a(n+1)\int_{-\infty}^{\infty} \cos 2\pi xX F_n(X, A)\,dX,$$

where $a = 1/A$. F-functions of all orders take the value unity at the origin: in the definition of the f-functions, the multiplier is so chosen that these functions of all orders also take the value unity at their origin. The fact that

$$(n+1)F_n(X, A) = 2A\int_{-\infty}^{\infty} \cos 2\pi Xx f_n(x, a)\,dx,$$

may also be recorded.

We first take s as "atomic" space and S as its reciprocal space and write

$$T(X, A, n) = F_n(X, A),$$

$$\rho(x, a, n) = 2f_n(x, a)/(n+1)a,$$

thereby obtaining a distribution of total weight unity. It then follows that

$$\rho_p(x, a, n) = \int_{-\infty}^{\infty} \cos 2\pi xX F_{2n}(X, A)\,dX = \rho(x, a, 2n).$$

To find the form of the ρ-functions, we first remark that

$$f_0(x, a) = \tfrac{1}{2}a\int_{-\infty}^{\infty} \cos 2\pi xX F_0(X, A)\,dX$$

$$= a\int_0^A \cos 2\pi xX\,dX = \frac{\sin(2\pi x/a)}{2\pi x/a}.$$

Thus the distribution

$$\rho(x, a, 0) = (2/a)(\sin u)/u,$$

where $u = 2\pi x/a$, is its own Patterson.

We next remark that

$$u^2 f_1(x, a) = 2\,!\,(1-\cos u),$$

and that the recurrence relation

$$u^2 f_{n+2}(x, a) = (n+2)(n+3)\{1-f_n(x, a)\},$$

permits the deduction that

$$u^{n+1}f_n(x, a) = (n+1)\,!\left\{\frac{u^{n-1}}{(n-1)\,!} - \frac{u^{n-3}}{(n-3)\,!} + \ldots - (-)^{\frac{1}{2}n}(u-\sin u)\right\},$$

when n is even and

$$u^{n+1}f_n(x, a) = (n+1)\,!\left\{\frac{u^{n-1}}{(n-1)\,!} - \frac{u^{n-3}}{(n-3)\,!} + \ldots + (-)^{\frac{1}{2}(n-1)}(1-\cos u)\right\},$$

when n is odd.

The distributions and their Pattersons, corresponding to $n=0, 1, \ldots,$ may be assembled as follows, the distributions on the left and the Pattersons on the right, u being written in place of $2\pi x/a$ as above :—

$n=0$	$(2/a) \sin u/u$	$(2/a) \sin u/u,$
$n=1$	$(2/a) 1\,!\,(1-\cos u)/u^2$	$(2/a) 2\,!\,(u-\sin u)/u^3,$
$n=2$	$(2/a) 2\,!\,(u-\sin u)/u^3$	$(2/a) 4\,!\,(\sin u-u+u^3/3\,!)/u^5,$
$n=3$	$(2/a) 3\,!\,(\cos u-1+u^2/2\,!)/u^4$	$(2/a) 6\,!\,(u-u^3/3\,!+u^5/5\,!-\sin u)/u^7,$

and so on. All the distributions are of infinite extent, though of finite—actually unit—weight. The transforms, on the other hand, are all of finite extent. As usual, increasing the scale of the transform by giving it an extent of $2\mathrm{A}(1+\mathrm{K})$ instead of $2\mathrm{A}$ contracts the scale of the distribution by the same factor, giving it a parameter $a(1-k)$ instead of a, where $(1-k)(1+\mathrm{K})=1$. Thus the distribution which is its own Patterson has its first zero at $2x=a$ when its transform extends from $-\mathrm{A}$ to $+\mathrm{A}$: as the extent of the transform is increased, the size of this zero is decreased in inverse proportion.

We may record the fact that at the origin the ratio of $\rho_p(x, a, n)$ to $\rho(x, a, n)$ is $n+1 : 2n+1$, and that the ratio of their second derivatives is $(n+1)(n+2)(n+3) : (2n+1)(2n+2)(2n+3)$ to indicate the relative behaviour of the distributions and their Pattersons in the neighbourhood of the origin.

We now take S as " atomic " space and s as its reciprocal space and write

$$\mathrm{T}(x, a, n)=f_n(x, a),$$

$$\rho(\mathrm{X}, \mathrm{A}, n)=\tfrac{1}{2}(n+1)\mathrm{F}_n(\mathrm{X}, \mathrm{A}, n)/\mathrm{A},$$

thereby obtaining a distribution of total weight unity. It then follows that

$$\rho_p(\mathrm{X}, \mathrm{A}, n)=\int_{-\infty}^{\infty} \cos 2\pi \mathrm{X} x f_n^2(x, a)dx.$$

To find the form of the ρ_p-functions, we remark that we already know that

$$\int_{-\infty}^{\infty} \cos 2\pi \mathrm{X} x f_m(x, b)dx=\tfrac{1}{2}(m+1)\mathrm{F}_m(\mathrm{X}, \mathrm{B})/\mathrm{B},$$

where m is any integer (or zero), b is any positive quantity, and $\mathrm{B}=1/b$. Hence to obtain the Patterson distributions, it is sufficient to throw the functions $f_n^2(x, a)$ into a linear series of $f_m(x, b)$-functions. This is readily accomplished in the following manner.

Since

$$f_0(x, a)=(\sin 2\pi x/a)/(2\pi x/a),$$

it follows that

$$f_0^2(x, a)=2\,!\,(1-\cos (4\pi x/a))/(4\pi x/a)^2=f_1(x, \tfrac{1}{2}a).$$

Similarly we find that

$$3f_1^2(x, a) = 4f_3(x, \tfrac{1}{2}a) - f_3(x, a),$$
$$5f_2^2(x, a) = 8f_5(x, \tfrac{1}{2}a) - 3f_4(x, a),$$
$$35f_3^2(x, a) = 64f_7(x, \tfrac{1}{2}a) - 28f_5(x, a) - f_7(x, a),$$
$$63f_4^2(x, a) = 128f_9(x, \tfrac{1}{2}a) - 60f_6(x, a) - 5f_8(x, a).$$

The general result, written with binomial coefficients, has the form

$$\frac{1}{2}\binom{2n+2}{n+1} f_n^2(x, a) = 2^{2n} f_{2n+1}(x, \tfrac{1}{2}a) - \binom{2n+2}{n-1} f_{n+2}(x, a) - \binom{2n+2}{n-3} f_{n+4}(x, a) - \dots - \left[\binom{2n+2}{1} f_{2n}(x, a) \text{ or } \binom{2n+1}{0} f_{2n+1}(x, a)\right],$$

according as n is even or odd. The fact that $f_n^2(0, a)$ is necessarily unity shows itself in the form of the well-known identity

$$2^{2n} = \binom{2n+2}{n-1} + \binom{2n+2}{n-3} + \dots + \left[\binom{2n+2}{1} \text{ or } \binom{2n+2}{0}\right].$$

The preliminaries are now completed and the Patterson distribution, obtained by integrating term by term, emerges in the form

$$\frac{(2n+1)\,!}{\{(n+1)\,!\}^2} A\rho_p(X, A, n) = 2^{2n-1} F_{2n+1}(X, 2A) - \binom{2n+1}{n-1} F_{n+2}(X, A) - \binom{2n+1}{n-3} F_{n+4}(X, A) - \dots - \left[\binom{2n+1}{1} F_{2n}(X, A) \text{ or } \binom{2n+1}{0} F_{2n+1}(X, A)\right],$$

according as n is even or odd. Since

$$2^{2n-1} = \binom{2n+1}{n-1} + \binom{2n+1}{n-3} + \dots + \left[\binom{2n+1}{1} \text{ or } \binom{2n+1}{0}\right] + \frac{1}{2}\binom{2n}{n},$$

it follows that

$$\rho_p(0, A, n) = \tfrac{1}{4}(n+1)^2/(2n+1)A = \{(n+1)/(2n+1)\}\rho(0, A, n),$$

since $\rho(0, A, n) = \frac{1}{2}(n+1)/A$.

Differentiating the expression for ρ_p term by term, we find that

$$\frac{(2n)\,!}{\{(n+1)\,!\}^2} A^2 \rho_p'(X, A, n) = 2^{2n-2} F_{2n}(X, 2A) - \binom{2n}{n-1} F_{n+1}(X, A) - \binom{2n}{n-3} F_{n+3}(X, A) - \dots - \left[\binom{2n}{1} F_{2n-1}(X, A) \text{ or } \binom{2n}{0} F_{2n}(X, A)\right],$$

so that $\rho_p'(0, \mathrm{A}, n)=0$ in view of the identity

$$2^{2n-2}=\binom{2n}{n-1}+\binom{2n}{n-3}+\ldots.+\left[\binom{2n}{1} \text{ or } \binom{2n}{0}\right].$$

Finally, we record that

$$\frac{(2n-1)\,!}{\{(n+1)\,!\}^2}\mathrm{A}^3\rho_p''(\mathrm{X}, \mathrm{A}, n)=2^{2n-3}\mathrm{F}_{2n-1}(\mathrm{X}, 2\mathrm{A})-\binom{2n-1}{n-1}\mathrm{F}_n(\mathrm{X}, \mathrm{A})$$

$$-\binom{2n-1}{n-3}\mathrm{F}_{n+2}(\mathrm{X}, \mathrm{A})-\ldots.-\left[\binom{2n-1}{1}\mathrm{F}_{2n-2}(\mathrm{X}, \mathrm{A})\right.$$

$$\left.\text{or}\quad\binom{2n-1}{0}\mathrm{F}_{2n-1}(\mathrm{X}, \mathrm{A})\right],$$

according as n is even or odd, and that, in view of the identity

$$2^{2n-3}=\binom{2n-1}{n-1}+\binom{2n-1}{n-3}+\ldots.+\left[\binom{2n-1}{1}\right.$$

$$\left.\text{or}\quad\binom{2n-1}{0}\right]+\frac{1}{2}\binom{2n-2}{n-1},$$

it follows that

$$\mathrm{A}^3\rho_p''(0, \mathrm{A}, n)=-\tfrac{1}{2}n^2(n+1)^2/(2n-1).$$

We may exhibit the results for $n=0, 1\ldots$ in their simplest forms, as in the previous case :—

$n=0$	$(1/2\mathrm{A})$	$(1/2\mathrm{A})(1-\lvert\mathrm{X}\rvert/2\mathrm{A})$,
$n=1$	$(2/2\mathrm{A})(1-\lvert\mathrm{X}\rvert/\mathrm{A})$	$(2/3\mathrm{A})\{2(1-\lvert\mathrm{X}\rvert/2\mathrm{A})^3-(1-\lvert\mathrm{X}\rvert/\mathrm{A})^3\}$,
$n=2$	$(3/2\mathrm{A})(1-\lvert\mathrm{X}\rvert\mathrm{A})^2$	$(3/10\mathrm{A})\{8(1-\lvert\mathrm{X}\rvert/2\mathrm{A})^5-5(1-\lvert\mathrm{X}\rvert/\mathrm{A})^4\}$,
$n=3$	$(4/2\mathrm{A})(1-\lvert\mathrm{X}\rvert/\mathrm{A})^3$	$(4/35\mathrm{A})\{32(1-\lvert\mathrm{X}\rvert/2\mathrm{A})^7-21(1-\lvert\mathrm{X}\rvert/\mathrm{A})^5$ $-(1-\lvert\mathrm{X}\rvert/\mathrm{A}^7\}$,
$n=4$	$(5/2\mathrm{A})(1-\lvert\mathrm{X}\rvert/\mathrm{A})^4$	$(5/126\mathrm{A})\{128(1-\lvert\mathrm{X}\rvert/2\mathrm{A})^9-84(1-\lvert\mathrm{X}\rvert/\mathrm{A})^6$ $-9(1-\lvert\mathrm{X}\rvert/\mathrm{A})^8\}$,

and so on. In all these expressions, a function $(1-|\mathrm{X}|/2\mathrm{A})^m$ is restricted to the range $-2\mathrm{A}$ to $2\mathrm{A}$ and a function $(1-|\mathrm{X}|/\mathrm{A})^m$ to the range $-\mathrm{A}$ to A. Thus all the distributions are of finite extent 2A and all the Pattersons of extent 4A. The transforms, on the other hand, are of infinite extent, though of finite weight.

Throughout this investigation, we have concerned ourselves only with distributions of unit weight, with the corresponding Pattersons also of unit weight. In this way the study of the comparative morphology of distributions and their Pattersons is facilitated. However, each and every result can be generalised by multiplying the distribution by any quantity and the corresponding Patterson by the square of this quantity.

LXXXIII. *Fading of Short-Wave Radio Signals and Space-Diversity Reception.*—Part I.

By Dr. S. S. Banerjee, D.Sc., and Prof. G. C. Mukerjee, M.Sc., M.A.I.E.E., Engineering College, Benares Hindu University *.

[Received June 21, 1947.]

1. *Introduction.*

The existence of ionized layers in the upper atmosphere (ionosphere) was proved quantitatively by Appleton and Barnett (1), (2) in 1925 from the observations of fading of radio signals received in an aerial situated away from the transmitter. Since that time, attention of various investigators (3)–(8) has been drawn to the study of variation of intensity of received signals from a remote transmitter with a view to explore the conditions of the ionosphere suitable for establishing radio cummunications between two stations situated apart. In most of such experiments, medium waves have been employed and observations have been made either in presence of the ground wave or after its suppression. In case of short-wave transmissions, however, the variations of signal intensity have been recorded outside the reach of ground waves. The nature of variations of intensity of received signals has generally been observed to be irregular and such variations have been suggested to be due to random scattering from diffracting centres in the ionosphere by Ratcliffe and Pawsey (5). Variations of intensity of short-wave signals have often been found to be very quick and occasionally possessing regular characteristics. Adequate explanations, however, for the various types of fading observed during such investigations, particularly for short-wave transmissions, appear to be still very meagre.

In view of the above considerations, a detailed study of the fading of short-wave signals was considered to be useful, and preliminary reports on such observations made by us have already been published (9), (10). In the present communication we have exhibited with explanations wherever possible the occurrence of various types of fading patterns recorded for short-wave transmissions on wave-bands of 41, 31, 25, 19 and 16 metres, generally from All India Radio, Delhi, and sometimes from Calcutta, Ceylon, Australia and B.B.C. (London). For studying the finer details of variations of received signal intensities, continuous records of the observations were made in a much extended time scale than the one usually employed for such observations, and for this purpose an automatic ink recorder was constructed.

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* Communicated by the Author.

In order to explain the various patterns of fading observed, they have been divided into three main types, namely,

(1) irregular, quick and random variations of peaky type ;
(2) slow and comparatively regular (quasi-periodic) variations of less peaky type ; and
(3) smooth and regular variations of periodic nature.

Various types of fading enumerated above have been explained by measuring the angles of arrival of downcoming waves and using the ionospheric data recorded by the Research Department of All India Radio, Delhi. It may be mentioned that such experimental observations on fading of signals are useful in the development and understanding of the principle of space-diversity reception of radio signals. From theoretical considerations and preliminary experimental observations, it has been shown that space-diversity reception for long distances may be accomplished with vertically spaced aerials placed fairly close to each other, as compared to larger distances required for horizontally spaced aerials.

2. *Theoretical considerations.*

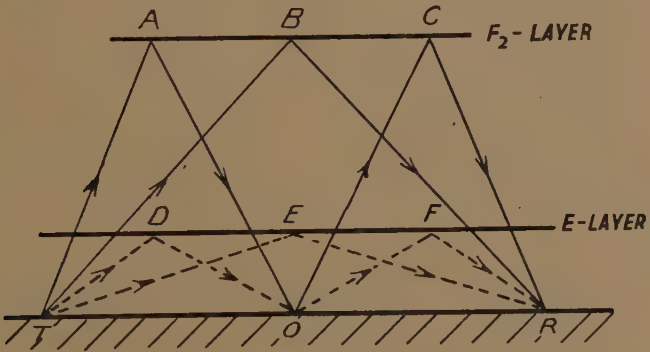
It is generally assumed that variations of intensity of radio signals received at a distance from the transmitter are of random nature and this has been explained by Ratcliffe and Pawsey with the help of distribution curves drawn for probable intensities of signal during a period of observations (5). It has been, however, reported by a few investigators (8) that Raleigh distribution for probable intensity does not agree very closely with the observed data on many occasions and particularly when the distance between the transmitter and receiver is great. The discrepancy between the observed and theoretical results obtained indicates that, during the total period of observation of these records, there have been occasions when the variation of signal intensity might not have remained random, but the intensity either remained fairly constant or varied with some regularity. It may be noted that such variations of intensity, which might have persisted only for a short period, can be observed if the records are made with an extended time scale as used in the present investigations.

As mentioned in the previous section, the fading patterns have been divded into three main types, namely, (1) peaky random, (2) slow random or quasi-periodic, and (3) periodic. It should be pointed out that all the above types of fading have been observed to persist not only for a short period, but also for fairly long periods extending sometimes to more than an hour. It has been further observed that the fading pattern of signals on a particular wave-band may change from one type to the other during the course of the day and sometimes within a very short time of a few minutes.

These types of fading may be explained to be due to single, double or multiple-spot reflections from the ionosphere. For example, the first or peaky random type of fading will be observed when there is only single-spot reflection from the ionosphere with no other reflections either from the same or any other layer. This is illustrated in fig. 1 by the paths of radio waves travelling between transmitter T and receiver R as TBR or TER. Second type of fading, namely, slow random or quasi-periodic, is possible when there are more than one-spot reflection either from one layer or from two different layers in the ionosphere as shown by other paths of travelling waves in fig. 1. In such cases the intensity of the signal at any moment will depend on the vector sum of components of intensities from different spots. Thus, this case may be compared with the resultant intensity which is obtained when two or more independent peaky random variations are combined together, as in space-diversity reception.

The last type of variation, which is fairly regular, can be explained to be due to interference fringes caused by either single and double reflections from one layer or two single reflections from two different layers, when one or both the layers are moving up or down very slowly. It should be mentioned, however, that in case the fading pattern is smooth and periodic, there will be, at a time, only two paths of received travelling waves from the ionosphere and the intensity of the waves sufficiently constant. This has been corroborated by measurement of angles of arrival of downcoming waves during the observations of fading of signals.

Fig. 1.



It may be pointed out that such slow movements of the ionospheric layers are difficult to observe by the usual " pulse method " of measuring the height of the ionospheric layers with cathode-ray oscillograph. Calculations show that the rate of movement of the layers for such cases may be of the order of 5 km./hr., whereas a fairly sensitive instrument for " pulse method " would measure the height of a layer with an approximation of ±10 km. Incidentally it may be observed that these fringes in fading records indicate the possibility of measuring fairly slow rate of change in height of the ionospheric layers.

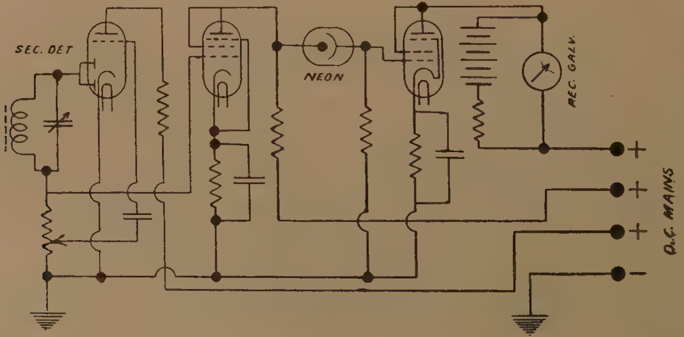
The above conditions of reflection for different types of fading of radio waves have been verified generally for 25- and 19-metre bands with vertically polarized waves. The first two patterns of fading can be observed at any time, particularly with higher wave-lengths, but the periodic nature of fading, more often observed either in the morning, after ground sunrise, or in the afternoon when the concentrations of the layers are either increasing or decreasing. It is more pronounced in the afternoon, presumably due to higher concentrations of the layers, permitting double reflections. Such patterns have been mostly found on 19-metre bands and occasionally on 25- and 16-metre bands. Typical records of various patterns of fading are shown in section 5.

3. *Experimental arrangements.*

During preliminary visual observations of variations of the intensity of received signals, a five-valve superhet receiver was built without incorporating automatic volume-control system. A sensitive low-resistance moving-coil mirror galvanometer was connected in the diode circuit of the second detector of the receiver for observing the fading of signals. Observations were made at the interval of 5 seconds and the deflection on scale could be measured for more than 50 cms.

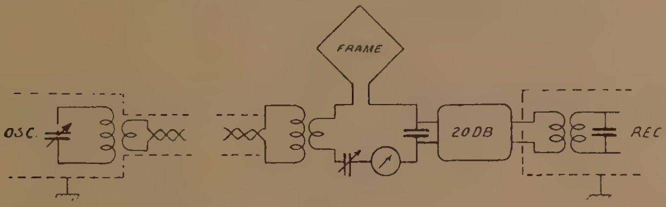
For continuous record of signal intensities an automatic recorder was constructed. A brief description of the apparatus is given below. The rectified voltage developed across the resistance in the diode circuit of the second detector of a seven-valve superhet receiver, without automatic volume control, was amplified by means of a two-stage direct-coupled amplifier with neon coupling. The amplified voltage was applied to a recording galvanometer attached with a writing-pen. The initial plate current in the galvanometer was neutralized by an external battery of about 30 volts. The circuit diagram of the neon-coupled amplifier with recording galvanometer and second detector of the receiver is shown in fig. 2. In the working position, the pen moved in a vertical plane over which 4-in. wide paper for recording was run by a self-governed electric motor. The vertical speed of paper could be varied from 2 to 6 cm./min. The deflection of the recording galvanometer was calibrated for field intensities by the following method :—

Fig. 2.



A shielded valve oscillator was linked to a system consisting of a frame aerial, a tuning condenser and a high-value fixed condenser, through a pair of loosely coupled shielded transmission lines. The circuit diagram of the arrangement is shown in fig. 3. The oscillator was kept at a distance so that it could not affect the receiver directly. The high-frequency current circulating in the aerial circuit was measured with the thermocouple-type radio-frequency milliammeter shown in the diagram.

Fig. 3.



The small high-frequency voltage, developed across the large-value condenser in the aerial circuit, was applied to the input coil of the receiver through a 20-db. attenuator, as shown in the diagram. The field-strengths developed were calculated from the knowledge of the constants of the frame aerial, and the deflections in the galvanometer were calibrated for different intensities. The induced effect of the frame aerial was reduced to minimum by adequate shielding and eliminated before calibration.

4. *The principle and method of measuring the angle of arrival of the downcoming waves.*

The principle of the method is based on the fact that the intensity of the received signal on a vertical aerial depends on the vertical component of the electric field associated with the downcoming wave and the intensity of the signal received on a frame aerial depends on the horizontal component of the magnetic field of the wave.

The above method was adopted by Appleton and Barnett (11) for measuring the angle of arrival of downcoming waves for medium-wave transmissions, in the presence of ground wave, when the signal was detected by a square-law detector. In the present method employed by us modification has been made for the linear detector used and the absence of ground wave, as shown below.

Let E and H be the electric and magnetic vectors respectively, associated with the downcoming radio wave as shown in fig. 4, and let ϕ be the angle of arrival of downcoming waves, that is, the angle subtended by the plane of propagation of the wave with the ground. It can be shown that the vertical aerial OV will be acted by an electric field $2E \cos \phi \sin wt$, where w is the pulsatance of the wave. A frame aerial directed towards the transmitter under the same conditions will be acted by a magnetic field $2H \sin wt$.

The instantaneous current flowing through a galvanometer in the second detector circuit of a superhet receiver connected with vertical aerial will therefore be given by

$$2\alpha K_v E \cos\phi \sin wt,$$

where α is constant of the detector characteristic and K_v is the constant of the vertical aerial.

The mean current (i_v) flowing through the galvanometer in the receiver with vertical aerial will, thus, be given by

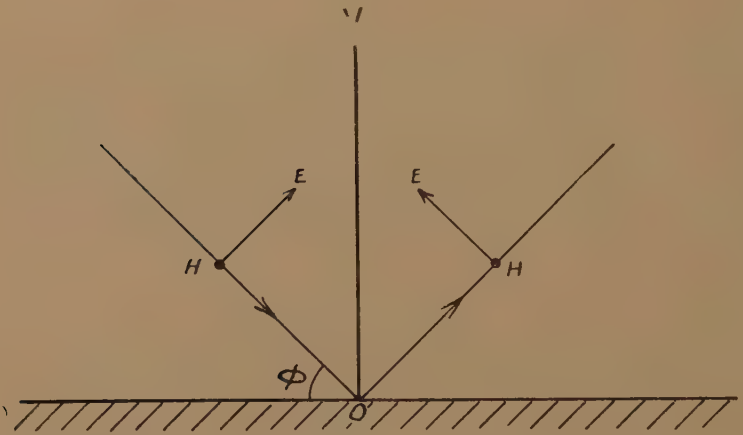
$$i_v = 2\alpha K_v E \cos\phi. \quad \dots \dots \dots \quad (1)$$

Similarly, it can be shown that the mean current in the galvanometer with a frame aerial will be given by

$$i_f = 2\alpha K_f H, \quad \dots \dots \dots \quad (2)$$

where suffix f indicates the values for frame aerial.

Fig. 4.



From equations (1) and (2) we get

$$\frac{i_v}{i_f} = \frac{K_v E \cos\phi}{K_f H} \quad \dots \dots \dots \quad (3)$$

If the sensitivities of the two receivers are made equal by arranging the two aerials in such a manner that the vertical electric vector from a wave travelling along the ground, without any sky-wave, gives the same current in both the galvanometers, then, as $E \equiv H$, we have from equation (3)

$$\frac{i_v}{i_f} = \cos\phi. \quad \dots \dots \dots \quad (4)$$

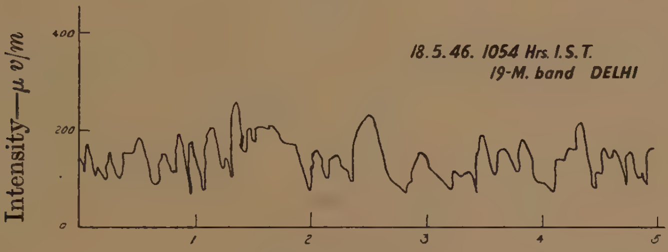
Thus, from equation (4), knowing the deflections in the two galvanometers simultaneously, the angle of arrival of the downcoming wave can be determined.

One five-valve superhet receiver was connected to a vertical aerial and another receiver of the same type was connected to a frame aerial. The sensitivities of the receivers were made equal by testing with a local oscillator kept fairly away at the same horizontal level. The deflections in the sensitive mirror galvanometers were observed simultaneously when a signal was received and the angle of arrival of the downcoming wave was calculated. Various fading patterns along with the angles of arrival of the downcoming waves are shown in the following section.

5. *Experimental observations and discussions.*

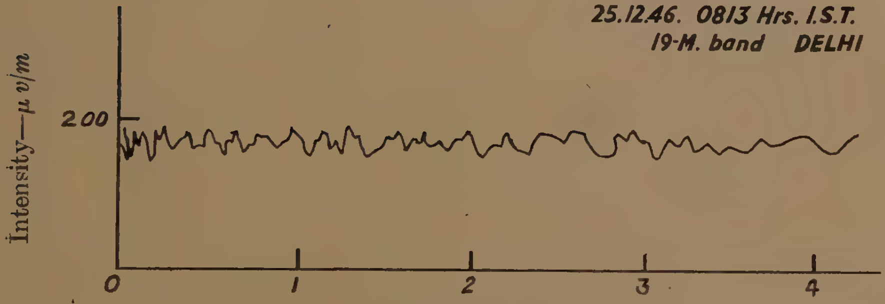
Typical automatic records of fading observations for transmissions on 25- and 19-metre bands from Delhi are shown in figs. 5–10 illustrating

Fig. 5.



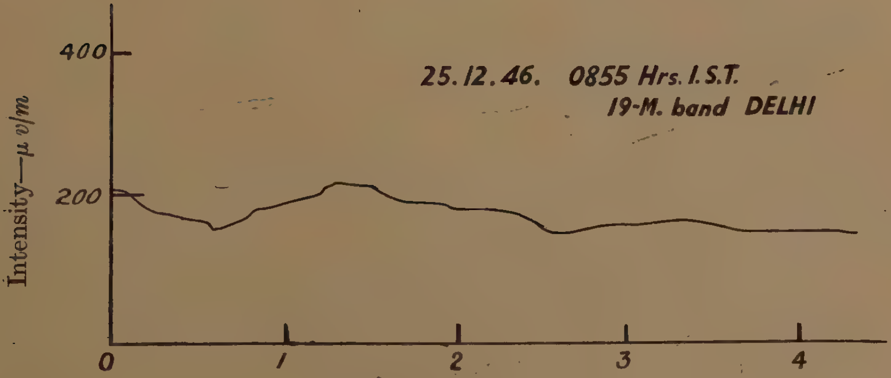
Time—mins.

Fig. 6.



Time—mins.

Fig. 7.



Time—mins.

Fig. 8.

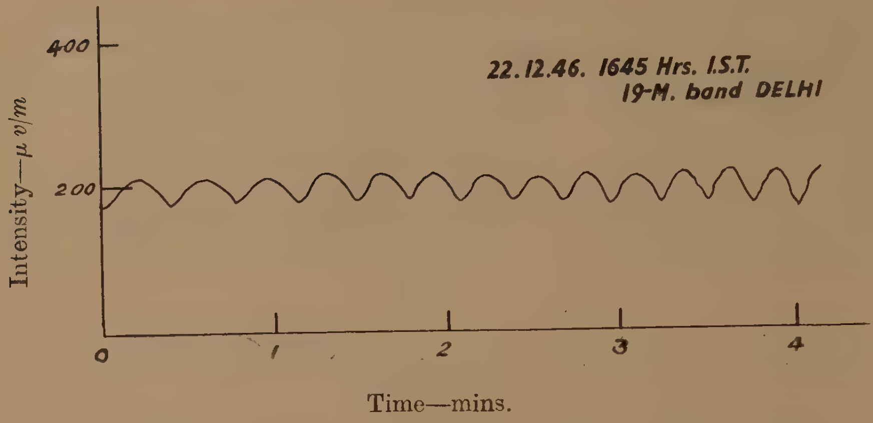


Fig. 9.

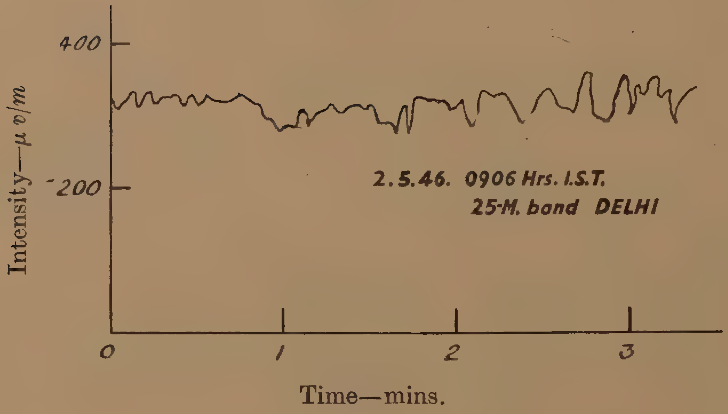
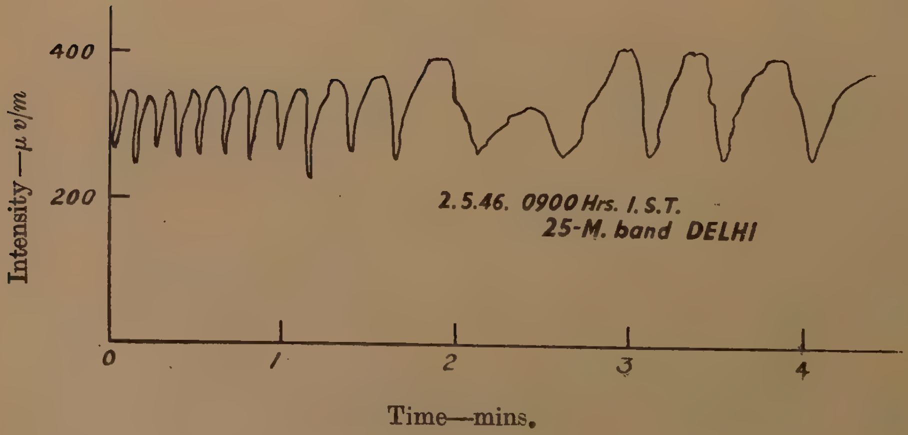


Fig. 10.



the three different types of fading, described in the previous section, and also, how the patterns change at different hours of the day. During these observations angles of arrival of the downcoming waves were not measured, but the observed results have been explained by the existing electronic concentrations in the ionospheric layers at those hours of observations, calculated from the ionospheric data obtained from the Research Department of All India Radio, Delhi.

For the sake of convenience, a brief summary of results of the above observations is given in Table I. The ionospheric data incorporated are calculated from observations made at Delhi by All India Radio. The electronic densities for E-layer, however, are from the observations at Peshawar (Lat. 34° 0′ N., Long. 71° 30′ E.), and the observed height of F_2-layer is 360 km., except where otherwise stated. In order to check the results in Table I., the electronic densities required for reflections of 25- and 19-metre bands for single and double reflections from E- and F_2-layers between Delhi (Lat. 28° 35′ N., Long. 77° 5′ E.) and Benares (Lat. 25° 16′ N., Long. 83° 2′ E.), situated at a distance of 678·4 km., are

TABLE I.

Fig.	Date	Time in I.S.T.	Wave-band	Type of Fading	Remarks
		Hrs.	Metres		
5	18. 5.46	1054	19	Moderately rapid random	Single reflection from E ($2·8\times10^5$ electrons/c.c.). Chance of single reflection from F_2($1·4\times10^6$ electrons/c.c.).
6	25.12.46	0813	19	Moderately rapid random	Single reflection from F_2($1·4\times10^6$ electrons/c.c.; height=300 km.). Generally no reflection from E($1·1\times10^5$ electrons/c.c.).
7	25.12.46	0855	19	Slow random	Single or double reflection from F_2($2·3\times10^6$ electrons/c.c.; height=330 km.). Generally no reflection from E($1·1\times10^5$ electrons/c.c.).
8	22.12.46	1645	19	Periodic	Single reflection from E($3·1\times10^5$ electrons/c.c.). Single reflection from F_2($1·5\times10^6$ electrons/cc.; height=390 km.).
9 and 10	2. 5.46	Between 0900 & 0910	25	Random, quasi-periodic and periodic	Single reflection from E($1·7\times10^5$ electrons/c.c.) and single or double reflection from F_2($1·2\times10^6$ electrons/c.c.).

given in Table II. The electronic densities have been calculated from the well-known relation

$$\cos^2 \beta = 1 - \frac{4\pi N e^2}{m w^2},$$

where β=angle of penetration of wave in the ionosphere,

N=electronic density,

e and m=charge and mass of electron respectively,

w=pulsatance of the wave.

TABLE II.

Layer and height	No. of reflections	Angle of radiation in degree	Required electronic density for reflection in electrons/c.c.	
			25-metre band	19-metre band
E—100 km.	1	13·7	$1{\cdot}359\times10^5$ to $1{\cdot}470\times10^5$	$2{\cdot}297\times10^5$ to $2{\cdot}545\times10^5$
	2	29·5	$4{\cdot}359\times10^5$ to $4{\cdot}714\times10^5$	$7{\cdot}365\times10^5$ to $8{\cdot}158\times10^5$
F_2—360 km.	1	44·5	$8{\cdot}937\times10^5$ to $9{\cdot}665\times10^5$	$1{\cdot}510\times10^6$ to $1{\cdot}673\times10^6$
	2	63·9	$1{\cdot}359\times10^6$ to $1{\cdot}470\times10^6$	$2{\cdot}297\times10^6$ to $2{\cdot}544\times10^6$

In view of the short wave-lengths used in these observations, the collisional frequency of electrons with neutral atoms and molecules has been neglected.

It will be observed from Table I. and figs. 5, 6 and 7, that more than one-spot reflection from the ionosphere will generally change the fading pattern from peaky random to less peaky type and, under favourable circumstances, will give rise to smooth periodic type, as shown in figs. 8 and 10. The periodic type, however, is obtained when there are only two-spot reflections from either single layer or two different layers. Fig. 11 shows the difference in the patterns of fading observed for transmission on 19-metre band from B.B.C. (London) and from Delhi at about the same time. Fig. 12 depicts the same type of observations for Australia and Delhi. It may be mentioned that the nature of random variations of intensity of the received signals from long distances, such as shown above, also changes their patterns under suitable conditions of the ionosphere. Such patterns of fading will be shown in Part II. of this paper which will be published later.

Figs. 13 and 14 show the automatic records of fading observed on 12.5.47 for prolonged periodic variations of intensity of received signals

Fig. 11.

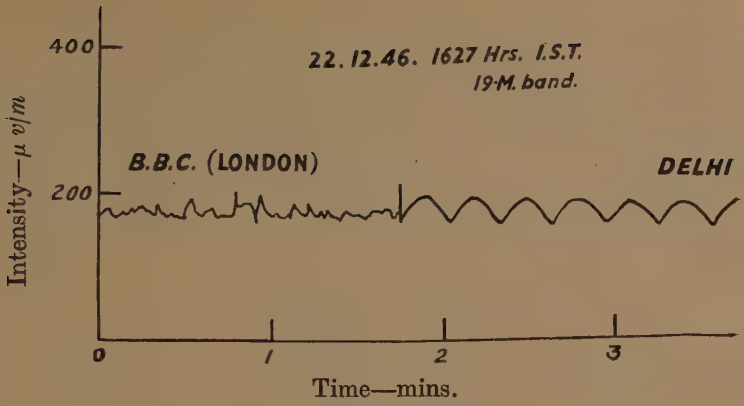


Fig. 12.

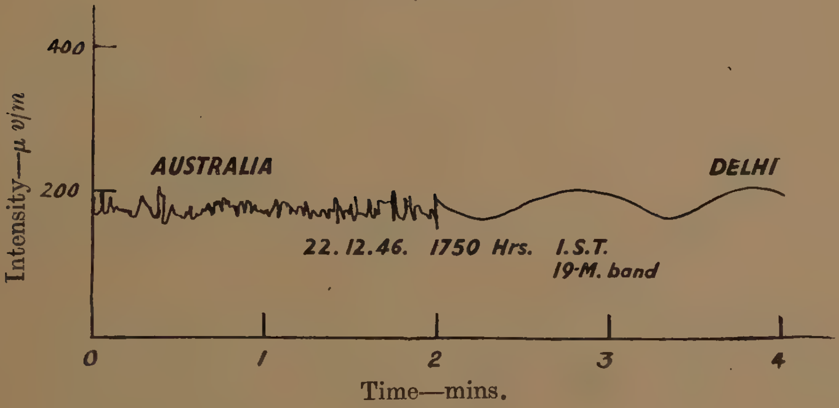


Fig. 13.

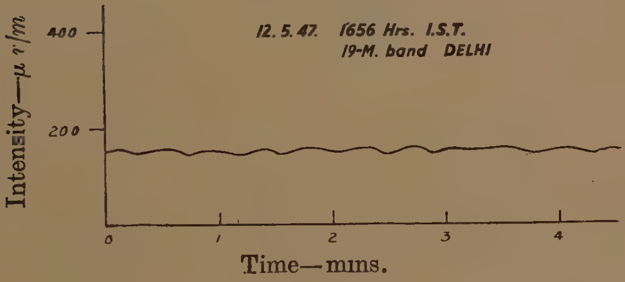
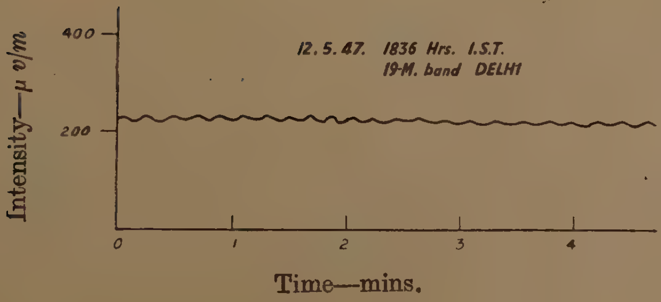


Fig. 14.



on 19-metre band from Delhi. The periodic variations on this occasion persisted for more than one hour during which, however, the periods of the variations changed. Simultaneous observations were recorded for the measurement of angles of arrival of the downcoming waves in order to find the layer from which the waves were being reflected. Table III. shows the ionospheric layers from which reflections occurred during the periods of observations recorded in the above diagrams. Column 6 of the table shows the approximate values around which the angles of arrival of the downcoming waves varied. These angles may be compared, for single or double reflections from the ionospheric layers, with the angles of radiation given in column 3 of Table II. It will be noted from the above table that single and double reflections occurred either from E- or F_2-layer which might have moved up or down very slowly. The rate

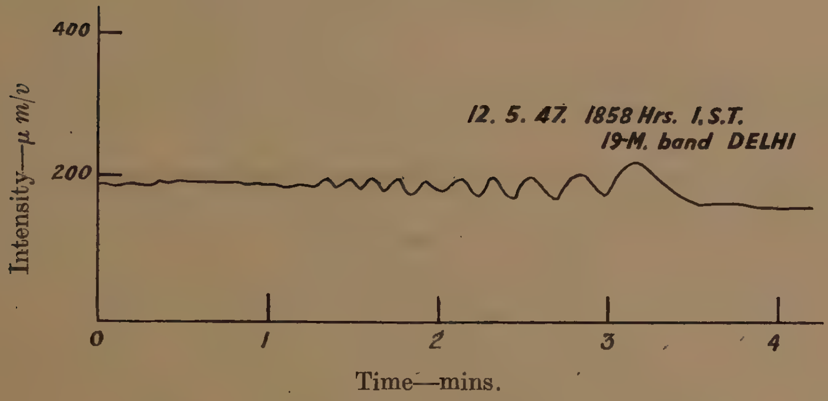
TABLE III.

Fig.	Date	Time in I.S.T.	Wave-band	Type of Fading	Angles of arrival of down-coming waves	Remarks
		Hrs.	Metres			
13	12.5.47	1656	19	Slow periodic	Around 15° or 33°	Single and double reflections from E (no data for electronic density). No reflection from F_2. Rate of movement of E=1·5 km./hr.
14	12.5.47	1836	19	Rapid periodic	Around 45° or 60°	Single and double reflections from F_2 ($4{\cdot}7\times10^6$ electrons/c.c.) *. No reflection from E. Rate of movement of F_2=2·9 km./hr.
15	12.5.47	1858	19	Periodic. Signal disappears at the end	Around 42° or 62°	Single and double reflections from F_2($4{\cdot}7\times10^6$ electrons/c.c.) *. Electronic concentration presumably reduced near sunset. No reflection from E. Rate of movement of F_2=3·1 km./hr. * Electronic density calculated from predicted curves of maximum usable frequencies for May, 1947, given by A.I.R., Delhi.

of movement calculated has also been included in the table. As it has been mentioned previously, such periodic fading patterns are more pronounced in the morning and afternoon during the transition periods of ionospheric layers when there may be slow variations in their equivalent heights, due to change in ionization.

Fig. 15 shows a typical record of fading, obtained on 12.5.47, near

Fig. 15.

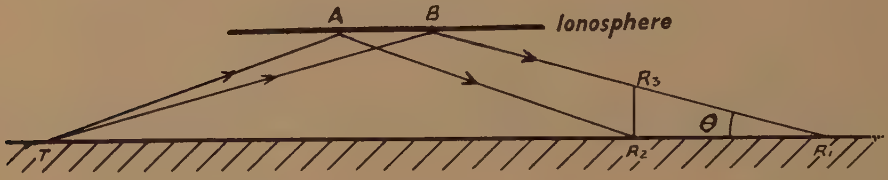


about sunset, when the received signal on 19-metre band from Delhi was observed to be periodic for some time and then it disappeared suddenly. Simultaneous measurement of the angles of arrival of downcoming waves at the time of observation indicated the presence of single and double reflections from F_2-layer and no reflection from E-layer, as shown in Table III. The cessation of signal which is presumably due to rapid decrease in ionization further corroborates the fact that the reflections were from F_2-layer only.

6. *Space-diversity reception of radio signals.*

Based on the principle of scattering of radio waves from the ionosphere as given above, it may be seen that, if we have two receiving aerials situated at horizontal distance of a few wave-lengths apart, it is likely that they will be actuated by signals reflected from two spots in the ionosphere and, consequently, the variations of intensity in the two aerials will be independent of each other. This fact is utilized in space-diversity reception, in which the outputs from two or more receivers connected to aerials separated by about 6 to 10 wave-lengths are mixed together to obtain a fairly constant strength of the received signal (12). It may be shown that such independent variations in the intensity of received signals may be brought about in two aerials vertically separated by a much smaller distance compared to the distance necessary in the horizontally spaced aerials. This has been explained in fig. 16. Suppose two spots in the ionosphere, A and B, reflect the radio waves which act on two aerials situated on the ground at R_1 and R_2 and produce variations in received signals in the two aerials which are independent of each other. Now, if the aerial at R_1 is moved along BR_1 and brought to R_3 vertically over R_2, the same two independent variations in signal intensities would be obtained in the aerials at R_2 and R_3 located at different heights. It will be observed that the ratio of the vertical and horizontal separations between the aerials in the two positions is given by

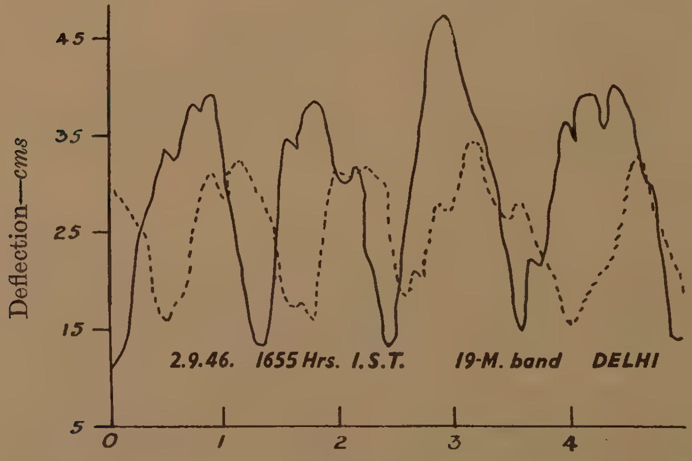
Fig. 16.



$$\frac{R_2R_3}{R_1R_2}=\tan\theta,$$

where θ is the angle of radiation.

Fig. 17.



Time—mins.

Therefore for long distances when θ is small, R_2R_3 is much smaller than R_1R_2. Thus the diversity reception is likely to be accomplished with two aerials vertically spaced with much smaller separation than with aerials horizontally spaced.

In order to verify the above results, two vertical aerials were installed at different heights with separation of about one-third wave-length, and were connected to two superhet sets of similar sensitivities. Simultaneous visual observations of variations of intensities of received signals in the two receivers were made by two observers with telephonic connections between them. Similar observations were also made keeping the two receivers near each other by connecting transmission lines from the aerials. Specimens of typical curves obtained from simultaneous visual observations with the two receivers are shown in fig. 17. Continuous line shows the variation of intensity of received signal with the elevated aerial and the dotted one shows the same with the aerial at lower level. It will be noticed that there is a substantial phase-difference between the variations in the intensity of signals in the two aerials. Further observations on this section of the present investigations will be published later in Part II. of this paper.

7. *Summary and conclusion.*

Observations have been made on various types of fading of short-wave radio signals received mainly from All India Radio, Delhi, and also from other Indian as well as foreign stations like Australia, Ceylon and B.B.C. (London). For convenience of explaining the different varieties of fading, they have been divided into three groups, namely, (1) peaky random type, (2) slow random or quasi-periodic type, and (3) smooth periodic type. Typical records of the above patterns of fading have been made visually and also with an automatic recorder constructed for the purpose.

The above types of fading have been explained on the assumption of random scattering of radio waves from the ionosphere. The first and second types of fading, mentioned above, have been attributed respectively to single-spot and double- or multiple-spot reflections of radio waves from the ionospheric layers. The third type of fading which is more frequent during the transition periods of the layers, either in the morning or in the afternoon, has been shown to be due to interference fringes caused by two waves reflected from either one or two layers, together with slow change in height of the layer or layers. Such slow movements of the layers may generally be caused by increase or decrease in their ionic densities. Whether a reflection was from E- or F_2-layer was determined by measuring the angle of arrival of the downcoming waves and calculating the electronic densities of the layers from the ionospheric data collected by Research Department, All India Radio, Delhi. It has also been shown from theoretical considerations and preliminary observations that diversity reception may be accomplished with much more closely situated aerials when vertically spaced than when they are horizontally spaced. Further work on this section of the present investigation is in progress and will be published in Part II. of this paper.

8. *Acknowledgments.*

We have great pleasure to record our sincere thanks to the Research Department of the All India Radio, Delhi, for providing us with the ionospheric data during the progress of the present investigations. Our thanks are also due to Principal M. Sengupta for his interest and helpful discussions, and to Mr. R. N. Singh, M.Sc., for checking the calculations in the paper.

Section of Communication
Engineering and Applied Physics,
Engineering College,
Benares Hindu University,
Benares (India).

14.6.47.

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LXXXIV. *The Analysis of Subsonic Flow and Constant Velocity Aerofoils.*

By A. R. MANWELL *.

[Received May 15, 1947.]

Summary.

It is shown that two-dimensional subsonic flow may be represented by an incompressible flow model. This contains as a particular case a result due to H. Bateman⁽¹⁾ for the pressure density law of von Kármán and Tsien⁽²⁾. This model is useful as an aid to the visualization of the boundary conditions in the hodograph plane and the interpretation of various approximations. In particular, the Kármán-Tsien representation is seen to be applicable for fairly high Mach numbers. The calculation of critical Mach numbers by existing methods is open to doubt. It is further shown that the terms occurring in Tsien's analysis are analogous to the first terms of the asymptotic expansions of the high order proper functions of the exact theory. Flow at great distances can therefore be represented by similar series.

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* Communicated by the Author.

It is suggested that boundary conditions be defined in the hodograph plane, rather than in physical space, when considerable simplification results. As an example the solution, on the Kármán-Tsien assumption for the flow over a constant velocity aerofoil is presented in a closed form.

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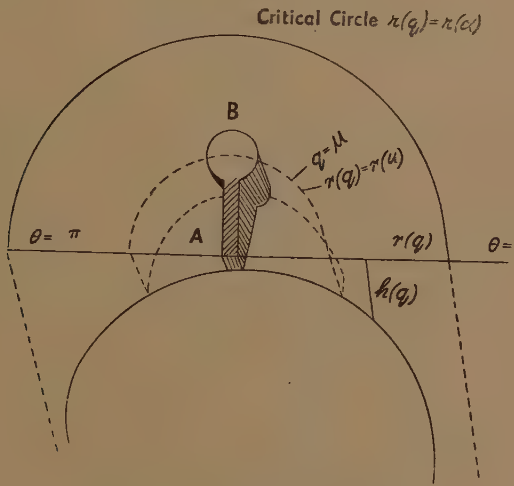
§ 1. *Introduction.*

THE transformation of two-dismensional compressible flow equations to the hodograph form is due to Molenbroek (3) and Tchapliguine (4). The latter and subsequent writers have suggested various approximations for slow speeds, in particular, Demtchenko (9) and Buseman (10). Tsien (2) uses an approximation to the adiabatic law which reduces the problem in principle to a certain incompressible flow in two dimensions. This idea is also due to von Kármán (2) and see also Bateman (1). It will now be shown that, whatever the relation between pressure and density, the equations satisfied by velocity potential and stream function in the hodograph plane are the same as those of an incompressible flow in a thin stratum of variable depth (h). This stratum is characterized, fig. 1, by having h almost constant over the low velocity region, but increasing to infinity on the circle at which the local critical speed is reached. Outside this range the representation has no meaning since h is imaginary. On the Kármán-Tsien law h is constant and the result is that of Bateman. The methods of the complex variable are then available to satisfy problems defined in the hodograph plane: that is to say, to determine flows for which the velocity profile is given in advance.

Fig. 1.



Sketch of model for subsonic flow.

§ 2. *A Model for Two-Dimensional Subsonic Flow.*

The equations of compressible flow may, after the usual contact transformation, be written

$$\frac{1}{q}\frac{\partial \Phi}{\partial q} = -\frac{(\rho q)'}{(\rho q)^2}\frac{\partial \Psi}{\partial \theta}, \qquad (1)$$

$$\frac{1}{\rho q}\frac{\partial \Psi}{\partial q} = \frac{1}{q^2}\frac{\partial \Phi}{\partial \theta}, \qquad (2)$$

where Φ, Ψ are potential and stream function respectively, $qe^{i\theta}$ is the velocity vector, and ρ is the density of the fluid which is a function of q in virtue of

$$\int \frac{dp}{\rho} + \frac{1}{2}q^2 = \text{constant}. \qquad (3)$$

Equivalent equations are

$$\frac{(\rho q)'}{\rho q}\left(\mathrm{X} - \frac{\partial \mathrm{Y}}{\partial \theta}\right) + \frac{\partial \mathrm{X}}{\partial q} = 0 \qquad (4)$$

$$\frac{\partial \mathrm{X}}{\partial \theta} + \frac{\partial}{\partial q}(q\mathrm{Y}) = 0, \qquad (5)$$

where x, y are cartesian coordinates of the physical plane and

$$\left.\begin{aligned} \mathrm{X} &= x\cos\theta + y\sin\theta, \\ \mathrm{Y} &= x\sin\theta - y\cos\theta. \end{aligned}\right\} \qquad (6)$$

The derivatives of equations (1) (2) (4) (5) are connected by

$$\frac{1}{q}\frac{\partial \Phi}{\partial q} = \frac{\partial \mathrm{X}}{\partial q}, \qquad (7)$$

$$\frac{1}{\rho q}\frac{\partial \Psi}{\partial q} = -\frac{\partial \mathrm{Y}}{\partial q}, \qquad (8)$$

$$\frac{1}{\rho q}\frac{\partial \Phi}{\partial \theta} = \frac{\partial \mathrm{X}}{\partial \theta} + \mathrm{Y} = -q\frac{\partial \mathrm{Y}}{\partial q}; \qquad (9)$$

$$\frac{1}{\rho q}\frac{\partial \Psi}{\partial \theta} = \mathrm{X} - \frac{\partial \mathrm{Y}}{\partial \theta} = -\frac{\rho q}{(\rho q)'}\frac{\partial \mathrm{Y}}{\partial q}. \qquad (10)$$

Differentiation with respect to q being denoted by a dash. A boundary in the hodograph is defined either by

$$\Psi = \text{constant for } q = q(\theta) \qquad (11)$$

or

$$\left(\frac{dy}{dx}\right)_c = \left(\frac{dy}{d\theta}\right)_c \Big/ \left(\frac{dx}{d\theta}\right)_c = \tan\theta \qquad (12)$$

or $\mathrm{X}_c = \left(\frac{dy}{d\theta}\right)$ where c denotes the curve $q = q(\theta)$. (13)

Equations (1) and (2) are equivalent to

$$hr\frac{\partial \Phi}{\partial r}=-\frac{\partial \Psi}{\partial \theta}, \qquad \ldots \ldots \ldots \quad (14)$$

$$\frac{h}{r}\frac{\partial \Phi}{\partial \theta}=\frac{\partial \Psi}{\partial r}, \qquad \ldots \ldots \ldots \quad (15)$$

which represents the flow in a thin stratum, h and r being functions of q as follows

$$h=\rho(1+q\rho'/\rho)^{-1/2}, \qquad \ldots \ldots \ldots \quad (16)$$

$$\log r=\int \frac{dq}{q}(1+q\rho'/\rho)^{+1/2}. \qquad \ldots \ldots \ldots \quad (17)$$

In general h is not constant and is in fact infinite when $1+q\rho'/\rho=0$, say $q=\alpha$, whilst r is finite and behaves like q at the origin. It will be seen later (30) that $r(q)$ reappears in the asymptotic expansion of certain proper functions. Calculation shows that on the adiabatic law

$$h=p(1-q^2/c^2)^{-1/2}, \qquad \ldots \ldots \ldots \quad (18)$$

where $c^2=\left(\frac{dp}{d\rho}\right)$ and $c^2+\frac{\gamma-1}{2}q^2=\text{constant}$ h becomes infinite for $q=c$ or $q^2=\frac{2}{\gamma+1}c_0^2$ where $c_0^2=\left(\frac{dp}{d\rho}\right)_{q=0}$.

The isothermal law gives

$$h=\rho(1-q^2/a^2)^{-1/2}, \qquad \ldots \ldots \ldots \quad (19)$$

where $a^2=\frac{dp}{d\rho}=$ a constant and

$$r=q\frac{\exp\sqrt{1-q^2/a^2}}{1+\sqrt{1-q^2/a^2}}. \qquad \ldots \ldots \ldots \quad (20)$$

The Kármán-Tsien assumption

$$p=k\left(\frac{1}{\rho_0}-\frac{1}{\rho}\right) \qquad \ldots \ldots \ldots \quad (21)$$

yields

$$h^2=\rho_0^2/(1-u^2/c_0^2)=\text{a constant} \qquad \ldots \ldots \quad (22)$$

and

$$r^2=\exp[(\rho_0/h)c/\sqrt{c_0^2-u^2}]\left\{\frac{\sqrt{q^2+c^2-u^2}-\sqrt{c^2-u^2}}{\sqrt{q^2+c^2-u^2}+\sqrt{c^2-u^2}}\right\}, \qquad . \quad (23)$$

where u is the velocity at great distances and

$$c_0^2=\left(\frac{dp}{d\rho}\right)_0.$$

Whilst in adiabatic or isothermal flow h/h_0 becomes infinite it is sufficiently near to unity over a range of velocities to make the Kármán-Tsien assumption a useful means of calculation, *cf.* Ref. (2).

Thus on the isothermal law

$$h/h_0=[\exp(-1/2q^2/a^2)](1-q^2/a^2)^{-1/2}$$

and the following table shows this function

q/a	h/ho
·1	1·00
·2	1·00
·3	1·00
·4	1·01
·5	1·02

q/a	h/ho
·6	1·03
·7	1·09
·8	1.20
·9	1·56
·95	1·95
1.00	∞

A similar result holds for adiabatic flow and h/h_0 becomes infinite for

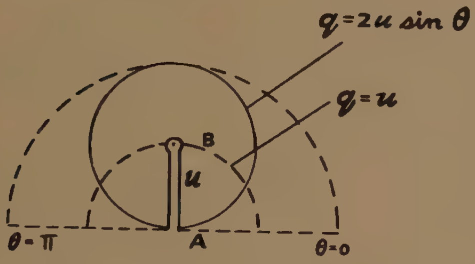
$$q^2=\frac{2}{\gamma+1}\left(\frac{dp}{d\rho}\right)_0. \qquad \text{(24)}$$

§ 3. *Boundary Conditions in the Hodograph Plane.*

By considering known cases of incompressible flow without circulation, it is readily seen that given one axis of symmetry the field to one side of the axis may be represented on a single region in the hodograph plane. Fig. 2 (*a*) (*b*) and (*c*). One point on the circle $q=u$, u being the velocity at great distances in the physical plane, say $q=u$, $\theta=\pi/2$, is a singularity, and the line joining the point B (fig. 2) to the origin A is a cut. The boundary conditions are those of an electric potential distribution due to a kind of doublet at B, the barrier AB and the velocity profile $q=q(\theta)$ being insulating. The distributions of potential and stream function over a small circle round B are given by the behaviour at great distances in the physical plane. This is the same in incompressible and compressible subsonic flows(9). A little consideration shows that provided the boundary $q=q(\theta)$ lies in the region for which h is real, *i. e.* the circle $q<\alpha$, subsonic flows may be represented similarly in the model. If there is not one axis of symmetry two sheets are needed to represent both halves of the field. Further, if q is everywhere small compared with α then h is nearly constant and the known incompressible flow solution appears as a special case.

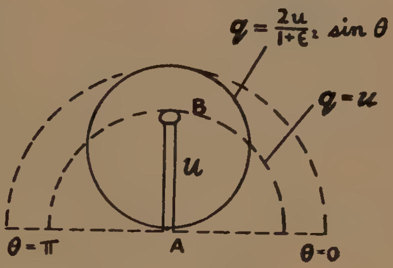
If boundary conditions are defined in the physical plane an immediate difficulty arises in that $q(\theta)$ is not known and, in fact, subsonic flow may not be possible. If, however, $q(\theta)$ is specified a direct solution can be carried out by means of the model and the physical boundary calculated by means of equations (6) (9) and (10). When this solution belongs to a range for which $h(q)$ is nearly constant the Kármán-Tsien approximation will give the solution in terms of the complex variable. It may also be remarked that the uniqueness of subsonic flow with boundary conditions in the hodograph plane reduces to that of electric fields, *i. e.* of Laplace's equation in three dimensions.

Fig. 2(*a*).



Circular cylinder $x^2+y^2=a^2$.

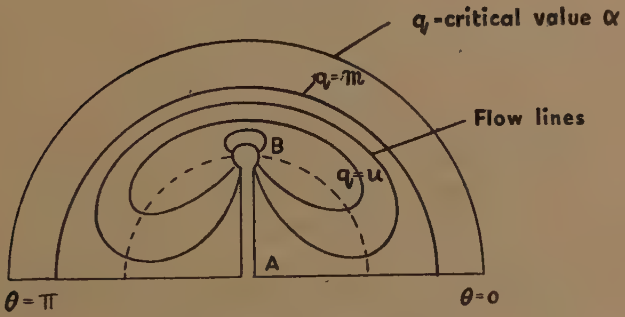
Fig. 2(*b*).



Elliptical cylinder

$$\left(\frac{x}{1+\epsilon^2}\right)^2+\left(\frac{y}{1-\epsilon^2}\right)^2=a^2.$$

Fig. 2(*c*).



A constant velocity aerofoil.

§ 4. *Asymptotic Expansion of the Proper Functions.*

The formal solution of (4), (5) and the associated partial differential equation

$$\frac{\partial^2 F}{\partial q^2}+(1/q+\rho'/\rho)\left(\frac{\partial F}{\partial q}+\frac{1}{q}\frac{\partial^2 F}{\partial \theta^2}\right)=0 \quad . \quad . \quad . \quad . \quad (25)$$

has been known since Tchapliguine's work [(4)] in the form

$$F(q,\theta)=\Sigma a_n F_n(q)\,\frac{\cos n\theta}{\sin n\theta},$$

where $$X_n=\frac{1}{n}\frac{dF_n}{dq}. \quad \text{(See equation (5).)}$$

If $1/q+\rho'/\rho$ is >0, *i. e.* if $q<\alpha$, then (25) has analytic solutions, whilst $X_n(q)$ increases steadily to its maximum at $q=\alpha$. For $q>\alpha$ (25) is of hyperbolic type and $F_n(q)$ is oscillatory, being, in fact, for the isothermal case, a degenerate Laguerre polynomial⁽¹⁵⁾. The Sturm Leonville expansion does not apply in general except for a function defined as identically zero in $0<q<\alpha$.

Let $X_n \cos n\theta$ and $Y_n(a) \sin n\theta$ be solutions of (4), (5) and write $V_n(q)=X_n/Y_n$.

Then writing $U_n=\rho V_n$

$$\frac{1}{n}U_n'-\rho\left(\frac{1}{q}+\rho'/\rho\right)+\frac{U^2}{\rho q}=0. \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Assume

$$U_n=U_0+\frac{1}{n}U_1+\frac{1}{n^2}U_2+\ldots\frac{1}{n^p}U_p. \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Then

$$U_0^2=\rho^2(1+q\rho'/\rho),$$

$$U_0'+\frac{2U_0U_1}{\rho q}=0,$$

$$U_1'+\frac{2U_0U_2+U_1^2}{\rho q}=0,$$

$$U_2'+\frac{2U_0U_3+2U_1U_2}{\rho q}=0.$$

Then

$$\log\{qY_n(q)\}=n\int\frac{U_n(q)\,dq}{\rho q}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and calculation shows that

$$F_n(q)\sim\frac{1}{\rho^{\frac{1}{2}}(1+q\rho'/\rho)^{\frac{1}{4}}}[r(q)]^n\left[1+0\left(\frac{1}{n}\right)\right], \quad . \quad . \quad . \quad (29)$$

where $r(q)$ is defined by (17).

Further calculations in the isothermal case yields, taking a as unity

$$F_n(q)\sim e^{\frac{1}{4}q^2}(1-q^2)^{-\frac{1}{4}}[r(q)]^{\pm n}\left[1+\Sigma\frac{Rp(q)}{n^p(1-q^2)^{3p/2}}\cdots\right], \quad . \quad (30)$$

where $R_p(q)$ is finite in $0<q^2<1$ and $r(q)=q\,\dfrac{\exp\sqrt{1-q^2}}{1+\sqrt{1-q^2}}$, which is (20).

The remainder terms are greater for larger q and the expansion requires $q^2<1$. For small q $F_n(q)\sim(q/2)^n$.

§ 5. *Various Exapansions.*

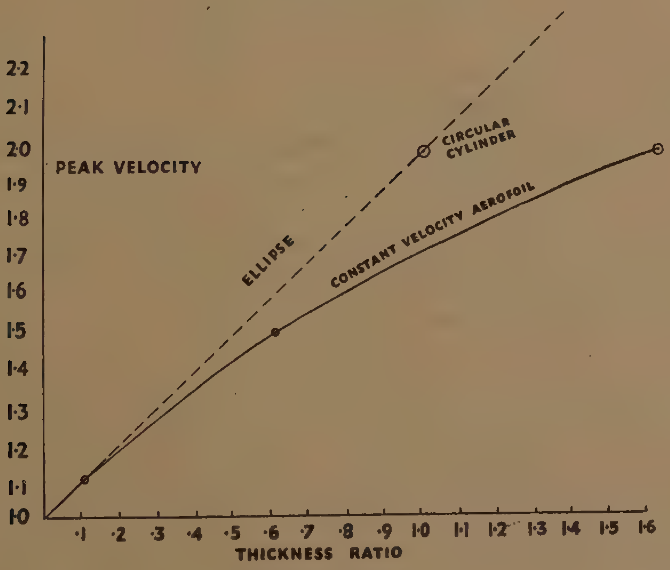
When q is very small $\Phi+i\Psi$ is a function of the complex variable $qe^{i\theta}$. Owing to the singularity at $q=iu$ the series expansions in the circle $q=u$ and outside of it have different forms. For example the series for the circular cylinder are a Taylor's series within $q=u$, and a series of terms of order $-n(+1/2)$ outside of this circle. The constant velocity aerofoil requires a Taylor's series within $q=u$ but the flow is represented by a Laurent series in the annular region $u<q<1/u$. A few attempts to satisfy the boundary conditions in these problems by means of series of the required type taking identical values on the circle $q=u$ shows great practical difficulty unless the elegant methods of the complex variable are used. This is only possible when $F_n(q)$ may be represented as $[r(q)]^n$ apart from trivial factors, that is to say, for very slow speeds, in the Kármán-Tsien method, and for high order functions. The last remark shows that the solution in the neighbourhood of B may be so represented. In physical language the flow very near B in the plane of the model is effectively two-dimensional. Again, in the physical plane at great distances, the flow is the same as for a certain incompressible flow. Provided that the whole solution lies in the subsonic range the asymptotic form shows the possibility of constructing series to represent the exact compressible flow in the regions $q<u$ and $q>u$ respectively analogous to those in the known incompressible flow. In the particular case of Kármán-Tsien flow only, can exact solutions be readily given owing to the difficulty of calculating the numerical coefficients in the expansions.

§ 6. *The Constant Velocity Aerofoil*⁽¹⁶⁾.

Using q to denote the complex variable

$$w=\frac{q-q^{-1}}{\sqrt{(1+u^2q^2)(1+a^2/q^2)}}, \quad \dots\dots \quad (32)$$

Fig. 3.



Comparison of elliptical cylinder with " constant velocity " shape (incompressible flow).

is the complex potential of the flow of an aerofoil having a flat nose $\theta=0$, $\theta=\pi$ and a constant velocity profile $q=1$ over most of its contour (in the case of slender bodies). The profile may be calculated in terms of standard elliptic integrals. Fig. 3 compares the velocity peak for this aerofoil with the ellipse of the same thickness/chord ratio.

Let
$$\mathrm{Q}=r(q)e^{i\theta}/r(m), \quad \ldots\ldots\ldots \quad (33)$$

$$\mathrm{U}=r(u)/r(m), \quad \ldots\ldots\ldots \quad (34)$$

and

$$\mathrm{W}=\Phi+i\Psi=\frac{\mathrm{Q}+\mathrm{Q}^{-1}}{\sqrt{(1+\mathrm{U}^2\mathrm{Q}^2)(1+\mathrm{U}^2/\mathrm{Q}^2)}} \quad \ldots\ldots \quad (35)$$

gives the complex potential of a similar type of flow in the plane of the model according to Tsien's approximation, $r(q)$ being defined by (23). In the case of incompressible flow the coordinates of the boundary are given by integrating $\frac{dz}{dq}=\frac{1}{q}\frac{dw}{dq}$ along the real axis $0<q<1$ and along the circle $q=1$. In particular the semichord in the direction of flow is given by

$$\mathrm{C}_x=\int_0^{\pi/2}\frac{\cos^2\theta}{(q^2+\cos^2\theta)^{1/2}}, \quad \ldots\ldots\ldots \quad (36)$$

and by

$$\mathrm{C}_y=\epsilon-\epsilon^2/(1+\epsilon^2)^{\frac{1}{2}}+4\epsilon^2\mathrm{J}, \quad \ldots\ldots\ldots \quad (37)$$

in the perpendicular direction where

$$\mathrm{J}=\int_0^1\frac{(1-q^2)dq}{[(1+q^2u^2)(1+q^2/u^2]^{3/2}} \quad \ldots\ldots \quad (38)$$

and $1/u-u=2\epsilon$.

Similarly for Tsien compressible flow

$$\frac{\partial\mathrm{W}}{\partial r}=\frac{\partial\Phi}{\partial r}+i\frac{\partial\Psi}{\partial r}, \quad \frac{\partial\mathrm{W}}{\partial\theta}=\frac{\partial\Phi}{\partial\theta}+i\frac{\partial\Psi}{\partial\theta}, \quad \ldots\ldots \quad (39)$$

and the boundary in the hodograph plane is given by $r(q)=1$ and $\theta=0$ or $\theta=\pi$ for which $r<1$.

Using equations (6), (9), (10) it is found that

$$\bar{\mathrm{C}}_x=\frac{1}{m}\int_0^{\pi/2}\frac{\cos^2\theta}{(\mathrm{E}^2+\cos^2\theta)^{1/2}}, \quad \ldots\ldots\ldots \quad (40)$$

$$\bar{\mathrm{C}}_y=\frac{1}{m}[\mathrm{E}-\mathrm{E}^2/(1+\mathrm{E}^2)^{1/2}]+4\mathrm{E}^2\bar{\mathrm{J}}(\mathrm{E}), \quad \ldots\ldots \quad (41)$$

where

$$\bar{\mathrm{J}}(\mathrm{E})=\int_0^1\frac{(1-r^2)dr}{[(1+r^2/\mathrm{U}^2)(1+r^2\mathrm{U}^2)^{3/2}]}\cdot\frac{r}{q(r)}, \quad \ldots\ldots \quad (42)$$

m being that value of q for which $|\mathrm{Q}|=1$ and $r(q)$ as in (23), whilst $2\mathrm{E}=u^{-1}-u$ where u is the velocity at great distances in the physical plane.

Similar expressions hold for other points on the boundary, which may be calculated in parametric form.

Conclusion.

The representation of subsonic flows by the model, together with the asymptotic formulæ, suggests that approximations starting with the incompressible flow as basis have no value except for very small speeds. Tsien's analysis seems, from both points of view, of much wider application, but it too, breaks down as the local speed of sound is approached. A detailed estimation of the possible accuracy of a given method is complicated by the necessity of knowing in advance the highest Mach numbers attained in the true solution. The advantage then of defining a solution in the plane of the model is two-fold since not only may a direct solution be attempted but the accuracy is well defined. Further, for moderate Mach numbers exact solutions can be given on the Tsien assumptions. It is suggested that the method of Poggi could be used to improve *this* solution rather than the incompressible solution.

Finally it should be pointed out that there is no particular connection between the model described above and the well-known experimental work of Taylor and Sharman (7). The present work is rather a generalization of Bateman's result (1).

Note added in proof.—Shortly after submitting the paper it was brought to my notice that H. S. Tsien and Y. H. Kuo (Ref. Tech. Notes, Nat. Adv. Comm. Aer. No. 995, p. 125, 1946) had already published an account of the asymptotic expansions and their applications. In preparing this paper I had no access to their work which was first reviewed in this country in April 1947 to the best of my knowledge. Whilst the subject has now been treated fairly exhaustively by M. J. Lighthill (Proc. Roy. Soc. A, cxci. pp. 323–369, 1947) and T. M. Cherry (Proc. Roy. Soc. A, cxcii, p. 44, *et seq.*) the present paper may still serve to give a picture of the nature of subsonic flow and to indicate that care is needed in continuing a solution into the supersonic region, whilst if the conditions at great distances are supersonic the whole solution is clearly of a different type.

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LXXXV. *The Dissipation of Internal Energy in Linear Dynamical Systems.*

By AUREL WINTNER *.

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INTRODUCTION.

THE purpose of this note is the proof of a criterion for the stability of a linear dynamical system which is of Hamilton's canonical type, but contains time explicitly. The criterion to be obtained occurs in the literature for the case of a single degree of freedom. However, the proof known for this particular case is such as to fail in case of systems, a case of more interest from the dynamical point of view. The obstacle arising in case of more than one degree of freedom is the circumstance that no appropriate analogue of the so-called oscillation theorem is then available.

The treatment of the general case is made possible by the replacement of the oscillation theorem by a simpler argument. It is based on a differential relation, expressing the dissipation of energy ; the latter being a variable function of the time by virtue of the non-conservative nature of the Hamiltonian.

Homogeneous Linear Systems.

Let x be a vector with $2n$ real components, n of which will represent coordinates, and n momenta, in a $2n$-dimensional phase-space. Let x' denote dx/dt, the dot in $x \,.\, y$ scalar multiplication, Ax the vector into which x is transformed by a ($2n$-rowed) matrix A, finally I the $2n$-rowed matrix

$$\mathrm{I}=\begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix}, \qquad \ldots \ldots \ldots \quad (1)$$

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* Communicated by the Author.

where ϵ is the n-rowed unit matrix and 0 the n-rowed zero matrix. Then the most general canonical system the Hamiltonian function of which is a quadratic form, say

$$h=\tfrac{1}{2}x \,.\, \mathrm{H}\, x, \quad \mathrm{H}=\mathrm{H}(t), \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

can be written in the form

$$x'=\mathrm{A}x, \quad \mathrm{A}=\mathrm{IH}, \quad \mathrm{H}=\mathrm{H}(t), \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

where $\mathrm{H}(t)$ is any real, symmetric, $2n$-rowed matrix of continuous functions on a t-interval, say $a \leqq t \leqq b$ (the energy, (2), is conserved only if $\mathrm{H}(t)=\text{const.}$).

It is *not* assumed that the $2n$-ary form (2) splits into kinetic and potential energies, *i. e.*, that the matrix $\mathrm{H}(t)$ is of the particular structure

$$\mathrm{H}(t)=\begin{pmatrix} \lambda(t) & 0 \\ 0 & \mu(t) \end{pmatrix}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

which would exclude " non-reversible " forces, such as (internal) frictions and Coriolis terms.

The Dissipation Law.

The fact that, if $\mathrm{H}(t)=\text{const.}$, the quadratic form (2) is independent of t along any solution $x=x(t)$ of (3), is the trivial instance of a general identity in t, expressing the law of energy dissipation in the non-conservative case, as follows :

For every continuous Hamiltonian matrix, $\mathrm{H}(t)$, *and for every solution vector* $x=x(t)$, *of the corresponding canonical system* (3), *the commutation rule*

$$d\{x(t) \,.\, \mathrm{H}(t)\, x(t)\}=x(t) \,.\, \{d\mathrm{H}(t)\}x(t) \quad . \; . \; . \; . \; . \; . \quad (4)$$

holds as an identity in t.

Since $\mathrm{H}(t)$ need not be differentiable or absolutely continuous, (4) must be interpreted in the sense of Stieltjes integrations, as follows : If u and v are arbitrary t-values on the t-interval on which the continuous matrix function $\mathrm{H}(t)$ is given, then

$$\int_u^v \phi(t)\, d\{x(t) \,.\, \mathrm{H}(t)\, x(t)\}=\int_u^v \phi(t)\, x(t) \,.\, \{d\mathrm{H}(t)\}x(t) \quad . \; . \quad (4\ bis)$$

holds for *every* (scalar) function, $\phi(t)$, of bounded variation, and for every solution vector, $x(t)$, of (3). The existence of these Stieltjes integrals is part of the statement. Actually, they exist even in the Riemann-Stieltjes sense. In fact, since $x(t)$ is a solution of (3), it is differentiable, hence continuous, and so (3), where $\mathrm{H}(t)$ is continouus, show that the derivative of $x(t)$ is continuous. Accordingly, $x(t)$ is of bounded variation on every interval $u \leqq t \leqq v$. Since the same is assumed of $\phi(t)$, and since $\mathrm{H}(t)$ is continuous, it follows that all integrals occurring in (4 *bis*) can be reduced to the form

$$\int_u^v \alpha(t)\, d\beta(t), \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

where $\alpha(t)$ is of bounded variation and $\beta(t)$ is continuous. But a partial integration (or, rather, summation) shows that this is just as sufficient for the existence of the Riemann-Stieltjes integral (5) as is the continuity of $\alpha(t)$ when $\beta(t)$ is of bounded variation.

The formal verification of (4) is simple indeed. First, if $h=h(x\,;t)$ is any scalar function of class C′ in a $(2n+1)$-dimensional $(x\,;t)$-domain, and if h_t denotes the partial derivative of h with respect to t, then

$$dh(x(t)\,;t)=h_t(x(t)\,;t)dt$$

is an identity in t along any solution $x(t)$ of the canonical system having the Hamiltonian function $h(x\,;t)$. In case of a linear system, (3), this identity can be written in the form (4), since the corresponding Hamiltonian function is (2).

The process of differentiation, used above, is justified if $\mathrm{H}(t)$ is of class C′. If $\mathrm{H}(t)$ is just continuous, the assertion is (4 *bis*), rather than (4). But (4 *bis*) follows in the same way as (4) did before, if the preceding differentiation is replaced by a partial integration. The latter is precisely the one which has been mentioned after (5).

A Criterion of Stability.

The simplest case of a Hamiltonian function (2) is

$$h=\tfrac{1}{2}(q'^2+f(t)q^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where $f(t)$ and q are scalars and x is the binary vector having the components q, q'; so that canonical equations, (3), are equivalent to the Lagrangian equation

$$q''+f(t)q=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If $f(t)$, which can be any real-valued, continuous function, is a positive constant, then every solution of (7) is an harmonic vibration (or the constant $q(t)\equiv 0$). This, along with Sturm's comparison theorem, might suggest that, if $f(t)$ is defined, for large positive t, as a real-valued, continuous function which tends to a finite, positive limit, $f(\infty)$, then $q(t)=\phi(1)$ as $t\rightarrow\infty$ will hold for every solution of (7). However, an example of Perron (4) shows that this is not in general true; not even if $\phi(t)$ is analytic. The existence of such a counter-example makes clear the content of the following theorem:

On a half-line, say $0\leq t<\infty$, *let* $\mathrm{H}(t)$ *be a* $2n$-*rowed symmetric matrix of real-valued, continuous functions. Suppose that the elements of this matrix are of bounded variation on the half-line, and that the limit matrix* $\mathrm{H}(\infty)$ (*which then exists by necessity*) *is positive definite. Then*

$$x(t)=O(1) \quad \text{as} \quad t\rightarrow\infty \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

holds for every solution of the Hamiltonian system (3). *What is more,*

$$\int^{\infty}|\,dh(t)\,|<\infty \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

holds for the energy, (2), *of every solution,* $x=x(t)$, *of* (3). *In particular, there exists a finite limit,*

$$h(\infty), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

for the energy, $h=h(t)$, *along every solution.*

First, if the trivial solution, $x(t)\equiv 0$, of (3) is excluded, then $x(t)=0$ cannot hold for any t (in fact, $x(t)\equiv 0$ is one, hence the only, solution of (3) satisfying the initial condition $x(t_0)=0$, where t_0 is arbitrary). Since $x(t)\neq 0$, it is seen from the definition, (2), of $h=h(t)$ that (4), or rather (4 *bis*), implies inequality

$$\int_u^v |\, dh(t)\,|/|\, x(t)\,|^2 \leqq \tfrac{1}{2}\int_u^v |\, x(t) \,.\, d\mathrm{H}(t)x(t)\,|/|\, x(t)\,|^2,$$

where $0\leqq u<v<\infty$. But every element of the matrix $\mathrm{H}(t)$ is supposed to be of bounded variation on the half-line $0\leqq t<\infty$. Hence it is clear, for reasons of homogeneity (of degree 0), that the integral on the right of the last inequality tends to 0 as $u\rightarrow\infty$, $v\rightarrow\infty$. Consequently, the same is true of the integral on the left. This means that

$$\int^\infty |\, dh(t)\,|/|\, x(t)\,|^2<\infty.$$

On the other hand, since $\mathrm{H}(\infty)$ is positive definite, there exists a positive constant, c, satisfying

$$c\,|\, x(t)\,|^2\leqq x(t) \,.\, \mathrm{H}(\infty)x(t).$$

Since $\mathrm{H}(t)\rightarrow\mathrm{H}(\infty)$ as $t\rightarrow\infty$, this inequality implies, for reasons of homogeneity, that

$$\tfrac{1}{2}c\,|\, x(t)\,|^2\leqq x(t) \,.\, \mathrm{H}(t)\, x(t)$$

holds from a certain $t=t^0$ onward. In view of (2), this means that

$$c\,|\, x(t)\,|^2\leqq 4h(t) \quad \text{if} \quad t_0<t<\infty.$$

Since c and $|\, x(t)\,|$ are positive, this implies that

$$\int^\infty |\, dh(t)\,|/|\, h(t)\,|\leqq 4c^{-1}\int^\infty |\, dh(t)\,|/|\, x(t)\,|^2,$$

if the (common) lower limit of the integrations exceeds t_0. But the integral on the right of this inequality was seen to be convergent. Consequently, the same is true of the integral on the left, that is, of the integral

$$\int^\infty |\, d\log h(t)\,|.$$

The convergence of the latter integral means that $\log h(t)$ is of bounded variation. Hence, $\log h(t)$ must tend to a finite limit as $t\rightarrow\infty$. In addition, $\log h(t)$ is real (or, more precisely, $h(t)$ has a real, continuous logarithm) from a certain t onward. This is clear from (2) and from the assumption that $\mathrm{H}(t)$ tends to $\mathrm{H}(\infty)$, a limit matrix which is supposed to be positive definite.

Consequently, the existence of a finite limit for $\log h(t)$ implies that $h(t)$ tends to a finite *positive* limit, (10). Since $\log h(t)$ has been shown to be of bounded variation, it follows that $h(t)$ is of bounded variation.

This proves (9). Finally, the existence of a finite limit (10) implies that

$$h(t)=O(1) \quad \text{as} \quad t\rightarrow\infty,$$

and so (8) follows from (2) and from the fact that the matrix, $\mathrm{H}(t)$, of (2) tends to a limit matrix, $\mathrm{H}(\infty)$, which is positive definite.

The Lower Limitation.

In the case of the trivial solution $x(t)\equiv 0$, excluded above, the energy, (2), vanishes for all t, and so $h(\infty)=0$. On the other hand, as shown above, $h(\infty)>0$ in case of any solution $x(t)\not\equiv 0$. Since the matrix of (2) tends to a positive definite limit, $\mathrm{H}(\infty)$, it follows that, if $x(t)$ is any solution of (3),

$$\liminf_{t\rightarrow\infty} |\,x(t)\,| \neq 0 \quad \text{unless} \quad x(t)\equiv 0. \quad . \;\; . \;\; . \quad (10\ bis)$$

Incidentally, two solutions of (3), say $x=x_1(t)$ and $x=x_2(t)$, belong to the same value of the asymptotic integration constant (10) if and only if

$$x_1(t)\,.\,\mathrm{H}(t)\,x_2(t)\rightarrow 0 \quad \text{as} \quad t\rightarrow\infty.$$

This follows by applying the definition (2) to both solutions as well as to their difference, $x=x_1(t)-x_2(t)$, which is again a solution of (3).

A criterion assuring (10 *bis*) for certain arbitrary (*i. e.*, not necessarily canonical) linear systems can be formulated as follows:

If $\mathrm{A}=\mathrm{A}(t)$, *where* $0\leq t<\infty$, *is a (not necessarily symmetric) matrix of* m *times* m *real-valued, continuous functions satisfying*

$$-\infty<\liminf_{t\rightarrow\infty}\int_0^t \min_{|x|=1}(x\,.\,\mathrm{A}(s)x)\,ds\leq\infty,$$

then (10 *bis*) *holds for all solutions,* $x=x(t)$, *of* $x'=\mathrm{A}(t)x$.

This is a dual of the following fact:

If the condition of the last formula line is replaced by

$$-\infty\leq \limsup_{t\rightarrow\infty}\int_0^t \max_{|x|=1}(x\,.\,\mathrm{A}(s)x)\,ds<\infty,$$

then (8) *holds for every solution of* $x'=\mathrm{A}(t)x$.

The proof of the latter theorem is simple indeed; cf. (5), p. 559. The proof of its preceding dual requires only a trivial transcription of the proof given loc. cit.

The case $n=1$.

When applied to the self-adjoint scalar differential equation

$$(g(t)q')'+f(t)q=0, \quad . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \quad (11)$$

the assumptions of (8) lead to the following result:

If $f(t)$ *and* $g(t)$ *are real-valued, continuous functions of bounded variation on a half-line, say* $0 \leq t < \infty$, *and if both the limits* $f(\infty)$, $g(\infty)$ (*which then exist*) *are positive, then*

$$q(t)=O(1) \quad \text{and} \quad q'(t)=O(1) \quad \text{as} \quad t\rightarrow\infty$$

hold for all solutions of (11).

This is known in the particular case (7) of (11), with a proof which could, with little effort, be transferred to the case $g(t)\not\equiv 1$ of (11). The situation is that the assertions of the last formula line are contained, for (7), in a consideration of A. Kneser (2), pp. 76–80. In fact, he assumes for the coefficient function of (7) more, but actually does not need more, than that $f(t)$ be continuous and of bounded variation, with $f(\infty)>0$. This observation has already been made by G. Ascoli (1).

On the other hand, Kneser's method of proof cannot be extended to the case $n>1$, since it involves, along the lines of Sturm's separation theorem, the distribution of successive zeros of the scalar function $q(t)$. The possibility of including the case of (3) for any n is due precisely to the fact that everything can be made to depend on the primitive fact (4) alone.

In order to reduce (11) to (3), it is sufficient to choose the matrix $\mathrm{H}(t)$ to be the binary diagonal matrix in which the diagonal elements are $1/g(t)$ and $f(t)$. Then, since $g(t)$ and $f(t)$ are of bounded variation and tend to positive limits, the same is true of both the diagonal elements of $\mathrm{H}(t)$. Hence, the assumptions of (8), and even those of (9) and (10), are satisfied. Since the components of the binary vector x become the coordinate q and its conjugate momentum, which is $g(t)$ times q', the assertions of the last formula line follow from (8) and from $1/g(\infty)\neq 0$.

It is also seen that, if $f(t)$ and $g(t)$ are arbitrary, real-valued functions on a finite t-interval (except that $g(t)$ must not have a zero, lest (11) becomes singular), the dissipation law, (4), reduces to

$$d\{q'^2(t)/g(t)+f(t)q^2(t)\}=-q'^2(t)/g^2(t)dg(t)+q^2(t)df(t),$$

where $g(t)$ is an arbitrary solution of (11). In the particular case (7), where $g(t)\equiv 1$, this becomes simply

$$dh(t)=\tfrac{1}{2}q^2(t)df(t), \quad \text{where} \quad 2h(t)=q'^2(t)+f(t)q^2(t). \quad . \quad . \quad . \quad (12)$$

The last relation, no matter how primitive, contains all the qualitative considerations dealing with the increase or decrease of a solution of (7) or of its local maxima and minima. For instance, it is seen from (12) that, if the coefficient function, $f(t)$, of (7) is increasing (decreasing) on a t-interval, then the energy, $h(t)$, of every solution $q(t)\not\equiv 0$ of (7) is increasing (decreasing) on the same t-interval. This in turn contains the facts pointed out by Osgood (3).

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LXXXVI. *The Metric of Space and Mass Deficiency.*

By A. M. MOSHARRAFA, Ph.D., D.Sc.,
Professor of Applied Mathematics in the Fuad I. University, Cairo*.

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§ 1. *Introduction and Aim.*

IN the theory of relativity the invariant quadratic differential form

$$g_{\mu\nu}\,dx^\mu\,dx^\nu\ ;\ \mu,\nu=1,2,3,4, \qquad (1)$$

defines the metric of space-time. The physical interval between two infinitesimally close events at a point is thus made to depend on the fundamental tensor $g_{\mu\nu}$ which defines the gravitational field at the point, but is quite independent of the electromagnetic field. The theory leads to the world-line of a gravitational particle as a geodesic, but fails to obtain the world-line of a charged particle as a geodesic. Attempts have been made at a unified field theory, notably by Weyl (1), who introduces a " gauge " factor in the metric, and by Kaluza (2), on the basis of a five-dimensional geometry. The aim of the present paper is to show that, by using a slightly generalized form of the Riemannian metric, it is possible to reduce the world-line of a charged particle to a geodesic; and furthermore, to show that this new theory provides a quantitative explanation of the phenomenon of mass deficiency (sometimes described as the " packing " effect) and a basis for measuring nuclear fields.

§ 2. *Introduction of a New Metric and Equations of a Geodesic.*

If we write the equation of the Riemannian metric, defining the element of length ds, in the form

$$g_{\mu\nu}\,dx^\mu\,dx^\nu-ds^2=0,$$

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* Communicated by the Author.

we observe that the left-hand side is of a general homogeneous form in the four differentials dx', dx^2, dx^3 and dx^4, but not in the five differentials dx', dx^2, dx^3, dx^4 and ds taken together. This suggests a generalization by the addition of terms in $dx^\mu ds$. We accordingly adopt a metric given by

$$g_{\mu\nu}\, dx^\mu\, dx^\nu + 2f_\mu\, dx^\mu\, ds - ds^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

There is no loss of generality in leaving the coefficient of ds^2 equal to -1, since this coefficient must be an invariant, and hence we can divide by it and include it in the coefficients of the other terms. The metric now contains the tensor $g_{\mu\nu}$ and the covariant vector f_μ. We proceed to obtain the equations of a geodesic. We have from (2), operating with δ by the usual method of variation,

$$dx^\mu\, dx^\nu\, \delta g_{\mu\nu} + g_{\mu\nu}\, dx^\mu\, \delta(dx^\nu) + g_{\mu\nu}\, dx^\nu\, \delta(dx^\mu) + 2dx^\mu\, ds\, \delta f_\mu$$
$$+ 2f_\mu\, ds\, \delta(dx^\mu) + 2f_\mu\, dx^\mu\, \delta(ds) - 2ds\, \delta(ds) = 0.$$

$$\therefore \quad 2\delta(ds)\left[1 - f_\mu \frac{dx^\mu}{ds}\right] = \left\{\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\delta x^\sigma + g_{\mu\nu}\frac{dx^\mu}{ds}\frac{d}{ds}(\delta x^\nu) + g_{\mu\nu}\frac{dx^\nu}{ds}\frac{d}{ds}(\delta x^\mu)\right.$$
$$\left. + 2\frac{dx^\mu}{ds}\frac{\partial f_\mu}{\partial x^\sigma}\delta x^\sigma + 2f_\mu \frac{d}{ds}(\delta x^\mu)\right\} ds.$$

The condition for a geodesic,

$$\int \delta(ds) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

now gives

$$\frac{1}{2}\int \left\{ \frac{\left[\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{\partial g_{\mu\nu}}{\partial x^\sigma} + 2\frac{ds}{dx^\mu}\frac{\partial f_\mu}{\partial x^\sigma}\right]\delta x^\sigma + \left[g_{\mu\nu}\frac{dx^\mu}{ds} + g_{\mu\nu}\frac{dx^\nu}{ds} + 2f_\sigma\right]\frac{d}{ds}(\delta x^\sigma)}{1 - f_\mu \frac{dx^\mu}{ds}} \right\} ds = 0,$$

or, on integrating by parts, using the formula

$$\int \frac{udv}{w} = \frac{uv}{w} - \int \frac{vdu}{w} + \int \frac{uv}{w^2} dw,$$

and discarding the integrated part, since δx^σ vanishes at the two limits, we have

$$\int \frac{\left[g_{\mu\sigma}\frac{dx^\mu}{ds} + g_{\sigma\nu}\frac{dx^\nu}{ds} + 2f_\sigma\right] d(\delta x^\sigma)}{1 - f_\mu \frac{dx^\mu}{ds}} = -\int \frac{\frac{d}{ds}\left[g_{\mu\sigma}\frac{dx^\mu}{ds} + g_{\sigma\nu}\frac{dx^\nu}{ds} + 2f_\sigma\right]}{1 - f_\mu \frac{dx_\mu}{ds}} \delta x^\sigma\, ds$$
$$- \int \frac{\left[g_{\mu\sigma}\frac{dx^\mu}{ds} + g_{\sigma\nu}\frac{dx^\nu}{ds} + 2f_\sigma\right]\frac{d}{ds}\left(f_\mu \frac{dx^\mu}{ds}\right)}{\left(1 - f_\mu \frac{dx^\mu}{ds}\right)^2} \delta x^\sigma\, ds.$$

$$\therefore \qquad \frac{1}{2}\int \frac{\dfrac{dx^\mu}{ds}\dfrac{dx^\nu}{ds}\dfrac{\partial g_{\mu\nu}}{\partial x^\sigma}+2\dfrac{dx^\mu}{ds}\dfrac{\partial f_\mu}{\partial x^\sigma}-\dfrac{d}{ds}\left[g_{\mu\nu}\dfrac{dx^\mu}{ds}+g_{\sigma\nu}\dfrac{dx^\nu}{ds}+2f_\sigma\right]}{\left(1-f_\mu\dfrac{dx^\mu}{ds}\right)^2}\,\delta x^\sigma ds$$

$$-\frac{1}{2}\int \frac{\left[g_{\mu\nu}\dfrac{\partial x_\mu}{ds}+g_{\sigma\nu}\dfrac{dx_\nu}{ds}+2f_\sigma\right]\dfrac{d}{ds}\left(f_\mu\dfrac{dx^\mu}{ds}\right)}{\left(1-f_\mu\dfrac{\partial x^\mu}{ds}\right)^2}\,\delta x^\sigma ds=0\,;$$

and since δx^σ is arbitrary, the integrand must vanish, giving

$$\frac{1}{2}\,\frac{\dfrac{dx^\mu}{ds}\dfrac{dx^\nu}{ds}\dfrac{\partial g_{\mu}}{\partial x^\sigma}+2\dfrac{dx^\mu}{ds}\dfrac{\partial f_\mu}{\partial x^\sigma}-\dfrac{d}{ds}\left[g_{\mu\nu}\dfrac{dx^\mu}{ds}+g_{\sigma\nu}\dfrac{dx^\nu}{ds}+2f_\sigma\right]}{1-f_\mu\dfrac{dx^\mu}{ds}}$$

$$-\frac{1}{2}\,\frac{\left[g_{\mu\sigma}\dfrac{dx^\mu}{ds}+g_{\sigma\nu}\dfrac{dx^\nu}{ds}+2f_\sigma\right]\dfrac{d}{ds}\left(f_\mu\dfrac{dx^\mu}{ds}\right)}{\left(1-f_\mu\dfrac{dx^\mu}{ds}\right)^2}=0. \qquad . \quad . \qquad (4)$$

The numerator of the first fraction on the left-hand side of equation (4) equals

$$\frac{1}{2}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{\partial g_{\mu\nu}}{\partial x_\sigma}+\frac{dx^\mu}{ds}\frac{\partial f_\mu}{\partial x^\sigma}-\frac{1}{2}g_{\mu\sigma}\frac{d^2x^\mu}{ds^2}-\frac{1}{2}g_{\sigma\nu}\frac{d^2x^\nu}{ds^2}$$

$$-\frac{1}{2}\frac{dx^\mu}{ds}\frac{\partial g_{\mu\sigma}}{\partial x^\nu}\frac{dx^\nu}{ds}-\frac{1}{2}\frac{dx^\nu}{ds}\frac{\partial g_{\sigma\nu}}{\partial x^\mu}\frac{dx^\mu}{ds}-\frac{\partial f_\sigma}{\partial x^\epsilon}\frac{dx^\epsilon}{ds}$$

$$=\frac{1}{2}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\left(\frac{\partial g_{\mu\nu}}{\partial x^\sigma}-\frac{\partial g_{\mu\tau}}{\partial x^\nu}-\frac{\partial g_{\sigma\nu}}{\partial x^\mu}\right)-\frac{1}{2}g_{\epsilon\tau}\frac{d^2x^\epsilon}{ds^2}$$

$$-\frac{1}{2}g_{\sigma\epsilon}\frac{d^2x^\epsilon}{ds^2}-\frac{dx^\epsilon}{ds}\left(\frac{\partial f_\sigma}{\partial x^\epsilon}-\frac{\partial f_\epsilon}{\partial x^\sigma}\right)$$

(since ϵ is a dummy suffix)

$$=-g_{\sigma\epsilon}\frac{d^2x^\epsilon}{ds^2}-[\mu\nu,\sigma]\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}-\frac{dx^\epsilon}{ds}\left(\frac{\partial f_\sigma}{\partial x^\epsilon}-\frac{\partial f_\epsilon}{\partial x^\sigma}\right),$$

which on multiplying by $g^{\sigma\alpha}$ and changing all signs becomes

$$\frac{d^2x^\alpha}{ds^2}+\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\{\mu\nu,\alpha\}+g^{\sigma\alpha}\left(\frac{\partial f_\sigma}{\partial x^\epsilon}-\frac{\partial f_\epsilon}{\partial x^\sigma}\right)\frac{dx^\epsilon}{ds}, \qquad . \quad . \quad . \qquad (A)$$

where $[\mu\nu,\sigma]$ and $\{\mu\nu,\alpha\}$ are Christoffel three-index symbols of the first and second kinds respectively. Similarly the numerator of the second fraction on the left-hand side of equation (4) may be written

$$-\frac{1}{2}g_{\epsilon\sigma}\left(\frac{dx^\epsilon}{ds}+g_{\sigma\epsilon}\frac{dx^\epsilon}{ds}+2f_\sigma\right)\frac{d}{ds}\left(f_\mu\frac{dx^\mu}{ds}\right)=-\left(g_{\sigma\epsilon}\frac{dx^\epsilon}{ds}+f_\sigma\right)\frac{d}{ds}\left(f_\mu\frac{dx_\mu}{ds}\right),$$

which when multiplied by $-g^{\sigma\alpha}$ becomes

$$\left(\frac{dx^\alpha}{ds}+f^\alpha\right)\frac{d}{ds}\left(f_\mu\frac{dx^\mu}{ds}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(B)}$$

Combining (A) and (B) with their respective denominators, we finally have as the equations of a geodesic :

$$\frac{\dfrac{d^2x^\alpha}{ds^2}+\dfrac{dx^\mu}{ds}\dfrac{dx^\nu}{ds}\{\mu\nu,\,\alpha\}+g^{\sigma\alpha}\left(\dfrac{\partial f_\sigma}{\partial x^\varepsilon}-\dfrac{\partial f_\varepsilon}{\partial x^\sigma}\right)\dfrac{dx^\varepsilon}{ds}}{1-f_\mu\dfrac{dx^\mu}{ds}}+\frac{\left(\dfrac{dx^\alpha}{ds}+f^\alpha\right)\dfrac{d}{ds}\left(f_\mu\dfrac{dx^\mu}{ds}\right)}{\left(1-f_\mu\dfrac{dx^\mu}{ds}\right)^2}=0. \quad \text{(5)}$$

§ 3. *Equations of Motion of a Charged Particle.*

We now consider the approximate equations of a geodesic by neglecting the second fraction on the left-hand side of equation (5). We have

$$\frac{d^2x^\alpha}{ds^2}+\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\{\mu\nu,\,\alpha\}+g^{\sigma\alpha}\left(\frac{\partial f_\sigma}{\partial x^\varepsilon}-\frac{\partial f_\varepsilon}{\partial x_\sigma}\right)\frac{dx^\varepsilon}{ds}=0. \quad . \quad . \quad . \quad \text{(6)}$$

Or, if we set

$$f^\lambda=\frac{e}{m_0c^2}\kappa^\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(7)}$$

where κ^λ is the four-vector combining the vector potential⁽³⁾ (F, G, H) and the scalar potential ϕ, so that

$$\kappa^\lambda=(\mathrm{F},\,\mathrm{G},\,\mathrm{H},\,\phi) \text{ in Galilean coordinates,} \quad . \quad . \quad . \quad . \quad \text{(8)}$$

we have

$$\frac{\partial\kappa_\sigma}{\partial x^\varepsilon}-\frac{\partial\kappa_\varepsilon}{\partial x^\sigma}=\mathrm{F}_{\sigma\varepsilon}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(9)}$$

where $\mathrm{F}_{\sigma\varepsilon}$ is the electromagnetic force tensor. Equation (6) may now be written

$$\frac{d^2x^\alpha}{ds^2}+\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\{\mu\nu,\,\alpha\}+\frac{e}{m_0c^2}g^{\sigma\alpha}\mathrm{F}_{\sigma\varepsilon}\frac{dx^\varepsilon}{ds}=0,$$

or

$$\frac{d^2x^\alpha}{ds^2}+\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\{\mu\nu,\,\alpha\}+\frac{e}{m_0c^2}\mathrm{F}_\varepsilon^{\ \alpha}\frac{dx^\varepsilon}{ds}=0, \quad . \quad . \quad . \quad . \quad \text{(10)}$$

which represents the equations of motion⁽⁴⁾ of a charged particle of charge e and mass m_0 when the radiation terms are neglected. We reserve a discussion of the radiation terms, represented by the second fraction on the left-hand side of (5), for another occasion.

§ 4. *Nature of the New Metric, Magnitude of a Vector and Angle between Two Vectors.*

Let l^μ be a contravariant vector, the magnitude l of the vector in Riemannian geometry is determined by

$$g_{\mu\nu}l^\mu l^\nu=l^2,$$

so that l has two equal and opposite values. Assuming these to be real, the negative value is discarded and thus

$$l=\sqrt{g_{\mu\nu}l^\mu l^\nu}.$$

We follow a similar procedure with the new metric. The magnitude l, assumed to be real, of a contravariant vector l^μ is now the positive root of the equation

$$g_{\mu\nu}l^\mu l^\nu+2f_\mu l^\mu l-l^2=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

so that

$$l=\sqrt{g_{\mu\nu}l^\mu l^\nu+(f_\mu l^\mu)^2}+f_\mu l^\mu. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Equation (12) may be written

$$l=\sqrt{g^{\mu\nu}l_\mu l_\nu+(f^\mu l_\mu)^2}+f^\mu l_\mu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

so that l is also the magnitude of the associated covariant vector l_μ. We observe that if $l=0$, then we have from (11)

$$g_{\mu\nu}l_\mu l_\nu=0 \text{ (for a null vector)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

which is the same as in Riemennian geometry. Similarly, if we put $ds=0$ in equation (2), we have

$$g_{\mu\nu}dx^\mu dx^\nu=0 \text{ (for } ds=0), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

so that the path of a ray of light remains the same as in Einstein's theory. Let l^μ and l'^ν be two contravariant vectors ; to define the angle θ between them two different lines of procedure suggest themselves. Either we write

$$\cos\theta=\frac{g_{\mu\nu}l^\mu l'^\nu}{ll'}+2f_\mu\frac{l^\mu+l'^\mu}{l+l'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.1)$$

or we solve the equation

$$\frac{g_{\mu\nu}l^\mu l'^\nu}{ll'}+2f_\mu\frac{l^\mu+l'^\mu}{l+l'}\sqrt{\cos\theta}-\cos\theta=0 \quad . \quad . \quad . \quad . \quad (16.2)$$

for $\sqrt{\cos\theta}$ and discard the negative value, so that

$$\sqrt{\cos\theta}=\sqrt{\frac{g_{\mu\nu}l^\mu l'^\nu}{ll'}+\left[\frac{f_\mu(l^\mu+l'^\nu}{l+l'}\right]^2}+\frac{f_\mu(l^\mu+l'^\mu}{l+l'}. \quad . \quad . \quad (17.2)$$

The latter alternative leaves the condition for perpendicularity of two vectors the same as in Riemannian geometry. Thus

$$g_{\mu\nu}l^\mu l'^\nu=0 \quad \text{(for two orthogonal non-null vectors)}. \quad . \quad . \quad (18.2)$$

§ 5. *Mass of a Particle, Dependence of Mass on the Potential Energy of the Particle.*

If in equation (2) we adopt Galilean values for the tensor $g_{\mu\nu}$ and use Galilean coordinates (x, y, z, t), we have, using (7) and (8),

$$\frac{1}{c}\frac{ds}{dt}=\pm\sqrt{1-\frac{u^2}{c^2}+\frac{e^2}{m_0{}^2c^4}\left(\phi-\frac{F\dot{x}+G\dot{y}+H\dot{z}}{c}\right)^2+\frac{e}{m_0c^2}\left(\phi-\frac{F\dot{x}+G\dot{y}+H\dot{z}}{c}\right)}, \qquad . \quad . \quad . \quad (19)$$

where u is the velocity. So that if the mass m is defined by the relation

$$\frac{m}{m_0}=\frac{cdt}{ds}, \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and we assume m to be positive, we have

$$m=\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}+\frac{e^2}{m_0{}^2c^4}\left(\phi-\frac{F\dot{x}+G\dot{y}+H\dot{z}}{c}\right)^2+\frac{e}{m_0c^2}\left(\phi-\frac{F\dot{x}+G\dot{y}+H\dot{z}}{c}\right)}}. \qquad . \quad . \quad . \quad . \quad (21)$$

If $$\frac{u}{c} \text{ and } \frac{e\phi}{m_0c^2}$$

are small, we have, to the first order,

$$m=m_0\left(1-\frac{e\phi}{m_0c^2}\right). \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

This offers an explanation of the deficiencies in the masses of atomic nuclei known as the "packing" effect. It also provides a quantitative measure of nuclear fields.

§ 6. *Application to Nuclear Fields.*

Consider a nucleus containing Z protons and (A−Z) neutrons, so that Z is the atomic number and A the mass number. If M is the atomic weight of the nucleus on the physical $^{16}0$ scale, we have

$$M=M_n(A-Z)+M_pZ(1-k),$$

where M_n and M_p are the atomic weights of a neutron and a proton respectively, and

$$k=\frac{e\phi}{m_0c^2}, \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

m_0 being now the free rest mass of a proton. Taking (5) for M_n the value 1·00893, and (6) for M_p the value 1·00813, we have

$$M-A=\cdot00893A-\cdot00080Z-1\cdot00813Zk. \qquad . \quad . \quad . \quad . \quad . \quad (24)$$

The following table gives the values of k calculated from equation (24), using determinations of M given by Livingston and Bethe (7) and by G. Gamow (8) based on mass spectrograph measurements by Aston (9), by Bainbridge and Jordan (10) and by Mattauch (11) and correlated with the data accruing from nuclear reactions to secure "best" values :—

Nucleus	A	Z	M–A	Authors	k	ϕ in millions of volts	r_0 in 10^{-13} cm.
^{2}H	2	1	0·014 73	L.B.	0·002 31	2·168	0·664
^{3}H	3	1	0·017 10	G.	0·008 82	8·273	0·174
^{3}Hl	3	2	0·017 10	G.	0·004 01	3·764	0·765
^{4}Hl	4	2	0·003 89	L.B.	0·014 99	14·07	0·405
^{6}Li	6	3	0·016 70	G.	0·011 40	10·70	0·404
^{7}Li	7	3	0·018 18	L.B.	0·013 86	13·01	0·332
^{8}Be	8	4	0·007 80	G.	0·014 99	14·06	0·410
^{9}Be	9	4	0·015 16	L.B.	0·015 38	14·43	0·399
^{10}Be	10	4	0·016 31	L.B.	0·017 31	16·24	0·355
^{10}B	10	5	0·016 10	G.	0·013 73	12·88	0·559
^{11}B	11	5	0·012 80	G.	0·016 15	15·16	0·475
^{12}C	12	6	0·003 98	L.B.	0·016 26	15·26	0·566
^{13}C	13	6	0·007 61	L.B.	0·016 74	15·71	0·550
^{14}N	14	7	0·007 50	L.B.	0·015 86	14·88	0·677
^{15}N	15	7	0·004 89	L.B.	0·017 49	16·41	0·614
^{16}O	16	8	0·000 00	—	0·016 92	15·88	0·725
^{17}O	17	8	0·004 60	G.	0·017 46	16·38	0·703
^{18}O	18	8	0·003 69	L.B.	0·018 68	17·52	0·657
^{19}F	19	9	0·004 52	L.B.	0·017 41	16·33	0·793
^{20}Ne	20	10	$\bar{1}$·998 81	L.B.	0·017 04	15·99	0·901
^{21}Ne	21	10	$\bar{1}$·999 68	L.B.	0·017 84	16·74	0·860
^{22}Ne	22	10	$\bar{1}$·998 64	L.B.	0·018 83	17·16	0·815
^{27}Al	27	13	$\bar{1}$·990 90	G.	0·018 30	17·17	1·09
^{23}Si	28	14	$\bar{1}$·986 80	L.B.	0·017 74	16·65	1·12
^{29}Si	29	14	$\bar{1}$·986 60	L.B.	0·018 51	17·37	1·16
^{31}P	31	15	$\bar{1}$·984 10	L.B.	0·018 57	17·42	1·24
^{32}S	32	16	$\bar{1}$·982 30	L.B.	0·018 02	16·91	1·36
^{35}Cl	35	17	$\bar{1}$·981 30	L.B.	0·018 53	17·39	1·41
^{37}Cl	37	17	$\bar{1}$·978 80	L.B.	0·019 72	18·50	1·32
^{36}A	36	18	$\bar{1}$·978 00	L.B.	0·018 15	17·03	1·52
^{40}A	40	18	$\bar{1}$·975 04	L.B.	0·020 26	19·01	1·36
^{45}Si	45	21	$\bar{1}$·968 00	G.	0·019 70	18·48	1·64
^{52}Cr	52	24	$\bar{1}$·948 00	G.	0·020 13	18·89	1·83
^{58}Ni	58	28	$\bar{1}$·942 00	G.	0·019 61	18·40	2·19
^{64}Zu	64	30	$\bar{1}$·937 00	G.	0·020 89	18·94	2·28
^{75}As	75	33	$\bar{1}$·934 00	G.	0·021 32	20·00	2·38
^{78}Se	78	34	$\bar{1}$·938 00	G.	0·021 33	20·02	2·45
^{80}Se	80	34	$\bar{1}$·941 00	G.	0·021 77	20·42	2·40
^{79}Br	79	35	$\bar{1}$·929 00	G.	0·021 22	19·90	2·53
^{81}Br	81	35	$\bar{1}$·926 00	G.	0·021 81	20·46	2·46
^{78}Kr	78	36	$\bar{1}$·926 00	G.	0·020 44	19·18	2·70
^{80}Kr	80	36	$\bar{1}$·926 00	G.	0·020 93	19·64	2·64
^{82}Kr	82	36	$\bar{1}$·927 00	G.	0·021 40	20·07	2·58
^{83}Kr	83	36	$\bar{1}$·927 00	G.	0·021 64	20·30	2·55
^{84}Kr	84	36	$\bar{1}$·928 00	G.	0·021 86	20·51	2·53
^{86}Kr	86	36	$\bar{1}$·929 00	G.	0·022 33	20·95	2·47
^{93}Nb	93	41	$\bar{1}$·926 00	G.	0·021 09	19·79	2·98
^{100}Mo	100	42	$\bar{1}$·945 00	G.	0·021 60	20·26	2·98
^{103}Rh	103	45	$\bar{1}$·920 00	G.	0·021 24	19·93	3·25

Nucleus	A	Z	M–A	Authors	k	Q in millions of volts	r_0 in 10^{-13} cm.
^{120}Sn	120	50	$\bar{1}$·912 00	G.	0·022 21	20·84	3·45
^{127}I	127	53	$\bar{1}$·932 00	G.	0·021 64	20·30	3·76
^{134}Xe	134	54	$\bar{1}$·929 00	G.	0·022 49	21·10	3·68
^{133}Cs	133	55	$\bar{1}$·933 00	G.	0·021 83	20·48	3·87
^{138}Bu	138	56	$\bar{1}$·916 00	G.	0·022 52	21·13	3·82
^{181}Ta	181	73	$\bar{1}$·928 00	G.	0·022 15	20·78	5·06
^{187}Re	187	75	$\bar{1}$·981 00	G.	0·021 54	20·21	5·34
^{190}Os	190	76	$\bar{1}$·980 00	G.	0·021 61	20·28	5·40
^{192}Os	192	76	$\bar{1}$·980 00	G.	0·021 85	20·50	5·34
^{200}Hg	200	80	0·016 00	G.	0·021 15	19·85	5·80
^{203}Tl	203	81	0·037 00	G.	0·020 95	19·66	5·93
^{205}Tl	205	81	0·037 00	G.	0·021 17	19·86	5·87

The data extend from Z=1 to Z=81 for 61 different nuclei of 37 different elements. The value of ϕ in millions of volts is calculated from the formula

$$\phi=\frac{m_0c^3k}{e}\times10^{-14}\text{ million volts,} \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and the effective radius r_0 from the equation

$$r_0=\frac{Ze^2}{m_0c^2k}\text{ cm.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

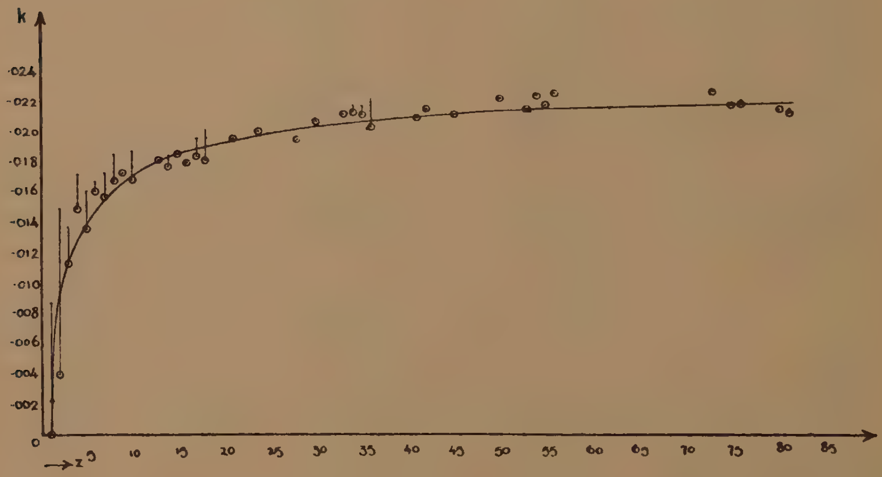
with the following data quoted by Taylor and Glasstone (12):—

$$m_0=1{\cdot}6725\times10^{-24}\text{ gm., } c=2{\cdot}9978\times10^{10}\text{ cm. sec.}^{-1},$$

$$e=4{\cdot}8025\times10^{-10}\text{ abs. e.s.u.}$$

The abbreviations L.B. stand for Livingston and Bethe, and G. for Gamow. With regard to the last element in the table, thallium, two isotopes are included in the data, namely ^{203}Tl and ^{205}Tl, and their effective radii are $5{\cdot}93\times10^{-13}$ cm. and $5{\cdot}87\times10^{-13}$ cm. respectively. The atomic number of thallium being 81, it is possible to compare our results with the results obtained by G. Gamow (13) in his theory of α-disintegration, which gives a satisfactory explanation of the Geiger-Nuttall relation between the range and the decay constant of α-radiators. Gamow obtains three values for nuclei of atomic number 81, namely $r_0=5{\cdot}7\times10^{-13}$ cm., $r_0=6{\cdot}0\times10^{-13}$ cm. and $r_0=6{\cdot}3\times10^{-13}$ cm., which are seen to be in rather striking agreement with our results.

In the diagram k is plotted against Z. The points with small circles round them correspond to the lowest or " ground " potential levels for each element as given by the available data. Other values, corresponding to different isotopes of the same element, are represented by small dots. The different levels for the same element are joined by a line parallel to the axis of k. The curve is drawn to represent the change in " ground " values of k with Z. It may prove possible with fuller data to draw a series of curves through the higher potential values, but owing to the insufficiency of the present data, this could not be done with any degree of certainty. In fact, some of the values which, being the lowest for each element, are now assumed to be "ground" levels may prove to be otherwise. The data are, however, just enough to indicate, unmistakably, the general trend of the curve, which is seen to rise steeply at first, then more gradually, reaching a value corresponding to about 20 million volts at Z=81.



§ 7. *Nuclear Potential Uniquely Determined by Mass Deficiency.*

It should be remarked that our determination of the nuclear potential is independent of the components of which the nucleus is conceived to be built up, provided all charged particles are massive particles. For instance, if the oxygen nucleus ^{16}O is considered to be built up of four α-particles instead of eight protons and eight neutrons, the value of ϕ and the mass of the nucleus are unaffected, since we must take into consideration the diminution in mass of each α-particle arising from its presence in a field of a higher potential than its own nuclear field. If k_0 is the value of k for ^{16}O and k_α for ^4He, the following equation which expresses the building up of ^{16}O from four ^4He nuclei, namely

$$16{\cdot}00000=4\times4{\cdot}00389\left[1-\frac{2\times1{\cdot}00813}{4{\cdot}00389}(k_0-k)_\alpha\right], \quad . \quad . \quad . \quad (27)$$

is seen to be satisfied if we substitute the values given in the table, namely $k_0=\cdot01692$ and $k_\alpha=\cdot01499$. The same is true for neutrons or other massive components. Thus the nuclear potential is uniquely determined by its mass deficiency, in contrast with certain current theories which make the so-called "binding energy" of the nucleus dependent on the proportion of protons, α-particles, etc. of which it is conceived to be built up.

§ 8. *Non-massive Particles.*

For non-massive particles the position is different. If μ_0 is the free rest mass of an electron, then $\frac{e\phi}{c^2}$ is large compared with μ_0 for nuclear fields. For a field of 20 million volts $\frac{e\phi}{\mu_0 c^2}$ is equal to about 40. If $\frac{\mu_0 c^2}{e\phi}$ is treated as a small quantity, equation (21) now gives to the first order

$$\mu=\frac{2e\phi}{c^2}, \qquad (28)$$

so that the mass μ of the electron becomes proportional to the potential of the field. For a positron the mass is diminished, being divided by $\frac{2e\phi}{\mu_0 c^2}$. Thus the mass of a positron in a field of 20 million volts is equal to about one-eightieth part of its free rest mass. A positron would accordingly dwindle, so to speak, to a very small mass in the neighbourhood of nuclei. This may be related to the observed short-livedness of the positron. Equation (28) is in one respect quite remarkable. The mass of an electron in a strong field seems to be quite independent of its free mass. Of course, this is only a first approximation, but it is nevertheless noteworthy. The mass of the electron appears to be completely " dissolved " in the field. The ejection of an electron from a nucleus becomes therefore more of the nature of an emission of field energy than an ejection of a particle. This is in agreement with the observed facts of β-disintegration.

Summary.

A new metric of space-time is introduced in the theory of relativity which is a generalized form of the Riemannian metric. The world-line of a charged particle is shown to be a geodesic, and the equations of motion of the particle are obtained. The metric gives the mass of a particle as a function of its potential energy. The mass deficiency of nuclei is thus explained as arising from the presence of their charged components in the nuclear field. The potential of nuclear fields is determined from exact mass measurements for 37 different elements. Calculations of effective radii agree with results obtained by Gamow from radioactive data. The mass of an electron in a nuclear field indicates that β-disintegration is more akin to emission of field energy than to particle ejection. The mass of a positron becomes very small in the neighbourhood of nuclei. A curve is drawn from the data to represent the growth of potential with atomic number Z, reaching a value of about 20 million volts at Z=81.

References.

(1) H. Weyl, ' Space—Time—Matter,' London (1922).
(2) M. Kaluza, *Sitzungsber. d. Berl. Akad.* p. 966 (1921).
(3) See Eddington, ' The Mathematical Theory of Relativity,' p. 171.

(4) See Eddington, *loc. cit.* p. 190; also G. Schott, 'Electromagnetic Radiation,' p. 283, § 278, Cambridge (1912).
(5) See H. A. Bethe, Phys. Rev. liii. p. 313 (1938).
(6) See J. Mattauch, Phys. Rev. lvii. p. 1115 (1940).
(7) M. S. Livingston and H. A. Bethe, Rev. Mod. Phys. ix. p. 336 (1937), quoted by Taylor and Glastone, reference below, p. 53.
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(10) K. T. Bainbridge and E. B. Jordan, Phys. Rev. xlix. p. 883 (1936); l. p. 98 (1936); li. pp. 384, 385 (1937).
(11) *Loc. cit.*
(12) H. S. Taylor and S. Glasstone, 'Atomistics and Thermodynamics,' New York, p. 671 (1943), quoted from R. T Birge.
(13) *Loc. cit.* p. 105, Table V.

LXXXVII. *Waves in Deep Water due to a concentrated Surface Pressure.*

By George Green, Lecturer in Physics in Glasgow University*.

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The problems of determining the wave-systems arising from the application of a stationary periodic, or moving surface pressure, might be regarded as fundamental problems of Hydrodynamics, in view of their relation to the important problem of Ship-Waves. Solutions of the problem of motion due to a stationary periodic surface pressure are given by Lamb in Edition 3 of 'Hydrodynamics,' pp. 375–7, and in his Presidential Address "On Deep Water Waves" to the London Mathematical Society, Nov. 10 (1904). Edition 6 of 'Hydrodynamics' does not include this problem, but it contains a full account of the related problem of the motion due to a moving surface pressure. The method employed to obtain a determinate result is the same in both cases, but these solutions are not in agreement one with the other.

In Edition 6, p. 406, reference is given to "a different treatment of the latter problem by Lord Kelvin in 'Deep Water Ship Waves,'" Proc. R.S.E. XXV. p. 562 (1905). In a still later paper, Proc. R.S.E. XXVI. p. 412 (1906), Lord Kelvin applied yet another different treatment to a group of cognate problems involving surface pressure, but the group did not include the case of a concentrated pressure which is that dealt with by Lamb. Some years ago the present writer had occasion to use Professor Lamb's result for the case of the stationary periodic pressure in connection with a practical problem, and found difficulty with it. He has thus been led to work out the solution of both problems referred to above by the method used in Lord Kelvin's last "Waves" paper, and has arrived at a solution that differs from that given by Lamb for the case of the periodic surface pressure. As each author claims that the method which he uses leads to a determinate result for the motion of every particle of the water due to the applied surface pressure, it is of importance to record the solution now obtained by Lord Kelvin's method in order to determine the discrepancy between it and the solution recorded by Lamb. Lord Kelvin intended to publish a solution of the Ship Waves problem by the method of the present paper (Proc. R.S.E. XXVI. p. 436).

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* Communicated by the Author.

To begin with, we consider the problem of the periodic surface pressure, and the notation which we use throughout is that used in Lamb's 'Hydrodynamics.' The fundamental theorem of 'Hydrodynamics' which we employ is that stated in § 18, that each impulse, P, applied to a fluid, gives rise to a motion represented by a velocity potential ϕ, such that

$$\rho\phi = \mathrm{P}, \text{ at the instant of application of P.} \quad . \quad . \quad . \quad (1)$$

For a concentrated impulse, P, applied at point $x=0$ of the surface at $t=0$, we have

$$\left.\begin{aligned} \rho\phi &= \frac{\mathrm{P}}{\pi}\int_0^\infty dk \cos\sigma t \cos kx\, e^{ky}, \\ \text{which gives} \qquad \rho\phi\begin{Bmatrix} x=0 \\ t=0 \end{Bmatrix} &= \frac{\mathrm{P}}{\pi}\int_0^\infty dk \cos kx = \mathrm{P} \end{aligned}\right\}, \quad . \quad . \quad . \quad (2)$$

as required. Herein the positive direction of y is upwards, and $\sigma^2=gk$. If the impulse P is that due to the action of a surface pressure $p\cos\omega\tau$ applied at time τ during an interval of time $d\tau$, we may write $\mathrm{P}=p\cos\omega\tau\, d\tau$ and

$$\rho\phi_0 = \frac{p}{\pi}\int_0^t\int_0^\infty d\tau\, dk \cos\omega\tau \cos\sigma(t-\tau)\cos kx \quad . \quad . \quad (3)$$

as value of $\rho\phi$ at the surface, $y=0$, due to the application of a periodic surface pressure, $p\cos\omega\tau$, during the interval from $\tau=0$ to $\tau=t$. To represent the effect, measured in terms of $\rho\phi$, of a periodic pressure applied for an indefinitely great interval of time, we require

$$\rho\phi_0 = \underset{t\to\infty}{\mathrm{L}t}\ \frac{p}{\pi}\int_0^\infty \frac{\sigma\sin\sigma t - \omega\sin\omega t}{\sigma^2-\omega^2}\cos kx\, dk. \quad . \quad . \quad . \quad . \quad (4)$$

The corresponding elevation of the upper surface of the fluid, η, is given by

$$\eta = \underset{t\to\infty}{\mathrm{L}t}\ \frac{p}{\pi g\rho}\int_0^\infty \frac{k}{k-\kappa}(\cos\sigma t - \cos\omega t)\cos kx\, dk, \quad . \quad (5)$$

in which, following Lamb, we have put $\kappa=\omega^2/g$. Equations (4) and (5), when evaluations are carried out, constitute the solution required in accordance with Lord Kelvin's method.

That the above solution is correct can be verified by the following test. According to Fourier's method a space periodic effect $p\cos kx$ is employed to yield a concentrated effect p at the point $x=0$ by means of the process represented by

$$\frac{p}{\pi}\int_0^\infty dk \cos kx,$$

as in (2) above. Similarly a time periodic effect, $\rho\phi_0$, due to the operation of a periodic surface pressure $p \cos \omega t$ being known, it may be employed to yield the effect, $(\rho\phi_0)$, due to the operation of an instantaneous pressure p applied at instant $t=0$. In short, by carrying out the process

$$\rho\phi_0 = \frac{1}{\pi}\int_0^\infty d\omega\, \rho\phi_0, \text{ as given in (4) above,} \quad . \quad . \quad . \quad (6)$$

we should recover equation (2), with p in place of P.

For convenience we may write (4) in the form $\rho\phi_0=(p/\pi)(\mathrm{I}_1-\mathrm{I}_2)$, where

$$\mathrm{I}_1 = \underset{t\to\infty}{\mathrm{Lt}} \int_0^\infty \frac{\sigma \sin \sigma t \cos kx}{\sigma^2-\omega^2}dk = \underset{t\to\infty}{\mathrm{Lt}}\ \frac{1}{2}\int_0^\infty \frac{\sigma\{\sin(\sigma t+kx)+\sin(\sigma t-kx)\}dk}{\sigma^2-\omega^2}. \quad . \quad . \quad . \quad (7)$$

Choose now a complex variable ζ defined by $\zeta=k+im$, and write $f=\sqrt{g\zeta}\,t+\zeta x$; the first term may be evaluated by means of the complex integrals

$$\int \frac{\sqrt{g\zeta}\, e^{if}}{\zeta-\kappa}d\zeta \quad \text{and} \quad \int \frac{\sqrt{g\zeta}\, e^{-if}}{\zeta-\kappa}$$

as indicated below.

For a contour consisting of the real axis indented above at $k=\kappa$, an infinite quadrant above the real axis, and a return to the origin by the imaginary axis, we obtain

$$\int_0^\infty \frac{\sqrt{gk}\, e^{i(\sigma t+kx)}dk}{k-\kappa} = \pi i\omega e^{i(\omega t+\kappa x)} + \int_0^\infty \frac{\sqrt{img}\, e^{i\{\sqrt{img}\,t+imx\}}dm}{m+i\kappa}. \quad . \quad (8)$$

For a contour consisting of the real axis indented below at $k=\kappa$, an infinite quadrant below the real axis and a return to the origin by the negative imaginary axis, we obtain

$$\int_0^\infty \frac{\sqrt{gk}\, e^{-i(\sigma t+kx)}dk}{k-\kappa} = -\pi i\omega e^{-i(\omega t+\kappa x)} + \int_0^\infty \frac{\sqrt{-img}\, e^{i\{\sqrt{img}\,t+imx\}}dm}{m+i\kappa}. \quad (9)$$

In each integration with respect to m the integrand contains the factor $e^{-\alpha t}$, where $\alpha=\sqrt{\frac{mg}{2}}$, so that in passing to $\mathrm{Lt}(t\to\infty)$, these integrals vanish. On taking into account similar results for the second term in (7), we obtain the evaluation

$$\left.\begin{aligned} \mathrm{I}_1 &= \frac{1}{2g}\{\pi\omega \cos(\omega t+\kappa x)+\pi\omega\cos(\omega t-\kappa x)\}. \\ &= \frac{\pi}{g}\,\omega \cos \omega t \cos \kappa x \end{aligned}\right\} \quad . \quad . \quad . \quad (10)$$

The same method of evaluation can be applied to evaluate I_2, or the evaluation may be written down from results given by Lamb (Presidential Address, p. 102). By inserting the required values in (4) we obtain

$$\left.\begin{aligned}\rho\phi_0&=\frac{p}{g}\left\{\omega\cos\omega t\cos\kappa x-\omega\sin\omega t\left[-\sin\kappa x+\frac{1}{\pi}\int_0^\infty\frac{me^{-mx}dm}{m^2+\kappa^2}\right]\right\}\\&=\frac{p}{g}\left\{\omega\cos(\omega t-\kappa x)-\frac{\omega\sin\omega t}{\pi}\int_0^\infty\frac{me^{-mx}dm}{m^2+\kappa^2}\right\}.\end{aligned}\right]\ .\ (11)$$

From this, or from (5) above, we may obtain the surface elevation η in the usual way.

We can now apply the test indicated in (6) above, using (11) as integrand. This process involves the evaluation of a double integral

$$I^l=\frac{p}{\pi^2 g}\int_0^\infty\int_0^\infty\frac{\omega\sin\omega t\, me^{-mx}dm\,d\omega}{m^2+\kappa^2}\quad .\ .\ .\ .\ .\ .\ .\ . \quad (12)$$

The integration with respect to ω yields

$$\left.\begin{aligned}I^l&=\frac{p}{2\pi}\int_0^\infty e^{-\sqrt{\frac{mg}{2}}t-mx}\sin\sqrt{\frac{mg}{2}}\,t\,dm\\&=\text{real part of }-\frac{p}{2\pi}i\int_0^\infty e^{i(\sqrt{img}\,t+imx)}\,dm\end{aligned}\right\}, \quad .\ .\ . \quad (13)$$

which, by means of the transformation $\kappa=im$ becomes

$$I^l=-\frac{p}{2\pi}\int_0^\infty\cos(\omega t+\kappa x)d\kappa.\quad .\ .\ .\ .\ .\ .\ .\ .\ . \quad (14)$$

We thus find

$$\left.\begin{aligned}\rho\phi_0&=\frac{1}{\pi}\int_0^\infty\rho\phi_0\,d\omega,\text{ where the integrand is (11) above}\\&=\frac{p}{2\pi}\int_0^\infty\{\cos(\omega t-\kappa x)+\cos(\omega t+\kappa x)\}d\kappa\\&=\frac{p}{\pi}\int_0^\infty\cos\omega t\cos\kappa x\,d\kappa,\text{ as required.}\end{aligned}\right\}\ .\quad (15)$$

The solution recorded by Lamb is given in terms of η only. It can be obtained from (5) by omitting the terms containing σ. On examination of (5) it therefore appears that Lamb's solution includes the effect of an initial surface elevation in addition to the effect due to the periodic surface pressure, whereas the solution by Lord Kelvin's method gives the effect due to the applied surface pressure alone. With respect to the test applied above, it will be observed that Lamb's solution recovers only half the value required.

Consider now the problem of a concentrated pressure, p, moving at a uniform speed c along the surface of the fluid in the direction of x positive.

Starting from equation (2) as before, we may write the solution of the problem in the form—

$$\left.\begin{aligned}\rho\phi_0&=\frac{p}{\pi}\int_0^t\int_0^\infty d\tau\, dk \cos\sigma(t-\tau)\cos k(x-c\tau)\\&=-\frac{p}{\pi c^2}\int\frac{c\sin k(x-ct)-c\cos\sigma t\sin kx+(\sigma/k)\sin\sigma t\cos kx}{k-\kappa}dk.\end{aligned}\right\}\quad .(16)$$

This applies to the case of a pressure applied during an interval t to fluid initially at rest, and κ represents (g/c^2). If the pressure is applied for an indefinitely great interval the limiting value $\underset{t\to\infty}{\mathrm{L}t}\ (\rho\phi_0)$ is the required solution; and the corresponding value η, the surface elevation, is given by

$$\eta=\underset{t\to\infty}{\mathrm{L}t}\ \frac{p\kappa}{\pi g\rho}\int_0^\infty\frac{\{\cos k(x-ct)-\cos\sigma t\cos kx-(k/\kappa)^{\frac{1}{2}}\sin\sigma t\sin kx\}dk}{k-\kappa}.\quad (17)$$

This equation is applicable at all values of x, except at the point, $x=ct$, just reached by the moving pressure. The values of η are in this problem required for points just behind or just in front of the moving pressure, so that the condition, $t\to\infty$, implies that both x and t become infinitely great while $(x-ct)$ may have any finite positive or negative value.

As in the case of the previous problem, we use the "principal value" of the integrals involved in (16). From an evaluation given in (11) we obtain, for the first term in (17),

$$\frac{p\kappa}{g\rho}\left[-\sin\kappa z+\frac{1}{\pi}\int_0^\infty\frac{me^{-m\alpha}\,dm}{m^2+\kappa^2}\right],\ \text{where } z=(x-ct) \text{ is positive.}\quad (18)$$

By expanding $\cos\sigma t$, and evaluating the separate terms, we obtain for the second term in (17) the value

$$-\frac{p\kappa}{g\rho}\left[-\sin\kappa x\cos\kappa ct+\frac{1}{\pi}\int_0^\infty\frac{e^{-m\alpha}\{\cos\alpha t\cosh\alpha t-\kappa\sin\alpha t\cosh\alpha t\}dm}{m^2+\kappa^2}\right],\quad .\ .\ .\ .\ (19)$$

where $\alpha=(mg/2)^{\frac{1}{2}}$, before applying the limiting condition. When t and x both become infinitely great, in the integral with respect to m the terms containing t are oscillatory, while the x term becomes negligible. The value of the second term in (17) reduces to $[p\kappa/(g\rho)\sin\kappa x\cos\kappa ct]$. Similarly, by expanding $\sin\sigma t$, and evaluating the separate terms, we obtain for the third term in (17) the ultimate value $-[p\kappa\cos\kappa x\sin\kappa ct/(g\rho)]$. These values inserted in (17) yield

$$\left.\begin{aligned}\eta&=\frac{p\kappa}{\pi g\rho}\int_0^\infty\frac{me^{-m\alpha}\,dm}{m^2+\kappa^2},\ z=x-ct\ \text{(positive)},\\\eta&=\frac{p\kappa}{\pi g\rho}\left[-2\pi\sin\kappa z'+\int_0^\infty\frac{me^{-m\alpha'}\,dm}{m^2+\kappa^2}\right],\ z'=ct-x\ \text{(positive)}.\end{aligned}\right\}\quad (20)$$

Equation (16) is the solution of the problem by Lord Kelvin's method. It represents the motion and all effects of the moving pressure at any instant t from the commencement of its motion. Equations (20) represent the displacement condition finally established when t becomes infinitely great. The result recorded in (20) applies equally to the cognate problem of steady fluid motion in which the pressure is applied at a fixed point, while a horizontal velocity c is imposed on every particle of fluid—which is the problem solved by Lamb: in this case the two solutions are in complete agreement.

One important difference between the solutions given above and those recorded by Lamb lies in the methods used to evaluate the integrals involved. In evaluating integral solutions obtained by Lord Kelvin's method we have used throughout the "principal values" of the integrals. In the solutions recorded by Lamb principal values are not used; and, as stated earlier, the evaluations given in the solutions of the two problems are not in agreement. The solution given in the case of the periodic pressure problem in fact raises mathematical questions the discussion of which is beyond the scope of the present paper.

The method used by Lamb involves the introduction of a frictional force operating throughout the fluid proportional to the velocity. This fictitious force is introduced in order to suppress all possible motions of the fluid which do not depend upon the application of the pressure for their maintenance, and in so doing to render the solution obtained determinate. The frictional force is finally made vanishingly small. The method is due to Lord Rayleigh.

Now it may be verified from equation (11) above, or from the solutions of similar problems *, that the motion ultimately established throughout the fluid due to the application of a stationary periodic pressure may be regarded as the resultant of two separate motions—one of which requires the continued application of the pressure for its maintenance—the other of which, when once established, would continue for ever without any surface pressure. This latter motion is an essential constituent of the whole motion due to the applied pressure, but does not depend on the application of the applied pressure for its continuance. It would therefore be suppressed by the continued operation of a frictional force, as in the present case. In short, the conclusion to be drawn from the above investigation is that the introduction of a fictitious force not only confuses the issue and renders the problem more difficult than before, but also leads into error by the suppression of essential terms.

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* Lord Kelvin, Proc. R.S.E. xxvi. p. 429 (1906).

LXXXVIII. *Notices respecting New Books.*

THE ROYAL SOCIETY.

Mathematical Tables for Science and Industry.

IT is considered that important investigations in the physical, chemical and engineering sciences, in mathematics and in industry may be held up by the absence of relevant mathematical tables. Several important tables of a fundamental nature have in the past been produced by the British Association through its Mathematical Tables Committee. The work in this field has now been transferred to the Royal Society on the invitation of the British Association and a special Royal Society Mathematical Tables Committee has been established to continue and extend the earlier work. This Committee would be glad to receive suggestions from industrial research groups, from individual investigators and from Service departments relating to existing needs in their special fields. Requests for assistance may be met either by advice as to the most economical way of producing the desired tables, or by an undertaking to produce the tables under the auspices of the Royal Society, or, in exceptional cases, by a grant towards the cost of tabulation or to make possible the publication of important tables which may exist only in manuscript. In the first instance inquiries should be addressed to the Assistant Secretary, The Royal Society, Burlington House, London, W. 1.

July 1948.

Electrode Processes. Discussions of the Faraday Society, No. 1. [Pp. 338.] (Gurney & Jackson: London and Edinburgh, 1947.) Price 30*s*.

THIS report on a general discussion held in April, 1947, is the first of a new series to be issued as separate publications by the Faraday Society, previous general discussions having appeared in the 'Proceedings of the Faraday Society.' The material is divided into sections entitled general and theoretical, hydrogen overvoltage, studies in electrolytic polarization, deposition of metals, oxygen overvoltage, and anodic processes and cell reactions in general. There are 40 communications as well as reports of discussions on them, so that no general review of the papers is possible. The section on deposition of metals deals mostly with problems of electroplating and reactions at mercury cathodes and the general tendency is experimental, although several theoretical papers are included. Much attention is given in the first sections to concentration polarization, the mechanism of which seems still to be imperfectly understood. A short paper on standard electrode potentials, giving selected values, shows that the figures appearing in text-books are substantially accurate and that no significant changes are called for in this field. A minor point of interest mentioned in one paper is that the hydrogen-oxygen coulometer is more accurate if the usual acid or alkali solution is replaced by a solution of sodium or potassium sulphate. The section on hydrogen overvoltage contains little which is new, but that on oxygen overvoltage makes it clear that some progress has been made in this region. In the metal deposition section there is an interesting paper reporting electron diffraction studies of deposited metals. The volume, while rather tedious reading for the non-specialist in any of the numerous sections, provides a useful survey of the subject and is of the high standard of earlier Faraday Society discussions. Most of the papers have summaries in French and German, but many of them have no summary in English, which is an omission to be deplored. There is a table of contents and an author index. The volume is in paper covers.

J. R. PARTINGTON.

The Strange Story of the Quantum. By BANESH HOFFMANN. [Pp. xi+239.] (New York: Harper and Brothers, 1947.) $3.00.

THOSE who would write on quantum mechanics for the general reader are faced with a pretty problem. For the ideas of quantum mechanics are intimately bound up with their mathematical formulation, while the one characterisitc which appears to be accepted as common to all species of that most varied genus, the general reader, is a distaste for mathematics; on some species, indeed, a mathematical symbol has the effect of a Gorgon's head. Accepting, and apparently welcoming, the handicap, tne author gives a lively account of his subject. The pre-quantum position is sketched in a Prologue, and Act I, in five chapters, covers the period from 1900 to the early twenties. An Intermezzo contains a warning of the rough going to come. Act II begins with de Broglie, deals with the work of Heisenberg, Dirac and Schrödinger and with the unification of the various schemes, and discusses at some length such matters as the indeterminacy principle, the probability interpretation of wave functions, and the exclusion principle. An Epilogue, which touches on problems of the nucleus, concludes with some remarks on the wonder of creation, and on the crisis of our civilization.

To readers with some knowledge of quantum mechanics, it would soon be apparent that the author has a thorough grasp of the subject, even in its more recondite aspects, and a good appreciation of the course of its historical development. He could have written a book which, so far as is possible in a non-mathematical treatment, would have conveyed to the non-expert but scientifically minded reader a clear impression of the character of quantum mechanics, and of the interest and excitement of its development. Unfortunately the essentials, which are there, are so cluttered up with extravagant imagery, puns and wisecracks, as to be almost indiscernable. Dealing with the wave and particle theories of light, the author writes of the particle: "It had to resort to cunning and seek out those obscure parts of the land where the power of the wave was practically non-existent: shabby, uncultivated wildernesses which the wave had found too forbidding to develop . . . In such neglected places did the particle find refuge, and there it doggedly built for itself a new existence, tapping rich veins of gold beneath the barren ground. In the beginning the power of the particle was to that of the wave as a pebble is to the ocean, but so vigorous and rapid was its revival that soon its might stood forth like huge continents in the seven seas of physics, and in defense of its realm it produced many new weapons to match the colossal power of the wave's interference." Writing such as this, with which the book abounds, might nave provided a happy hunting ground for the authors of *The King's English*, but is hardly likely to appeal to any of those who wish to know something of quantum mechanics. In a quotation on the jacket this book is placed "in the same class with the Popular Science Classics of Jeans, Eddington and E. T. Bell." The author has a long way to go back before he can begin to approach any of those writers.

E. C. S.

Tables of Physical and Chemical Constants. By G. W. C. KAYE and T. H. LABY. [Pp. vii+194.] (Longmans, Green & Co.) Price 21*s*.

THE passage of 37 years since the first edition has left "Kaye and Laby" almost as firmly established in every physical laboratory—and as necessary—as the furniture. To discuss the usefulness of this tenth edition would therefore verge on impertinence. It is sufficient to note the continued vitality of the book despite a steady increase of girth—the ninth edition added some 20 pages, the tenth adds another 13, and the price increases in similar proportion. In this new edition the opportunity has been taken to sift again the available values of e, e/m, and h; and a revised value of g is adopted. One section—that on optical glasses—has been rewritten. The style of the ninth edition is retained; and in spite of the slight increase in size, the book is still (happily) the familiar "K. and L."

One cannot but remember that the shadow cast over the publication of the ninth edition by the death of Dr. Kaye has been deepened by the passing of his collaborator as the tenth edition was going to press. J. P. A.

Unified Calculus. By E. S. Smith, M. Salkover and H. K. Justice. [Pp. v+507.] (John Wiley and Sons, Inc. and Chapman and Hall, Ltd., 1947.) Price 21*s.* 0*d.*

This book is a sequel to an earlier text on the calculus and is intended to embody the subject-matter of a first course. The Differential and Integral Calculus is developed as a single subject, and applications to Physics and Mechanics are numerous. It is a book which may be easily read by all students, and should be of particular value to those without a teacher or a tutor. Diagrams illustrating the theory are numerous, and the drawings are excellent. Practically every aspect of the theory is illustrated by the detailed working of examples. The book is not too condensed, and indeed the length of the book is sufficient to show that the authors have not been sparing in their use of words, when these made for greater clarity. Important formulae and expressions are printed in heavy type, and tables of integrals and numerical tables of some special functions are given.

There is a single chapter only on differential equations; and a short chapter on approximate integration. The student is thereby merely introduced to these topics and will need to consult other texts for a more detailed treatment. But the chapters on hyperbolic functions, partial differentiation and multiple integration are more complete and it will probably be unnecessary to look elsewhere for additions to the theory. It should perhaps be pointed out that the present book does not treat functions of a complex variable; and, for example, the integrations discussed are confined to standard forms.

Until now there has been a need for a clear and lucid account of the calculus, and the book under review fulfils this need. It should appeal to students in the Technical Colleges as well as in the Universities. L. S. G.

Frontiers in Chemistry. Vol. V, Chemical Architecture. By R. E. Burk and O. Grummitt. [Pp. 202.] (Interscience Publishers Ltd., New York and London, 1948.) Price 27*s.* 0*d.*

This volume of the series contains articles on applications of molecular geometry in the field of reaction mechanism (H. S. Taylor); dipole moment, resonance, and molecular structure (C. P. Smyth); structure of inorganic compounds (W. C. Fernelius); X-ray studies of randomness in various materials (B. E. Warren); light scattering in polymer solutions (H. Mark); and the nature of inorganic gels (M. W. Tamlee). The articles are informative reviews of the subjects, the one on light scattering being notable in containing descriptions of experimental methods. The articles are suitably illustrated and each is provided with a bibliographly of recent publications. The paper, printing and binding are very good. Although, as in previous parts of the series, the standard of the various articles is uneven, and the price is very high for a slender volume of this character, the book will be useful to students and lecturers who wish for a rapid and easy survey of the field of study with which it deals.

J. R. Partington

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

LXXXIX. *The Gravitational Field of a Rotating Nearly Spherical Body.*

By G. L. CLARK, Fellow of Trinity College, Cambridge *.

[Received January 27, 1946.]

Summary.

IN a previous paper (1941) I have described how by considering the field at great distances from a gravitating system the equations of motion of classical dynamics can be derived in the relativity theory. Both the linear and angular equations of motion were seen to be conditions of integrability of the equations $G_{\mu\nu}=0$. Owing to the complexity of the equations it is not practicable, in general, to obtain a solution for the external field which is valid right up to the boundary of the system, but the case of a rotating nearly spherical body can be treated without undue difficulty as far as terms of the first order in the mass are concerned. The present paper is devoted to a discussion of the external and internal fields of such a system. It is assumed that the angular velocities about the principal axes are small so that cubes and higher powers may be neglected. For the sake of clearness we write down the solution and then determine the system it represents. The external field yields the angular equations of motion in the manner described in the paper quoted above. By making the $g_{\mu\nu}$ and their derivatives continuous on the boundary the form of the energy tensor in the interior is determined. It is found to consist of two parts, one representing the mass motion, the other the internal stresses which vanish at the boundary. The solutions for particular simple systems, as for example a rotating sphere, are then written down merely by omitting several of the terms of the general solution. In all cases the coordinate system is taken to be Galilean at infinity.

1. *The Energy Tensor.*

We set

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.1)$$

where $\eta_{\mu\nu}$ are the Galilean values of the $g_{\mu\nu}$ and the $h_{\mu\nu}$ are small. The components of the energy tensor to the first approximation are then given by (Eddington, 1924, p. 129, eqn. 57·5).

$$16\pi T=-\Box(h_{\mu}^{\nu}-\tfrac{1}{2}\delta_{\mu}^{\nu}h), \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.2)$$

where the operator

$$\Box\equiv\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.3)$$

and

$$\frac{\partial}{\partial x_{\alpha}}(h_{\mu}^{\alpha}-\tfrac{1}{2}\delta_{\mu}^{\alpha}h)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.4)$$

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* Communicated by Sir Edmond Whittaker, F.R.S.

Writing
$$\gamma_{\mu\nu}=h_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}\eta^{\sigma\varrho}h_{\sigma\varrho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{1.5}$$
the equations (1.2) and (1.4) become
$$16\pi T_{\mu\nu}=\gamma_{\mu\nu/ss}-\gamma_{\mu\nu/44}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{1.6}$$
provided
$$\gamma_{4\mu/4}=\gamma_{\mu s/s}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{1.7}$$

In the above equations we have used the conventions of the 1941 paper, namely, the ordinary derivative has been denoted by a line followed by the appropriate suffix and the Latin indices take only the spatial values 1, 2, 3, whereas Greek indices refer to both space and time, running over the values 1, 2, 3, 4.

2. *The Equation of a Slowly Rotating Nearly Spherical Ellipsoid.*

The equation of a nearly spherical ellipsoid referred to principal axes fixed in the body is
$$\frac{x'^2}{a^2(1-\epsilon_1)}+\frac{y'^2}{a^2(1-\epsilon_2)}+\frac{z'^2}{a^2(1-\epsilon_3)}=1, \quad . \quad . \quad . \quad . \tag{2.1}$$
and the squares of the ϵ may be neglected. If ω_s, $\dot{\omega}_s$ are the components of the angular velocity and angular acceleration of these axes about axes fixed in space and instantaneously coinciding with them, then the direction cosines of the moving system are given by the set of equations (7.3) of the 1938 paper, *i, e.*,
$$\left.\begin{aligned} l_1&=1-\tfrac{1}{2}(\omega_2{}^2+\omega_3{}^2)t^2,\\ l_2&=-\omega_3 t-\tfrac{1}{2}\dot{\omega}_3 t^2+\tfrac{1}{2}\omega_1\omega_2 t^2,\\ l_3&=\omega_2 t+\tfrac{1}{2}\dot{\omega}_2 t^2+\tfrac{1}{2}\omega_3\omega_1 t^2, \end{aligned}\right\}, \quad . \quad . \tag{2.2}$$
and similar expressions for (m_1, m_2, m_3) and (n_1, n_2, n_3). We have
$$\left.\begin{aligned} x'&=l_1x+m_1y+n_1z,\\ y'&=l_2x+m_2y+n_2z,\\ z'&=l_3x+m_3y+n_3z. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \tag{2.3}$$
Using (2.2) and (2.3) the equation (2.1) may be written
$$\begin{aligned} r&=a\left[1-\frac{1}{2}(\epsilon_1x'^2+\epsilon_2y'^2+\epsilon_3z'^2)/a^2\right]\\ &=a\left[1-\frac{1}{2a^2}(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)-\frac{1}{2a^2}(\epsilon_3-\epsilon_2)\omega_1{}^2t^2(y^2-z^2)\right.\\ &\quad-\frac{1}{2a^2}(\epsilon_1-\epsilon_3)\omega_2{}^2t^2(z^2-x^2)-\frac{1}{2a^2}(\epsilon_2-\epsilon_1)\omega_3{}^2t^2(x^2-y^2)\\ &\quad+(\epsilon_2-\epsilon_1)(\omega_3t+\tfrac{1}{2}\dot{\omega}_3t^2)\frac{xy}{a^2}+(\epsilon_3-\epsilon_2)(\omega_1t+\frac{1}{2}\dot{\omega}_1t^2)\frac{yz}{a^2}\\ &\quad+(\epsilon_1-\epsilon_3)(\omega_2t+\frac{1}{2}\dot{\omega}_2t^2)\frac{zx}{a^2}+\frac{1}{2}\omega_1\dot{\omega}_2t^2(2\epsilon_3-\epsilon_1-\epsilon_2)\frac{xy}{a^2}\\ &\quad\left.+\frac{1}{2}\omega_2\omega_3t^2(2\epsilon_1-\epsilon_2-\epsilon_3)\frac{yz}{a^2}+\frac{1}{2}\omega_3\omega_1t^2(2\epsilon_2-\epsilon_3-\epsilon_1)\frac{zx}{a^2}\right]. \end{aligned} \tag{2.4}$$

We shall find in the next section that $\dot{\omega}_s$ is of the order ϵ so that terms involving $\epsilon_r\dot{\omega}_s$ are of the order ϵ^2 and may accordingly be neglected. The internal and external values of the $\gamma_{\mu\nu}$ and $\gamma_{\mu\nu/\sigma}$ must be continuous at the surface (2.4).

3. *The External Field.*

We consider the field given by

$$\gamma_{44}=\gamma_0+\gamma_u. \qquad (3.1)$$

$$\begin{aligned}
\gamma_0= & -\frac{16\pi\sigma a^3}{3r}+\frac{8\pi\sigma a^3(\epsilon_1+\epsilon_2+\epsilon_3)}{3r}+\frac{8\pi\sigma a^5}{15}\left\{\epsilon_1\left(\frac{3x^2}{r^5}-\frac{1}{r^3}\right)+\epsilon_2\left(\frac{3y^2}{r^5}-\frac{1}{r^3}\right)\right. \\
& \left.+\epsilon_3\left(\frac{3z^2}{r^5}-\frac{1}{r^3}\right)\right\}-\frac{16\pi\sigma^5}{5}\left\{\frac{\omega_3t(\epsilon_2-\epsilon_1)yx}{r^5}+\frac{\omega_1t(\epsilon_3-\epsilon_2)zy}{r^5}\right. \\
& \left.+\frac{\omega_2t(\epsilon_1-\epsilon_3)xz}{r^5}\right\}+\frac{8\pi\sigma a^5}{5}\left\{\frac{(\epsilon_2-\epsilon_1){\omega_3}^2t^2(x^2-y^2)}{r^5}+\frac{(\epsilon_3-\epsilon_2){\omega_1}^2t^2(y^2-z^2)}{r^5}\right. \\
& \left.+\frac{(\epsilon_1-\epsilon_3){\omega_2}^2t^2(z^2-x^2)}{r^5}\right\}-\frac{8\pi\sigma a^5}{15}\left\{\frac{(\epsilon_2-\epsilon_1){\omega_3}^2(x^2-y^2)}{r^3}\right. \\
& \left.+\frac{(\epsilon_3-\epsilon_2){\omega_1}^2(y^2-z^2)}{r^3}+\frac{(\epsilon_1-\epsilon_3){\omega_2}^2(z^2-x^2)}{r^3}\right\} \\
& +\frac{8\pi\sigma a^7}{35}\left\{\frac{(\epsilon_2-\epsilon_1){\omega_3}^2(x^2-y^2)}{r^5}+\frac{(\epsilon_3-\epsilon_2){\omega_1}^2(y^2-z^2)}{r^5}\right. \\
& \left.+\frac{(\epsilon_1-\epsilon_3){\omega_2}^2(z^2-x^2)}{r^5}\right\}-\frac{8\pi\sigma a^5}{5}\left\{\frac{(\epsilon_3-{}^1{}_1)\dot{\omega}_3t^2yx}{r^5}+\frac{(\epsilon_3-\epsilon_2)\dot{\omega}_1t^2zy}{r^5}\right. \\
& \left.+\frac{(\epsilon_1-\epsilon_3)\dot{\omega}_2t^2xz}{r^5}\right\}+\frac{8\pi\sigma a^5}{15}\left\{(\epsilon_2-\epsilon_1)\dot{\omega}_3\left(\frac{xy}{r^3}-\frac{3xy}{7r^5}\right)\right. \\
& \left.+(\epsilon_3-\epsilon_2)\dot{\omega}_1\left(\frac{zy}{r^5}-\frac{3}{7}\frac{zy}{r^5}\right)+(\epsilon_1-\epsilon_3)\dot{\omega}_2\left(\frac{xz}{r^5}-\frac{3xz}{7r^5}\right)\right\} \\
& -\frac{8\pi\sigma a^5}{5}\left\{\frac{(2\epsilon_3-\epsilon_1-\epsilon_2)\omega_1\omega_2t^2xy}{r^5}+\frac{(2\epsilon_1-\epsilon_2-\epsilon_3)\omega_2\omega_3t^2zy}{r^5}\right. \\
& \left.+\frac{(2\epsilon_2-\epsilon_3-\epsilon_1)\omega_3\omega_1t^2xz}{r^5}\right\}+\frac{8\pi\sigma a^5}{15}\left\{(2\epsilon_3-\epsilon_1-\epsilon_2)\omega_1\omega_2\left(\frac{xy}{r^3}-\frac{3}{7}\frac{xya^2}{r^5}\right)\right. \\
& \left.+(2\epsilon_1-\epsilon_2-\epsilon_3)\omega_2\omega_3\left(\frac{zy}{r^3}-\frac{3}{7}\frac{zya^2}{r^5}\right)+(2\epsilon_2-\epsilon_3-\epsilon_1)\omega_3\omega_1\left(\frac{xz}{r^3}-\frac{3}{7}\frac{xza^2}{r^5}\right)\right\}.
\end{aligned} \qquad (3.2)$$

$$\begin{aligned}
\gamma_{41}= & -\frac{16\pi\sigma a^5}{15}\frac{(1-\epsilon_2)(\omega_3+\dot{\omega}_3t)y}{r^3}+\frac{8\pi\sigma a^5}{15}\frac{(\epsilon_1+\epsilon_2+\epsilon_3)\omega_3y}{r^3} \\
& +8\pi\sigma a^7\omega_3\left\{\epsilon_1\left(-\frac{y}{35r^5}+\frac{x^2y}{7r^7}\right)+\epsilon_2\left(-\frac{3y}{35r^5}+\frac{y^3}{7r^7}\right)\right. \\
& \left.+\epsilon_3\left(-\frac{y}{35r^5}+\frac{z^2y}{7r^7}\right)\right\}+\frac{16\pi\sigma a^5}{15}\frac{(1-\epsilon_3)(\omega_2+\dot{\omega}_2t)z}{r^3}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{8\pi\sigma a^5}{15}\,\frac{(\epsilon_1+\epsilon_2+\epsilon_3)\omega_2 z}{r^3}-8\pi\sigma a^7\omega_2\left\{\epsilon_1\left(-\frac{z}{35r^5}+\frac{x^2z}{7r^7}\right)\right.\\
&\left.+\epsilon_2\left(-\frac{z}{35r^5}+\frac{y^2z}{7r^7}\right)+\epsilon_3\left(-\frac{3z}{35r^5}+\frac{z^3}{7r^7}\right)\right\}-\frac{16\pi\sigma a^5}{15}\,\frac{(\epsilon_2-\epsilon_1)\omega_3{}^2tx}{r^3}\\
&+16\pi\sigma a^7(\epsilon_2-\epsilon_1)\omega_3{}^2t\left(\frac{x}{35r^5}-\frac{xy^2}{7r^7}\right)-\frac{16\pi\sigma a^5}{15}\,\frac{(\epsilon_3-\epsilon_1)\omega_2{}^2tx}{r^3}\\
&+16\pi\sigma a^7(\epsilon_3-\epsilon_1)\omega_2{}^2t\left(\frac{x}{35r^5}-\frac{xz^2}{7r^7}\right)+\frac{16\pi\sigma a^5}{15}\,(\epsilon_3-\epsilon_2)\left(\frac{\omega_1\omega_2ty}{r^3}\right.\\
&\left.-\frac{\omega_3\omega_1tz}{r^3}\right)+16\pi\sigma a^7(\epsilon_3-\epsilon_2)\left\{\omega_1\omega_2t\left(\frac{yz^2}{7r^7}-\frac{y}{35r^5}\right)-\omega_3\omega_1t\left(\frac{y^2z}{7r^7}-\frac{z}{35r^5}\right)\right\}\\
&+16\pi\sigma a^7\,\frac{(\epsilon_2+\epsilon_3-2\epsilon_1)\omega_2\omega_3txyz}{7r^7}\,. \qquad \ldots\ldots\ldots\ldots \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{42}=&-\frac{16\pi\sigma a^5}{15}\,\frac{(1-\epsilon_3)z(\omega_1+\dot{\omega}_1t)}{r^3}+\frac{8\pi\sigma a^5}{15}\,\frac{(\epsilon_1+\epsilon_2+\epsilon_3)\omega_1 z}{r^3}\\
&+8\pi\sigma a^7\omega_1\left\{\epsilon_1\left(-\frac{z}{35r^5}+\frac{x^2z}{7r^7}\right)+\epsilon_2\left(-\frac{z}{35r^5}+\frac{y}{7r^7}\right)+\epsilon_3\left(-\frac{3z}{35r^5}+\frac{z^3}{7r^7}\right)\right\}\\
&+\frac{16\pi\sigma a^5}{15}\,\frac{(1-\epsilon_1)x(\omega_3+\dot{\omega}_3t)}{r^3}-\frac{8\pi\sigma a^5}{15}\,\frac{(\epsilon_1+\epsilon_2+\epsilon_3)\omega_3 x}{r^3}\\
&-8\pi\sigma a^7\omega_3\left\{\epsilon_1\left(-\frac{3x}{35r^5}+\frac{x^3}{7r^7}\right)+\epsilon_2\left(-\frac{x}{35r^5}+\frac{y^2x}{7r^7}\right)+\epsilon_3\left(-\frac{x}{35r^5}+\frac{z^2x}{7r^7}\right)\right\}\\
&-\frac{16\pi\sigma a^5}{15}\,\frac{(\epsilon_3-\epsilon_2)\omega_1{}^2ty}{r^3}+16\pi\sigma a^7(\epsilon_3-\epsilon_2)\omega_1{}^2t\left(\frac{y}{35r^5}-\frac{yz^2}{7r^7}\right)\\
&-\frac{16\pi\sigma a^5}{15}\,\frac{(\epsilon_1-\epsilon_2)\omega_3{}^2ty}{r^3}+16\pi\sigma a^7(\epsilon_1-\epsilon_2)\omega_3{}^2t\left(\frac{y}{35r^5}-\frac{yx^2}{7r^7}\right)\\
&+\frac{16\pi\sigma a^5}{15}\,(\epsilon_1-\epsilon_3)\left(\frac{\omega_2\omega_3tz}{r^3}-\frac{\omega_2\omega_1tx}{r^3}\right)\\
&+16\pi\sigma a^7(\epsilon_1-\epsilon_3)\left[\omega_2\omega_3t\left(\frac{zx^3}{7r^7}-\frac{z}{35r^5}\right)-\omega_1\omega_2t\left(\frac{z^2x}{7r^7}-\frac{x}{35r^5}\right)\right]\\
&+16\pi\sigma a^7\,\frac{(\epsilon_3+\epsilon_1-2\epsilon_2)\omega_3\omega_1txyz}{7r^7}\,. \qquad \ldots\ldots\ldots\ldots \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{43}=&-\frac{16\pi\sigma a^5}{15}\,\frac{(1-\epsilon_1)x(\omega_2+\dot{\omega}_2t)}{r^3}+\frac{8\pi\sigma a^5}{15}\,\frac{(\epsilon_1+\epsilon_2+\epsilon_3)\omega_2 x}{r^3}\\
&+8\pi\sigma a^7\omega_2\left\{\epsilon_1\left(-\frac{3x}{35r^5}+\frac{x^3}{7r^7}\right)+\epsilon_2\left(-\frac{x}{35r^5}+\frac{y^2x}{7r^7}\right)\right.\\
&\left.+\epsilon_3\left(-\frac{x}{35r^5}+\frac{z^2x}{7r^7}\right)\right\}+\frac{16\pi\sigma a^5}{15}\,\frac{(1-\epsilon_2)y(\omega_1+\dot{\omega}_1t)}{r^3}\\
&-\frac{8\pi\sigma a^5}{15}\,\frac{(\epsilon_1+\epsilon_2+\epsilon_3)\omega_1 y}{r^9}-8\pi\sigma a^7\omega_1\left\{\epsilon_1\left(-\frac{y}{35r^5}+\frac{x^2y}{7r^7}\right)\right.
\end{aligned}
$$

$$+\epsilon_2\left(-\frac{3y}{35r^5}+\frac{y^3}{7r^7}\right)+\epsilon_3\left(-\frac{y}{35r^5}+\frac{z^2y}{7r^7}\right)\Big\}-\frac{16\pi\sigma a^5}{15}\frac{(\epsilon_1-\epsilon_3)\omega_2{}^2tz}{r^3}$$

$$+16\pi\sigma a^7(\epsilon_1-\epsilon_3)\omega_2{}^2t\left(\frac{z}{35r^5}-\frac{zx^2}{7r^7}\right)-\frac{16\pi\sigma a^5}{15}\frac{(\epsilon_2-\epsilon_3)\omega_1{}^2tz}{r^3}$$

$$+16\pi\sigma a^7(\epsilon_2-\epsilon_3)\omega_1{}^2t\left(\frac{z}{35r^5}-\frac{zy^2}{7r^7}\right)+\frac{16\pi\sigma a^5}{15}(\epsilon_2-\epsilon_1)\left(\frac{\omega_3\omega_1tx}{r^3}-\frac{\omega_3\omega_2ty}{r^3}\right)$$

$$+16\pi\sigma a^7(\epsilon_2-\epsilon_1)\left\{\omega_3\omega_1t\left(\frac{xy^2}{7r^7}-\frac{x}{35r^5}\right)-\omega_2\omega_3t\left(\frac{x^2y}{7r^7}-\frac{y}{35r^5}\right)\right\}$$

$$+16\pi\sigma a^7(\epsilon_1+\epsilon_2-2\epsilon_3)\omega_1\omega_2t\,\frac{xyz}{7r^7}. \qquad \text{(3.5)}$$

$$\gamma_{11}=\frac{16\pi\sigma a^7\omega_3{}^2}{105}\left\{1-\left(\frac{3}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_3\right)\right\}\left(\frac{1}{r^3}-\frac{3y^2}{r^5}\right)+\frac{16\pi\sigma a^5\omega_3{}^2}{15}\frac{(\epsilon_2-\epsilon_1)}{r}$$

$$+\frac{16\pi\sigma a^7\omega_3{}^2}{105}(\epsilon_2-\epsilon_1)\left(-\frac{1}{r^3}+\frac{3y^2}{r^5}\right)+\frac{8\pi\sigma\omega_3{}^2a^9}{9}\Big\{\epsilon_1\left(\frac{1}{35r^5}-\frac{1}{7}\frac{x^2+y^2}{r^7}+\frac{x^2y^2}{r^9}\right)$$

$$+\epsilon_2\left(\frac{3}{35r^5}-\frac{6}{7}\frac{y^2}{r^7}+\frac{y^4}{r^9}\right)+\epsilon_3\left(\frac{1}{35r^5}-\frac{1}{7}\frac{y^2+z^2}{r^7}+\frac{y^2z^2}{r^9}\right)\Big\}$$

$$+\frac{16\pi\sigma a^7\omega_2{}^2}{105}\left\{1-\left(\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_2+\frac{3}{2}\epsilon_3\right)\right\}\left(\frac{1}{r^3}-\frac{3z^2}{r^5}\right)+\frac{16\pi\sigma a^5\omega_2{}^2}{15}\frac{(\epsilon_3-\epsilon_1)}{r}$$

$$+\frac{16\pi\sigma a^7\omega_2{}^2}{105}(\epsilon_3-\epsilon_1)\left(-\frac{1}{r^3}+\frac{3z^2}{r^5}\right)$$

$$+\frac{8\pi\sigma a^9\omega_2{}^2}{9}\Big\{\epsilon_1\left(\frac{1}{35r^5}-\frac{1}{7}\frac{z^2+x^2}{r^7}+\frac{z^2x^2}{r^9}\right)+\epsilon_2\left(\frac{1}{35r^5}-\frac{1}{7}\frac{z^2+y^2}{r^7}+\frac{z^2y^2}{r^9}\right)$$

$$+\epsilon_3\left(\frac{3}{35r^5}-\frac{6}{7}\frac{z^2}{r_4}+\frac{z^4}{r^9}\right)\Big\}+\frac{16\pi\sigma a^7\omega_2\omega_3}{35}\frac{(2-\epsilon_1-3\epsilon_2-3\epsilon_3)\,zy}{r^5}$$

$$+\frac{16\pi\sigma a^9\omega_2\omega_3}{9}\left\{\epsilon_1\left(\frac{yz}{7r^7}-\frac{yzx^2}{r}\right)+\epsilon_2\left(\frac{3}{7}\frac{yz}{r^7}-\frac{y^3z}{r^9}\right)+\epsilon_3\left(\frac{3yz}{7r^7}-\frac{yz^3}{r^9}\right)\right\}. \qquad \text{(3.6)}$$

$$\gamma_{22}=\frac{16\pi\sigma a^7\omega_1{}^2}{105}\left\{1-\left(\frac{1}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{3}{2}\epsilon_3\right)\right\}\left(\frac{1}{r^3}-\frac{3z^2}{r^5}\right)+\frac{16\pi\sigma a^5\omega_1{}^2}{15}\frac{(\epsilon_3-\epsilon_2)}{r}$$

$$+\frac{16\pi\sigma a^7\omega_1{}^2}{105}(\epsilon_3-\epsilon_2)\left(-\frac{1}{r^3}+\frac{3z^2}{r^5}\right)$$

$$+\frac{8\pi\sigma a^9\omega_1{}^2}{9}\Big\{\epsilon_1\left(\frac{1}{35r^5}-\frac{1}{7}\frac{z^2+x^2}{r^7}+\frac{z^2x^2}{r^9}\right)+\epsilon_2\left(\frac{1}{35r^5}-\frac{1}{7}\frac{y^2+z^2}{r^7}+\frac{y^2z^2}{r^9}\right)$$

$$+\epsilon_3\left(\frac{3}{35r^5}-\frac{6z^2}{7r^7}+\frac{z^4}{r^9}\right)\Big\}+\frac{16\pi\sigma a^7\omega_3{}^2}{105}\left\{1-\left(\frac{3}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_3\right)\right\}$$

$$\times\left(\frac{1}{r^3}-\frac{3x^2}{r^5}\right)+\frac{16\pi\sigma a^5\omega_3{}^2}{15}\frac{(\epsilon_1-\epsilon_2)}{r}+\frac{16\pi\sigma a^7\omega_3{}^2}{105}(\epsilon_1-\epsilon_2)\left(-\frac{1}{r^3}+\frac{3x^2}{r^5}\right)$$

$$+\frac{8\pi\sigma a^9\omega_3{}^2}{9}\biggl\{\epsilon_1\biggl(\frac{3}{35r^5}-\frac{6x^2}{7r^7}+\frac{x^4}{r^9}\biggr)+\epsilon_2\biggl(\frac{1}{35r^5}-\frac{x^2+y^2}{7r^7}+\frac{x^2y^2}{r^9}\biggr)$$

$$+\epsilon_3\biggl(\frac{1}{35r^5}-\frac{x^2+z^2}{7r^7}+\frac{x^2z^2}{r^9}\biggr)\biggr\}+\frac{16\pi\sigma a^7\omega_3\omega_1}{35}\,\frac{(2-3\epsilon_1-\epsilon_2-3\epsilon_3)\,xz}{r^5}$$

$$+\frac{16\pi\sigma a^9\omega_3\omega_1}{9}\biggl\{\epsilon_1\biggl(\frac{3zx}{r^7}-\frac{zx^3}{r^9}\biggr)+\epsilon_2\biggl(\frac{zx}{7r^7}-\frac{zxy^2}{r^9}\biggr)+\epsilon_3\biggl(\frac{3zx}{7r^7}-\frac{z^3x}{r^9}\biggr)\biggr\}. \qquad \text{. . . . (3.7)}$$

$$\gamma_{33}=\frac{16\pi\sigma a^7\omega_2{}^2}{105}\biggl\{1-\biggl(\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_2+\frac{3}{2}\epsilon_3\biggr)\biggr\}\biggl(\frac{1}{r^3}-\frac{3x^2}{r^5}\biggr)+\frac{16\pi\sigma a^5\omega_2{}^2}{15}\,\frac{(\epsilon_1-\epsilon_3)}{r}$$

$$+\frac{16\pi\sigma a^7\omega_2{}^2}{105}(\epsilon_1-\epsilon_3)\biggl(-\frac{1}{r^3}+\frac{3x^2}{r^5}\biggr)+\frac{8\pi\sigma a^9\omega_2{}^2}{9}\biggl\{\epsilon_1\biggl(\frac{3}{35r^5}-\frac{6x^2}{7r^7}+\frac{x^4}{r^9}\biggr)$$

$$+\epsilon_2\biggl(\frac{1}{35r^5}-\frac{x^2+y^2}{7r^7}+\frac{x^2y^2}{r^9}\biggr)+\epsilon_3\biggl(\frac{1}{35r^5}-\frac{z^2+x^2}{7r^7}+\frac{z^2x^2}{r^9}\biggr)\biggr\}$$

$$+\frac{16\pi\sigma a^7\omega_1{}^2}{105}\biggl\{1-\biggl(\frac{1}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{3}{2}\epsilon_3\biggr)\biggr\}\biggl(\frac{1}{r^3}-\frac{3y^2}{r^5}\biggr)$$

$$+\frac{16\pi\sigma a^5\omega_1{}^2}{15}\,\frac{(\epsilon_2-\epsilon_3)}{r}+\frac{16\pi\sigma a^7\omega_1{}^2}{105}(\epsilon_2-\epsilon_3)\biggl(-\frac{1}{r^3}+\frac{3y^2}{r^5}\biggr)$$

$$+\frac{8\pi\sigma a^9\omega_1{}^2}{9}\biggl\{\epsilon_1\biggl(\frac{1}{35r^5}-\frac{y^2+x^2}{7r^7}+\frac{y^2x^2}{r^9}\biggr)+\epsilon_2\biggl(\frac{3}{35r^5}-\frac{6y^2}{7r^7}+\frac{y^4}{r^9}\biggr)$$

$$+\epsilon_3\biggl(\frac{1}{35r^5}-\frac{y^2+z^2}{7r^7}+\frac{y^2z^2}{r^9}\biggr\}\biggr)+\frac{16\pi\sigma a^7\omega_1\omega_2}{35}\,\frac{(2-\epsilon_3-3\epsilon_2-3\epsilon_1)yx}{r^5}$$

$$+\frac{16\pi\sigma a^9\omega_1\omega_2}{9}\biggl\{\epsilon_1\biggl(\frac{3}{7}\frac{xy}{r^7}-\frac{x^3y}{r^9}\biggr)+\epsilon_2\biggl(\frac{3xy}{7r^7}-\frac{xy^3}{r}\biggr)+\epsilon_3\biggl(\frac{xy}{7r^7}-\frac{xyz^2}{r}\biggr)\biggr\}. \qquad \text{. . . . (3.8)}$$

$$\gamma_{12}=\frac{16\pi\sigma a^7\omega_3{}^2}{35}\biggl\{1-\biggl(\frac{3}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_3\biggr)\biggr\}\frac{xy}{r^5}+\frac{8\pi\sigma a^9\omega_3{}^2}{9}\biggl\{\epsilon_1\biggl(\frac{3}{7}\frac{xy}{r^7}-\frac{x^3y}{r^9}\biggr)$$

$$+\epsilon_2\biggl(\frac{3xy}{7r^7}-\frac{xy^3}{r^9}\biggr)+\epsilon_3\biggl(\frac{xy}{7r^7}-\frac{xyz^2}{r^9}\biggr)\biggr\}+\frac{8\pi\sigma a^5}{15}\,\frac{(\epsilon_1-\epsilon_2)\omega_3}{r}$$

$$+\frac{16\pi\sigma a^5}{15}\,\frac{(\frac{1}{2}\epsilon_1+\frac{1}{2}\epsilon_2-\epsilon_3)\omega_1\omega_2}{r}+\frac{16\pi\sigma a^7\omega_1\omega_2}{35}\biggl(\frac{z^2}{r^5}-\frac{1}{3r^3}\biggr)$$

$$+\frac{16\pi\sigma a^7\omega_1\omega_2}{35}\biggl(\frac{5}{2}\epsilon_3+\frac{1}{2}\epsilon_1+\frac{1}{2}\epsilon_2\biggr)\biggl(\frac{1}{3r^3}-\frac{z^2}{r^5}\biggr)$$

$$+\frac{8\pi\sigma a^9\omega_1\omega_2}{9}\biggl\{\epsilon_1\biggl(-\frac{1}{35r^5}+\frac{x^2+z^2}{7r^7}-\frac{x^2z^2}{r^9}\biggr)$$

$$+\epsilon_2\biggl(-\frac{1}{35r^5}+\frac{z^2+y^2}{7r^7}-\frac{y^2z^2}{r^9}\biggr)+\epsilon_3\biggl(-\frac{3}{35r^5}+\frac{6z^2}{7r^7}-\frac{z^4}{r^9}\biggr)\biggr\}$$

$$+\frac{16\pi\sigma a^7\omega_2\omega_3}{35}\biggl(-1+\frac{3}{2}\epsilon_3+\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_2\biggr)\frac{xz}{r^5}$$

$$+\frac{8\pi\sigma a^9\omega_2\omega_3}{9}\left\{\epsilon_1\left(-\frac{3xz}{7r^7}+\frac{x^3z}{r^9}\right)+\epsilon_2\left(-\frac{xz}{7r^7}+\frac{xzy^2}{r^9}\right)\right.$$

$$\left.+\epsilon_3\left(-\frac{3xz}{7r^7}+\frac{xz^3}{r^9}\right)\right\}+\frac{16\pi\sigma a^7\omega_3\omega_1}{35}\left(-1+\frac{3}{2}\epsilon_3+\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_1\right)\frac{yz}{r^5}$$

$$+\frac{8\pi\sigma a^9\omega_3\omega_3}{9}\left\{\epsilon_1\left(-\frac{yz}{7r^7}+\frac{yzx^2}{r^9}\right)+\epsilon_2\left(-\frac{3yz}{7r^7}+\frac{y^3z}{r^9}\right)\right.$$

$$\left.+\epsilon_3\left(-\frac{3yz}{7r^7}+\frac{yz^3}{r^9}\right)\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.9)$$

$$\gamma_{23}=\frac{16\pi\sigma a^7\omega_1{}^2}{35}\left\{1-\left(\frac{1}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{3}{2}\epsilon_3\right)\right\}\frac{yz}{r^5}+\frac{8\pi\sigma a^9\omega_1{}^2}{9}\left\{\epsilon_1\left(\frac{yz}{7r^7}-\frac{yzx^2}{r^2}\right)\right.$$

$$\left.+\epsilon_2\left(\frac{3}{7}\frac{yz}{r^7}-\frac{y^3z}{r^9}\right)+\epsilon_3\left(\frac{3}{7}\frac{yz}{r^7}-\frac{yz^3}{r^9}\right)\right\}+\frac{8\pi\sigma a^5(\epsilon_2-\epsilon_3)}{15}\frac{\dot{\omega}_1}{r}$$

$$+\frac{16\pi\sigma a^5}{15}\left(\frac{1}{2}\epsilon_2+\frac{1}{2}\epsilon_3-\epsilon_1\right)\frac{\omega_2\omega_3}{r}+\frac{16\pi\sigma a^7\omega_2\omega_3}{35}\left(\frac{x^2}{r^5}-\frac{1}{3r^3}\right)$$

$$+\frac{16\pi\sigma a^7\omega_2\omega_3}{35}\left(\frac{5}{2}\epsilon_1+\frac{1}{2}\epsilon_2+\frac{1}{2}\epsilon_3\right)\left(\frac{1}{3r^3}-\frac{x^2}{r^5}\right)$$

$$+\frac{8\pi\sigma a^9\omega_2\omega_3}{9}\left\{\epsilon_1\left(-\frac{3}{35r^5}+\frac{6x^2}{7r^7}-\frac{x^4}{r^9}\right)+\epsilon_2\left(-\frac{1}{35r^5}+\frac{y^2+x^2}{7r^7}-\frac{y^2x^2}{r^9}\right)\right.$$

$$\left.+\epsilon_3\left(-\frac{1}{35r^5}+\frac{x^2+z^2}{7r^7}-\frac{x^2z^2}{r^9}\right)\right\}$$

$$+\frac{16\pi\sigma a^7\omega_3\omega_1}{35}\left(-1+\frac{3}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_3\right)\frac{yx}{r^5}$$

$$+\frac{8\pi\sigma a^9\omega_3\omega_1}{9}\left\{\epsilon_1\left(-\frac{3yx}{7r^7}+\frac{yx^3}{r^9}\right)+\epsilon_2\left(-\frac{3yx}{7r^7}+\frac{y^3x}{r^9}\right)\right.$$

$$\left.+\epsilon_3\left(-\frac{yx}{7r^7}+\frac{yxz^2}{r^9}\right)\right\}+\frac{16\pi\sigma a^7\omega_1\omega_2}{35}\left(-1+\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_2+\frac{3}{2}\epsilon_3\right)\frac{zx}{r^5}$$

$$+\frac{8\pi\sigma a^9\omega_2{}^2}{9}\left\{\epsilon_1\left(-\frac{3zx}{7r^7}+\frac{zx^3}{r^9}\right)+\epsilon_2\left(-\frac{zx}{7r^7}+\frac{zxy^2}{r^9}\right)+\epsilon_3\left(-\frac{3zx}{7r^7}+\frac{z^3x}{r^9}\right)\right\}.$$

$$. \quad . \quad . \quad . \quad (3.10)$$

$$\gamma_{31}=\frac{16\pi\sigma a^7\omega_2{}^2}{35}\left\{1-\left(\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_2+\frac{3}{2}\epsilon_3\right)\right\}\frac{zx}{r^5}+\frac{8\pi\sigma a^9\omega_2{}^2}{9}\left\{\epsilon_1\left(\frac{3zx}{7r^7}-\frac{zx^3}{r^9}\right)\right.$$

$$\left.+\epsilon_2\left(\frac{zx}{7r^7}-\frac{zxy^2}{r^9}\right)+\epsilon_3\left(\frac{3}{7}\frac{zx}{r^7}-\frac{z^3x}{r^9}\right)\right.+\frac{8\pi\sigma a^5(\epsilon_3-\epsilon_1)}{15}\frac{\dot{\omega}_2}{r}$$

$$+\frac{16\pi\sigma a^5}{15}\left(\frac{1}{2}\epsilon_3+\frac{1}{2}\epsilon_1-\epsilon_2\right)\frac{\omega_3\omega_1}{r}+\frac{16\pi\sigma a^7\omega_3\omega_1}{35}\left(\frac{y^2}{r^5}-\frac{1}{3r^3}\right)$$

$$+\frac{16\pi\sigma a^7\omega_3\omega_1}{35}\left(\frac{1}{2}\epsilon_1+\frac{5}{2}\epsilon_2+\frac{1}{2}\epsilon_3\right)\left(\frac{1}{3r^3}-\frac{y^2}{r^5}\right)$$

$$
+\frac{8\pi\sigma a^9\omega_3\omega_1}{9}\left\{\epsilon_1\left(-\frac{1}{35r^5}+\frac{y^2+x^2}{7r^7}-\frac{y^2x^2}{r^9}\right)+\epsilon_2\left(-\frac{3}{35r^5}+\frac{6y^2}{7r^7}-\frac{y^4}{r^9}\right)\right.
$$

$$
\left.+\epsilon_3\left(-\frac{1}{35r^5}+\frac{z^2+y^2}{7r^7}-\frac{z^2y^2}{r^9}\right)\right\}
$$

$$
+\frac{16\pi\sigma a^7\omega_1\omega_2}{35}\left(-1+\frac{3}{2}\epsilon_2+\frac{3}{2}\epsilon_3+\frac{1}{2}\epsilon_1\right)\frac{zy}{r^5}
$$

$$
+\frac{8\pi\sigma a^9\omega_1\omega_2}{9}\left\{\epsilon_1\left(-\frac{zy}{7r^7}+\frac{zyx^2}{r^9}\right)+\epsilon_2\left(-\frac{3zy}{7r^7}+\frac{zy^3}{r^9}\right)\right.
$$

$$
\left.+\epsilon_3\left(-\frac{3zy}{7r^7}+\frac{z^3y}{r^9}\right)\right\}+\frac{16\pi\sigma a^7\omega_2\omega_3}{35}\left(-1+\frac{3}{2}\epsilon_2+\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_3\right)\frac{xy}{r^5}
$$

$$
+\frac{8\pi\sigma a^9\omega_2\omega_3}{9}\left\{\epsilon_1\left(-\frac{3}{7}\frac{xy}{r^7}+\frac{x^3y}{r^9}\right)+\epsilon_2\left(-\frac{3xy}{7r^7}+\frac{xy^3}{r^9}\right)\right.
$$

$$
\left.+\epsilon_3\left(-\frac{xy}{7r^7}+\frac{xyz^2}{r^9}\right)\right\}. \qquad (3.11)
$$

We have

$$
\gamma_{0/4}=-\frac{16\pi\sigma a^5}{5}\left\{\frac{\omega_3(\epsilon_2-\epsilon_1)yx}{r^5}+\frac{\omega_1(\epsilon_3-\epsilon_2)zy}{r^5}+\frac{\omega_2(\epsilon_1-\epsilon_3)xz}{r^5}\right\}
$$

$$
+\frac{16\pi\sigma a^5}{5}\left\{\frac{(\epsilon_2-\epsilon_1)\omega_3{}^2t(x^2-y^2)}{r^5}+\frac{(\epsilon_3-\epsilon_2)\omega_1{}^2t(y^2-z^2)}{r^5}\right.
$$

$$
\left.+\frac{(\epsilon_1-\epsilon_3)\omega_2{}^2t(z^2-x^2)}{r^5}\right\}-\frac{16\pi\sigma a^5}{5}\left\{\frac{(\epsilon_2-\epsilon_1)\dot{\omega}_3tyx}{r^5}+\frac{(\epsilon_3-\epsilon_2)\dot{\omega}_1tzy}{r^5}\right.
$$

$$
\left.+\frac{(\epsilon_1-\epsilon_3)\dot{\omega}_2txz}{r^5}\right\}-\frac{16\pi\sigma a^5}{5}\left\{\frac{(2\epsilon_3-\epsilon_1-\epsilon_2)\omega_1\omega_2txy}{r^5}\right.
$$

$$
\left.+\frac{(2\epsilon_1-\epsilon_2-\epsilon_3)\omega_2\omega_3tzy}{r^5}+\frac{(2\epsilon_2-\epsilon_3-\epsilon_1)\omega_3\omega_1txz}{r^5}\right\}.
$$

$$
\gamma_{4s/s}=\frac{16\pi\sigma a^5}{5}\left\{\frac{(\epsilon_1-\epsilon_2)(\omega_3+\dot{\omega}_3t)xy}{r^5}+\frac{(\epsilon_2-\epsilon_3)(\omega_1+\dot{\omega}_1t)zy}{r^5}\right.
$$

$$
\left.+\frac{(\epsilon_3-\epsilon_1)(\omega_2+\dot{\omega}_2t)xz}{r^5}\right\}+\frac{16\pi\sigma a^5}{5}\left\{\frac{(\epsilon_2-\epsilon_1)\omega_3{}^2t(x^2-y^2)}{r^5}\right.
$$

$$
\left.+\frac{(\epsilon_3-\epsilon_2)\omega_1{}^2t(y^2-z^2)}{r^5}+\frac{(\epsilon_1-\epsilon_3)\omega_2{}^2t(z^2-x^2)}{r^5}\right\}
$$

$$
+\frac{16\pi\sigma a^5}{5}\left\{\frac{(\epsilon_1+\epsilon_2-2\epsilon_3)\omega_1\omega_2txy}{r^5}+\frac{(\epsilon_2+\epsilon_3-2\epsilon_1)\omega_2\omega_3tzy}{r^5}\right.
$$

$$
\left.+\frac{(\epsilon_1+\epsilon_3-2\epsilon_2)\omega_3\omega_1txz}{r^5}\right\}.
$$

$$
\gamma_{mn/4}=0.
$$

Therefore

$$\gamma_{44/4}-\gamma_{48/8}=0. \qquad (3.12)$$

Also

$$\begin{aligned}\gamma_{41/4}=&-\frac{16\pi\sigma a^5}{15}\frac{(1-\epsilon_2)y\dot{\omega}_3}{r^3}+\frac{16\pi\sigma a^5(1-\epsilon_3)}{15}\frac{z\dot{\omega}_2}{r^3}-\frac{16\pi\sigma a^5}{15}\frac{(\epsilon_2-\epsilon_1)\omega_3{}^2x}{r^3}\\&+16\pi\sigma a^7(\epsilon_2-\epsilon_1)\omega_3{}^2\left(\frac{x}{35r^5}-\frac{xy^2}{7r^7}\right)-\frac{16\pi\sigma a^5}{15}\frac{(\epsilon_3-\epsilon_1)\omega_2{}^2x}{r^3}\\&+16\pi\sigma a^7(\epsilon_3-\epsilon_1)\omega_2{}^2\left(\frac{x}{35r^5}-\frac{xz^2}{7r^7}\right)+\frac{16\pi\sigma a^5}{15}(\epsilon_3-\epsilon_2)\left(\frac{(\omega_1\omega_2y)}{r^3}\right.\\&\left.-\frac{\omega_3\omega_1z}{r^3}\right)+16\pi\sigma a^7(\epsilon_3-\epsilon_2)\left\{\omega_1\omega_2\left(\frac{yz^2}{7r^7}-\frac{y}{35r^5}\right)\right.\\&\left.-\omega_3\omega_1\left(\frac{y^2z}{7r^7}-\frac{z}{35r^5}\right)\right\}+16\pi\sigma a^7\frac{(\epsilon_2+\epsilon_3-2\epsilon_1)\omega_2\omega_3xyz}{7r^7},\end{aligned}$$

and

$$\begin{aligned}\gamma_{18/8}=&16\pi\sigma a^7\left\{\omega_3{}^2(\epsilon_2-\epsilon_1)\left(\frac{x}{35r^5}-\frac{1}{7}\frac{xy^2}{r^7}\right)+\omega_2{}^2(\epsilon_3-\epsilon_1)\left(\frac{x}{35r^5}-\frac{1}{7}\frac{xz^2}{r^7}\right)\right.\\&+\frac{16\pi\sigma a^5}{15}\left\{\frac{\omega_3{}^2(\epsilon_1-\epsilon_2)x}{r^3}+\frac{\omega_2{}^2(\epsilon_1-\epsilon_3)x}{r^3}\right\}\\&+\frac{8\pi\sigma a^5}{15}\left\{\frac{(\epsilon_2-\epsilon_1)\dot{\omega}_3y}{r^3}+\frac{(\epsilon_1-\epsilon_3)\dot{\omega}_2z}{r^3}\right\}\\&+\frac{16\pi\sigma a^7}{7}\frac{(\epsilon_2+\epsilon_3-2\epsilon_1)\omega_2\omega_3xyz}{r^7}+\frac{16\pi\sigma a^5}{15}\frac{(\epsilon_3-\frac{1}{2}\epsilon_1-\frac{1}{2}\epsilon_2)\omega_1\omega_2y}{r^3}\\&+16\pi\sigma a^7(\epsilon_3-\epsilon_2)\omega_1\omega_2\left(-\frac{y}{35r^5}+\frac{1}{7}\frac{z^2y}{r^7}\right)+\frac{16\pi\sigma a^5}{15}\frac{(\epsilon_2-\frac{1}{2}\epsilon_3-\frac{1}{2}\epsilon_1)\omega_1\omega_3z}{r^3}\\&+16\pi\sigma a^7(\epsilon_3-\epsilon_2)\omega_1\omega_3\left(\frac{z}{35r^5}-\frac{1}{7}\frac{y^2z}{r^7}\right).\end{aligned}$$

Therefore

$$\begin{aligned}\gamma_{4/14}-\gamma_{18/8}=&-\frac{16\pi\sigma a^5}{15}\frac{(1-\frac{1}{2}\epsilon_1-\frac{1}{2}\epsilon_2)\dot{\omega}_3y}{r^3}+\frac{16\pi\sigma a^5}{15}\frac{(1-\frac{1}{2}\epsilon_1-\frac{1}{2}\epsilon_3)\dot{\omega}_2z}{r^3}\\&+\frac{8\pi\sigma a^5}{15}\frac{(\epsilon_1-\epsilon_2)\omega_1\omega_2y}{r^3}+\frac{8\pi\sigma a^5}{15}\frac{(\epsilon_1-\epsilon_3)\omega_1\omega_3z}{r^3}.\end{aligned}$$

The condition (1.7) requires

$$\left.\begin{aligned}(1-\tfrac{1}{2}\epsilon_1-\tfrac{1}{2}\epsilon_2)\dot{\omega}_3-\tfrac{1}{2}(\epsilon_1-\epsilon_2)\omega_1\omega_2=0,\\(1-\tfrac{1}{2}\epsilon_1-\tfrac{1}{2}\epsilon_3)\dot{\omega}_2-\tfrac{1}{2}(\epsilon_3-\epsilon_1)\omega_1\omega_3=0.\end{aligned}\right\} \qquad (3.13)$$

The other equations $\gamma_{42/4}-\gamma_{28/8}=0$

and $\gamma_{43/4}-\gamma_{38/8}=0$

lead to a third condition

$$(1-\tfrac{1}{2}\epsilon_2-\tfrac{1}{2}\epsilon_3)\dot{\omega}_1-\tfrac{1}{2}(\epsilon_2-\epsilon_3)\omega_2\omega_3=0. \qquad (3.14)$$

Since terms of the second order in ϵ have been neglected, the conditions (3.13) and (3.14) may be written

$$\left.\begin{aligned}\dot{\omega}_1&=\tfrac{1}{2}(\epsilon_2-\epsilon_3)\omega_2\omega_3,\\ \dot{\omega}_2&=\tfrac{1}{2}(\epsilon_3-\epsilon_1)\omega_3\omega_1,\\ \dot{\omega}_3&=\tfrac{1}{2}(\epsilon_1-\epsilon_2)\omega_1\omega_2.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3.15)$$

These equations are simply the form taken by Euler's equations $A\dot{\omega}_1-(B-C)\omega_1\omega_2=0$ if squares of ϵ are neglected, for in this case

$$A=\frac{2Ma^2}{5}\,(1-\tfrac{1}{2}\epsilon_2-\tfrac{1}{2}\epsilon_3)$$

and

$$B-C=\frac{Ma^2}{5}\,(\epsilon_2-\epsilon_3).$$

4. *Values of the Field at the Boundary.*

The values of the $\gamma_{\mu\nu}$ and $\gamma_{\mu\nu/\sigma}$ at the boundary are obtained by substituting the expression on the right-hand side of (2.4) for r in (3.2)–(3.12). We have on omitting terms of order $\epsilon_r\dot{\omega}_s$

$$\begin{aligned}(\gamma_0)_{bdry}=&-\frac{16\pi\sigma a^2}{3}-\frac{8\pi\sigma}{3}(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)-\frac{8\pi\sigma}{3}\{(\epsilon_3-\epsilon_2)\omega_1{}^2t^2(y^2-z^2)\\ &+(\epsilon_1-\epsilon_3)\omega_2{}^2t^2(z^2-x^2)+(\epsilon_2-\epsilon_1)\omega_3{}^2t^2(x^2-y^2)\}\\ &+\frac{16\pi\sigma}{3}\{(\epsilon_2-\epsilon_1)\omega_3txy+(\epsilon_3-\epsilon_2)\omega_1tyz+(\epsilon_1-\epsilon_3)\omega_2tzx\}\\ &+\frac{8\pi\sigma}{3}\{(2\epsilon_3-\epsilon_1-\epsilon_2)\omega_1\omega_2t^2xy+(2\epsilon_1-\epsilon_2-\epsilon_3)\omega_2\omega_3t^2yz\\ &+(2\epsilon_2-\epsilon_3-\epsilon_1)\omega_3\omega_1t^2zx\}+\frac{8\pi\sigma a^2}{3}(\epsilon_1+\epsilon_2+\epsilon_3)\\ &+\frac{8\pi\sigma}{5}(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)-\frac{8\pi\sigma a^2}{15}(\epsilon_1+\epsilon_2+\epsilon_3)\\ &-\frac{16\pi\sigma}{5}\{\omega_3t(\epsilon_2-\epsilon_1)yx+\omega_1t(\epsilon_3-\epsilon_2)zy+\omega_2t(\epsilon_1-\epsilon_3)xz\}\\ &+\frac{8\pi\sigma}{5}\{(\epsilon_2-\epsilon_1)\omega_3{}^2t^2(x^2-y^2)+(\epsilon_3-\epsilon_2)\omega_1{}^2t^2(y^2-z^2)\\ &+(\epsilon_1-\epsilon_3)\omega_2{}^2t^2(z^2-x^2)\}-\frac{8\pi\sigma a^2}{15}\{(\epsilon_2-\epsilon_1)\omega_3{}^2(x^2-y^2)\\ &+(\epsilon_3-\epsilon_2)\omega_1{}^2(y^2-z^2)+(\epsilon_1-\epsilon_3)\omega_2{}^2(z^2-x^2)\}\\ &+\frac{8\pi\sigma}{35}\{(\epsilon_2-\epsilon_1)\omega_3{}^2(x^2-y^2)+(\epsilon_3-\epsilon_2)\omega_1{}^2(y^2-z^2)\\ &+(\epsilon_1-\epsilon_3)\omega_2{}^2(z^2-x^2)\}-\frac{8\pi\sigma}{5}\{(2\epsilon_3-\epsilon_1-\epsilon_2)\omega_1\omega_2t^2xy\\ &+(2\epsilon_1-\epsilon_2-\epsilon_3)\omega_2\omega_3t^2zy+(2\epsilon_2-\epsilon_3-\epsilon_1)\omega_3\omega_1t^2xz\}\end{aligned}$$

$$
\begin{aligned}
&+\frac{8\pi\sigma}{15}\left\{(2\epsilon_3-\epsilon_1-\epsilon_2)\omega_1\omega_2\left(a^2xy-\frac{3}{7}xy\right)\right.\\
&+(2\epsilon_1-\epsilon_2-\epsilon_3)\omega_2\omega_3\left(a^2zy-\frac{3}{7}zy\right)\\
&\left.+(2\epsilon_2-\epsilon_3-\epsilon_1)\omega_3\omega_1\left(a^2xz-\frac{3}{7}xz\right)\right\}.
\end{aligned}
\tag{4.1}
$$

$$
\begin{aligned}
(\gamma_{0/1})_{bd^{ry}}=&\frac{16\pi\sigma x}{3}-\frac{16\pi\sigma}{15}(\epsilon_1+\epsilon_2+\epsilon_3)x+\frac{16\pi\sigma\epsilon_1 x}{5}+\frac{16\pi\sigma}{5}\{\omega_3t(\epsilon_1-\epsilon_2)y\\
&+\omega_2t(\epsilon_3-\epsilon_1)z\}+\frac{16\pi\sigma}{5}\{(\epsilon_2-\epsilon_1)\omega_3{}^2t^2+(\epsilon_3-\epsilon_1)\omega_2{}^2t^2\}x\\
&+\left(\frac{16}{15}-\frac{16}{35}\right)\pi\sigma a^2\{(\epsilon_1-\epsilon_2)\omega_3{}^2+(\epsilon_1-\epsilon_3)\omega_2{}^3\}x\\
&+\left(\frac{8}{5}-\frac{8}{7}\right)\pi\sigma\{(\epsilon_2-\epsilon_1)\omega_3{}^2(x^2-y^2)+(\epsilon_3-\epsilon_2)\omega_1{}^2(y^2-z^2)\\
&+(\epsilon_1-\epsilon_3)\omega_2{}^2(z^2-x^2)\}+\frac{8\pi\sigma}{5}\{(\epsilon_1+\epsilon_2-2\epsilon_3)\omega_1\omega_2t^2y\\
&+(\epsilon_3+\epsilon_1-2\epsilon_2)\omega_3\omega_1t^2z\}+\frac{32\pi\sigma a^2}{105}\{(2\epsilon_3-\epsilon_1-\epsilon_2)\omega_1\omega_2y\\
&+(2\epsilon_2-\epsilon_3-\epsilon_1)\omega_3\omega_1z\}+\frac{16\pi\sigma}{35}\{(\epsilon_1+\epsilon_2-2\epsilon_3)\omega_1\omega_2x^2y\\
&+(\epsilon_2+\epsilon_3-2\epsilon_1)\omega_2\omega_3xyz+(\epsilon_1+\epsilon_3-2\epsilon_2)\omega_3\omega_1x^2z\}.
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
(\gamma_{41})_{bd^{ry}}=&-\frac{16\pi\sigma a^2}{15}(1-\epsilon_2)(\omega_3+\dot{\omega}_3t)y-\frac{8\pi\sigma}{5}(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)\omega_3y\\
&+\frac{16\pi\sigma}{5}\{(\epsilon_2-\epsilon_1)\omega_3{}^2txy^2+(\epsilon_3-\epsilon_2)\omega_1\omega_3ty^2z+(\epsilon_1-\epsilon_3)\omega_2\omega_3txyz\}\\
&+\frac{8\pi\sigma a^2}{15}(\epsilon_1+\epsilon_2+\epsilon_3)\omega_3y+8\pi\sigma\omega_3\left\{\epsilon_1\left(-\frac{a^2y}{35}+\frac{x^2y}{7}\right)\right.\\
&\left.+\epsilon_2\left(-\frac{3}{35}a^2y+\frac{y^3}{7}\right)+\epsilon_3\left(-\frac{ya^2}{35}+\frac{z^2y}{7}\right)\right\}\\
&+\frac{16\pi\sigma a^2}{15}(1-\epsilon_3)z(\omega_2+\dot{\omega}_2t)+\frac{8\pi\sigma}{5}(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)z\omega_2\\
&-\frac{16\pi\sigma}{5}\{(\epsilon_2-\epsilon_1)\omega_2\omega_3txyz+(\epsilon_3-\epsilon_2)\omega_1\omega_2tyz^2+(\epsilon_1-\epsilon_3)\omega_2{}^2txz^2\}\\
&-\frac{8\pi\sigma a^2}{15}(\epsilon_1+\epsilon_2+\epsilon_3)\omega_2z+8\pi\sigma\omega_2\left\{\epsilon_1\left(\frac{za^2}{35}-\frac{x^2z}{7}\right)+\epsilon_2\left(\frac{za^2}{35}-\frac{y^2z}{7}\right)\right.\\
&\left.+\epsilon_3\left(-\frac{3}{35}za^2+\frac{z^3}{7}\right)\right\}+16\pi\sigma\omega_3{}^2t(\epsilon_2-\epsilon_1)x\left\{\left(-\frac{1}{15}+\frac{1}{35}\right)a^2-\frac{y^2}{7}\right\}\\
&+16\pi\sigma\omega_2{}^2t(\epsilon_3-\epsilon_1)x\left\{\left(-\frac{1}{15}+\frac{1}{35}\right)a^2-\frac{z^2}{7}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+16\pi\sigma\omega_1\omega_2 t(\epsilon_3-\epsilon_2)y\left\{\left(\frac{1}{15}-\frac{1}{35}\right)a^2+\frac{1}{7}z^2\right\}\\
&+16\pi\sigma\omega_1\omega_3 t(\epsilon_3-\epsilon_2)z\left\{\left(-\frac{1}{15}+\frac{1}{35}\right)a^2-\frac{y^2}{7}\right\}\\
&+\frac{16\pi\sigma}{7}\omega_2\omega_3 t(\epsilon_2+\epsilon_3-2\epsilon_1)xyz. \qquad \ldots\ldots\ldots \quad (4.3)
\end{aligned}
$$

$$
\begin{aligned}
(\gamma_{41/1})_{bd^{ry}}=&\frac{16\pi\sigma}{5}(1-\epsilon_2)(\omega_3+\dot{\omega}_3 t)xy-\frac{8\pi\sigma}{5}(\epsilon_1+\epsilon_2+\epsilon_3)\omega_3 xy\\
&+\frac{8\pi\sigma}{7}(3\epsilon_1+3\epsilon_2+\epsilon_3)\omega_3 xy-\frac{16\pi\sigma}{5}(1-\epsilon_3)(\omega_2+\dot{\omega}_2 t)zx\\
&+\frac{8\pi\sigma}{5}(\epsilon_1+\epsilon_2+\epsilon_3)\omega_2 zx-\frac{8\pi\sigma}{7}(3\epsilon_1+\epsilon_2+3\epsilon_3)\omega_2 zx\\
&-\frac{16\pi\sigma}{15}(\epsilon_2-\epsilon_1)\omega_3{}^2ta^2+\frac{16\pi\sigma}{5}(\epsilon_2-\epsilon_1)\omega_3{}^2tx^2\\
&+16\pi\sigma(\epsilon_2-\epsilon_1)\omega_3{}^2t\left(\frac{a^2}{35}-\frac{(y^2+x^2)}{7}\right)-\frac{16\pi\sigma}{15}(\epsilon_3-\epsilon_1)\omega_2{}^2ta^2\\
&+\frac{16\pi\sigma}{5}(\epsilon_3-\epsilon_1)\omega_2{}^2tx^2+16\pi\sigma(\epsilon_3-\epsilon_1)\omega_2{}^2t\left(\frac{a^2}{35}-\frac{(z^2+x^2)}{7}\right)\\
&+\frac{16\pi\sigma}{5}(\epsilon_2-\epsilon_3)(\omega_1\omega_2 txy-\omega_3\omega_1 txz)\\
&+\frac{16\pi\sigma}{7}(\epsilon_3-\epsilon_2)(\omega_1\omega_2 txy-\omega_3\omega_1 tzx)+\frac{16\pi\sigma}{7}(\epsilon_2+\epsilon_3-2\epsilon_1)\omega_2\omega_3 tyz.\\
&\qquad\qquad\ldots\ldots \quad (4.4)
\end{aligned}
$$

$$
\begin{aligned}
(\gamma_{11})_{bd^{ry}}=&\frac{16\pi\sigma a^2\omega_3{}^2}{105}(a^2-3y^2)+\frac{8\pi\sigma}{35}(\epsilon_1 x^2+\epsilon_2 y^2+\epsilon_3 z^2)a^2\omega_3{}^2\\
&-\frac{16\pi\sigma}{105}\left(\frac{3}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_3\right)a^2\omega_3{}^2(a^2-3y^2)\\
&-\frac{8\pi\sigma y^2}{7}(\epsilon_1 x^2+\epsilon_2 y^2+\epsilon_3 z^2)+\frac{16\pi\sigma\omega_3{}^2}{15}(\epsilon_2-\epsilon_1)a^4\\
&+\frac{16\pi\sigma}{105}(\epsilon_2-\epsilon_1)a^2\omega_3{}^2(-a^2+3y^2)\\
&+\frac{8\pi\sigma a^2\omega_3{}^2}{9}\left\{\epsilon_1\left(\frac{a^2}{35}-\frac{1}{7}(x^2+y^2)+\frac{x^2y^2}{a^2}\right)+\epsilon_2\left(\frac{3a^2}{35}-\frac{6}{7}y^2+\frac{y^4}{a^2}\right)\right.\\
&\left.+\epsilon_3\left(\frac{a^2}{35}-\frac{1}{7}(y^2+z^2)+\frac{y^2z^2}{a^2}\right)\right\}+\frac{16\pi\sigma a^2\omega_2{}^2}{105}(a^2-3z^2)\\
&+\frac{8\pi\sigma}{35}(\epsilon_1 x^2+\epsilon_2 y^2+\epsilon_3 z^2)a^2\omega_2{}^2\\
&-\frac{16\pi\sigma}{105}\left(\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_2+\frac{3}{2}\epsilon_3\right)a^2\omega_2{}^2(a^2-3z^2)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{8\pi\sigma\omega_2{}^2z^2}{7}(\epsilon_1x^2+\epsilon_3y^2+\epsilon_3z^2)+\frac{16\pi\sigma\omega_2{}^2}{15}(\epsilon_3-\epsilon_1)a^4\\
&+\frac{16\pi\sigma}{105}(\epsilon_3-\epsilon_1)a^2\omega_2{}^2(-a^2+3z^2)\\
&+\frac{8\pi\sigma a^2\omega_2{}^2}{9}\Big\{\epsilon_1\left(\frac{a^2}{35}-\frac{1}{7}(z^2+x^2)+\frac{z^2x^2}{a^2}\right)\\
&+\epsilon_2\left(\frac{a^2}{35}-\frac{1}{7}(z^2+y^2)+\frac{z^2y^2}{a^2}\right)\\
&+\epsilon_3\left(\frac{3}{35}a^2-\frac{6}{7}z^2+\frac{z^4}{a^2}\right)\Big\}+\frac{32\pi\sigma a^2\omega_2\omega_3}{35}zy\\
&+\frac{16\pi\sigma\omega_2\omega_3}{7}(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)zy-\frac{16\pi\sigma a^2\omega_2\omega_3}{35}(\epsilon_1+3\epsilon_2+3\epsilon_3)zy\\
&+\frac{16\pi\sigma a^2\omega_2\omega_3}{63}(\epsilon_1+3\epsilon_2+3\epsilon_3)yz-\frac{16\pi\sigma\omega_2\omega_3}{9}yz(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2).\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad .\quad .\quad .\quad .\quad (4.5)
\end{aligned}
$$

$$
\begin{aligned}
(\gamma_{11/1})_{bd^{ry}}=&-\frac{16\pi\sigma a^2\omega_3}{35}x-\frac{8\pi\sigma\omega_3{}^2}{7}x(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)+\frac{16\pi\sigma a^2\omega_3{}^2}{7}xy^2\\
&+\frac{16\pi\sigma a^2\omega_3{}^2}{35}x\left(\frac{3}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_3\right)+\frac{16\pi\sigma a^2\omega_3{}^2}{15}(\epsilon_1-\epsilon_2)x\\
&+\frac{16\pi\sigma a^2\omega_3{}^2}{35}(\epsilon_2-\epsilon_1)x-\frac{16\pi\sigma\omega_3{}^2}{7}(\epsilon_2-\epsilon_1)y^2x\\
&-\frac{16\pi\sigma\omega_3{}^2}{7}xy^2\left(\frac{3}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_3\right)\\
&+\frac{8\pi\sigma\omega_3{}^2}{9}x\Big\{\epsilon_1\left(-\frac{3}{7}a^2+x^2+3y^2\right)+\epsilon_2\left(-\frac{3}{7}a^2+6y^2\right)\\
&+\epsilon_3\left(-\frac{a^2}{7}+y^2+z^2\right)\Big\}-\frac{16\pi\sigma a^2\omega_2}{35}x-\frac{8\pi\sigma\omega_2{}^2}{7}x(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)\\
&+\frac{16\pi\sigma a^2\omega_2{}^2}{7}xz^2+\frac{16\pi\sigma a^2\omega_2{}^2}{35}x\left(\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_2+\frac{3}{2}\epsilon_3\right)\\
&+\frac{16\pi\sigma a^2\omega_2{}^2}{15}(\epsilon_1-\epsilon_3)x+\frac{16\pi\sigma a^2\omega_2{}^2}{35}(\epsilon_3-\epsilon_1)x\\
&-\frac{16\pi\sigma\omega_2{}^2}{7}(\epsilon_3-\epsilon_1)xz^2-\frac{16\pi\sigma\omega_2{}^2}{7}xz^2\left(\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_2+\frac{3}{2}\epsilon_3\right)\\
&+\frac{8\pi\sigma\omega_2{}^2}{9}x\Big\{\epsilon_1\left(-\frac{3}{7}a^2+x^2+3z^2\right)\\
&+\epsilon_2\left(-\frac{a^2}{7}+z^2+y^2\right)+\epsilon_3\left(-\frac{3}{7}a^2+6z^2\right)\Big\}-\frac{32\pi\sigma\omega_2\omega_3}{7}xyz\\
&+\frac{16\pi\sigma\omega_2\omega_3}{7}(\epsilon_1+3\epsilon_2+3\epsilon_3)xyz-\frac{16\pi\sigma\omega_2\omega_3}{3}(\epsilon_1+\epsilon_2+\epsilon_3)xyz.\quad (4.6)
\end{aligned}
$$

$$
\begin{aligned}
(\gamma_{12})_{bd^{ry}} = {} & \frac{16\pi\sigma a^2\omega_3{}^2}{35}xy - \frac{8\pi\sigma a^2\omega_3{}^2}{35}(3\epsilon_1+3\epsilon_2+\epsilon_3)xy \\
& + \frac{8\pi\sigma\omega_3{}^2}{7}(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)xy + \frac{8\pi\sigma a^2\omega_3{}^2}{21}\left(\epsilon_1+\epsilon_2+\frac{1}{3}\epsilon_3\right)xy \\
& - \frac{8\pi\sigma\omega_3{}^2}{9}xy(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2) + \frac{16\pi\sigma a^4}{15}\left(\frac{1}{2}\epsilon_1+\frac{1}{2}\epsilon_2-\epsilon_3\right)\omega_1\omega_2 \\
& + \frac{16\pi\sigma a^2\omega_1\omega_2}{35}\left(z^2-\frac{1}{3}a^2\right) \\
& + \frac{16\pi\sigma a^2\omega_1\omega_2}{35}\left(\frac{5}{2}\epsilon_3+\frac{1}{2}\epsilon_1+\frac{1}{2}\epsilon_2\right)\left(\frac{a^2}{3}-z^2\right) \\
& + \frac{8\pi\sigma a^2\omega_1\omega_2}{9}\left\{\epsilon_1\left(-\frac{a^2}{35}+\frac{(x^2+z^2)}{7}-\frac{x^2z^2}{a^2}\right)\right. \\
& \left.+\epsilon_2\left(-\frac{a^2}{35}+\frac{(z^2+y^2)}{7}-\frac{y^2z^2}{a^2}\right)+\epsilon_3\left(-\frac{3}{35}a^2+\frac{6}{7}z^2-\frac{z^4}{a^2}\right)\right\} \\
& - \frac{16\pi\sigma a^2\omega_2\omega_3}{35}xz + \frac{8\pi\sigma\omega_2\omega_3}{35}(3\epsilon_3+3\epsilon_1+\epsilon_2)xza^2 \\
& - \frac{8\pi\sigma\omega_2\omega_3}{7}xz(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2) - \frac{8\pi\sigma\omega_2\omega_3}{21}\left(\epsilon_1+\frac{1}{3}\epsilon_2+\epsilon_3\right)a^2xz \\
& + \frac{8\pi\sigma\omega_2\omega_3}{9}xz(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2) - \frac{16\pi\sigma\omega_3\omega_1}{35}a^2yz \\
& + \frac{8\pi\sigma\omega_3\omega_1}{35}(3\epsilon_3+3\epsilon_2+\epsilon_1)a^2yz - \frac{8\pi\sigma\omega_3\omega_1}{7}yz(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2) \\
& - \frac{8\pi\sigma\omega_3\omega_1}{21}\left(\frac{1}{3}\epsilon_1+\epsilon_2+\epsilon_3\right)a^2yz + \frac{8\pi\sigma\omega_3\omega_1}{9}yz(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2).
\end{aligned}
\qquad . \quad . \quad . \quad . \quad (4.7)
$$

$$
\begin{aligned}
(\gamma_{12/1})_{bd^{ry}} = {} & \frac{16\pi\sigma\omega_3{}^2}{35}a^2y + \frac{8\pi\sigma\omega_3{}^2}{7}y(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2) - \frac{16\pi\sigma\omega_3{}^2}{7}x^2y \\
& - \frac{16\pi\sigma\omega_3{}^2}{35}\left(\frac{3}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_3\right)ya^2 + \frac{16\pi\sigma\omega_3{}^2}{7}\left(\frac{3}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_3\right)x^2y \\
& + \frac{8\pi\sigma}{9}\left\{\epsilon_1\left(\frac{3}{7}ya^2-6x^2y\right)+\epsilon_2\left(\frac{3}{7}a^2y-3x^2y-y^3\right)\right. \\
& \left.+\epsilon_3\left(\frac{ya^2}{7}-x^2y-yz^2\right)\right\} - \frac{16\pi\sigma\omega_1\omega_2}{15}\left(\frac{1}{2}\epsilon_1+\frac{1}{2}\epsilon_2-\epsilon_3\right)a^2x \\
& + \frac{16\pi\sigma\omega_1\omega_2}{35}a^2x + \frac{8\pi\sigma\omega_1\omega_2}{7}x(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2) \\
& - \frac{16\pi\sigma\omega_1\omega_2}{7}xz^2 - \frac{16\pi\sigma\omega_1\omega_2}{35}a^2x\left(\frac{5}{2}\epsilon_3+\frac{1}{2}\epsilon_1+\frac{1}{2}\epsilon_2\right) \\
& + \frac{16\pi\sigma\omega_1\omega_2}{7}\left(\frac{5}{2}\epsilon_3+\frac{1}{2}\epsilon_1+\frac{1}{2}\epsilon_2\right)xz^2
\end{aligned}
$$

$$+\frac{8\pi\sigma\omega_1\omega_2}{9}\left\{\epsilon_1\left(\frac{3}{7}xa^2-x^3-3xz^2\right)+\epsilon_2\left(\frac{xa^2}{7}-xz^2-xy^2\right)\right.$$

$$\left.+\epsilon_3\left(\frac{3}{7}+xa^2-6xz^2\right)\right\}-\frac{16\pi\sigma\omega_2\omega_3}{35}a^2z$$

$$-\frac{8\pi\sigma\omega_2\omega_3}{7}z(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)+\frac{16\pi\sigma\omega_2\omega_3}{7}x^2z$$

$$+\frac{16\pi\sigma\omega_2\omega_3}{35}\left(\frac{3}{2}\epsilon_3+\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_2\right)za^2$$

$$-\frac{16\pi\sigma\omega_2\omega_3}{7}\left(\frac{3}{2}\epsilon_3+\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_2\right)x^2z$$

$$+\frac{8\pi\sigma\omega_2\omega_3}{9}\left\{\epsilon_1\left(-\frac{3}{7}za^2+6x^2z\right)+\epsilon_2\left(-\frac{za^2}{7}+x^2z+zy^2\right)\right.$$

$$\left.+\epsilon_3\left(-\frac{3}{7}za^2+3x^2z+z^3\right)\right\}+\frac{16\pi\sigma\omega_3\omega_1}{7}xyz$$

$$-\frac{16\pi\sigma\omega_3\omega_1}{7}\left(\frac{3}{2}\epsilon_3+\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_1\right)xyz+\frac{8\pi\sigma\omega_3\omega_1}{3}(\epsilon_1+\epsilon_2+\epsilon_3)xyz. \quad \ldots\ldots \quad (4.8)$$

5. *The Internal Field.*

The field discussed in section 3 is a solution of the equation $G_{\mu\nu}=0$ which is of the order -1 in r at infinity, and satisfies the conditions

$$\gamma_{4\mu/4}-\gamma_{\mu s/s}=0. \quad \ldots\ldots \quad (5.1)$$

We have seen that the condition obtained by putting $\mu=4$ in (5.1) is identically satisfied, but when μ is equal to 1, 2 or 3 the condition leads to the angular equations of motion.

In this section we consider a field which is finite at the origin, satisfies (5.1) and which, with its derivatives, is continuous with the above external field at the boundary represented by the equation (2.4). The field is :—

$$\gamma_{44}=\gamma_0+\gamma_{II}, \quad \ldots\ldots \quad (5.2)$$

where

$$\gamma_0=-8\pi\sigma a^2+\frac{8\pi\sigma a^2}{3}(\epsilon_1+\epsilon_2+\epsilon_3)+\frac{8\pi\sigma r^2}{3}-\frac{8\pi\sigma}{15}(\epsilon_1+\epsilon_2+\epsilon_3)r^2$$

$$+\frac{8\pi\sigma}{5}(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)+\frac{16\pi\sigma}{5}\{(\epsilon_1-\epsilon_2)\omega_3txy$$

$$+(\epsilon_2-\epsilon_3)\omega_1tyz+(\epsilon_3-\epsilon_1)\omega_2tzx\}+\frac{8\pi\sigma}{5}\{(\epsilon_2-\epsilon_1)\omega_3{}^2t^2(x^2-y^2)$$

$$+(\epsilon_3-\epsilon_2)\omega_1{}^2t^2(y^2-z^2)+(\epsilon_1-\epsilon_3)\omega_2{}^2t^2(z^2-x^2)\}-\frac{8\pi\sigma}{15}\{(\epsilon_2-\epsilon_1)\omega_3{}^2(x^2-y^2)$$

$$+(\epsilon_3-\epsilon_2)\omega_1{}^2(y^2-z^2)+(\epsilon_1-\epsilon_3)\omega_2{}^2(z^2-x^2)\}+\frac{8\pi\sigma r^2}{35a^2}\{(\epsilon_2-\epsilon_1)\omega_3{}^2(x^2-y^2)$$

$$+(\epsilon_3-\epsilon_2)\omega_1^2(y^2-z^2)+(\epsilon_1-\epsilon_3)\omega_2^2(z^2-x^2)\}+\frac{8\pi\sigma}{5}\{(\epsilon_1+\epsilon_2-2\epsilon_3)\omega_1\omega_2t^2yx$$

$$+(\epsilon_2+\epsilon_3-2\epsilon_1)\omega_2\omega_3t^2zy+(\epsilon_3+\epsilon_1-2\epsilon_2)\omega_3\omega_1t^2xz\}$$

$$+\frac{8\pi\sigma}{15}\left\{(2\epsilon_3-\epsilon_1-\epsilon_2)\omega_1\omega_2\left(a^2xy-\frac{3}{7}xyr^2\right)\right.$$

$$+(2\epsilon_1-\epsilon_2-\epsilon_3)\omega_2\omega_3\left(a^2zy-\frac{3}{7}zyr^2\right)$$

$$\left.+(2\epsilon_2-\epsilon_3-\epsilon_1)\omega_3\omega_1\left((a^2zx-\frac{3}{7}xzr^2\right)\right\}. \quad \ldots \ldots \ldots \quad (5.3)$$

$$\gamma_{41}=\frac{8\pi\sigma}{15}(1-\epsilon_2)(\omega_3+\dot{\omega}_3t)(-5a^2y+3yr^2)+\frac{8\pi\sigma}{15}(1-\epsilon_3)(\omega_2+\dot{\omega}_2t)(5za^2-3zr^2)$$

$$+\frac{4\pi\sigma}{3}(\epsilon_1+\epsilon_2+\epsilon_3)\left\{\omega_3\left(a^2y-\frac{3}{5}yr^2\right)-\omega_2\left(a^2z-\frac{3}{5}zr^2\right)\right\}$$

$$+8\pi\sigma\omega_3\left\{\epsilon_1\left(-\frac{1}{10}a^2y+\frac{1}{14}yr^2+\frac{1}{7}x^2y\right)+\epsilon_2\left(-\frac{3}{10}ya^2+\frac{3}{14}yr^2+\frac{1}{7}y^3\right)\right.$$

$$\left.+\epsilon_3\left(-\frac{1}{10}a^2y+\frac{1}{14}yr^2+\frac{1}{7}z^2y\right)\right\}+8\pi\sigma\omega_2\left\{\epsilon_1\left(\frac{1}{10}a^2z-\frac{1}{14}zr^2-\frac{1}{7}x^2z\right)\right.$$

$$\left.+\epsilon_2\left(\frac{1}{10}a^2z-\frac{1}{14}zr^2-\frac{1}{7}y^2z\right)+\epsilon_3\left(\frac{3}{10}za^2-\frac{3}{14}zr^2-\frac{1}{7}z^3\right)\right\}$$

$$+8\pi\sigma\left\{\omega_3^2t(\epsilon_2-\epsilon_1)\left(-\frac{2}{15}a^2x+\frac{2}{35}xr^2-\frac{2}{7}xy^2\right)\right.$$

$$\left.+\omega_2^2t(\epsilon_1-\epsilon_3)\left(\frac{2}{15}a^2x-\frac{2}{35}xr^2+\frac{2}{7}xz^2\right)\right\}$$

$$+\frac{16\pi\sigma}{15}\left\{(\epsilon_3-\epsilon_2)\omega_1\omega_2t\left(a^2y-\frac{3}{7}yr^2\right)+(\epsilon_2-\epsilon_3)\omega_3\omega_1t\left(a^2z-\frac{3}{7}zr^2\right)\right\}$$

$$+\frac{16\pi\sigma}{7}\{(\epsilon_3-\epsilon_2)\omega_1\omega_2tyz^2+(\epsilon_2+\epsilon_3-2\epsilon_1)\omega_2\omega_3txyz+(\epsilon_2-\epsilon_3)\omega_3\omega_1tzy^2\}.$$

$$\ldots \quad (5.4)$$

$$\gamma_{42}=\frac{8\pi\sigma}{15}(1-\epsilon_3)(\omega_1+\dot{\omega}_1t)(-5za^2+3zr^2)+\frac{8\pi\sigma}{15}(1-\epsilon_1)(\omega_3+\dot{\omega}_3t)(5xa^2-3xr^2)$$

$$+\frac{4\pi\sigma}{3}(\epsilon_1+\epsilon_2+\epsilon_3)\left\{\omega_1\left(+a^2z-\frac{3}{5}zr^2\right)-\omega_3\left(a^2x-\frac{3}{5}xr^2\right)\right\}$$

$$+8\pi\sigma\omega_1\left\{\epsilon_1\left(-\frac{1}{10}za^2+\frac{1}{14}zr^2+\frac{1}{7}x^2z\right)\right.$$

$$\left.+\epsilon_2\left(-\frac{1}{10}za^2+\frac{1}{14}zr^2+\frac{1}{7}y^2z\right)+\epsilon_3\left(-\frac{3}{10}za^2+\frac{3}{14}zr^2+\frac{1}{7}z^3\right)\right\}$$

$$+8\pi\sigma\omega_3\left\{\epsilon_1\left(\frac{3}{10}xa^2-\frac{3}{14}xr^2-\frac{1}{7}x^3\right)+\epsilon_2\left(\frac{1}{10}xa^2-\frac{1}{14}xr^2-\frac{1}{7}xy^2\right)\right.$$

$$+\epsilon_3\left(\frac{1}{10}xa^2-\frac{1}{14}xr^2-\frac{1}{7}z^2x\right)$$

$$+8\pi\sigma\Big\{\dot{\omega}_1^{\,2}t(\epsilon_3-\epsilon_2)\Big(-\frac{2}{15}ya^2+\frac{2}{35}yr^2-\frac{2}{7}yz^2\Big)$$

$$+\omega_3^{\,2}t(\epsilon_2-\epsilon_1)\Big(\frac{2}{15}ya^2-\frac{2}{35}yr^2+\frac{2}{7}x^2y\Big)\Big\}$$

$$+\frac{16\pi\sigma}{15}\Big\{(\epsilon_1-\epsilon_3)\omega_2\omega_3t\Big(za^2-\frac{3}{7}zr^2\Big)+(\epsilon_3-\epsilon_1)\omega_1\omega_2t\Big(xa^2-\frac{3}{7}xr^2\Big)\Big\}$$

$$+\frac{16\pi\sigma}{7}\{(\epsilon_1-\epsilon_3)\omega_2\omega_3tzx^2+(\epsilon_3+\epsilon_1-2\epsilon_2)\omega_3\omega_1txyz+(\epsilon_3-\epsilon_1)\omega_1\omega_2txz^2\}. \quad \text{. . . . } (5.5)$$

$$\gamma_{43}=\frac{8\pi\sigma}{15}(1-\epsilon_1)(\omega_2+\dot{\omega}_2t)(-5xa^2+3xr^2)+\frac{8\pi\sigma}{15}(1-\epsilon_2)(\omega_1+\dot{\omega}_1t)(5ya^2-3yr^2)$$

$$+\frac{4\pi\sigma}{3}(\epsilon_1+\epsilon_2+\epsilon_3)\Big\{\omega_2(+a^2x-\frac{3}{5}xr^2\Big)-\omega_1\Big(+a^2y-\frac{3}{5}yr^2\Big)\Big\}$$

$$+8\pi\sigma\omega_2\Big\{\epsilon_1\Big(-\frac{3}{10}xa^2+\frac{3}{14}xr^2+\frac{1}{7}x^3\Big)+\epsilon_2\Big(-\frac{1}{10}xa^2+\frac{1}{14}xr^2+\frac{1}{7}y^2x\Big)$$

$$+\epsilon_3\Big(-\frac{1}{10}xa^2+\frac{1}{14}xr^2+\frac{1}{7}z^2x\Big)\Big\}+8\pi\sigma\omega_1\Big\{\epsilon_1\Big(\frac{1}{10}ya^2-\frac{1}{14}yr^2-\frac{1}{7}x^2y\Big)$$

$$+\epsilon_2\Big(\frac{3}{10}ya^2-\frac{3}{14}yr^2-\frac{1}{7}y^3\Big)+\epsilon_3\Big(\frac{1}{10}ya^2-\frac{1}{14}yr^2-\frac{1}{7}yx^2\Big)\Big\}$$

$$+8\pi\sigma\Big\{\omega_2^{\,2}t(\epsilon_1-\epsilon_3)\Big(-\frac{2}{15}za^2+\frac{2}{35}zr^2-\frac{2}{7}zx^2\Big)$$

$$+\omega_1^{\,2}t(\epsilon_3-\epsilon_2)\Big(\frac{2}{15}za^2-\frac{2}{35}zr^2+\frac{2}{7}y^2z\Big)\Big\}$$

$$+\frac{16\pi\sigma}{15}\Big\{\epsilon_2-\epsilon_1)\omega_3\omega_1t\Big(xa^2-\frac{3}{7}xr^2\Big)+(\epsilon_1-\epsilon_2)\omega_2\omega_3t\Big(ya^2-\frac{3}{7}yr^2\Big)\Big\}$$

$$+\frac{16\pi\sigma}{7}\Big\{(\epsilon_2-\epsilon_1)\omega_3\omega_1txy^2+(\epsilon_1+\epsilon_2-2\epsilon_3)\omega_1\omega_2txyz+(\epsilon_1-\epsilon_2)\omega_2\omega_3tyx^2\Big\}. \quad \text{. . . . } (5.6)$$

$$\gamma_{11}=4\pi\sigma a^2\omega_3^{\,2}\Big\{\frac{a^2}{6}-\frac{r^2}{5}+\frac{r^4}{14a^2}+y^2\Big(-\frac{2}{5}+\frac{7}{2}\frac{r^2}{a^2}\Big)\Big\}$$

$$+4\pi\sigma a^2\omega_3^{\,2}\Big(\frac{3}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_3\Big)\Big\{-\frac{a^2}{6}+\frac{r^2}{5}-\frac{r^4}{14a^2}+y^2\Big(\frac{2}{5}-\frac{2}{7}\frac{r^2}{a^2}\Big)\Big\}$$

$$+\frac{8\pi\sigma a^2\omega_3^{\,2}}{9}\Big\{\epsilon_1\Big(\frac{9}{40}a^2-\frac{9}{14}(x^2+y^2)+\frac{1}{2}(x^2+y^2)\frac{r^2}{a^2}+\frac{x^2y^2}{a^2}$$

$$-\frac{9}{28}r^2+\frac{1}{8}\frac{r^4}{a^2}\Big)+\epsilon_2\Big(\frac{27}{40}a^2-\frac{27}{7}y^2-\frac{27}{28}r^2+\frac{y^1}{a^2}+\frac{3y^2r^2}{a^2}+\frac{3}{8}\frac{r^2}{a^2}\Big)$$

$$+\epsilon_3\Big(\frac{9}{40}a^2-\frac{27}{28}(y^2+z^2)-\frac{9}{28}x^2+\frac{1}{8}\frac{x^4}{a^2}+\frac{5}{8a^2}(y^4+z^4)+\frac{3}{4a}(y^2+z^2)x^2$$

$$+\frac{9}{4}\frac{y_2}{z^2}\Big\}+\frac{8\pi\sigma\omega_3^{\,2}}{35}(\epsilon_2-\epsilon_1)a^2\Big(\frac{35}{6}a^2-\frac{7}{3}r^2+\frac{1}{2}\frac{r^4}{a^2}+7y^2-\frac{5y^2r^2}{a^2}\Big)$$

$$+4\pi\sigma a^2\omega_2{}^2\biggl\{\frac{a^2}{6}-\frac{r^2}{5}+\frac{1}{14}\frac{r^4}{a^2}+z^2\biggl(-\frac{2}{5}+\frac{2}{7}\frac{r^2}{a^2}\biggr)\biggr\}$$

$$+4\pi\sigma a^2\omega_2{}^2\biggl(\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_2+\frac{3}{2}\epsilon_3\biggr)\biggl\{-\frac{a^2}{6}+\frac{r^2}{5}-\frac{r^4}{14a^2}+z^2\biggl(\frac{2}{5}-\frac{2}{7}\frac{r^2}{a^2}\biggr)\biggr\}$$

$$+\frac{8\pi\sigma a^2\omega_2{}^2}{9}\biggl\{\epsilon_1\biggl(\frac{9}{40}a^2-\frac{9}{14}(x^2+z^2)+\frac{1}{2}(x^2+z^2)\frac{r^2}{a^2}+\frac{x^2z^2}{a^2}+\frac{1}{8}\frac{r^4}{a^2}-\frac{9r^2}{28}\biggr)$$

$$+\epsilon_2\biggl(\frac{9}{40}a^2-\frac{27}{28}(y^2+z^2)-\frac{9}{28}x^2+\frac{1}{8}\frac{x^4}{a^2}+\frac{5}{8a^2}(x^4+z^4)+\frac{3}{4a^2}(x^2+z^2)y^2$$

$$+\frac{9}{4a^2}z^2x^2\biggr)+\epsilon_3\biggl(\frac{27}{40}a^2-\frac{27}{7}z^2-\frac{27}{28}r^2+\frac{x^2}{a^2}+\frac{3x^2r^2}{a^2}+\frac{3}{8}\frac{r^4}{a^2}\biggr)\biggr\}$$

$$+\frac{8\pi\sigma\omega_2{}^2}{35}(\epsilon_1-\epsilon_3)a^2\biggl(-\frac{35}{6}a^2+\frac{7}{3}r^2-\frac{1}{2}\frac{r^4}{a^2}-7z^2+\frac{5z^2r^2}{a^2}\biggr)$$

$$+\frac{16\pi\sigma a^2\omega_2\omega_3}{35}\biggl\{7\biggl(1-\frac{1}{2}\epsilon_1-\frac{3}{2}\epsilon_3-\frac{3}{2}\epsilon_2\biggr)zy$$

$$-5\biggl(1-\frac{1}{2}\epsilon_1-\frac{3}{2}\epsilon_3-\frac{3}{2}\epsilon_2\biggr)\frac{zyr^2}{a^2}\biggr\}$$

$$+\frac{16\pi\sigma a^2\omega_2\omega_3}{9}\biggl\{\epsilon_1\biggl(\frac{9}{14}yz-\frac{1}{2}\frac{yzr^2}{a^2}-\frac{yzx^2}{a^2}\biggr)+\epsilon_2\biggl(\frac{27}{14}yz-\frac{y^3z}{a^2}-\frac{3}{2}\frac{yzr^2}{a^2}\biggr)$$

$$+\epsilon_3\biggl(\frac{27}{14}yz-\frac{yz^3}{a^2}-\frac{3}{2}\frac{yzr^2}{a^2}\biggr)\biggr\}\quad\text{. . . . .}\quad(5.7)$$

$$\gamma_{22}=4\pi\sigma a^2\omega_1{}^2\biggl\{\frac{a^2}{6}-\frac{r^2}{5}+\frac{r^4}{14a^2}+z^2\biggl(-\frac{2}{5}+\frac{2r^2}{7a^2}\biggr)\biggr\}$$

$$+4\pi\sigma a^2\omega_1{}^2\biggl(\frac{1}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{3}{2}\epsilon_3\biggr)\biggl\{-\frac{a^2}{6}+\frac{r^2}{5}-\frac{r^4}{14a^2}+z^2\biggl(\frac{2}{5}-\frac{2}{7}\frac{r^2}{a^2}\biggr)\biggr\}$$

$$+\frac{8\pi\sigma a^2\omega_1{}^2}{9}\biggl\{\epsilon_1\biggl(\frac{9}{40}a^2-\frac{27}{28}(z^2+x^2)-\frac{9}{28}y^2+\frac{1}{8}\frac{y^4}{a^2}+\frac{5}{8a^2}(z^4+x^4)$$

$$+\frac{3}{4a^2}(z^2+x^2)y^2+\frac{9}{4}\frac{z^2x^2}{a^2}\biggr)$$

$$+\epsilon_2\biggl(\frac{9}{40}a^2-\frac{9}{14}(y^2+z^2)+\frac{1}{2}(y^2+z^2)\frac{r^2}{a^2}+\frac{y^2z^2}{a^2}-\frac{9}{28}r^2+\frac{1}{8}\frac{r^4}{a^2}\biggr)$$

$$+\epsilon_3\biggl(\frac{27}{40}a^2-\frac{27}{7}z^2-\frac{27}{28}r^2+\frac{z^4}{a^2}+\frac{3z^3r^2}{a^2}+\frac{3}{8}\frac{r^4}{a^2}\biggr)\biggr\}$$

$$+\frac{8\pi\sigma\omega_1{}^2}{35}(\epsilon_3-\epsilon_2)a^2\biggl(\frac{35}{6}a^2-\frac{7}{3}r^2+\frac{1}{2}\frac{r^4}{a^2}+7z^2-\frac{5z^2r^2}{a^2}\biggr)$$

$$+4\pi\sigma a^2\omega_3{}^2\biggl\{\frac{a^2}{6}-\frac{r^2}{5}+\frac{1}{14}\frac{r^4}{a^2}+x^2\biggl(-\frac{2}{5}+\frac{2}{7}\frac{r^2}{a^2}\biggr)\biggr\}$$

$$+4\pi\sigma a^2\omega_3{}^2\biggl(\frac{3}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_3\biggr)\biggl\{-\frac{a^2}{6}+\frac{r^2}{5}-\frac{r^4}{14a^2}+x^2\biggl(\frac{2}{5}-\frac{2}{7}\frac{r^2}{a^2}\biggr)\biggr\}$$

$$+\frac{8\pi\sigma a^2\omega_3{}^2}{9}\biggl\{\epsilon_1\biggl(\frac{27}{40}a^2-\frac{27}{7}x^2-\frac{27}{28}r^2+\frac{x^4}{a^2}+\frac{3x^2r^2}{a^2}+\frac{3r^4}{8a^2}\biggr)$$

$$+\epsilon_2\left(\frac{9}{40}a^2-\frac{9}{14}(x^2+y^2)+\frac{1}{2}(x^2+y^2)\frac{r^2}{a^2}+\frac{x^2y^2}{a^2}+\frac{1}{8}\frac{r^2}{a^2}-\frac{9}{28}r^2\right)$$

$$+\epsilon_3\left(\frac{9}{40}a^2-\frac{27}{28}(x^2+z^2)-\frac{9}{28}y^2+\frac{1}{8}\frac{y^4}{a^2}+\frac{5}{8a^2}(x^4+z^4)+\frac{3}{4a^2}(x^2+z^2)y^2\right.$$

$$\left.+\frac{9}{4a^2}x^2z^2\right)\Bigg\}+\frac{8\pi\sigma\omega_3{}^2}{35}(\epsilon_2-\epsilon_1)a^2\left(-\frac{35}{6}a^2+\frac{7}{3}r^2-\frac{1}{2}\frac{r^4}{a^2}-7x^2+\frac{5x^2r^2}{a^2}\right)$$

$$+\frac{16\pi\sigma a^2\omega_3\omega_1}{35}\Bigg\{7\left(1-\frac{1}{2}\epsilon_2-\frac{3}{2}\epsilon_1-\frac{3}{2}\epsilon_3\right)xz$$

$$-5\left(1-\frac{1}{2}\epsilon_2-\frac{3}{2}\epsilon_1-\frac{3}{2}\epsilon_3\right)\frac{xzr^2}{a^2}\Bigg\}$$

$$+\frac{16\pi\sigma a^2\omega_3\omega_1}{9}\Bigg\{\epsilon_1\left(\frac{27}{14}zx-\frac{zx^3}{a^2}-\frac{3}{2}\frac{zxr^2}{a^2}\right)+\epsilon_2\left(\frac{9}{14}zx-\frac{1}{2}\frac{zxr^2}{a^2}-\frac{zxy^2}{a^2}\right)$$

$$+\epsilon_3\left(\frac{27}{14}zx-\frac{z^3x}{a^2}-\frac{3}{2}\frac{xzr^2}{a^2}\right)\Bigg\}. \qquad \qquad (5.8)$$

$$\gamma_{33}=4\pi\sigma a^2\omega_2{}^2\left\{\frac{a^2}{6}-\frac{r^2}{5}+\frac{r^4}{14a^2}+x^2\left(-\frac{2}{5}+\frac{2}{7}\frac{r^2}{a^2}\right)\right\}$$

$$+4\pi\sigma a^2\omega_2{}^2\left(\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_2+\frac{3}{2}\epsilon_3\right)\left\{-\frac{a^2}{6}+\frac{r^2}{5}-\frac{r^4}{14a^2}+x^2\left(\frac{2}{5}-\frac{2}{7}\frac{r^2}{a^2}\right)\right\}$$

$$+\frac{8\pi\sigma a^2\omega_2{}^2}{9}\Bigg\{\epsilon_1\left(\frac{27}{40}a^2-\frac{27}{7}x^2-\frac{27}{28}r^2+\frac{x^4}{a^2}+\frac{3x^2r^2}{a^2}+\frac{3r^4}{8a^2}\right)$$

$$+\epsilon_2\left(\frac{9}{40}a^2-\frac{27}{28}(x^2+y^2)-\frac{9}{28}z^2+\frac{1}{8}\frac{z^4}{a^2}\right.$$

$$\left.+\frac{5}{8a^2}(x^4+y^4)+\frac{3}{4a^2}(x^2+y^2)z^2+\frac{9x^2y^2}{4a^2}\right)$$

$$+\epsilon_3\left(\frac{9}{40}a^2-\frac{9}{14}(z^2+x^2)+\frac{1}{2}(z^2+x^2)\frac{r^2}{a^2}+\frac{z^2x^2}{a^2}-\frac{9r^2}{28}+\frac{r^1}{8a^2}\right)\Bigg\}$$

$$+\frac{8\pi\sigma\omega_2{}^2x^2}{35}(\epsilon_1-\epsilon_3)\left(\frac{35}{6}a^2-\frac{7}{3}r^2+\frac{1}{2}\frac{r^4}{a^2}+7y^2-5y^2\frac{r^2}{a^3}\right)$$

$$+4\pi\sigma a^2\omega_1{}^2\left\{\frac{a^2}{6}-\frac{r^2}{5}+\frac{1}{14}\frac{r^4}{a^2}+y^2\left(-\frac{2}{5}+\frac{2}{7}\frac{r^2}{a^2}\right)\right\}$$

$$+4\pi\sigma a^2\omega_1{}^2\left(\frac{1}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{3}{2}\epsilon_3\right)\left\{-\frac{a^2}{6}+\frac{r^2}{5}-\frac{r^4}{14a^2}+y^2\left(\frac{2}{5}-\frac{2}{7}\frac{r^2}{a^2}\right)\right\}$$

$$+\frac{8\pi\sigma a^2\omega_1{}^2}{9}\Bigg\{\epsilon_1\left(\frac{9}{40}a^2-\frac{27}{28}(x^2+y^2)-\frac{9}{28}z^2+\frac{1z^4}{8a^2}+\frac{5}{8a^2}(y^4+x^4)\right.$$

$$\left.+\frac{3}{4a^2}(y^2+x^2)z^2+\frac{9}{4a^2}y^2x^2\right)$$

$$+\epsilon_2\left(\frac{27}{40}a^2-\frac{27}{7}y^2-\frac{27}{28}r^2+\frac{y^4}{a^2}+\frac{3y^2r^2}{a^2}+\frac{3r^4}{8a^2}\right)$$

$$+\epsilon_3\left(\frac{9}{40}a^2-\frac{9}{14}(z^2+y^2)+\frac{1}{2}(z^2+y^2)\frac{r^2}{a^2}+\frac{y^2z^2}{a^2}+\frac{1r^4}{8a^2}-\frac{9}{28}r^2\right)\Bigg\}$$

$$+\frac{8\pi\sigma\omega_1{}^2}{35}(\epsilon_3-\epsilon_2)a^2\left(-\frac{35}{6}a^2+\frac{7}{3}r^2-\frac{1}{2}\frac{r^4}{a^2}-7y^2+\frac{5y^2r^2}{a^2}\right)$$

$$+\frac{16\pi\sigma a^2\omega_1\omega_2}{35}\biggl\{7\left(1-\frac{1}{2}\epsilon_3-\frac{3}{2}\epsilon_2-\frac{3}{2}\epsilon_1\right)yx$$

$$-5\left(1-\frac{1}{2}\epsilon_3-\frac{3}{2}\epsilon_2-\frac{3}{2}\epsilon_1\right)yx\frac{r^2}{a^2}\biggr\}$$

$$+\frac{16\pi\sigma a^2\omega_1\omega_2}{9}\biggl\{\epsilon_1\left(\frac{27}{14}xy-\frac{x^3y}{a^2}-\frac{3}{2}\frac{xyr^2}{a^2}\right)+\epsilon_2\left(\frac{27}{14}xy-\frac{xy^3}{a^2}-\frac{3}{2}\frac{xyr^2}{a^2}\right)$$

$$+\epsilon_3\left(\frac{9}{14}xy-\frac{1}{2}\frac{xyr^2}{a^2}-\frac{xyz^2}{a^2}\right)\biggr\}. \qquad (5.9)$$

$$\gamma_{12}=8\pi\sigma a^2\omega_3{}^2xy\left(\frac{1}{5}-\frac{r^2}{7a^2}\right)+8\pi\sigma a^2\omega_3{}^2\left(\frac{3}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_3\right)xy\left(-\frac{1}{5}+\frac{r^2}{7a^2}\right)$$

$$+\frac{8\pi\sigma a^2\omega_3{}^2}{9}\biggl\{\epsilon_1\left(\frac{27}{14}xy-\frac{x^3y}{a^2}-\frac{3xyr^2}{2a^2}\right)+\epsilon_2\left(\frac{27}{14}xy-\frac{xy^3}{a^2}-\frac{3xyr^2}{2a^2}\right)$$

$$+\epsilon_3\left(\frac{9}{14}xy-\frac{xyr^2}{2a^2}-\frac{xyz^2}{a^2}\right)\biggr\}$$

$$+\frac{16\pi\sigma a^2\omega_1\omega_2}{35}\left(-\frac{35}{24}a^2+\frac{7}{4}r^2-\frac{5}{8}\frac{r^4}{a^2}+\frac{7}{2}z^2-\frac{5}{2}\frac{z^2r^2}{a^2}\right)$$

$$+\frac{16\pi\sigma a^2\omega_1\omega_2}{35}\frac{(5\epsilon_3+\epsilon_1+\epsilon_2)}{2}\left(\frac{35}{24}a^2-\frac{7}{4}r^2+\frac{5}{8}\frac{r^4}{a^2}-\frac{7}{2}z^2+\frac{5}{2}\frac{z^2r^2}{a^2}\right)$$

$$+\frac{16\pi\sigma a^2}{15}\left(\frac{1}{2}\epsilon_1+\frac{1}{2}\epsilon_2-\epsilon_3\right)\omega_1\omega_2\left(\frac{15}{8}a^2-\frac{5}{4}r^2+\frac{3}{8}\frac{r^4}{a^2}\right)$$

$$+\frac{8\pi\sigma a^2\omega_1\omega_2}{9}\biggl\{\epsilon_1\biggl(-\frac{9}{40}a^2+\frac{9}{14}(x^2+z^2)-\frac{1}{2}(x^2+z^2)\frac{r^2}{a^2}-\frac{x^2z^2}{a^2}$$

$$-\frac{1}{8}\frac{r^4}{a^2}+\frac{9}{28}r^2\biggr)$$

$$+\epsilon_2\left(-\frac{9}{40}a^2+\frac{9}{14}(y^2+z^2)-\frac{1}{2}(y^2+z^2)\frac{r^2}{a^2}-\frac{y^2z^2}{a^2}-\frac{1}{8}\frac{r^4}{a^2}+\frac{9}{28}r^2\right)$$

$$+\epsilon_3\left(-\frac{27}{40}a^2+\frac{27}{7}z^2+\frac{27}{28}r^2-\frac{z^4}{a^2}-\frac{3z^2r^2}{a^2}-\frac{3}{8}\frac{r^4}{a^2}\right)\biggr\}$$

$$+\frac{8\pi\sigma a^2\omega_3\omega_1}{35}\biggl\{7\left(-1+\frac{3}{2}\epsilon_3+\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_1\right)yz$$

$$-5\left(-1+\frac{3}{2}\epsilon_3+\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_1\right)\frac{yzr^2}{a^2}\biggr\}$$

$$+\frac{8\pi\sigma a^2\omega_3\omega_1}{9}\biggl\{\epsilon_1\left(-\frac{9}{14}yz+\frac{1}{2}\frac{yzr^2}{a^2}+\frac{yzx^2}{a^2}\right)$$

$$+\epsilon_2\left(-\frac{27}{14}yz+\frac{y^3z}{a^2}+\frac{3}{2}\frac{yzr^2}{a^2}\right)+\epsilon_3\left(-\frac{27}{14}yz+\frac{z^3y}{a^2}+\frac{3yzr^2}{2a^2}\right)\biggr\}$$

$$+\frac{8\pi\sigma a^2\omega_2\omega_3}{35}\biggl\{7\left(-1+\frac{3}{2}\epsilon_3+\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_2\right)xz$$

$$-5\left(-1+\frac{3}{2}\epsilon_3+\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_2\right)\frac{xzr^2}{a^2}\Big\}$$

$$+\frac{8\pi\sigma a^2\omega_2\omega_3}{9}\Big\{\epsilon_1\left(-\frac{27}{14}xz+\frac{x^3z}{a^2}+\frac{3}{2}\frac{zxr^2}{a^2}\right)+\epsilon_2\left(-\frac{9}{14}xz+\frac{xzr^2}{2a^2}+\frac{xzy^2}{a^2}\right)$$

$$+\epsilon_3\left(-\frac{27}{14}xz+\frac{z^3x}{a^2}+\frac{3xzr^2}{2a^2}\right)\Big\}. \qquad (5.10)$$

$$\gamma_{23}=8\pi\sigma\omega_1{}^2yz\left(\frac{1}{5}-\frac{r^2}{7a^2}\right)+8\pi\sigma a^2\omega_1{}^2\left(\frac{1}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{3}{2}\epsilon_3\right)yz\left(-\frac{1}{5}+\frac{r^2}{7a^2}\right)$$

$$+\frac{8\pi\sigma\omega_1{}^2a^2}{9}\Big\{\epsilon_1\left(\frac{9}{14}yz-\frac{yzr^2}{2a^2}-\frac{yzx^2}{2a^2}\right)+\epsilon_2\left(\frac{27}{14}yz-\frac{y^3x}{a^2}-\frac{3yzr^2}{2a^2}\right)$$

$$+\epsilon_3\left(\frac{27}{14}yz-\frac{yz^3}{a^2}-\frac{3yzr^2}{2a^2}\right)\Big\}$$

$$+\frac{16\pi\sigma a^2\omega_2\omega_3}{35}\left(-\frac{35}{24}a^2+\frac{7}{4}r^2-\frac{5}{8}\frac{r^4}{a^2}+\frac{7}{2}x^2-\frac{5}{2}\frac{x^2r^2}{a^2}\right)$$

$$+\frac{16\pi\sigma a^2\omega_2\omega_3}{35}\frac{(5\epsilon_1+\epsilon_2+\epsilon_3)}{2}\left(\frac{35}{24}a^2-\frac{7}{4}r^2+\frac{5r^4}{8a^3}-\frac{7}{2}x^2+\frac{5}{2}\frac{x^2r^2}{a^2}\right)$$

$$+\frac{16\pi\sigma a^2}{15}\left(\frac{1}{2}\epsilon_2+\frac{1}{2}\epsilon_3-\epsilon_1\right)\omega_2\omega_3\left(\frac{15}{8}a^2-\frac{5}{4}r^2+\frac{3r^4}{8a^2}\right)$$

$$+\frac{8\pi\sigma a^2\omega_2\omega_3}{9}\Big\{\epsilon_1\left(-\frac{27}{40}a^2+\frac{27}{7}x^2+\frac{27}{28}r^2-\frac{x^4}{a^2}-\frac{3x^2r^2}{a^2}-\frac{3r^4}{8a^2}\right)$$

$$+\epsilon_2\left(-\frac{9}{40}a^2+\frac{9}{14}(y^2+x^2)-\frac{1}{2}(y^2+x^2)\frac{r^2}{a^2}-\frac{y^2x^2}{a^2}-\frac{r^4}{8a^2}+\frac{9}{28}r^2\right)$$

$$+\epsilon_3\left(-\frac{9}{40}a^2+\frac{9}{14}(z^2+x^2)-\frac{1}{2}(z^2+x^2)\frac{r^2}{a^2}-\frac{z^2x^2}{a^2}-\frac{r^4}{8a^3}+\frac{9}{28}r^2\right)\Big\}$$

$$+\frac{8\pi\sigma a^2\omega_1\omega_2}{35}\Big\{7\left(-1+\frac{3}{2}\epsilon_1+\frac{3}{2}\epsilon_3+\frac{1}{2}\epsilon_2\right)zx$$

$$-5\left(-1+\frac{3}{2}\epsilon_1+\frac{3}{2}\epsilon_3+\frac{1}{2}\epsilon_2\right)\frac{zxr^2}{a^2}\Big\}$$

$$+\frac{8\pi\sigma a^2\omega_1\omega_2}{9}\Big\{\epsilon_1\left(-\frac{27}{14}zx+\frac{x^3z}{a^2}+\frac{3zxr^2}{2a^2}\right)$$

$$+\epsilon_2\left(-\frac{9}{14}zx+\frac{1}{2}\frac{zxr^2}{a^2}+\frac{zxy^2}{a^2}\right)+\epsilon_3\left(-\frac{27}{14}zx+\frac{z^3x}{a^2}+\frac{3xzr^2}{2a^2}\right)\Big\}$$

$$+\frac{8\pi\sigma a^2\omega_1\omega_3}{35}\Big\{7\left(-1+\frac{3}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_3\right)yx$$

$$-5\left(-1+\frac{3}{2}\epsilon_1+\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_3\right)\frac{xyr^2}{a^2}\Big\}$$

$$+\frac{8\pi\sigma a^2\omega_1\omega_3}{9}\Big\{\epsilon_1\left(-\frac{27}{14}xy+\frac{x^3y}{a^2}+\frac{3yxr^2}{2a^2}\right)$$

$$+\epsilon_2\left(-\frac{27}{14}yx+\frac{y^3x}{a^2}+\frac{3xyr^2}{2a^2}\right)+\epsilon_3\left(-\frac{9}{14}yx+\frac{yxr^2}{2a^2}+\frac{yxz^2}{2a^2}\right)\Big\}. \qquad (5.11)$$

$$
\begin{aligned}
\gamma_{31}= & \, 8\pi\sigma\omega_2{}^2 zx\biggl(\frac{1}{5}-\frac{1}{7}\frac{r^2}{a^2}\biggr)+8\pi\sigma a^2\omega_2{}^2\biggl(\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_2+\frac{3}{2}\epsilon_3\biggr)zx\biggl(-\frac{1}{5}+\frac{1}{7}\frac{r^2}{a^2}\biggr)\\
& +\frac{8\pi\sigma a^2\omega_2{}^2}{9}\biggl\{\epsilon_1\biggl(\frac{27}{14}zx-\frac{zx^3}{a^2}-\frac{3zxr^2}{2a^2}\biggr)+\epsilon_2\biggl(\frac{9}{14}zx-\frac{zxr^2}{2a^2}-\frac{zxy^2}{2a^2}\biggr)\\
& +\epsilon_3\biggl(\frac{27}{14}zx-\frac{z^3y}{a^2}-\frac{3zxr^2}{2a^2}\biggr)\biggr\}\\
& +\frac{16\pi\sigma a^2\omega_3\omega_1}{35}\biggl(-\frac{35}{24}a^2+\frac{7}{4}r^2-\frac{5r^4}{8a^2}+\frac{7}{2}y^2-\frac{5}{2}\frac{y^2r^2}{a^2}\biggr)\\
& +\frac{16\pi\sigma a^2\omega_3\omega_1}{35}\biggl(\frac{5}{2}\epsilon_2+\frac{1}{2}\epsilon_3+\frac{1}{2}\epsilon_1\biggr)\biggl(\frac{35}{24}a^2-\frac{7}{4}r^2+\frac{5r^4}{8a^2}-\frac{7y^2}{2}+\frac{5y^2r^2}{2a^2}\biggr)\\
& +\frac{16\pi\sigma a^2\omega_3\omega_1}{15}\biggl(\frac{1}{2}\epsilon_3+\frac{1}{2}\epsilon_1-\epsilon_2\biggr)\biggl(\frac{15a^2}{8}-\frac{5r^2}{4}+\frac{3r^4}{8a^2}\biggr)\\
& +\frac{8\pi\sigma a^2\omega_3\omega_1}{9}\biggl\{\epsilon_1\biggl(-\frac{9}{40}a^2+\frac{9}{14}(x^2+y^2)-\frac{1}{2}(x^2+y^2)\frac{r^2}{a^2}\\
& -\frac{x^2y^2}{a^2}-\frac{r^4}{8a^2}+\frac{9r^2}{28}\biggr)+\epsilon_2\biggl(-\frac{27}{40}a^2+\frac{27}{7}y^2+\frac{27}{28}r^2-\frac{y^4}{a^2}-\frac{3y^2r^2}{a^2}-\frac{3r^4}{8a^2}\biggr)\\
& +\epsilon_3\biggl(-\frac{9}{40}a^2+\frac{9}{14}(z^2+y^2)-\frac{1}{2}(z^2+y^2)\frac{r^2}{a^2}-\frac{z^2y^2}{a^2}-\frac{r^4}{8a^2}+\frac{9r^2}{28}\biggr)\biggr\}\\
& +\frac{8\pi\sigma a^2\omega_2\omega_3}{35}\biggl\{7\biggl(-1+\frac{3}{2}\epsilon_2+\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_3\biggr)xy\\
& -5\biggl(-1+\frac{3}{2}\epsilon_2+\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_3\biggr)\frac{xyr^2}{a^2}\biggr\}\\
& +\frac{8\pi\sigma a^2\omega_2\omega_3}{9}\biggl\{\epsilon_1\biggl(-\frac{27}{14}xy+\frac{x^3y}{a^2}+\frac{3yxr^2}{2a^2}\biggr)+\epsilon_2\biggl(-\frac{27}{14}xy+\frac{y^3x}{a^2}+\frac{3xyr^2}{2a^2}\biggr)\\
& +\epsilon_2\biggl(-\frac{9}{14}xy+\frac{xyr^2}{2a^2}+\frac{xyz^2}{a^2}\biggr)\biggr\}\\
& +\frac{8\pi\sigma a^2\omega_1\omega_2}{35}\biggl\{7\biggl(-1+\frac{3}{2}\epsilon_2+\frac{3}{2}\epsilon_3+\frac{1}{2}\epsilon_1\biggr)zy\\
& -5\biggl(-1+\frac{3}{2}\epsilon_2+\frac{3}{2}\epsilon_3+\frac{1}{2}\epsilon_1\biggr)\frac{zyr^2}{a^2}\biggr\}\\
& +\frac{8\pi\sigma a^2\omega_1\omega_2}{9}\biggl\{\epsilon_1\biggl(-\frac{9}{14}zy+\frac{zyr^2}{2a^2}+\frac{zyx^2}{a^2}\biggr)+\epsilon_2\biggl(-\frac{27}{14}zy+\frac{y^3z}{a^2}+\frac{3zyr^2}{2a^2}\biggr)\\
& +\epsilon_3\biggl(-\frac{27}{14}zy+\frac{z^3y}{a^2}+\frac{3yzr^2}{2a^2}\biggr)\biggr\}. \qquad \text{(5.12)}
\end{aligned}
$$

We have from (5.3)–(5.6)

$$
\begin{aligned}
\gamma_{0/4}= & \frac{16\pi\sigma}{5}\{(\epsilon_1-\epsilon_2)\omega_3xy+(\epsilon_2-\epsilon_3)\omega_1yz+(\epsilon_3-\epsilon_1)\omega_2zx\}\\
& +\frac{16\pi\sigma}{5}\{(\epsilon_2-\epsilon_1)\omega_3{}^2t(x^2-y^2)+(\epsilon_3-\epsilon_2)\omega_1{}^2t(y^2-z^2)
\end{aligned}
$$

$$+(\epsilon_1-\epsilon_3)\omega_2^2t(z^2-x^2)\}+\frac{16\pi\sigma}{5}\{(\epsilon_1+\epsilon_2-2\epsilon_3)\omega_1\omega_2tyx$$
$$+(\epsilon_2+\epsilon_3-2\epsilon_1)\omega_2\omega_3tzy+(\epsilon_3+\epsilon_1-2\epsilon_2)\omega_3\omega_1tzx\}$$
$$=\gamma_{4s/8}. \quad \ldots \quad (5.13)$$

Also $\gamma_{mn/4}=0$.

$$\gamma_{41/4}=\frac{8\pi\sigma\dot{\omega}_3}{15}(-5ya^2+3yr^2)+\frac{8\pi\sigma\dot{\omega}_2}{15}(5za^2-3zr^2)$$
$$+8\pi\sigma\left\{\omega_3^2(\epsilon_2-\epsilon_1)\left(-\frac{2}{15}xa^2+\frac{2}{35}xr^2-\frac{2}{7}xy^2\right)\right.$$
$$\left.+\omega_2^2(\epsilon_1-\epsilon_3)\left(\frac{2}{15}xa^2-\frac{2}{35}xr^2+\frac{2}{7}z^2x\right)\right\}$$
$$+\frac{16\pi\sigma}{15}\left\{(\epsilon_3-\epsilon_2)\omega_1\omega_2\left(ya^2-\frac{3}{7}yr^2\right)+(\epsilon_2-\epsilon_3)\omega_3\omega_1\left(za^2-\frac{3}{7}zr^2\right)\right\}$$
$$+\frac{16\pi\sigma}{7}\{(\epsilon_3-\epsilon_2)\omega_1\omega_2yz^2+(\epsilon_2+\epsilon_3-2\epsilon_1)\omega_2\omega_3xyz+(\epsilon_2-\epsilon_3)\omega_3\omega_1zy^2\}. \quad \ldots \quad (5.14)$$

Again the expressions (5.7)–(5.12) give

$$\gamma_{1s/8}=8\pi\sigma\omega_3^2(\epsilon_2-\epsilon_1)\left(-\frac{2}{15}xa^2+\frac{2}{35}xr^2-\frac{2}{7}xy^2\right)$$
$$+8\pi\sigma\omega_2^2(\epsilon_1-\epsilon_3)\left(\frac{2}{15}xa^2-\frac{2}{35}xr^2+\frac{2}{7}z^2x\right)$$
$$+\frac{16\pi\sigma}{7}\omega_2\omega_3(\epsilon_2+\epsilon_3-2\epsilon_1)xyz$$
$$+\frac{16\pi\sigma\omega_1\omega_2}{15}\left(\frac{1}{2}\epsilon_1+\frac{1}{2}\epsilon_2-\epsilon_3\right)\left(-\frac{5}{2}ya^2+\frac{3}{2}yr^2\right)$$
$$+\frac{8\pi\sigma\omega_1\omega_2}{35}(\epsilon_2-\epsilon_3)(7ya^2-5yr^2-10yz^2)$$
$$+\frac{16\pi\sigma\omega_1\omega_3}{15}\left(\frac{1}{2}\epsilon_1+\frac{1}{2}\epsilon_3-\epsilon_2\right)\left(-\frac{5}{2}za^2+\frac{3}{2}zr^2\right)$$
$$+\frac{8\pi\sigma\omega_1\omega_3}{35}(\epsilon_3-\epsilon_2)(7za^2-5zr^2-10zy^2), \quad \ldots \quad (5.15)$$

whence

$$\gamma_{41/4}-\gamma_{1s/8}=\frac{8\pi\sigma}{15}\left(\dot{\omega}_3-\frac{\omega_1\omega_2}{2}(\epsilon_1-\epsilon_2)\right)(-5ya^2+3yr^2)$$
$$+\frac{8\pi\sigma}{15}\left(\dot{\omega}_2-\frac{1}{2}(\epsilon_3-\epsilon_1)\omega_1\omega_3\right)(5za^2-3zr^2)$$
$$=0, \quad \ldots \quad (5.16)$$

if the conditions (3.15) are staisfied.

Similarly we find $\gamma_{42/4}-\gamma_{2s/8}=0$

and $\gamma_{43/4}-\gamma_{3s/8}=0.$

6. *The Energy Tensor.*

The next step in the calculation is to determine the components of the energy tensor. Since the conditions (5.1) are satisfied the energy tensor is given by (1.6). From (5.3) we find

$$\begin{aligned}\gamma_{0/ss}=16\pi\sigma&+\frac{16\pi\sigma}{15a^2}(\epsilon_2-\epsilon_1)\omega_3{}^2(x^2-y^2)+\frac{16\pi\sigma}{15a^2}(\epsilon_3-\epsilon_2)\omega_1{}^2(y^2-z^2)\\&+\frac{16\pi\sigma}{15a^2}(\epsilon_1-\epsilon_3)\omega_2{}^2(z^2-x^2)+\frac{16\pi\sigma}{5a^2}(\epsilon_1+\epsilon_2-2\epsilon_3)\omega_1\omega_2xy\\&+\frac{16\pi\sigma}{5a^2}(\epsilon_2+\epsilon_3-2\epsilon_1)\omega_2\omega_3yz+\frac{16\pi\sigma}{5a^2}(\epsilon_3+\epsilon_1-2\epsilon_2)\omega_3\omega_1zx. \qquad (6.1)\end{aligned}$$

$$\begin{aligned}\gamma_{0/44}=&\frac{16\pi\sigma}{15a^2}\{(\epsilon_2-\epsilon_1)\omega_3{}^2(x^2-y^2)+(\epsilon_3-\epsilon_2)\omega_1{}^2(y^2-z^2)+(\epsilon_1-\epsilon_3)\omega_2{}^2(z^2-x^2)\}\\&+\frac{16\pi\sigma}{5a^2}\{(\epsilon_1+\epsilon_2-2\epsilon_3)\omega_1\omega_2xy+(\epsilon_2+\epsilon_3-2\epsilon_1)\omega_2\omega_3yz\\&+(\epsilon_3+\epsilon_1-2\epsilon_2)\omega_3\omega_1zx\}. \qquad (6.2)\end{aligned}$$

Therefore $$\gamma_{0/ss}-\gamma_{0/44}=16\pi\sigma. \qquad (6.3)$$

Again, since $$\gamma_{mn/ss}-\gamma_{mn/44}=16\pi\mathrm{T}_{mn},$$

we have $$\gamma_{ll/ss}-\gamma_{ll/44}=16\pi T_{ll}. \qquad (6.4)$$

Consequently $$\mathrm{T}_{44}=\sigma+\mathrm{T}_{ll} \qquad (6.5)$$

on using (5.2). The proper density T is given by

$$\begin{aligned}\mathrm{T}&=g^{\alpha\beta}\mathrm{T}_{\alpha\beta}\\&=\mathrm{T}_{44}-\mathrm{T}_{ll}\\&=\sigma \qquad (6.6)\end{aligned}$$

to the order of approximation considered.

The expressions for the components of the momentum are obtained from (5.4)–(5.6). We find

$$\gamma_{41/ss}=16\pi\sigma\{(\omega_3+\dot\omega_3t)y-(\omega_2+\dot\omega_2t)z\}, \qquad (6.7)$$

$$\gamma_{41/44}=0, \qquad (6.8)$$

so (1.6) gives $$\mathrm{T}_{14}=\sigma\{(\omega_3+\dot\omega_3t)y-(\omega_2+\dot\omega_2t)z\}. \qquad (6.9)$$

By a suitable interchange of indices it is evident that the other components of momentum are :—

$$\mathrm{T}_{24}=\sigma\{(\omega_1+\dot\omega_1t)z-(\omega_3+\dot\omega_3t)x\} \qquad (6.10)$$

and $$\mathrm{T}_{34}=\sigma\{(\omega_2+\dot\omega_2t)x-(\omega_1+\dot\omega_1t)y\}. \qquad (6.11)$$

The components T_{mn} of the energy tensor are found by substituting (5.7)–(5.12) in (1.6). Typical expressions for $\gamma_{mn/ss}$ are :—

$$\begin{aligned}\gamma_{11/ss}=&16\pi\sigma\omega_3{}^2\{y^2-\tfrac{1}{2}(a^2-r^2)(1-\epsilon_1)+\tfrac{1}{2}(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)\}\\&+16\pi\sigma\omega_2{}^2\{z^2-\tfrac{1}{2}(a^2-r^2)(1-\epsilon_1)+\tfrac{1}{2}(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)\}\\&+16\pi\sigma\omega_2\omega_3(-2yz), \qquad (6.12)\end{aligned}$$

$$\gamma_{12/88}=16\pi\sigma(-\omega_3{}^2xy+\omega_3\omega_1yz+\omega_2\omega_3zx-\omega_1\omega_2z^2)$$
$$+8\pi\sigma\omega_1\omega_2\{(a^2-r^2)(1-\tfrac{1}{2}(\epsilon_1+\epsilon_2))-(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)\}. \quad . \quad (6.13)$$

The components T_{mn}, therefore, consist of two parts; the first T'_{mn} gives the mass motion and the second the stresses T''_{mn}. The actual values of T'_{mn} are :—

$$\left.\begin{aligned}
&T'_{11}=\sigma(\omega_3{}^2y^2-2\omega_2\omega_3zy+\omega_2{}^2z^2) ;\\
&\qquad\qquad T'_{23}=\sigma(\omega_1\omega_2zx-\omega_1{}^2yz-\omega_2\omega_3x^2+\omega_3\omega_1yx) ;\\
&T'_{22}=\sigma(\omega_1{}^2z^2-2\omega_3\omega_1xz+\omega_3{}^2x^2) ;\\
&\qquad\qquad T'_{31}=\sigma(\omega_2\omega_3xy-\omega_2{}^2zx-\omega_3\omega_1y^2+\omega_1\omega_2zy) ;\\
&T'_{33}=\sigma(\omega_2{}^2x^2-2\omega_1\omega_2yx+\omega_1{}^2y^2) ;\\
&\qquad\qquad T'_{12}=\sigma(\omega_3\omega_1yz-\omega_3{}^2xy-\omega_1\omega_2z^2+\omega_2\omega_3xz).
\end{aligned}\right\} . (6.14)$$

That is,

$$\begin{matrix}\downarrow\!\xrightarrow{}m\\ n\end{matrix}\qquad T'^{n}_{m}=\left.\begin{matrix}-\rho u^2 & -\rho uv & -\rho uv\\ -\rho uv & -\rho v^2 & -\rho vw\\ -\rho uw & -\rho vw & -\rho w^2\end{matrix}\right\}, \quad . \quad . \quad (6.15)$$

where

$$\left.\begin{aligned}
u&=(\omega_2+\dot{\omega}_2t)z-(\omega_3+\dot{\omega}_3t)y,\\
v&=(\omega_3+\dot{\omega}_3t)x-(\omega_1+\dot{\omega}_1t)z,\\
w&=(\omega_1+\dot{\omega}_1t)y-(\omega_2+\dot{\omega}_2t)x.
\end{aligned}\right\} . \quad . \quad . \quad . \quad (6.16)$$

The stresses are given by

$$\left.\begin{matrix}T''_{mn}\equiv P_{xx} & P_{xy} & P_{xz},\\ P_{xy} & P_{yy} & P_{yz},\\ P_{xz} & P_{yz} & P_{zz},\end{matrix}\right\}, \quad . \quad . \quad (6.17)$$

where

$$P_{xx}=\tfrac{1}{2}\sigma\{\omega_3{}^2[(r^2-a^2)(1-\epsilon_1)+(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)]$$
$$+\omega_2{}^2[(r^2-a^2)(1-\epsilon_1)+(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)]\},$$

$$P_{yy}=\tfrac{1}{2}\sigma\{\omega_1{}^2[(r^2-a^2)(1-\epsilon_2)+(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)]$$
$$+\omega_3{}^3[(r^2-a^2)(1-\epsilon_2)+(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)]\},$$

$$P_{zz}=\tfrac{1}{2}\sigma\{\omega_2{}^2[(r^2-a^2)(1-\epsilon_3)+(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)]$$
$$+\omega_1{}^2[(r^2-a^2)(1-\epsilon_3)+(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)]\},$$

$$P_{yz}=-\tfrac{1}{2}\sigma\omega_2\omega_3\left\{(r^2-a^2)\left(1-\frac{\epsilon_2+\epsilon_3}{2}\right)+(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)\right\},$$

$$P_{\text{[illegible]}}=-\tfrac{1}{2}\sigma\omega_3\omega_1\left\{(r^2-a^2)\left(1-\frac{\epsilon_3+\epsilon_1}{2}\right)+(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)\right\},$$

$$P_{xy}=-\tfrac{1}{2}\sigma\omega_1\omega_2\left\{(r^2-a^2)\left(1-\frac{\epsilon_1+\epsilon_2}{2}\right)+(\epsilon_1x^2+\epsilon_2y^2+\epsilon_3z^2)\right\}.$$

It will be noticed that the stresses vanish at the boundary (2.4). A further check on the arithmetical details can be made by verifying that $(T^{\nu}_{\mu})_{\nu}=0$.

To the order of approximation considered

$$T^{s}_{m}=g^{s\alpha}T_{m\alpha}=-T_{ms},$$

$$T^{4}_{m}=g^{4\alpha}T_{m\alpha}=T_{m4}.$$

So

$$(T^{\nu}_{\mu})_{\nu}=-T_{ms/s}+T_{m4/4}.$$

But (6.14), (6.17) give

$$T_{1s/s}=\tfrac{1}{2}\sigma\{\omega_1\omega_2 y(\epsilon_1-\epsilon_2)-\omega_3\omega_1 z(\epsilon_3-\epsilon_1)\}.$$

Also from (6.9) we have

$$T_{14/4}=\sigma(\dot{\omega}_3 y-\dot{\omega}_2 z).$$

Therefore

$$-T_{1s/s}+T_{14/4}=\sigma\{\dot{\omega}_3-\tfrac{1}{2}\omega_2\omega_1(\epsilon_1-\epsilon_2)\}y-\sigma\{\dot{\omega}_2-\tfrac{1}{2}\omega_3\omega_1(\epsilon_3-\epsilon_1)\}z. \quad . \quad . \quad . \quad . \quad (6.18)$$

The expression on the right-hand side of (6.18) vanishes if the condition (3.15) is satisfied and the divergence $(T^{\nu}_{\mu})_{\nu}$ is zero. The invariant mass, m, of the system is given by

$$m=\int\sigma\frac{dt}{ds}\,dV$$

$$=\int\sigma\{1+\tfrac{1}{2}(\omega_2 z-\omega_3 y)^2+\tfrac{1}{2}(\omega_3 x-\omega_1 z)^2+\tfrac{1}{2}(\omega_1 y-\omega_2 x)^2\}dV$$

$$=\frac{4\pi\sigma a^3}{3}\left(1-\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3)\right)+\frac{4\pi\sigma a^5}{15}\left\{\left(1-\frac{1}{2}\epsilon_2-\frac{1}{2}\epsilon_3\right)\omega_1{}^2\right.$$

$$\left.+\left(1-\frac{1}{2}\epsilon_3-\frac{1}{2}\epsilon_1\right)\omega_2{}^2+\left(1-\frac{1}{2}\epsilon_1-\frac{1}{2}\epsilon_2\right)\omega_3{}^2\right\}. \quad . \quad . \quad . \quad (6.19)$$

It is interesting to note that if, in (6.19), we put

$$\left.\begin{array}{ll}\epsilon_1=\epsilon_2=\epsilon, & \epsilon_3=0,\\ \omega_1=\omega_2=0, & \omega_3=\omega\end{array}\right\} \quad . \quad . \quad . \quad (6.20)$$

and neglect terms of order $\epsilon\omega^2$, the expression for the invariant mass becomes

$$m=\frac{4\pi\sigma a^3}{3}+\frac{4\pi\sigma a^3}{3}\left(-\epsilon+\frac{a^2\omega^2}{5}\right). \quad . \quad . \quad . \quad . \quad (6.21)$$

Consequently, if

$$\epsilon=\frac{a^2\omega^2}{5}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.22)$$

the invariant mass is

$$\frac{4\pi\sigma a^3}{3}.$$

7. *The Field due to a Rotating Sphere.*

The rather complicated expressions which occur in the preceding investigation assume quite simple forms in certain special cases. For example, the field due to a sphere rotating with constant angular velocity ω about the z-axis is independent of the time and in this case the equations (1.6) become

$$16\pi T_{\mu\nu}=\gamma_{\mu\nu/ss}, \qquad (7.1)$$

provided

$$\gamma_{\mu s/s}=0. \qquad (7.2)$$

The External Field.

When the ϵ_i are zero, the external field is :—

$$\gamma_{44}=-\frac{16\pi\sigma a^3}{r}+\gamma_{ll}, \qquad (7.3)$$

$$\left.\begin{aligned}\gamma_{41}&=-\frac{16\pi\sigma a^5}{15}\frac{\omega y}{r^3},\\ \gamma_{42}&=\frac{16\pi\sigma a^5}{15}\frac{\omega x}{r^3},\end{aligned}\right\}, \qquad (7.4)$$

$$\gamma_{43}=0,$$

$$\left.\begin{aligned}\gamma_{11}&=\frac{16\pi\sigma a^7\omega^2}{105}\left(\frac{1}{r^3}-\frac{3y^2}{r^5}\right), & \gamma_{12}&=\frac{16\pi\sigma a^7\omega^2}{35}\frac{xy}{r^5},\\ \gamma_{22}&=\frac{16\pi\sigma a^7\omega^2}{105}\left(\frac{r^3}{1}-\frac{3z^2}{r^5}\right), & \gamma_{23}&=0,\\ \gamma_{33}&=0, & \gamma_{31}&=0.\end{aligned}\right\} \qquad (7.5)$$

The $\gamma_{\mu\nu}$ satisfy $\gamma_{\mu\nu/ss}=0$ and the condition (7.2) so the energy tensor vanishes everywhere.

The Internal Field.

For the internal field we have

$$\gamma_{44}=-8\pi\sigma a^2+\frac{8\pi\sigma r^2}{3}, \qquad (7.6)$$

$$\left.\begin{aligned}\gamma_{41}&=\frac{8\pi\sigma\omega}{15}(-5ya^2+3yr^2),\\ \gamma_{42}&=\frac{8\pi\sigma\omega}{15}(5xa^2-3xr^2),\end{aligned}\right\}, \qquad (7.7)$$

$$\gamma_{43}=0,$$

$$\left.\begin{aligned}\gamma_{11}&=4\pi\sigma a^2\omega^2\left\{\frac{a^2}{6}-\frac{r^2}{5}+\frac{r^4}{14a^2}+y^2\left(-\frac{2}{5}+\frac{2}{7}\frac{r^2}{a^2}\right)\right\},\\ \gamma_{12}&=8\pi\sigma a^2\omega^2xy\left(\frac{1}{5}-\frac{r^2}{7a^2}\right),\\ \gamma_{22}&=4\pi\sigma a^2\omega^2\left\{\frac{a^2}{6}-\frac{r^2}{5}+\frac{r^4}{14a^2}+x^2\left(-\frac{2}{5}+\frac{2}{7}\frac{r^2}{a^2}\right)\right\},\\ \gamma_{23}&=0,\\ \gamma_{33}=0,\qquad \gamma_{31}&=0.\end{aligned}\right\} \qquad (7.8)$$

The $\gamma_{\mu\nu}$, $\gamma_{\mu\nu/s}$ are continuous at $r=a$ and the condition (7.2) is satisfied. The equations (7.1) give for the non-vanishing components of $T_{\mu\nu}$:—

$$\left.\begin{aligned}
&T_{44}=\sigma+T_{ll}=\sigma+\sigma\omega^2(r^2-a^2)+\sigma\omega^2(x^2+y^2),\\
&T_{41}=\sigma\omega y,\\
&T_{42}=-\sigma\omega x,\\
&T'_{11}=\sigma\omega^2 y^2,\quad T'_{22}=\sigma\omega^2 x^2,\quad T'_{12}=-\sigma\omega^2 xy,\\
&P_{xx}=\tfrac{1}{2}\sigma\omega^2(r^2-a^2),\\
&P_{yy}=\tfrac{1}{2}\sigma\omega^2(r^2-a^2).
\end{aligned}\right\} \quad . \quad . \quad (7.9)$$

Also the proper density $T=T_{44}-T_{ll}=\sigma$.

8. *The Field due to a Rotating Spheroid of Small Ellipticity.*

The expressions for the potentials are also fairly simple in the case of a spheroid

$$r=a\left(1-\frac{\epsilon(x^2+y^2)}{2a^2}\right),\quad (\epsilon \text{ small}) \quad . \quad . \quad . \quad . \quad . \quad (8.1)$$

rotating with constant angular velocity ω about the z-axis.

The External Field.

The external field given by (3.1)—(3.11) reduces to :—

$$\gamma_{44}=\gamma_0+\gamma_{ll}, \quad . \quad . \quad . \quad . \quad . \quad (8.2)$$

$$\gamma_0=-\frac{16\pi\sigma a^3}{3r}+\frac{16\pi\sigma a^3\epsilon}{3r}+\frac{8\pi\sigma a^4\epsilon}{15}\left\{\frac{3(x^2+y^2)}{r^5}-\frac{2}{r^3}\right\}, \quad . \quad . \quad . \quad (8.3)$$

$$\left.\begin{aligned}
\gamma_{41}&=-\frac{16\pi\sigma a^5}{15}(1-2\epsilon)\frac{\omega y}{r^3}+8\pi\sigma a^7\omega\epsilon\left\{-\frac{4y}{35r^5}+\frac{x^2y+y^3}{7r^7}\right\},\\
\gamma_{42}&=\frac{16\pi\sigma a^5}{15}(1-2\epsilon)\frac{\omega x}{r^3}+8\pi\sigma a^7\omega\epsilon\left\{\frac{4x}{35r^5}-\frac{x^3+xy^2}{7r^7}\right\},\\
\gamma_{43}&=0,
\end{aligned}\right\}, \quad (8.4)$$

$$\left.\begin{aligned}
\gamma_{11}&=\frac{16\pi\sigma a^7\omega^2}{105}(1-3\epsilon)\left(\frac{1}{r^3}-\frac{3y^2}{r^5}\right)\\
&\quad+\frac{8\pi\sigma a^9\omega^2\epsilon}{9}\left\{\frac{4}{35r^5}-\frac{y^2}{r^7}-\frac{x^2}{7r^7}+\frac{x^2y^2+y^4}{r^9}\right\},\\
\gamma_{22}&=\frac{16\pi\sigma a^7\omega^2}{105}(1-3\epsilon)\left(\frac{1}{r^3}-\frac{3x^2}{r^5}\right)\\
&\quad+\frac{8\pi\sigma a^9\omega^2\epsilon}{9}\left\{\frac{4}{35r^5}-\frac{x^2}{r^7}-\frac{y^2}{7r^7}+\frac{x^2y^2+x^4}{r^9}\right\},\\
\gamma_{12}&=\frac{16\pi\sigma a^7\omega^2}{35}(1-3\epsilon)\frac{xy}{r^5}+\frac{8\pi\sigma a^9\omega^2\epsilon}{9}\left\{\frac{6xy}{7r^7}+\frac{x^3y+xy^3}{r^9}\right\},\\
\gamma_{33}&=\gamma_{13}=\gamma_{23}=0.
\end{aligned}\right\}, \quad . \quad (8.5)$$

As in the case of the sphere, we have $\gamma_{\mu\nu/4}=0$ and $\gamma_{\mu\nu/ss}=0$.

The Internal Field.

The internal field given by (5.1)–(5.12) reduces to :—

$$\gamma_{44}=\gamma_0+\gamma_{ll} \qquad \text{(8.6)}$$

$$\gamma_0=-8\pi\sigma a^2+\frac{16\pi\sigma a^2\epsilon}{3}+\frac{8\pi\sigma r^2}{3}-\frac{16\pi\sigma\epsilon r^2}{15}+\frac{8\pi\sigma\epsilon}{5}(x^2+y^2), \qquad \text{(8.7)}$$

$$\left.\begin{aligned}
\gamma_{41}&=\frac{8\pi\sigma}{15}(1-2\epsilon)\omega(-5ya^2+3yr^2)+8\pi\sigma\epsilon\omega\Big\{-\frac{2}{5}a^2y+\frac{2}{7}yr^2+\frac{x^2y+y^3}{7}\Big\},\\
\gamma_{42}&=\frac{8\pi\sigma}{15}(1-2\epsilon)\omega(5xa^2-3xr^2)+8\pi\sigma\epsilon\omega\Big\{\frac{2}{5}a^2x-\frac{2}{7}xr^2-\frac{xy^2+x^3}{7}\Big\},\\
\gamma_{43}&=0,
\end{aligned}\right\}, \qquad \text{(8.8)}$$

$$\left.\begin{aligned}
\gamma_{11}&=4\pi\sigma a^2\omega^2(1-3\epsilon)\Big\{\frac{a^2}{6}-\frac{r^2}{5}+\frac{r^1}{14a^2}+y^2\Big(-\frac{2}{5}+\frac{2}{7}\frac{r^2}{a^2}\Big)\Big)\Big\}\\
&\quad+\frac{8\pi\sigma a^2\omega^2\epsilon}{9}\Big(\frac{9a^2}{10}-\frac{9}{14}x^2-\frac{9}{2}y^2-\frac{9}{7}r^2\\
&\quad+\frac{1}{2}\frac{r^4}{a^2}+\frac{x^2y^2}{a^2}+\frac{1}{2}\frac{x^2r^2}{a^2}+\frac{7}{2}\frac{y^2r^2}{a^2}+\frac{y^4}{a^2}\Big),\\
\gamma_{22}&=4\pi\sigma a^2\omega^2(1-3\epsilon)\Big(\frac{a^2}{6}-\frac{r^2}{5}+\frac{r^4}{14a^2}\Big)+x^2\Big(-\frac{2}{5}+\frac{2}{7}\frac{r^2}{a^2}\Big)\Big\}\\
&\quad+\frac{8\pi\sigma a^2\omega^2\epsilon}{9}\Big(\frac{9}{10}a^2-\frac{9}{2}x^2-\frac{9}{14}y^2-\frac{9}{7}r^2\\
&\quad+\frac{1}{2}\frac{r^4}{a^2}+\frac{x^2y^2}{a^2}+\frac{1}{2}\frac{y^2r^2}{a^2}+\frac{7x^2r^2}{2a^2}+\frac{x^4}{a^2}\Big),\\
\gamma_{12}&=8\pi\sigma a^2\omega^2(1-3\epsilon)xy\Big(\frac{1}{5}-\frac{r^2}{7a^2}\Big)+\frac{8\pi\sigma a^2\omega^2\epsilon}{9}\Big\{\frac{27}{7}xy-\frac{3xyr^2}{a^2}-\frac{x^3y+xy^3}{a^2}\Big\},
\end{aligned}\right\}, \qquad \text{(8.9)}$$

$$\gamma_{33}=\gamma_{13}=\gamma_{23}=0.$$

The $\gamma_{\mu\nu}$ and their derivatives $\gamma_{\mu\nu/s}$ are continuous on the surface (8.1) and the condition $\gamma_{\mu s/s}=0$ is satisfied. The non-vanishing components of $T_{\mu\nu}$ are then found to be :—

$$\left.\begin{aligned}
&T_{44}=\sigma+T_{ll},\\
&T_{14}=\sigma\omega y, \qquad T_{24}=-\sigma\omega x,\\
&T_{mn}=T'_{mn}+T''_{mn},\\
&\text{where}\\
&T'_{11}=\sigma\omega^2y^2, \qquad T''_{11}=P_{xx}=\tfrac{1}{2}\sigma\omega^2\{(r^2-a^2)(1-\epsilon)+\epsilon(x^2+y^2)\},\\
&T'_{22}=\sigma\omega^2x^2, \qquad T''_{22}=P_{yy}=\tfrac{1}{2}\sigma\omega^2\{(r^2-a^2)(1-\epsilon)+\epsilon(x^2+y^2)\},\\
&T'_{12}=-\sigma\omega^2xy, \quad T''_{12}=P_{xy}=-\tfrac{1}{2}\sigma\omega^2\{(r^2-a^2)(1-\epsilon)+\epsilon(x^2+y^2)\}.
\end{aligned}\right\}. \qquad \text{(8.10)}$$

9. *Alternative Solutions for Static Fields.*

The static fields discussed in sections 7 and 8 satisfy the condition $\gamma_{ms/s}=0$, and the proper densities of the systems considered were constant. Other solutions can, of course, be found for various density distributions and, in general, $\gamma_{ms/s}$ will not be zero. When this happens the energy tensor will be given by the equations :—

$$\left.\begin{aligned}16\pi T_{44}&=\gamma_{44/ss}-\gamma_{rs/rs},\\16\pi T_{4n}&=\gamma_{4n/ss},\\16\pi T_{mn}&=\gamma_{mn/ss}-\gamma_{ms/sn}-\gamma_{ns/sm}+\gamma_{rs/rs}\delta_{mn}\end{aligned}\right\}\quad . \; . \; . \; . \; . \quad (9.1)$$

subject to the condition $\gamma_{4s/s}=0$. As an example, we will consider a solution for which $\gamma_{ms/s}$ only vanishes at the boundary and in the external field. We do not consider the terms γ_{4n} which are the same as those in paragraph 7.

The External Field.

For the external field we take

$$\gamma_{44}=-\frac{4m}{r}+A'(x^2+y^2-2z^2)\frac{a^2}{r^4},\quad . \; . \; . \; . \; . \; . \; . \; . \quad (9.2)$$

$$\left.\begin{aligned}\gamma_{11}&=\frac{1}{3}B'(x^2+y^2-2z^2)\frac{a^5}{r^5}, & \gamma_{23}&=B'\frac{yza^5}{r^5},\\\gamma_{22}&=\frac{1}{3}B'(x^2+y^2-2z^2)\frac{a^5}{r^5}, & \gamma_{31}&=B'\frac{zxa^5}{r^5},\\\gamma_{33}&=\frac{1}{3}B'(-x^2-y^2+2z^2)\frac{a^5}{r^5}, & \gamma_{12}&=0.\end{aligned}\right\}\quad . \; . \quad (9.3)$$

It is readily verified that

$$\gamma_{44/ss}=0,\quad \gamma_{mn/ss}=0,\quad \gamma_{ms/s}=0.\quad . \; . \; . \; . \; . \quad (9.4)$$

The Internal Field.

For the internal field we take

$$\gamma_{44}=-\frac{m}{a}+\frac{2mr^2}{a^3}+\frac{7A'}{2}(x^2+y^2-2z^2)-\frac{5A'}{2}(x^2+y^2-2z^2)\frac{r^2}{a^2},\quad . \; . \quad (9.5)$$

$$\left.\begin{aligned}\gamma_{11}&=\left\{\frac{7}{6}B'(x^2+y^2-2z^2)-\frac{5}{6}B'(x^2+y^2-2z^2)\frac{r^2}{a^2}\right\},\\\gamma_{23}&=\frac{7}{2}B'yz-\frac{5}{2}B'\frac{yzr^2}{a^2},\\\gamma_{22}&=\left\{\frac{7}{6}B'(x^2+y^2-2z^2)-\frac{5}{9}B'(x^2+y^2-2z^2)\frac{r^2}{a^2}\right\},\\\gamma_{13}&=\frac{7}{2}B'xz-\frac{5}{2}B'\frac{xzr^2}{a^2},\\\gamma_{33}&=\left\{\frac{7}{6}B'(-x^2-y^2+2z^2)-\frac{5}{6}B'(-x^2-y^2+2z^2)\frac{r^2}{a^2}\right\},\\\gamma_{12}&=0.\end{aligned}\right\}\quad . \quad (9.6)$$

We find that the $\gamma_{\mu\nu}$ and their derivatives $\gamma_{\mu\nu/s}$ are continuous at $r=a$. Also

$$\left.\begin{aligned}
\gamma_{1s/s}&=\frac{35}{6}\mathrm{B}'x\left(1-\frac{r^2}{a^2}\right),\\
\gamma_{2s/s}&=\frac{35}{6}\mathrm{B}'y\left(1-\frac{r^2}{a^2}\right),\\
\gamma_{3s/s}&=\frac{35}{3}\mathrm{B}'z\left(1-\frac{r^2}{a^2}\right).
\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad (9.7)$$

The expressions $\gamma_{ms/s}$, therefore, only vanish at the boundary $r=a$. From (9.7) we have

$$\left.\begin{aligned}
&\gamma_{1s/s1}=\frac{35}{6}\mathrm{B}'\left(1-\frac{r^2}{a^2}-\frac{2x^2}{a^2}\right),\qquad \gamma_{2s/s1}=-\frac{35}{3a^2}\mathrm{B}'xy,\\
&\qquad\qquad \gamma_{3s/s1}=-\frac{70}{3}\mathrm{B}'\frac{zx}{a^2},\\
&\gamma_{1s/s2}=-\frac{35}{3a^2}\mathrm{B}'xy,\qquad \gamma_{2s/s2}=\frac{35}{6}\mathrm{B}'\left(1-\frac{r^2}{a^2}-\frac{2y^2}{a^2}\right),\\
&\qquad\qquad \gamma_{3s/s2}=-\frac{70}{3}\mathrm{B}'\frac{zy}{a^2},\\
&\gamma_{1s/s3}=-\frac{35}{3a^2}\mathrm{B}'xz,\qquad \gamma_{2s/s3}=-\frac{35}{3}\mathrm{B}'\frac{yz}{a^2},\\
&\qquad\qquad \gamma_{3s/s3}=\frac{35}{3}\mathrm{B}'\left(1-\frac{r^2}{a^2}-\frac{2z^2}{a^2}\right),\\
&\gamma_{rs/sr}=\frac{70}{3}\mathrm{B}'\left\{1-\frac{3a^2}{2}(x^2+y^2)-\frac{2z^2}{a^2}\right\}.
\end{aligned}\right\}\ (9.8)$$

Moreover,

$$\left.\begin{aligned}
\gamma_{11/ss}&=\frac{35}{3a^2}\mathrm{B}'(-x^2-y^2+2z^2), & \gamma_{23/ss}&=-35\mathrm{B}'\frac{yz}{a^2},\\
\gamma_{22/ss}&=\frac{35}{3a^2}\mathrm{B}'(-x^2-y^2+2z^2), & \gamma_{31/ss}&=-35\mathrm{B}'\frac{xz}{a^2},\\
\gamma_{33/ss}&=\frac{35}{3a^2}\mathrm{B}'(x^2+y^2-2z^2), & \gamma_{12/ss}&=0.
\end{aligned}\right\}\ .\ (9.9)$$

Writing $\mathrm{A}'=16\pi\mathrm{A}$, $\mathrm{B}'=16\pi\mathrm{B}$, the equations (9.1) give, on using (9·8), (9.9)

$$\mathrm{T}_{44}=\frac{12m}{16\pi a^3}-\frac{70\mathrm{B}}{3}+(-35\mathrm{A}+35\mathrm{B})\left(\frac{x^2+y^2}{a^2}\right)+\left(70\mathrm{A}+\frac{140}{3}\mathrm{B}\right)\frac{z^2}{a^2},\quad (9.10)$$

$$\left.\begin{aligned} &T_{11}=\frac{70B}{3a^2}\left\{-y^2+\frac{1}{2}(a^2-r^2)\right\}, && T_{23}=0,\\ &T_{22}=\frac{70B}{3a^2}\left\{-x^2+\frac{1}{2}(a^2-r^2)\right\}, && T_{13}=0,\\ &T_{33}=0, && T_{12}=\frac{70}{3a^2}Bxy,\end{aligned}\right\} \quad . \qquad (9.11)$$

$$T_{ll}=\frac{70}{3}B\{a^2-2(x^2+y^2)-z^2\}.$$

The proper density ρ_0 has the value

$$\rho_0=T_{44}-T_{ll}=\frac{12m}{16\pi a^3}-\frac{140B}{3}+\left(-35A+\frac{245}{3}B\right)\frac{(x^2+y^2)}{a^2}+(70A+70B)\frac{z^2}{a^2} \quad . \; . \; . \; . \; . \quad (9.12)$$

Putting $A=-\frac{3}{70}a^2\omega^2\alpha\sigma$, $B=-\frac{3}{70}a^2\omega^2\sigma$, the expressions (9.10), (9.11) and (9.12) become

$$T_{44}=\frac{12m}{16\pi a^3}+\sigma\omega^2\left\{\left(\frac{3}{2}\alpha-\frac{3}{2}\right)(x^2+y^2)-(3\alpha+2)z^2+a^2\right\}, \qquad (9.13)$$

$$\left.\begin{aligned} &T_{11}=\sigma\omega^2y^2-\tfrac{1}{2}\sigma\omega^2(a^2-r^2),\\ &T_{22}=\sigma\omega^2x^2-\tfrac{1}{2}\sigma\omega^2(a^2-r^2),\\ &T_{12}=-\sigma\omega xy,\end{aligned}\right\}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9.14)$$

$$\rho_0=T=\frac{12m}{16\pi a^3}+\sigma\omega^2\left\{\left(\frac{3}{2}\alpha-\frac{7}{2}\right)(x^2+y^2)-(3\alpha+3)z^2+2a^2\right\}, \quad (9.15)$$

where the parameter α is arbitrary. It will be noticed that this series of systems does not include the case of constant proper density ρ_c.

References.

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XC. *An Exact Formula for* $Q_k(n)$, *the Probable Number of k-aggregates in a Random Distribution of n Points.*

By C. Mack, M.A. *

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Dr. Silberstein's paper on " Aggregates in Random Distributions of Points " in your issue of May 1945, raises some points of interest. He deals with the problem by an approximate analysis. However, it is possible to find an exact formula for $Q_k(n)$ which in the one-dimensional

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* Communicated by the Author.

case is easy to evaluate. Also, it is possible to obtain some idea of the probability P_m of m k-aggregates occurring simultaneously ($Q_k(n)=\sum_m mP_m$ by definition), at least in most practical cases.

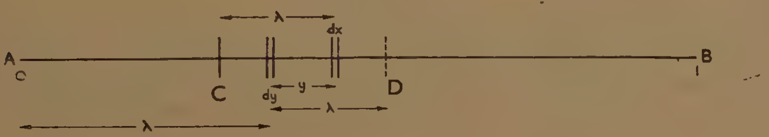
1. *One-Dimensional Formula for* $Q_k(n)$, *neglecting End-effects.*

Though the formula is straightforward it is not immediately obvious, so it is advisable to consider an analogy of the problem. Suppose we have n balls which may be placed at random in N pigeon-holes where N is very large compared with n. We wish to find the probable number of k-aggregates. If k balls are so placed that the number of pigeon-holes separating the first from the kth is some given number P or less, they form a k-aggregate provided that the k balls are not part of some higher $(k+1)$ aggregate.

This problem may be solved by considering a particular pigeon-hole and finding the chance that it is the right-hand member of a k-aggregate. This probability may be arrived at by considering the chance that one of the n balls lies in this particular pigeon-hole, multiplying it by the chance that $(k-1)$ of the remainder are distributed in the previous $(P-1)$ pigeon-holes and that the remainder are distributed elsewhere. It can be seen that this probability also includes the probability that a second k-aggregate occurs somewhere else at the same time as one occurs with the particular right-hand member we have chosen. However, if we now sum the probabilities taking each pigeon-hole in turn, if two k-aggregates occur simultaneously they will be included twice, 3 simultaneous k-aggregates three times and so on. Thus, by summing the probabilities, taking each pigeon-hole in turn as right-hand member, we obtain $Q_k(n)$ *i.e.* the "probable number" of k-aggregates.

We now apply this theory to a continuous distribution of n random points along a line of unit length and determine $Q_k(n)$ the probable number of k-aggregates, where a k-aggregate is now a set of k points lying within a length λ but not forming part of a higher aggregate.

Fig. 1.



Consider a line AB of unit length and a small length dx at a distance x from A. We shall suppose that dx contains the right-hand point of a k-aggregate. Suppose the left-hand point lies in a small length dy at a distance y from dx.

The chance that two points lie in dx and dy is $ndx(n-1)dy$. The chance that a further $(k-2)$ points lie between these two is ${}^{n-2}C_{k-2}y^{k-2}$. Since they form part of a pure k-aggregate no other point must lie between CD (a length of $(2\lambda-y)$). Therefore the total probability that a k-aggregate exists with its right-hand point between x and $x+dx$ and its left-hand point between $x-y$ and $x-y-dy$ is

$$n(n-1)\,{}^{n-2}\mathrm{C}_{k-2}\,dx\,dy\,y^{k-2}(1-2\lambda+y)^{n-k}. \quad . \quad . \quad . \quad (1.1)$$

Now the limits for y are 0 and λ so, integrating as y goes from 0 to λ, then integrating over the limits for x (whose nature will be discussed later), we have as a first value for

$$\mathrm{Q}_k(n)=(x_1-x_0)\,n(n-1)\,{}^{\lambda-2}\mathrm{C}_{k-2}\int_0^\lambda y^{k-2}(1-2\lambda+y)^{n-k}\,dy. \quad . \quad (1.2)$$

If end effects can be neglected x_1 can be made 1 and x_0 0, and we obtain, provided $k\geqq 2$

$$\mathrm{Q}_k(n)=n(n-1)\,{}^{n-2}\mathrm{C}_{k-2}\int_0^\lambda y^{k-2}(1-2\lambda+y)^{n-k}\,dy. \quad . \quad . \quad (1.3)$$

$\mathrm{Q}_k(n)$ can be expressed in various ways depending on the magnitude of some of the parameters.

Usually n is very large while k is moderate and we may replace $(1-2\lambda+y)^{n-k}$ by $e^{(n-k)(y-2\lambda)}$.

$\therefore$ If $(n-k)$ is very large

$$\mathrm{Q}_k(n)=n(n-1)\,{}^{\lambda-2}\mathrm{C}_{k-2}\int_0^\lambda y^{k-2}e^{(n-k)(y-2\lambda)}\,dy; \quad . \quad . \quad . \quad (1.4)$$

$\therefore$ we have that

$$n\,{}^{n-1}\mathrm{C}_{k-1}\lambda^{k-1}e^{-(n-k)\lambda}>\mathrm{Q}_k(n)>n\,{}^{n-1}\mathrm{C}_{k-1}\lambda^{k-1}e^{-2(n-k)\lambda}. \quad . \quad (1.5)$$

(1.5) may give $\mathrm{Q}_k(n)$ to the required accuracy if $(n-k)\lambda$ is small enough. Alternative expressions are :

$$\mathrm{Q}_k(n)=n\,{}^{n-1}\mathrm{C}_{k-1}\lambda^{k-1}e^{-(n-k)\lambda}$$
$$\times\left[1-\frac{(n-k)\lambda}{k}+\frac{(n-k)^2\lambda^2}{k(k+1)}-\frac{(n-k^3)\lambda^3}{k(k+1)(k+2)}+\ldots\right], \quad . \quad . \quad (1.6)$$

an infinite series useful if $\dfrac{(n-k)\lambda}{k}\leqq 0{\cdot}5$, and which can be used for any higher value though more slowly convergent.

Or,

$$\mathrm{Q}_k(n)=n(n-1)\,{}^{n-2}\mathrm{C}_{k-2}e^{-(n-k)\lambda}\frac{\lambda^{k-2}}{n-k}$$
$$\times\left[1-\frac{k-2}{(n-k)\lambda}+\frac{(k-2)(k-3)}{(n-k)^2\lambda^2}-\ldots\pm\frac{(k-2)\,!\,(1-e^{-(n-k)\lambda})}{(n-k)^{k-2}\lambda^{k-2}}\right],$$
$$. \quad . \quad . \quad . \quad . \quad (1.7)$$

a finite series most useful when $(n-k)\lambda\gg(k-2)$.

If $k\ll n$ we may further simplify matters by replacing $n-1$, $n-2$, etc., by n, thus obtaining :

$$\mathrm{Q}_k(n)=\frac{n(\lambda n)^{k-1}}{(k-1)\,!}e^{-n\lambda}\left[1-\frac{n\lambda}{k}+\frac{(n\lambda)^2}{k(k+1)}-\ldots\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.8)$$

$$\mathrm{Q}_k(n)=\frac{n(\lambda n)^{k-2}}{(k-2)\,!}\left[1-\frac{k-2}{n\lambda}+\frac{(k-2)(k-3)}{(n\lambda)^2}-\ldots\pm\frac{(k-2)\,!\,(1-e^{-(n-k)\lambda})}{(n\lambda)^{k-2}}\right].$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.9)$$

If we express the formula in powers of (λn) and consider the first two terms only, from 1.8 we obtain

$$Q_k(n) \doteqdot \frac{n(\lambda n)^{k-1}}{(k-1)!}\left[1-\lambda n\frac{k+1}{k}\right]. \quad . \quad . \quad . \quad . \quad . \qquad (1.10)$$

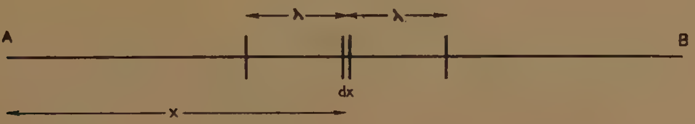
Dr. Silberstein's formula (p. 327 equation (17 *a*), May, 1945) is

$$\frac{n(\lambda n)^{k-1}}{k}\left(1-\frac{4+K}{k+1}\lambda n\right),$$

where $K=k-1+1/2+1/3+\ldots\ldots+1/k$. The difference is due, I think, to Dr. Silberstein's assumption that the probabilities he calls π_k (equation (9), p. 323) are independent of n.

In the above formulæ k has been assumed to be $\geqq 2$, we now consider $k=1$.

Fig. 2.



Suppose we have a point between x and $x+dx$. No further point is required to make a 1-aggregate and all other points must lie at a distance $>(x+\lambda)$ or $<(x-\lambda)$ from A.

$$\therefore \quad Q_1(n)=n(x_1-x_0)\,(1-2\lambda)^{n-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1.11)$$

$$=n(1-2\lambda)^{n-1} \text{ if end effects are neglected} \quad . \quad . \quad . \qquad (1.12)$$

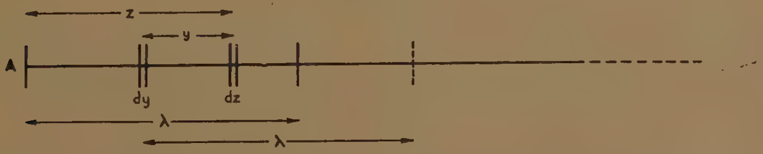
$$=ne^{-2(n-1)\lambda} \text{ if } n \text{ very large}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1.13)$$

2. *Rigorous Formula for $Q_k(n)$ including End-effects.*

End-effects only affect the situation when the right-hand point of the k-aggregate is less than λ from A or nearer than λ to B.

Consider the end near A first.

Fig. 3.



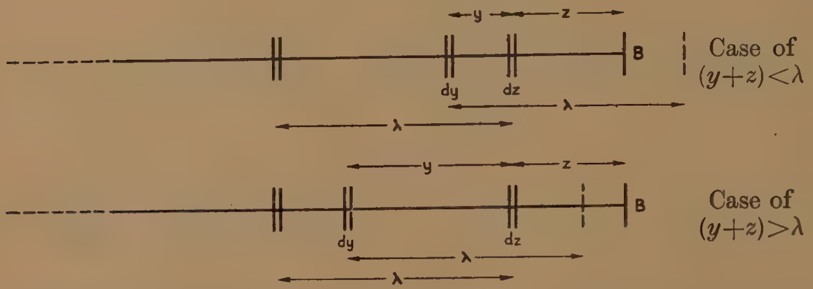
Suppose the right-hand point lies between Z and Z$+d$Z from A while the left-hand point lies between $(Z-y)$ and $(Z-y-dy)$ from A. No further point can lie on the range 0 to $(Z-y+\lambda)$ without turning the k-aggregate into a higher aggregate.

Thus the contribution towards $Q_k(n)$ from the left-hand end near A is

$$\int_0^\lambda n(n-1)^{n-2}C_{k-2}\left[\int_0^Z y^{k-2}(1-Z+y-\lambda)^{n-k}\,dy\right]dz. \qquad (2.1)$$

The right-hand end is more complicated.

Fig. 4.



Suppose the right-hand point is a distance z from B. The formula depends on whether the left-hand point is at a smaller distance than λ from B or not, and we have to take the two cases separately (see above figure).

The contribution to $Q_k(n)$ from this end is

$$n(n-1)^{n-2}C_{k-2}\int_0^\lambda\left\{\frac{(\lambda-z)^{k-1}}{k-1}(1-z-\lambda)^{n-k}+\int_{\lambda-z}^{\lambda} y^{k-2}(1-2\lambda+y)^{n-k}dy\right\}dz. \qquad (2.2)$$

The main contribution to $Q_k(n)$ is obtained from (1.2), the limits x_1 and x_0 being $1-\lambda$ and λ. Thus, adding all contributions together, we obtain

$$Q_k(n)=n(n-1)^{n-2}C_{k-2}\Bigg[(1-2\lambda)\int_0^\lambda y^{k-2}(1-2\lambda+y)^{n-k}\,dy$$
$$+\int_0^\lambda\left\{\int_0^z y^{k-2}(1-z+y-\lambda)^{n-k}\,dy+\frac{(\lambda-z)^{k-1}}{k-1}(1-z-\lambda)^{n-k}\right.$$
$$\left.+\int_{\lambda-z}^{\lambda} y^{k-2}(1-2\lambda+y)^{n-k}\,dy\right\}dz\Bigg]. \qquad (2.3)$$

(2.3) only holds if $k\geqq 2$. If $k=1$ we obtain by similar methods the formula

$$Q_1(n)=(n-2)(1-2\lambda)^n+2(1-\lambda)^n. \qquad (2.4)$$

When $(n-k)$ is large we can write $(1-x)^{n-k}=e^{-(n-k)x}$ and, in this case, we have

$$Q_k(n)=n(n-1)^{n-2}C_{k-2}e^{-(n-k)\lambda}\Bigg[(1-2\lambda)e^{-(n-k)\lambda}\int_0^\lambda y^{k-2}e^{(n-k)y}\,dy$$
$$+\int_0^\lambda\left\{\int_0^z y^{k-2}e^{(n-k)(y-z)}\,dy+\frac{(\lambda-z)^{k-1}}{k-1}e^{-(n-k)z}\right.$$
$$\left.+\int_{\lambda-z}^{\lambda} y^{k-2}e^{(y-\lambda)(n-k)}dy\right\}dz\Bigg], \qquad (2.5)$$

where $k\geqq 2$, and

$$Q_1(n)=(n-2)e^{-2n\lambda}+2e^{-n\lambda}. \qquad (2.6)$$

Dr. Silberstein gives values of $Q_k(n)$ in some special cases where they can be deduced rigorously.

Our formula (2.3) gives the right answer in all these cases.

Thus $Q_k(k)$ by our formula

$$=k(k-1)\left[(1-2\lambda)\frac{\lambda^{k-1}}{k-1}+\frac{\lambda^k}{k(k-1)}+\frac{\lambda^k}{k-1}\right]$$

$=k\lambda^{k-1}-(k-1)\lambda^k$ agreeing with Dr. Silberstein's formula (8).

Again, Dr. Silberstein analyses the case $n=3$ thoroughly. Though he does not give $Q_2(3)$ explicitly he gives the probability $P_2(3)$ of a single doublet (*i. e.* a k-aggregate where $k=2$) as $P_2(3)=2(1-\lambda)^3$ if $\lambda\geqq\frac{1}{2}$ or $6\lambda\left[(1-\lambda)(1-2\lambda)+\frac{\lambda^2}{3}\right]$ if $\lambda<\frac{1}{2}$, and the probability of two doublets as

$$P_{2,2}(3)=(1-\lambda)^2(4\lambda-1) \text{ if } \lambda>\tfrac{1}{2} \text{ or } \lambda^2(3-4\lambda) \text{ if } \lambda<\tfrac{1}{2}.$$

Now $Q_2(3)=1\times P_2(3)+2P_{2,2}(3)$ (by definition) $=6\lambda(1-\lambda)^2$, whether $\lambda>\frac{1}{2}$ or $\lambda<\frac{1}{2}$. Formula (2.3) gives

$$Q_2(3)=6\lambda\left[1-3\,.\,5\lambda+3\lambda^2+(1-\lambda)\frac{\lambda}{2}-\frac{\lambda^2}{6}+(1-2\lambda)\lambda+\frac{\lambda^2}{2}+\frac{\lambda^2}{6}\right]$$

$=6\lambda(1-\lambda)^2$, agreeing with the previous result.

It is incidentially proved in obtaining formula (2.3) that if we express $Q_k(n)$ as a polynomial in λ its coefficients are independent of the size of λ. Though, as seen in the case we have just considered ($n=3$), the probabilities of just one doublet and just two doublets have polynomial expressions which may change as λ alters.

It is of some interest for the sake of comparison with Dr. Silberstein's paper to consider the expansion of (2.3) in powers of λ.

We obtain

$$Q_k(n)=k^nC_k\lambda^{k-1}\left[1-\left\{(n-k)\frac{(k+1)}{k}+\frac{k-1}{k}\right\}\lambda+\text{terms in }\lambda^2,\text{ etc.}\right]$$

and this is to be compared with Dr. Siberstein's formula, equation (17), p. 326 of

$$Q_k(n)=k^nC_k\lambda^{k-1}\left\{1-\frac{4+K}{k+1}\lambda(n-k)-\frac{k-1}{k}\lambda\right\}.$$

Again, Dr. Siberstein's factor $\frac{4+K}{k+1}$ is replaced by $1+\frac{1}{k}$ in our formula.

The difference is not great, however, for example, when $k=1, 2, 3, 4$

$$\frac{4+K}{k+1}=2, \frac{11}{6}, \frac{41}{24}, \frac{97}{60}; \quad 1+\frac{1}{k}=2, 1\tfrac{1}{2}, 1+1/3, 1+1/4.$$

As $k\rightarrow\infty$ both expressions $\rightarrow 1$.

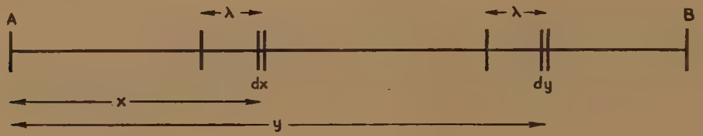
3. *The Distribution of Probabilities* $P_m(k, n)$ *where* P_m *is the chance of* m *Simultaneous* k*-aggregates.*

As we have noted $Q_k(n)=\sum\limits_m mP_m(k, n)$ and though $Q_k(n)$ can be found rigorously and exactly the individual P_m's are more difficult to determine and, as seen in the special case $n=3$, $k=2$, the formula for P_m may change as λ increases.

We shall assume that λ is very small and that λn is quite small ($\lambda n \leqq \cdot 1$) and that end effects and overlapping effects (where some of the points of one k-aggregate form part of another k-aggregate) are negligible, also we assume n very large and k moderate.

Again, as before, we consider a case in which the right-hand point of a k-aggregate lies between x and $x+dx$.

Now let us consider the probability $P_1(k, n)$ of obtaining a single k-aggregate (excluding cases where two or more aggregates exist simultaneously).



The chance that a k-aggregate occurs at x is $n\,dx\,{}^{n-1}C_{k-1}\lambda^{k-1}(1-\lambda)^{n-k}$. Now we must exclude from the factor $(1-\lambda)^{n-k}$ the probability of another k-aggregate occurring elsewhere than at x.

We shall now make the assumption that $k \ll n$ so that

$$n\,{}^{n-1}C_{k-1}\lambda^{k-1}(1-\lambda)^{n-k} \doteqdot \frac{n(\lambda n)^{k-1}e^{-\lambda n}}{(k-1)\,!}=X \text{ (say)}. \quad . \quad . \quad (3.1)$$

The probability that another k-aggregate occurs at y is

$$(n-k)\,dy\,{}^{n-k-1}C_{k-1}\lambda^{k-1}(1-\lambda)^{n-2k}.$$

The total probability that one or more k-aggregates occur somewhere outside the range $(x-\lambda, x)$ among the $(n-k)$ remaining points is

$$\left(\int_0^{x-\lambda}+\int_x^1\right)(n-k)\,{}^{n-k-1}C_{k-1}\lambda^{k-1}(1-\lambda)^{n-2k}\,dy \doteqdot X.$$

$\therefore$ a better approximation to $P_1(k, n)$, the probability that only one k-aggregate occurs is $X(1-X)$.

Again, the probability of two or more k-aggregates occurring outside the range $(x-\lambda, x)$ among the $(n-k)$ points available is $\doteqdot X^2/2$. The factor of 2 occurs because with our method of summation each pair is counted twice, once when our infinitesimal length dy passes over the right-hand point of the first of the pair, and secondly when dy passes over the right-hand point of the second k-aggregate.

Finally we obtain

$$\begin{aligned} P_1(k,n) &\doteq X\left[1-X\left(1-\frac{X}{2}\right)\left(1-\frac{X}{3}\right)(1-\ldots .)\right], \quad . \quad . \quad . \quad (3.2) \\ &\doteq Xe^{-X}. \end{aligned}$$

Now, by a similar argument, we obtain

$$P_2(k,n)=\frac{X^2}{2}e^{-X}, \quad P_3(k,n)=\frac{X^2}{2}\cdot\frac{X}{3}e^{-X}=\frac{X^3}{3\,!}e^{-X},$$

and so on.

$\therefore$ in general $$P_m(k,n)\doteq\frac{X^m}{m\,!}e^{-X}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.3)$$

Let us now see whether $\Sigma m P_m(k,n)=Q_k(n)$ with the above values of $P_m(k,n)$.

Consider $\Sigma m P_m(k,n)$. Strictly speaking m does not run from 1 to ∞ but, assuming the contribution due to large values of m to be very small,

$$\begin{aligned} \sum_1^\infty m P_m(k,n) &= e^{-X}\left[X+X^2+\frac{X^3}{2!}+\frac{X^4}{3!}+\ldots .\right] \\ &= X = Q_k(n) \text{ as expected.} \end{aligned}$$

One point of interest arises as to the number M of simultaneous aggregates which has the highest probability. It is given by $M=[X]$, that is, M is the largest integral number less than $X=Q_k(n)$.

4. *Two-Dimensional Distributions.*

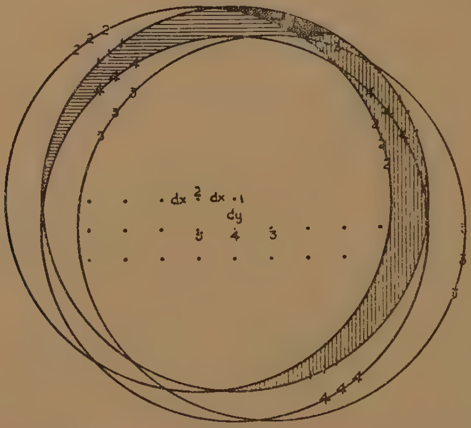
We now consider the problem in which n points are distributed at random over an area (which we shall assume of unit size) and we have to determine the probable number $Q_k(n)$ of k-aggregates, where a k-aggregate consists of k points but no more inside a circle of radius R (area$=\pi R^2=\sigma$). We shall not attempt to obtain an exact formula for $Q_k(n)$ but we shall obtain a good approximation which will be sufficient for most practical purposes. Our formula differs from Dr. Silberstein's in that ours is general and holds for all values of k. There is also a difference between our formula and Dr. Silberstein's formula (24) in the special case $n=3$, $k=3$ which results from some errors Dr. Silberstein has made.

This time we shall consider a large number of circles each of radius R whose centres are separated by distances dx in the x direction and distances dy in the y direction. The points 1, 2, 3, 4, 5, are examples of such circles. Beginning with one such circle on the boundary of the large area of unit size we find the probability that such a circle contains k points which do not form part of a higher aggregate. We then consider the next circle displaced a distance dx from the first and find the probability that a k-aggregate occurs in this circle which has *not been included* in the first. Then we take the next circle displaced infinitesimally from the second and find the probability of a new k-aggregate. Repeating the process until the whole of the unit area is covered and then summing the probabilities we obtain the value of $Q_k(n)$.

We now attempt to find a general formula for the probability arising from the addition of one of these infinitesimally displaced circles to that portion of the unit large area which has been already covered. Suppose all circles with centres marked as dots in our figure have been included with the exception of the point 1. The addition of the circle centre 1 means that a small area (shaded in the figure) has been added to the already covered area. This small area is to a good approximation *dxdy* (since the sum of all such areas must equal the total area, neglecting edge effects). Therefore one new type of k-aggregate consists of a point lying in this shaded area with $(k-1)$ other points lying in the remainder of the circle 1. The probabliity of such a k-aggregate occurring is

$$n\,dx\,dy^{n-1}C_{k-1}(\pi R^2)^{k-1}(1-A)^{n-k}, \quad . \quad . \quad . \quad . \quad . \quad (4.1)$$

where A is the area in which no other point may lie. In fact it is a matter of great difficulty to evaluate A exactly. We can state that it lies between πR^2 and $4\pi R^2$ however, and in general, in practical problems $(1-A)^{n-k}$ is near enough to 1.



Unfortunately such aggregates are not the only type introduced by the new circle with centre 1. There is a type consisting of one point inside the area bounded by the circles 1, 4 and 2 (shaded horizontally in the figure), of a second point in the area bounded by the circumferences of the circles 2 and 1 (shaded vertically in our figure) and $(k-2)$ in the rest of the area of circle 1. These two areas are $\frac{R\,dy}{2}$ and $R\,dx$ in magnitude and the probability of k-aggregates of this type is

$$n(n-1)^{n-2}C_{k-2}(\pi R^2)^{k-2}\frac{R^2}{2}dx\,dy(1-A)^{n-k}. \quad . \quad . \quad . \quad (4.2)$$

There are, unfortunately, an infinite series of similar types to the last, *e. g.* those with one point lying in the area outside circle 3, but within circle 1, and circle 2, with a second point inside circle 1 and outside circles 2 and 3. All we can say is that their sum is

$$Kn(n-1)^{n-2}C_{k-2}(\pi R^2)^{k-1}\,dx\,dy(1-A)^{n-k}, \quad . \quad . \quad . \quad (4.3)$$

where K is some factor which can be considered constant provided that $(1-A)^{n-k}=1$. We can evaluate K by considering a special value of n.

Summing (4.1) and (4.3) over all circles we obtain, neglecting end effects

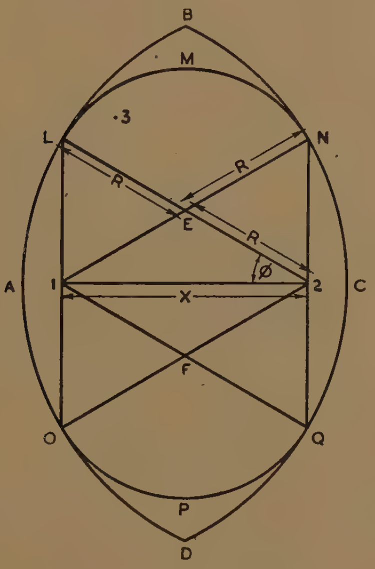
$$Q_k(n)=k\,{}^nC_k(1+(k-1)K)(\pi R^2)^{k-1}(1-A)^{n-k}. \quad . \quad . \quad . \quad (4.4)$$

Now the special case $n=2$, $k=2$ is easy to evaluate as Dr. Silberstein has shown. The first point may fall anywhere in the unit area. The second point must fall within a distance 2R of the first if they are to form a doublet. The probability of a doublet being formed by two points, is, therefore,

$$P_2(2)=4\pi R^2=4\sigma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.5)$$

but $Q_2(2)=P_2(2)$ in this case, $\therefore\ Q_2(2)=4\sigma$.

Fig. 5.



From (4.4) and (4.5) we obtain $2\,{}^2C_2[1+K]=4$,

$$\therefore\ K=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.6)$$

Now let us test this value of K in another special case which Dr. Silberstein evaluates (incorrectly as it happens), namely $n=3$, $k=3$. As before, point 1 may fall anywhere and the second point 2 must fall within a distance 2R from the first. The chance that it is at a distance between x and $x+dx$ is $2\pi x\,dx$.

Now we have to determine the limits within which point 3 may lie. In our figure the arc DAB is part of a circle, radius 2R, centre 2, while BCD is an arc of a circle centre 1, radius 2R. Dr. Silberstein assumed that the third point could lie anywhere inside the area enclosed by these two arcs. However, the third point cannot lie inside the areas LMNB and OPQD, for as is easily seen, no circle of radius R could include a point in these areas and both 1 and 2. The area in which point 3 may lie is, therefore,

$$R^2[6\phi-\sin 2\phi+\pi] \text{ where } \cos\phi=\frac{x}{2R}\cdot$$

The probability $P_3(3)$ (which $=Q_3(3)$ here) of a triplet being formed by 3 points is, therefore, writing $2\phi=\alpha$,

$$Q_3(3)=2\pi R^4\int_0^\pi \sin\alpha\,[3\alpha-\sin\alpha+\pi]d\alpha=9\pi^2R^4=9\sigma^2. \quad . \quad (4.7)$$

Dr. Silberstein obtains the result $16\sigma^2(1-3\sqrt{3})/4\pi$ which $=9{\cdot}37\sigma^2$ (not $6{\cdot}616\sigma^2$ as he gives) slightly greater than our value of $9\sigma^2$ as would be expected. Substituting the values $n=3$, $k=3$, $K=1$ in (4.4) we also obtain $9\pi^2R^4$ thus confirming our formula for $Q_k(n)$ which we now write as

$$Q_k(n)=k^2\,{}^nC_k(\pi R^2)^{k-1}(1-A)^{n-k}, \quad . \quad . \quad . \quad . \quad . \quad (4.8)$$

where A lies between πR^2 and $4\pi R^2$, *i. e.* between σ and 4σ.

Usually, since $(n-k)$ is large, $(1-A)^{n-k}\doteqdot e^{-(n-k)A}$ and we may rewrite (4.8) as

$$Q_k(n)=k^2\,{}^nC_k\sigma^{k-1}F \text{ where } e^{-(n-k)\sigma}>F>e^{-4(n-k)\sigma}. \quad . \quad . \quad (4.9)$$

This formula also neglects end effects.

As in section 3, the probability $P_m(k, n)$ of m simultaneous k-aggregates is

$$P_m(k, n)\doteqdot\frac{X^m e^{-X}}{m\,!} \text{ where } X=Q_k(n). \quad . \quad . \quad . \quad . \quad (4.10)$$

5. *Exact Formula for* $Q_2(n)$ *in Two Dimensions, neglecting End-effects.*

It is possible to obtain a rigorous formula for $Q_2(n)$ which is relatively simple, but formulæ for $Q_k(n)$, where $k\geqslant 3$, are complicated and difficult to evaluate. Even the formula for $Q_2(n)$ can only be represented as an integral which, as far as the writer knows, cannot be evaluated in terms of known functions.

We evaluate $Q_2(n)$ by a similar, though different, method from our previous investigation where we considered centres of circles which were displaced small distances from each other. Now consider a small area $dxdy$ at A (figure 5 *a*). We find the probability that one point falls inside this area $dxdy$ and a second point P falls in such a position that it forms a doublet with A, with the proviso that P must be *lower* than A, that is, the y coordinate of P must be lower than that of A. Take a similar small area $dxdy$ at A′ adjoining the first one at A, displaced from A by a small distance dx parallel to the x axis. Again calculate the probability of a doublet being formed with one point at A and a second point below it.

If we repeat this process until we reach the right-hand boundary, then start again at the left-hand boundary (each area $dxdy$ now being directly above the previous row), eventually the whole area will have been covered and the sum of the probabilities over each of the areas $dxdy$ will give $Q_k(n)$.

To a first approximation the chance that one point out of n falls in $dxdy$ at A, a second inside a semicircle of radius 2R centre A (obviously the necessary condition for a doublet to be formed) and the rest elsewhere is

$$n\,dxdy(n-1)2\pi R^2=n(n-1)2\sigma\,dxdy. \quad . \quad . \quad . \quad . \quad (5.1)$$

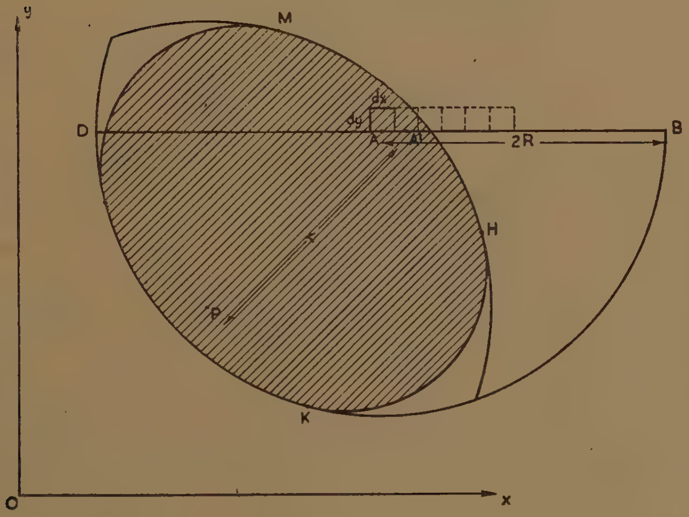
Summing this over the whole area we obtain

$$n(n-1)2\sigma=2^2\,{}^nC_2\sigma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.2)$$

agreeing with our formula (4.9).

$$n\,dxdy(n-1)\pi r dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.3)$$

Fig. 5 (a).



Now we shall consider the problem more exactly. Suppose the second point P is at a distance between r and $r+dr$ from A (see figure 5 (a)). The probability of two points forming a doublet at A and P is

Now all other points must be excluded from the area LMHK (shaded in figure 5 (a)). Otherwise the points at A and P will form a triplet or higher aggregate. Thus the probability that two points fall at A and P where P lies between r and $r+dr$ from A and that all other points fall outside the area LMHK is

$$n(n-1)dxdy\int_{r=0}^{2R}\pi r[1-R^2(6\phi+\pi-\sin 2\phi)]^{n-2}\,dr, \quad . \quad . \quad (5.4)$$

where $\cos\phi=\frac{r}{2R}$ and the total area in which the points may be is of unit magnitude. Integrating over all the small areas *dxdy* and writing $2\phi=\gamma$ we obtain as the exact formula, apart from end effects,

$$Q_2(n)=n(n-1)\sigma\int_0^{\pi}\sin\gamma[1-R^2(3\gamma+\pi-\sin\gamma)]^{n-2}\,d\gamma. \quad . \quad . \quad . \quad . \qquad (5.4)$$

$$=n(n-1)\sigma\int_0^{\pi}\sin\gamma e^{-(n-2)R^2(3\gamma+\pi-\sin\gamma)}\,d\gamma \text{ if } n \text{ is large.} \quad . \qquad (5.5)$$

$$\doteqdot n(n-1)\sigma\left[2-\frac{(n-2)9\sigma}{2}+\text{terms in }\sigma^2\ldots.\right]. \quad . \quad . \quad . \quad . \qquad (5.6)$$

The integral (5.5) does not seem to be evaluable in terms of elementary functions. Similar exact formulæ could be worked out for $Q_3(n)$, $Q_4(n)$, etc., but would involve double and triple and higher integrals and it would be extremely laborious to evaluate them.

XCI. *Relativistic Hamiltonian System of a Particle and Relativistic Heisenberg's Equation.*

By K. H. Tzou, Chekiang University, Meitan, Kweichow, China *.

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Abstract.

In this paper, as suggested by the classical Hamiltonian equations, a new set of corresponding relativistic equations is set up. Therefrom a relativistic form of Heisenberg's equation is obtained. The relativistic Hamiltonian system of a particle according to the formulation here established has been fully discussed and by following it, however, Dirac's equation appears only naturally as a necessary form of relativistic wave equation for electron in quantum mechanics. The process of taking square root in Dirac's theory is seen to have its classical analogy. Finally, some applications of the relativistic Heisenberg's equation to Dirac's theory have been discussed and it has thereby been pointed out that this equation brings some quantities to being more symmetrical in the relativity sense and also some more general than those the non-relativistic equation can give.

1. *The Relativistic Hamiltonian System of a Particle.*

It is well known that, in the special theory of relativity, the motion of a particle can be described by the Hamiltonian equations

$$\frac{dx_i}{dt}=\frac{\partial H}{\partial p_i},\quad \frac{dp_i}{dt}=-\frac{\partial H}{\partial x_i},\quad (i=1,2,3), \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

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with the Hamiltonian function

$$\mathrm{H}=c\sqrt{m_0^2c^2+p_1^2+{}_2^2+p_3^2}-m_0c^2+\mathrm{W}(x_1,x_2,x_3), \quad . \quad . \quad . \quad (2)$$

if the particle moves in a field of force with potential energy function $\mathrm{W}(x_1, x_2, x_3)$.

In equations (1) the quantity H is generally understood as the total energy of the conservative system (the particle in a conservative field) and thus constitutes only the fourth component of a 4-vector, namely, the energy-momentum vector according to the special theory of relativity, so also does the time t (in the form ict) occurring in these equations. Therefore the equations (1) are not themselves relativistically invariant. They are invariant only in the classical mechanics. Moreover, with equations (1) and (2), the relation corresponding to the law of conservation of energy cannot be obtained. With these relativistical asymmetries we are led to a more symmetrical relativistical form of the Hamiltonian equations analogous to equations (1), *i. e.*,

$$\frac{dx_\mu}{d\tau}=\frac{\partial\mathscr{H}}{\partial p_\mu}, \quad \frac{dp_\mu}{d\tau}=-\frac{\partial\mathscr{H}}{\partial x_\mu}, \quad (\mu=1,2,3,4), \quad . \quad . \quad . \quad (3)$$

where $d\tau$ (τ being called the proper time) is defined by

$$c^2d\tau^2=-\sum_1^4 dx_\mu^2, \quad x_\mu=(x, y, z, ict),$$

and $\mathscr{H}$ is expected to be a relativistic scalar.

In the case of a particle, however, as will soon be shown, we have

$$\mathscr{H}=-c\sqrt{-\sum_1^4 p_\mu^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

if the particle is free, where p_μ is, according to the special theory of relativity, the energy-momentum-vector $(p_x, p_y, p_z, i\mathrm{W}/c)$; and

$$\mathscr{H}=-c\sqrt{-\sum_1^4\left(p_\mu-\frac{i}{c}\mathrm{V}\delta_{\mu4}\right)^2}, \quad \delta_{\mu4}=\begin{cases}0 & \mu\neq4\\ 1 & \mu=4\end{cases}, \quad i=\sqrt{-1}, \quad . \quad (5)$$

when it moves in a scalar field of force with potential energy function $\mathrm{V}(x_1, x_2, x_3, x_4)$, such as an electrostatic or a gravitational field ; and

$$\mathscr{H}=-c\sqrt{-\sum_1^4\left(p_\mu-\frac{e}{c}\mathrm{A}_\mu-\frac{i}{c}\mathrm{V}\delta_{\mu4}\right)^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

if it is charged (of charge e) and moves in an electromagnetic field defined by the 4-potential $\mathrm{A}_\mu=(\mathrm{A}_x, \mathrm{A}_y, \mathrm{A}_z, i\phi)$ and another scalar field, such as a gravitational field, with potential energy function $\mathrm{V}(x_1, x_2, x_3, x_4)$.

(a) *General constancy of* $\mathscr{H}$:

$\mathscr{H}$ is in general a function of x_μ and p_μ, thus, by equations (3),

$$\frac{d\mathscr{H}}{d\tau}=\frac{\partial\mathscr{H}}{\partial x_\mu}\frac{dx_\mu}{d\tau}+\frac{\partial\mathscr{H}}{\partial p_\mu}\frac{dp_\mu}{d\tau}=\frac{\partial\mathscr{H}}{\partial x_\mu}\frac{\partial\mathscr{H}}{\partial p_\mu}-\frac{\partial\mathscr{H}}{\partial p_\mu}\frac{\partial\mathscr{H}}{\partial x_\mu}=0,$$

or
$$\frac{\partial \mathscr{H}}{dt}=0.$$

Hence $\mathscr{H}$ is a constant of motion with respect to every system, and since $\mathscr{H}$ is a scalar, we have
$$\mathscr{H}=\text{constant},$$
this constant being independent of coordinate systems.

Therefore, if a system can be associated with such a function $\mathscr{H}$ and its motion described by equations (3), the function $\mathscr{H}$ must be a constant independent of coordinate systems.

In the case of a particle, we may put
$$\mathscr{H}=-m_0c^2,$$
completely irrespective of the condition of the particle, where m_0, as it will be shown later, may be understood as the rest of the particle, and c is the light speed in vacuum.

(b) *The equations of motion of a particle (proof of correctness of the above assumed Hamiltonian system)*:

Let us take the most general case given by (6) and discuss it as follows:

At first, we have
$$\mathscr{H}=-c\sqrt{-\sum_1^4\left(p_\mu-\frac{e}{c}\mathrm{A}_\mu-\frac{i}{c}\mathrm{V}\delta_{\mu4}\right)^2}=-m_0c^2, \quad . \quad . \quad . \quad (7)$$
as has already been discussed in (*a*); and from equations (3),
$$\frac{dx_\mu}{d\tau}=\frac{\partial\mathscr{H}}{\partial p_\mu}=\frac{1}{m_0}\left(p_\mu-\frac{e}{c}\mathrm{A}_\mu-\frac{i}{c}\mathrm{V}\delta_{\mu4}\right)$$
or
$$p_\mu=m_0\frac{dx_\mu}{d\tau}+\frac{e}{c}\mathrm{A}_\mu+\frac{i}{c}\mathrm{V}\delta_{\mu4}, \quad (\mu=1, 2, 3, 4), \quad . \quad . \quad . \quad (8)$$
and
$$\frac{dp_\mu}{d\tau}=-\frac{\partial\mathscr{H}}{\partial x_\mu}=\frac{e}{c}\frac{\partial\mathrm{A}_\nu}{\partial x_\mu}\frac{dx_\mu}{d\tau}+\frac{i}{c}\frac{\partial\mathrm{V}}{\partial x_\mu}\frac{dx_\nu}{d\tau}\delta_{\mu4}, \quad (\mu=1, 2, 3, 4),$$
combining this with (8), we have
$$\frac{d}{dt}\left[\frac{m_0\dfrac{dx_\mu}{dt}}{\sqrt{1-\beta^2}}\right]=\frac{e}{c}\left(\frac{\partial\mathrm{A}_\nu}{\partial x_\mu}-\frac{\partial\mathrm{A}_\mu}{dx_\nu}\right)\frac{dx_\nu}{dt}+\frac{i}{c}\left(\frac{\partial\mathrm{V}}{\partial x_\mu}\delta_{\nu4}-\frac{\partial\mathrm{V}}{\partial x_\nu}\delta_{\mu4}\right)\frac{dx_\nu}{dt}, \quad (\mu=1, 2, 3, 4).$$
$$. \quad . \quad . \quad . \quad . \quad (9)$$

Putting
$$\mathrm{F}_{\mu\nu}=\frac{\partial\mathrm{A}_\nu}{\partial x_\mu}-\frac{\partial\mathrm{A}_\mu}{\partial x_\nu},$$
we have
$$(\mathrm{F}_{23}, \mathrm{F}_{31}, \mathrm{F}_{12})=(\mathrm{H}_x, \mathrm{H}_y, \mathrm{H}_z),$$
$$(\mathrm{F}_{14}, \mathrm{F}_{24}, \mathrm{F}_{34})=(i\mathrm{E}_x, i\mathrm{E}_y, i\mathrm{E}_z),$$

then equations (9) become

$$\frac{d}{dt}\left[\frac{m_0\dfrac{dx_i}{dt}}{\sqrt{1-\beta^2}}\right]=e\left[\vec{E}+\frac{1}{c}(\vec{v}\times\vec{H})\right]_i-\frac{\partial V}{\partial x_i}, \quad (i=1,2,3) \quad . \quad (10)$$

and

$$\frac{d}{dt}\left(\frac{(m_0c^2}{\sqrt{1-\beta^2}}\right)=e\vec{v}\cdot\vec{E}-\vec{v}\cdot\text{grad } V \quad . \quad . \quad . \quad . \quad . \quad (11)$$

for $\mu=4$.

Here equations (10) represent the relativistic equations of motion of a particle in an electromagnetic field and another scalar field, *e.g.* a gravitational field, and equation (11) means, however, the law of conservation of energy.

m_0 is now obviously the rest mass of the particle.

(c) *The Einstein energy-momentum relation* :

The equation (7) gives, when squared out,

$$m_0{}^2c^2=-\sum_1^4\left(p_\mu-\frac{e}{c}A_\mu-\frac{i}{c}V\delta_{\mu4}\right)^2,$$

or, explicitly,

$$\frac{(W-e\phi-V)^2}{c^2}=\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2+m_0{}^2c^2, \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and, for a free particle,

$$\frac{W^2}{c^2}=p^2+m_0{}^2c^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Equations (12) and (13) are here Einstein's relativistic energy-momentum relations for a particle at different conditions, p_μ being given by

$$p_\mu=\frac{m_0}{\sqrt{1-\beta^2}}\frac{dx_\mu}{dt}+\frac{e}{c}A_\mu+\frac{i}{c}V\delta_{\mu4}, \quad (\mu=1,2,3,4), \quad . \quad . \quad (14)$$

and, for a free particle,

$$p_\mu=\frac{m_0}{\sqrt{1-\beta^2}}\frac{dx_\mu}{dt}, \quad (\mu=1,2,3,4). \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The corresponding Lagrangian variational principle is given by

$$\delta\int\mathscr{L}d\tau=0,$$

where

$$\mathscr{L}=p_\mu\frac{dx_\mu}{d\tau}-\mathscr{H}$$

$$=m_0c\sqrt{-\sum_1^4\left(\frac{dx_\mu}{d\tau}\right)^2}+m_0\sum_1^4\left(\frac{dx_\mu}{d\tau}\right)^2+\frac{1}{c}\sum_1^4(eA_\mu-iV\delta_{\mu4})\frac{dx_\mu}{d\tau},$$

with the condition

$$\sqrt{-\sum_1^4\left(\frac{dx_\mu}{d\tau}\right)^2}=c,$$

which is actually an identity.

2. *Dirac's Equation as Naturally a Necessary Form of Relativistic Wave Equation for Electron from the above Hamiltonian System.*

For the motion of an electron in an electromagnetic field and another scalar field, as classically described by equations (3) and (6), the wave equation is natuarlly of the form

$$\mathscr{H}\psi=-c\sqrt{-\sum_1^4\left(p_\mu-\frac{e}{c}A_\mu-\frac{i}{c}V\delta_{\mu4}\right)^2}\,\psi,$$

or, by equation (7),

$$\sqrt{-\sum_1^4\left(p_\mu-\frac{e}{c}A_\mu-\frac{i}{c}V\delta_{\mu4}\right)^2}\,\psi=m_0c\psi. \qquad . \quad . \quad . \quad . \quad (15)$$

It may here be quite reasonable to carry out the square root by putting

$$\sqrt{-\sum_1^4\left(p_\mu-\frac{e}{c}A_\mu-\frac{i}{c}V\delta_{\mu4}\right)^2}=i\sum_1^4\beta_\mu\left(p_\mu-\frac{e}{c}A_\mu-\frac{i}{c}V\delta_{\mu4}\right),$$

where $$\beta_\mu\beta_\nu+\beta_\nu\beta_\mu=2\delta_{\mu\nu}, \quad (\mu,\nu=1,2,3,4), \qquad . \quad . \quad . \quad (16)$$

Thus equation (15) reduces to

$$i-\sum_1^4\beta_\mu\left(p_\mu-\frac{e}{c}A_\mu-\frac{i}{c}V\delta_{\mu4}\right)\psi=m_0c\,\psi$$

or

$$\left\{\sum_1^4\beta_\mu\left(p_\mu-\frac{e}{c}A_\mu-\frac{i}{c}V\delta_{\mu4}\right)+im_0c\right\}\psi=0, \qquad . \quad . \quad . \quad . \quad (17)$$

and, for a free particle,

$$\left(\sum_1^4\beta_\mu p_\mu+im_0c\right)\psi=0. \qquad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Equations (17) and (18) are Dirac's equations for the electron, with the ordinary quantum mechanical substitions,

$$p_\mu=\frac{\hbar}{i}\frac{\partial}{\partial x_\mu} \qquad (\mu=1,2,3,4).$$

In its explicit form, equation (18) reads

$$\sum_1^4\beta_\mu\frac{\partial\psi}{\partial x_\mu}+\frac{m_0c}{\hbar}\psi=0, \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

with its conjugate equation (1),

$$\sum_1^4\frac{\partial\phi}{\partial x_\mu}\beta_\mu-\frac{m_0c}{\hbar}\phi=0, \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where $$\phi=-\psi^+\beta_4. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The operator $\mathscr{H}=-ic\sum_1^4\beta_\mu\left(p_\mu-\frac{e}{c}A_\mu-\frac{i}{c}V\delta_{\mu4}\right)$ is obviously the rest-mass operator, which has arbitrary eigenvalue in flat space-time according to the special theory of relativity. The problem of its quantization with reference to various curved space-time has been discussed by Podolsky and Branson (2).

3. *The Relativistic Heisenberg's Equation.*

It is generally true, that, in non-commutative matrix algebra, if f is a function of x_l and p_l,

$$\left.\begin{aligned}\frac{i}{\hbar}(x_lf-fx_l)&=-\frac{\partial f}{\partial p},\\ \frac{i}{\hbar}(p_lf-fp_l)&=\frac{\partial f}{\partial x_l},\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (22)$$

For classical Hamiltonian mechanics, equations (1) are valid (with $l=1, 2, 3$ in the case of a particle); thus, by comparing them with equations (22), we have

$$\left.\begin{aligned}\frac{dx_l}{dt}&=\frac{i}{\hbar}(\mathrm{H}x_l-x_l\mathrm{H})=[\mathrm{H}, x_l],\\ \frac{dp_l}{dt}&=\frac{i}{\hbar}(\mathrm{H}p_l-p_l\mathrm{H})=[\mathrm{H}, p_l].\end{aligned}\right\} \quad \ldots\ldots \quad (23)$$

It may easily be proved, that, if P is any dynamic variable and is a function of x_l and $p_l(l=1, 2, 3$ in the case of a particle), but not of time t explicitly,

$$\frac{d\mathrm{P}}{dt}=[\mathrm{H}, \mathrm{P}]. \quad \ldots\ldots\ldots \quad (24)$$

This is the ordinary non-relativistic Heisenberg's equation in quantum mechanics.

In the case of a relativistic dynamical system, however, equations (3) are in general valid, thus, by comparing them with equations (22), we have

$$\frac{dx_\mu}{d\tau}=\frac{i}{\hbar}(\mathscr{H}x_\mu-x_\mu\mathscr{H})=[\mathscr{H}, x_\mu],$$

$$\frac{dp_\mu}{d\tau}=\frac{i}{\hbar}(\mathscr{H}p_\mu-p_\mu\mathscr{H})=[\mathscr{H}, p_\mu],$$

($\mu=1, 2, 3, 4$ in the case of a particle), and, similarly, if Q is a dynamic variable of the system considered and is a function of x_μ and p_μ,

$$d\mathrm{Q}/d\tau=[\mathscr{H}, \mathrm{Q}], \quad \ldots\ldots\ldots \quad (25)$$

which is the relativistic form of Heisenberg's equation.

It is obvious, by this equation, that the relativistic scalar $\mathscr{H}$ is a constant of motion of the dynamic system considered also in the sense of quantum mechanics.

4. *Some Applications of the Relativistic Heisenberg's Equation to Dirac's Theory of Electron.*

Up to the present, Dirac's equation for electron serves as the only relativistic wave equation in quantum mechanics. It would thus be expected that some applications of the relativistic Heisenberg's equation may only be found in Dirac's theory. It will be seen in the following discussion that this equation will bring some quantities to being more symmetrical in the sense of relativity and also some more general than those the original non-relativistic equation can give.

(a) *Velocity-vector operator* :

For an electron moving in an electromagnetic field and another scalar field,

$$\mathscr{H}=-c\sqrt{-\sum_{1}^{4}\left(p_\mu-\frac{e}{c}\mathrm{A}_\mu-\frac{i}{c}\mathrm{V}\delta_{\mu 4}\right)^2},$$

and in Dirac's theory,

$$\mathscr{H}=-ic\sum_{1}^{4}\beta_\mu\left(p_\mu-\frac{e}{c}\mathrm{A}_\mu-\frac{i}{c}\mathrm{V}\delta_{\mu 4}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Thus, by equation (25)

$$\frac{dx_\nu}{d\tau}=[\mathscr{H}, x_\nu]=-ic\beta_\nu, \quad (\nu=1, 2, 3, 4). \quad . \quad . \quad . \quad . \quad (27)$$

The average of it is given by

$$\phi\frac{dx_\nu}{d\tau}\psi=-ic\phi\beta_\nu\psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

which is identical with that obtained directly from equations (19) and (20).

By considering the relativistic transformation properties of Dirac's equation, both (27) and (28) are 4-vectors, since β_ν and $\phi\psi$ transform respectively as a 4-vector and a scalar (3).

Equation (27) may also be written

$$\frac{1}{\sqrt{1-\beta^2}}\frac{dx_\nu}{dt}=-ic\beta_\nu,$$

the fourth component of which gives

$$\frac{ic}{\sqrt{1-\beta^2}}=-ic\beta_4,$$

or

$$\beta_4=-\frac{1}{\sqrt{1-\beta^2}}.$$

Thus

$$\frac{dx_k}{dt}=ic\beta_4\beta_k=-c\alpha_k, \quad (k=1, 2, 3),$$

where $\alpha_k=i\beta_k\beta_4$. This is just the velocity operator obtained directly from the non-relativistic Heisenberg's equation.

(b) *Einstein's mass-energy-relation* :

The 4-velocity in Dirac's theory has already been discussed and is given by equation (28), *i. e.*

$$\mathrm{U}'_{av}=\left(\frac{dx}{d\tau}\right)_{av}=-ic\phi\beta_1\psi=-c\psi^+\alpha_1\psi, \text{ etc.,}$$

$$\mathrm{U}^4_{av}=ic\left(\frac{dt}{d\tau}\right)_{av}=-ic\phi\beta_4\psi=ic\psi^+\psi,$$

we obtain therefrom the three-dimensional velocity components,

$$V_{x_{av}}=\left(\frac{dx}{dt}\right)_{av}=\left(\frac{dx}{d\tau}\right)_{av}\Big/\left(\frac{dt}{d\tau}\right)_{av}=-c\,\frac{\psi^{+}\alpha_1\psi}{\psi^{+}\psi}\text{, etc.}$$

These relations are identical with those obtained ordinarily⁽⁴⁾.

For a free electron, we have

$$V_{x_{av}}^{\pm}=c^2p_x/E_{\pm}\text{, etc.,}$$

where p_x, p_y and p_z are the momentum components of the particle, and $E_{\pm}$ its energies corresponding to positive and negative energy states. From these relations we have, as it is quite obvious⁽⁵⁾,

$$E_{\pm}=\pm\frac{m_0c^2}{\sqrt{1-\frac{1}{c^2}V_{av}^2}},$$

or

$$M_{\pm}=\pm\frac{m_0}{\sqrt{1-\frac{1}{c^2}V_{av}^2}},$$

where

$$V_{av}^2=(V_{av}^{+})^2=(V_{av}^{-})^2.$$

(c) *The motion of a free election. Oscillatory motion:*

The velocity operator has been already obtained, and is given by

$$dx_\nu/d\tau=-ic\beta_\nu,\quad(\nu=1,2,3,4).$$

Thus, the acceleration operator follows as

$$\frac{d^2x_\nu}{d\tau^2}=-ic\frac{d\beta_\nu}{d\tau}.$$

Now,

$$i\hbar\frac{d\beta_\nu}{d\tau}=\beta_\nu\mathscr{H}-\mathscr{H}\beta_\nu,$$

and

$$\beta_\nu\mathscr{H}+\mathscr{H}\beta_\nu=-2icp_\nu,$$

hence

$$i\hbar\frac{d\beta_\nu}{d\tau}=2\beta_\nu\mathscr{H}+2icp_\nu,\quad(\nu=1,2,3,4).\quad\ldots\ldots\quad(29)$$

By a similar calculation, we have

$$i\hbar\frac{d^2\beta_\nu}{d\tau^2}=2\frac{d\beta_\nu}{d\tau}\mathscr{H},$$

hence

$$\frac{d\beta_\nu}{d\tau}=\left(\frac{d\beta_\nu}{d\tau}\right)_0\exp\left(-\frac{4\pi i}{h}\mathscr{H}\tau\right).\quad\ldots\ldots\quad(30)$$

Substituting (30) into (29), we have

$$\beta_\nu = -icp_\nu \mathscr{H}^{-1} + \frac{\hbar i}{2}\left(\frac{d\beta_\nu}{d\tau}\right)_0 \exp\left(-\frac{2i}{\hbar}\mathscr{H}\tau\right)\mathscr{H}^{-1},$$

hence

$$\frac{dx_\nu}{d\tau} = -c^2 p_\nu \mathscr{H}^{-1} + \frac{\hbar c}{2}\left(\frac{d\beta_\nu}{d\tau}\right)_0 \exp\left(-\frac{2i}{\hbar}\mathscr{H}\tau\right)\mathscr{H}^{-1}. \quad . \quad . \quad . \quad . \quad (31)$$

The first term on the right of equation (31) means the ordinarily observed velocity, since

$$-c^2 p_\nu \mathscr{H}^{-1} = -c^2 p_\nu \frac{1}{-m_0c^2} = \frac{p_\nu}{m_0},$$

while the second term is called the oscillatory motion owing to the occurrence of the exponential factor. The frequency of oscillation is given by

$$V_z = \left|\frac{2\mathscr{H}}{h}\sqrt{1-\beta^2}\right| = \frac{2m_0c^2}{h}\sqrt{1-\beta^2},$$

hence

$$2m_0c^2 = \frac{hV_z}{\sqrt{1-\beta^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

the energy difference between positive and negative energy levels $(2m_0c^2)$ may therefore be measured by the frequency of the oscillatory motion according to the relation (32).

The results obtained with the help of the non-relativistic Heisenberg's equation have some deviations from those discussed here (6). But no judgment can be made here, since the motion is non-observable.

(d) *Law of conservation of total angular momentum. The spin tensor.*

For electrons moving in a four-dimensional central field defined by

$$A_1 = A_2 = A_3 = 0, \quad A_4 = A_4(s), \quad \left(ds^2 = \sum_1^4 dx_\mu^2\right),$$

the relativistic Hamiltonian function is

$$\mathscr{H} = ic\beta_\mu p_\mu - ie\beta_4 A_4(s). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Now defining

$$m_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \quad (\mu, \nu = 1, 2, 3, 4). \quad . \quad . \quad . \quad . \quad (34)$$

we have, since by a very simple calculation $A_4(s)$ commutes with both x and p_μ,

$$\frac{dm_{\mu\nu}}{d\tau} = [\mathscr{H}, m_{\mu\nu}] = -ic(\beta_\mu p_\nu - \beta_\nu p_\mu). \quad . \quad . \quad . \quad . \quad (35)$$

The three spatial components of the tensor (*i. e.* m_{23}, m_{31}, m_{12}) are generally understood as the components of orbital angular momentum of the electron . The physical meaning of the three time components (m_{14}, m_{24}, m_{34}) has not yet been investigated.

By equation (35), the orbital angular momentum is not a constant of motion, we can thus expect that the total angular momentum should be composed of, apart from the orbital one, also another component.

Let us come to the tensor $\beta_\mu\beta_\nu$. The time variation of it is given by

$$\frac{d}{d\tau}(\beta_\mu\beta_\nu)=[\mathscr{H}, \beta_\mu\beta_\nu]=-\frac{2c}{\hbar}(\beta_\mu p_\nu-\beta_\nu p_\mu), \quad . \quad . \quad . \quad . \quad (36)$$

hence, by combining equations (35) and (36),

$$\frac{d}{d\tau}\left(m_{\mu\nu}-\frac{\hbar}{2}i\beta_\mu\beta_\nu\right)=0,$$

or

$$\frac{d}{dt}\left(m_{\mu\nu}-\frac{\hbar}{2}i\beta_\mu\beta_\nu\right)=0, \quad (\mu, \nu=1, 2, 3, 4). \quad . \quad . \quad . \quad (37)$$

Thus, the tensor $m_{\mu\nu}-\hbar i\beta_\mu\beta_\nu/2$ is a constant of motion, and is called the total angular momentum of the electron. We denote it as usually by $J_{\mu\nu}$:

$$J_{\mu\nu}=m_{\mu\nu}-\frac{\hbar}{2}i\beta_\mu\beta_\nu,$$

hence $J_{\mu\nu}$ is conserved or is a constant of motion.

The term $-\frac{\hbar}{2}i\beta_\mu\beta_\nu$ occuring in equation (38) is generally called the spin angular momentum of the electron and denoted by $S_{\mu\nu}$:

$$S_{\mu\nu}=-\frac{\hbar}{2}i\beta_\mu\beta_\nu,$$

The three time components of it (S_{14}, S_{24}, S_{34}) have not, however, been investigated. They must obviously behave physically like those of the orbital angular momentum (m_{14}, m_{24}, m_{34}).

By means of the ordinary non-relativistic Heisenberg's equation, only the three spatial component of orbital and spin angular moments are exhibited. This is obviously not symmetrical in the sense of relativity⁽⁷⁾.

This law of conservation of total angular momentum is true, of course, also in the case of a free electron.

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XCII. *Constitution of Iodic Acid.*—Part IV. *Rheochor, Parachor and Molecular Refraction.*

By M. R. Nayar, B.A., A.I.I.Sc., and L. N. Srivastava, M.Sc.*

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The abnormal behaviour of iodic acid in aqueous solutions especially compared with its analogues chloric and bromic acids, has been the subject of investigation by several authors since 1874 (1–11). According to some the acid is dibasic, and should be represented by the formula $H_2I_2O_6$, while according to others there is polymerization but no change in valency of the iodate ion. A certain amount of confusion of thought exists regarding polymerization and change in valency. There is general agreement that there is a change in the properties of the acid with change in concentration. A 6N solution seems to consist of polymerized trimeric molecules which break up on dilution, and at very low concentrations, below 0·1N, the solution exhibits the properties of a simple monobasic acid. A systematic study of the various physico-chemical properties like density, viscosity, surface tension, refractive index, conductivity, Raman spectra and magnetic susceptibility attempted in this laboratory has revealed two breaks in the various curves at identical concentrations, viz., at 0·1N and 0·04N. In this paper we report the rheochor values of aqueous iodic acid at different concentrations ranging from 0·01N to 4·0N, and compare them with corresponding values of parachor and molecular refraction (9).

Friend and Hargreaves have determined the rheochor of a number of pure liquids at their boiling points, making use of the modified equation

$$\frac{\mathrm{M.}\,(10^3\times\eta)^{\frac{1}{8}}}{\mathrm{D}+2d}=\mathrm{R}.$$

We have, however, employed the original equation of Friend, viz.

$$\frac{\mathrm{M}\eta^{\frac{1}{8}}}{\mathrm{D}}=\mathrm{R},$$

where R=rheochor, M=molecular weight, η=viscosity and D=density.

Table I. gives the comparative values for rheochor, parachor and molecular refraction and their relation with each other.

One noteworthy feature is the approximate constancy of the values in column (6), Table I., that is, proportionality between rheochor and parachor. These two being more or less additive functions should further be proportional to other additive molecular properties, for example, molecular refraction. The values in column (8) indicate that

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* Communicated by the Authors.

TABLE I.
Temperature 30° C.

(1) Concn. (Normality)*	(2) Density D	(3) Viscosity	(4) Rheochor. R (soln.)	(5) Parachor† P	(6) P/R	(7) Mol. refraction† L	(8) R/L
0·00	1·00000	$8{\cdot}000 \times 10^{-3}$	9·8435	52·324	5·315	3·6899	2·668
0·01	1·00152	8·0473 ,,	9·8513	52·315	5·310	3·69267	2·667
0·02	1·00310	8·0177 ,,	9·8483	52·380	5·319	3·69750	2·664
0·04	1·00610	8·0599 ,,	9·8560	52·306	5·308	3·71010	2·656
0·06	1·00910	8·0795 ,,	9·8599	52·320	5·306	3·70746	2·661
0·08	1·01200	8·1173 ,,	9·8689	52·343	5·304	3·71253	2·659
0·10	1·01515	8·2073 ,,	9·9366	52·890	5·323	3·73804	2·659
0·20	1·02990	8·3631 ,,	9·9160	52·435	5·288	3·74462	2·648
0·40	1·05940	8·7188 ,,	9·9930	52·575	5·261	3·80007	2·629
0·60	1·08950	9·0708 ,,	10·065	52·745	5·241	3·85519	2·610
0·80	1·11815	9·4426 ,,	10·141	52·913	5·230	3·90990	2·592
1·00	1·14720	9·8422 ,,	10·221	53·165	5·201	3·96534	2·577
1·50	1·21950	11·1850 ,,	10·463				
2·00	1·29115	12·5175 ,,	10·704				
4·00	1·57415	23·3075 ,,	11·844				

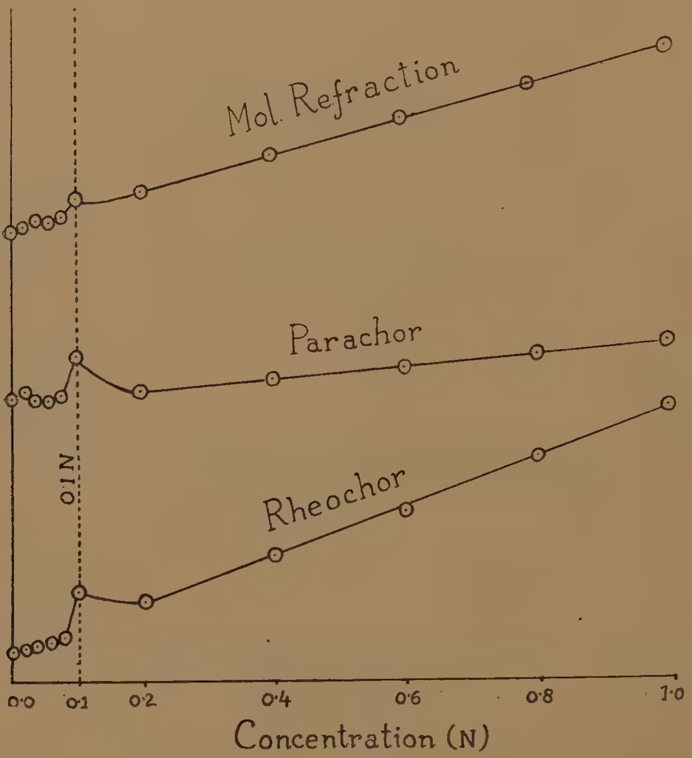
* Refers to volume concentration. For the application of the "Mixture Law" the wt. composition was also determined.

† The values of surface tension and refractive index are given in Part III. of this paper (9).

such indeed is the case, especially with solutions up to concentration 0·1N, after which there is a slow drift.

The practical importance of this proportionality between three such properties is that if we know one we can calculate the other two with the aid of the constants; for example, from rheochor we can calculate parachor and molecular refraction, and hence surface tension and refractive index of the solution at any concentration. This was implicit in the original equation of Friend for rheochor.

Fig. 1.



Regarding the particular liquid under investigation, the various values obtained are represented graphically in fig. 1. The point of interest is the abnormality at concentration 0·1N.

The break in the curve at 0·1N appears to be attributable to a change in the molecular species due to polymerization, and hence may be termed a "transition point." If there are any other similar points they must lie below 0·1N, probably masked by the predominating influence of the solvent. To eliminate the latter the following device, viz., the mixture law equation, was applied:

$$P_{\text{soln.}} = (1-x)\ P_{\text{solvent}} + (x)\ P_{\text{solute}},$$

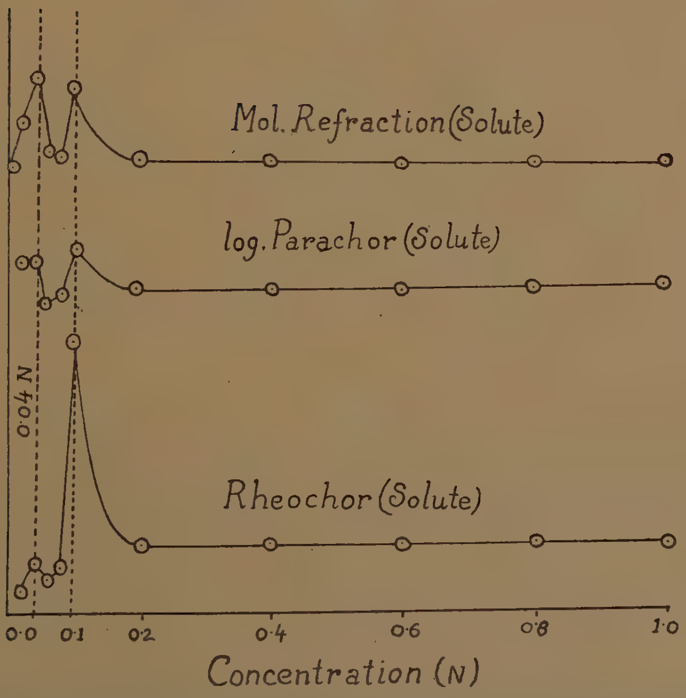
where P is any specific property studied, and (x) and $(1-x)$ are molecular fractions of the solute and solvent respectively. Since the mixture law is not obeyed, the absolute values of $P_{(solute)}$ have little significance ; it is the relative values that are important.

The following table (Table II.) gives the values for rheochor (R), parachor (P) and molecular refraction (L) of the solute corresponding to various concentrations.

Table II.

Concn.	R_{solute}	P_{solute}	L_{solute}
0·01N	53·56	—	18·776
0·02	22·92	204·4	24·296
0·04	27·19	220·8	31·612
0·06	24·85	49·7	19·878
0·08	27·32	64·9	19·251
0·10	61·24	346·6	30·235
0·20	29·83	83·52	18·759
0·40	30·39	86·90	18·840
0·60	30·07	90·78	18·794
0·80	30·21	92·64	19·044
1·00	30·42	98·20	18·654
1·50	32·22	95·32	—
2·00	32·91	96·17	—

Fig. 2.



When these values or their logs are plotted the abnormalities become pronounced, as is evident from the graphs in fig. 2. All the curves indicate one break at 0·1N as observed before, and in addition there is one more at 0·04N, which had been masked in the solution-curves by the preponderating influence of the solvent. This "transition point" has been obtained with a number of other properties studied (9). The same two "transition points" have also been obtained with experiments where there was no necessity for the application of the mixture law equation, *e. g.*, temperature coefficient of conductivity (9). It is suggested that the following reactions take place

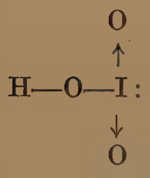
$$\tfrac{1}{3}(\mathrm{HIO_3})_3 \xrightarrow[0.1\mathrm{N}]{} \tfrac{1}{2}(\mathrm{HIO_3})_2 \xrightarrow[0.04\mathrm{N}]{} \mathrm{HIO_3} \rightarrow \mathrm{H^+} + \mathrm{IO_3^-}.$$

Thus a concentrated solution of iodic acid contains mainly trimeric molecules, which change into dimers below 0·1N. It is only below 0·04N that monomeric molecules uncontaminated by polymers exist. But in this region the dilution has become so great that a large proportion of the molecules probably exists in the ionized condition.

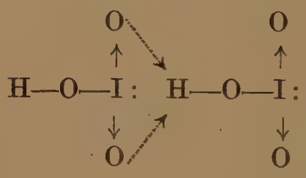
It is possible that other molecular species than trimers exist above 0·1N. The significance of the "transition point" is that trimers do not exist below 0·1N; and similarly no polymers exist below 0·04N. Raman Effect data support these conclusions (11).

Electronic Constitution.

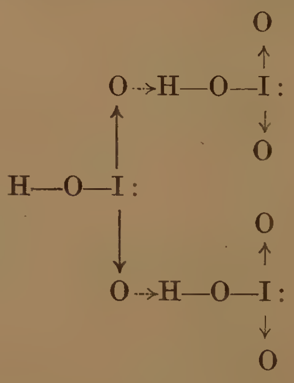
If three types of molecules of iodic acid exist it is interesting to consider their electronic constitution. In an earlier paper straight chain structures were postulated which are evidently not so satisfactory as the following:



I.



II.



III.

In structure II. the H-bond is a bifurcated one (13, 14).

The structural formula II. brings out the difference between the two H-atoms, one of which (covalently linked) is ionizable and hence replaceable by an alkali ion; the other, which forms the H-bridge, is not. The acid $(\mathrm{HIO_3})_2$ is therefore monobasic. When the second H-atom is substituted by an alkali atom the H-bond breaks; the metallic atom is incapable of forming bonds corresponding to H-bonds; hence the polymerized acid molecule breaks up, and we obtain a salt corresponding to the monomeric acid only. In other words, the $(\mathrm{IO_3})^-$ ion does not polymerize, hence the divalent $(\mathrm{I_2O_6})^=$ ion does not exist.

Polymerization of the acid is possible only through the H-bond; in the absence of the latter, as in normal salts, *e. g.*, iodates, there is no polymerization.

Similar remarks hold good for structure III. Even though there are three H-atoms in the molecule $(HIO_3)_3$ only one is ionizable, while the other two form H-bonds. This suggests that we can prepare the salt $KH_2(IO_3)_3$, but not $K_2H(IO_3)_3$, nor $K_3(IO_3)_3$, because either of the latter involves the breaking up of the H-bond.

Formula III. also implies that in the polymer $(HIO_3)_n$, n can have any value up to 3 but not more.

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Chemistry Department,
Lucknow University,
Lucknow, India.

XCIII. *Variation of Rheochor with Temperature.*

By L. N. Srivastava, M.Sc.*

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With a view to determine the extent of variation of rheochor ⁽¹⁾ with temperature, the rheochors have been calculated from existing data for the hydrocarbons, benzene, seven esters, three alcohols, acetic acid and water, from values of viscosity and density as given in Landolt–Bornstein Tabellen.

Description of the Results.

(Tabulated values omitted for economy of space.)

Hydrocarbons (pentane to octane).—The difference between the extreme values of R(rheochor) is about 1 per cent., and the deviation from the mean is not more than 0·5 per cent., very often it is less. On closer inspection, starting from 0° C., R has a tendency to decrease at first with rise in temperature and then to increase, the highest value being near the B.P.

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* Communicated by the Author.

Fatty esters (methyl acetate, methyl propionate, ethyl acetate, propyl formate, methyl butyrate, ethyl propionate and propyl acetate).—The slight variation in the value of R is of the same order as in the case of hydrocarbons, but the magnitude is less.

Benzene.—If the value at 0° C. is omitted the rheochor has a constant value within 0·5 per cent. up to the B.P.

Alcohols (methyl, ethyl and propyl alcohols).—The difference between the extreme values is more than 1 per cent., more than in the case of hydrocarbons. In every case there is a gradual decrease in the value of R with rise in temperature.

Acetic acid.—The value of R from 20° C. to 70° C. follows the same course as in the case of alcohols, that is, decreases ; between 80° and 110° C., R has a constant value. The maximum difference is 0·3 per cent.

Water.—Water shows the largest variation of R with temperature, the value falling from 10·9 at 0° C. to 9·03 at 100° C., making use of the formula $R = \frac{m\eta^{\frac{1}{8}}}{D}$

Even a superficial study of rheochor values at different temperatures brings out the distinction between associated and non-associated or "normal" liquids. In the case of the former, *e.g.*, alcohols, acetic acid and water, rheochor continuously decreases with temperature, while in "normal" liquids it shows a slight tendency first to decrease and then to increase as the B.P. is approached, the magnitude of variation being not more than 1 per cent. in the majority of cases examined. This fact was recognized by J. N. Friend (2) who proposed, therefore, the B.P. of the liquid as the standard temperature for the evaluation of rheochor. Whatever conditions are chosen as the standard, agreement on atomic rheochors can probably never be expected, since constitutional influence is an important factor affecting rheochor as well as parachor (3).

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Department of Chemistry,
Lucknow University,
Lucknow, India.

XCIV. *A Method for producing a Random Sequence of Numbers.*

By J. R. Tillman *, F. H. Reynolds * and P. J. Hilton † ‡.

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1. INTRODUCTION.

KENDALL and Babington Smith (K. and B. S.) discussed Randomness and Random Sampling Numbers in two papers to the Royal Statistical Society (1); they paid most attention to the meaning of the word random, the production of tables of random sampling numbers and tests for determining randomness and local randomness, *i. e.* the randomness of a small set of numbers taken from the tables. Earlier, Tippett had compiled a table of random numbers by taking digits "at random" from census reports (the quotation marks are K. and B. S.'s). K. and B. S., while not approving of Tippett's method, find nothing to criticize in the results obtained; Yule (2), on the other hand, considers Tippett's series to be "patchy."

In K. and B. S.'s method of producing tables, a disk, divided into 10 equal sectors and numbered 0–9 consecutively, was rotated by an electric motor; a neon lamp giving flashes of very short duration was so placed as to illuminate the disk. An observer moved a stylus over a resistor consisting of a close maze; the contacts set up charged a capacitor which eventually discharged through the neon lamp. At each flash the observer noted that number on the apparently stationary disk which appeared opposite a fixed pointer.

It is not an aim of this article to criticize K. and B. S.'s method, even in so far as the human element enters into the method. Their tables (3) to some extent justify the means. But the method has an obvious limitation apart from the need for a trustworthy observer; the recommended rate of production of random numbers is only 25 per minute. The present aim is to remove the human element and much to increase the rate of production.

2. PHYSICAL PHENOMENA IN WHICH RANDOMNESS EXISTS.

In the physical world we distinguish two kinds of macroscopic effects; first, that which is merely the aggregate of numerous effects on the molecular or atomic scale, each substantially as precise as the macroscopic effects (*e. g.* line spectra excepting natural and Doppler widths, mass and most chemical reactions), and, second, that which on the molecular or atomic scale obeys statistical laws (*e. g.* radioactivity, resistance noise and many phenomena of the gaseous state). It is with the second kind that we are concerned here and we confine ourselves to the consideration of radioactivity.

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* P.O. Research Station.

† Queen's College, Oxford.

‡ Communicated by Prof. Sir George Thomson, F.R.S.

Analysis of the statistical laws of radioactivity suggests that the element of randomness exists in the disintegration of individual radioactive atoms—a suggestion which has gained strength because the results of numerous experiments have been compatible with it while no evidence refutes it. Thus no basis exists for predicting the exact time of an individual disintegration; but the chance of n_1 atoms (out of a much larger initial number n_2) of a particular isotope disintegrating in a given time interval can be stated with mathematical accuracy.

3. MATHEMATICAL BASIS OF THE METHOD.

If the probability of an event occurring is p and the probability of it not occurring is $(1-p)$, the probability, $p(k)$, of the event occurring k times in n independent trials is given by $p(k)=\dfrac{p^k(1-p)^{n-k}\underline{|n}}{\underline{|n-k}\,\underline{|k}}$:

If $n\rightarrow\infty$ and $p\rightarrow 0$, but the mean np $(=x)$ remains finite, $p(k)=\dfrac{e^{-x}x^k}{\underline{|k}}$.

(In other words, the binomial distribution degenerates to the Poisson distribution when n is large, p is small and np is finite.)

Now suppose the number of occurrences of the event be divided into the s subsets $s_0, s_1, s_2, \ldots s_k \ldots s_{s-1}$ such that s_k includes all numbers of the form $sa+k$ where a is any positive integer or zero.

Then the probability, p_k, that the number of occurrences of the event falls into subset s_k, is given by $p_k=e^{-x}\left(\dfrac{x^k}{\underline{|k}}+\dfrac{x^{s+k}}{\underline{|s+k}}+\dfrac{x^{2s+k}}{\underline{|2s+k}}+\ldots\right)$ and $p_0=e^{-x}\left(1+\dfrac{x^s}{\underline{|s}}+\dfrac{x^{2s}}{\underline{|2s}}+\ldots\right)=\dfrac{1}{s}e^{-x}\Sigma e^{\lambda x}$, where $\lambda^s=1$, the summation being taken over the s roots of $\lambda^s=1$.

$$\therefore \qquad p_k=e^{-x}\frac{d^{s-k}}{dx^{s-k}}(e^x p_0)=\frac{1}{s}e^{-x}\Sigma\lambda^{-k}e^{\lambda x}.$$

$\therefore\ p_k-\dfrac{1}{s}=\dfrac{1}{s}e^{-x}\Sigma\lambda^{-k}e^{\lambda x}$, where $\lambda^s=1, \lambda\neq 1$. Now $\lambda^s=e^{2m\pi i}$.

$$\therefore \qquad \lambda^{-k}=e^{-\frac{2mk\pi i}{s}}, \quad e^{\lambda x}=e^{x\left(\cos\frac{2m\pi}{s}+i\sin\frac{2m\pi}{s}\right)}.$$

Since p_k is real, we may ignore the imaginary part, giving

$$p_k-\frac{1}{s}=\frac{1}{s}e^{-x}\sum_{m=1}^{m=s-1}e^{x\cos\frac{2m\pi}{s}}\cos\left(x\sin\frac{2m\pi}{s}-\frac{2mk\pi}{s}\right).$$

Now the mth term equals the $(s-m)$th term. $\therefore$ when s is odd,

$$p_k-\frac{1}{s}=\frac{2}{s}e^{-x}\sum_{m=1}^{m=\frac{s-1}{2}} e^{x\cos\frac{2m\pi}{s}}\cos\left(x\sin\frac{2m\pi}{s}-2m\frac{k\pi}{s}\right)$$

while, when s is even,

$$p_k-\frac{1}{s}=\frac{2}{s}e^{-x}\left[\frac{1}{2}(-1)^k e^{-x}+\sum_{m=1}^{m=\frac{s-2}{2}} e^{x\cos\frac{2m\pi}{s}}\cos\left(x\sin\frac{2m\pi}{s}-\frac{2m\pi k}{s}\right)\right],$$

the first term in [ ] being half the $s/2$th term.

The expressions for (p_k-1/s) give the errors involved in assuming the p_k's equal. As a practical example consider $s=10$ and $x=40$. Then

$$p_k-\frac{1}{10}=\frac{1}{5}e^{-x}\left[\sum_{m=1}^{m=4} e^{x\cos\frac{m\pi}{5}}.\cos\left(x\sin\frac{m\pi}{5}-\frac{mk\pi}{5}\right)+\frac{1}{2}(-1)^k e^{-x}\right].$$

$$\therefore\ p_k-\frac{1}{10}=\frac{1}{5}\left\{e^{40\left(\cos\frac{\pi}{5}-1\right)}.\cos\left(40\sin\frac{\pi}{5}-\frac{k\pi}{5}\right)\right.$$

$$\left.+e^{40\left(\cos\frac{2\pi}{5}-1\right)}.\cos\left(40\sin\frac{2\pi}{5}-\frac{2k\pi}{5}\right)\dots\right\}$$

$$=\frac{1}{5}\left\{e^{-7\cdot64}\cos\left(40\sin\frac{\pi}{5}-\frac{k\pi}{5}\right)\right.$$

$$\left.+e^{-27\cdot641}\cos\left(40\sin\frac{2\pi}{5}-\frac{2k\pi}{5}\right)\dots\right\},$$

$|p_k-\frac{1}{10}|$ is therefore $<\frac{1}{5}e^{-7\cdot64}$ and $\left|\dfrac{p_k-\frac{1}{10}}{\frac{1}{10}}\right|<0\cdot001.$

If x is reduced to 20, $\left|\dfrac{p_k-\frac{1}{10}}{\frac{1}{10}}\right|$ is in general greater, but is still $<0\cdot05$.

4. USE OF RADIOACTIVITY.

Radioactivity affords a method of using the results given in (3). The disintegration of an individual atom of, say, radium, occurring in some given time interval, corresponds to the event postulated above. One-tenth of a microgram of radium provides sufficient possible events ($\sim 3\times10^{14}$) to justify the assumption that $n\rightarrow\infty$. During a time interval of 1 second, p is small ($\sim 10^{-11}$); during one-tenth of a second p is proportionately smaller.

In order to obtain a series of digits in the scale of 10 it would be sufficient therefore to count the number of disintegrations occurring in successive intervals of time each of duration 1/75 second (corresponding to $x=40$). If successive counts were 37, 29, 41, 42, 39, . . ., the digits 7, 9, 1, 2, 9, . . . would form the series. Any lack of randomness would be very small—of little or no significance in any resultant set of digits less than 10^7 in number.

Practical difficulties prevent such a straightforward application ; it is impossible to detect every disintegration of radium and it is inconvenient to remove the radon which is produced by the disintegration of radium and whose later disintegration would be confusing. But other applications, giving effectively the same results, can be carried out without difficulties. Thus a small quantity of radium, in equilibrium with its products, is a source of γ rays. The generation of each γ ray can be taken as the occurrence of a discrete event and the rays which are propagated within a small solid angle from a small enclosed source of radium can be detected individually by a Geiger-Muller counter. The closeness with which the G.M. counter and the radium enclosure can be associated determines the maximum solid angle which can be used. Although disintegrations from some of the successive products of radium play a big part and the limitation of solid angle restricts the counted disintegrations to but a small fraction of the total possible, the experiment suffers nothing provided the mean count-per-unit-time-interval is maintained at the value which ensures a sufficiently small value of $(P_k-\frac{1}{s})$, by the use either of a new, longer unit time interval or of more radium.

Vickery $^{(4)}$ has suggested the use of a radioactive substance and a Rutherford-Geiger counter to control the drawing of a random sample from a set of punched cards. He aimed primarily at obtaining a series of timing impulses having a controlled average rate but randomly spaced in time. The series of timing impulses could replace the manual operations which control the times of flashing of the neon lamp in the K. and B. S. method of producing a random series of numbers ; bias due to imperfections of the disk, motor hunting and the operator's sight would remain. Vickery here restricted radioactive substances to emitters of α particles, but clearly emitters of β or γ rays can equally well be used with an appropriate counter.

5. EXPERIMENTAL METHOD.

In the application of the method suggested above, a rapid rate of production of the series has been essential. The equipment used is more elaborate than would be necessary for a very leisurely rate ; it is unnecessary to detail the possible simplifications. The description which follows refers to an equipment producing a sequence of numbers in the scale of four.

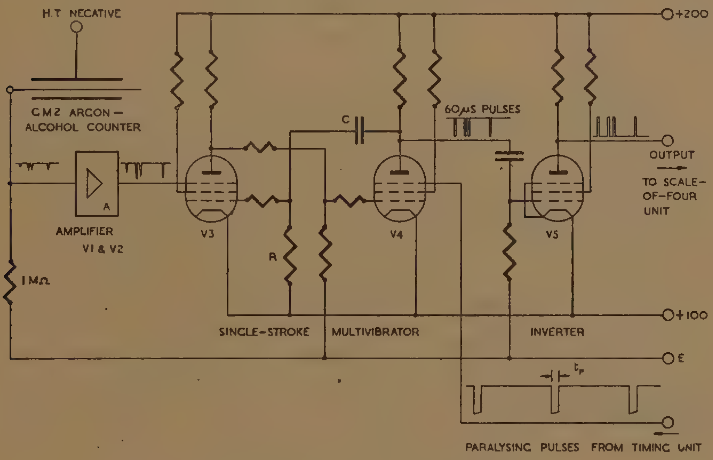
5.1. *Time-random Pulse Generating Circuit.*

The G.M.2, a recently produced argon-alcohol filled Geiger-Muller counter, detects the γ rays emitted by 20 μg of radium enclosed in a small, thick-walled glass, cylindrical capsule. The axes of the counter and capsule are about 1 in. apart. A stabilized h.t. supply incorporating a control of output voltage energizes the G.M.2 : the control enables the counter to be operated within the limits of its plateau. The counter is self-quenching ; a small-valued load resistor is therefore permissible and a short time-constant for the load circuit possible.

The h.t. supply has a negative polarity with respect to earth. Each disintegration therefore produces a small negative voltage ($\sim 0{\cdot}2$ V.) pulse of short duration—across the load resistor—which is linearly amplified by A about 100 times (see fig. 1), but neither the magnitude nor the duration is constant from pulse to pulse. It will be seen later that the equipment which determines the successive values of k must receive pulses of adequate size and constant shape. A single stroke multivibrator, triggered by the amplified pulses, produces negative pulses of a desired constancy of shape and size corresponding therefore to the events utilized in section 3.

The single-stroke multivibrator consists of valves V_3 and V_4, the former being normally conducting and the latter cut off. A short-duration negative pulse applied to the suppressor grid of V_3 cuts off the anode current; the anode potential of V_3 rises, carrying the grid of V_4 into the conducting region. The consequent drop in the anode potential of V_4 cuts off the space current in V_3, thus setting up the alternative condition, *i. e.* V_3 cuts off and V_4 conducting. The potential of the first grid of V_3 is not held down permanently however; it rises at a rate determined by RC, and eventually V_3 draws space current. A sequence of events now restores the initial conditions. A negative rectangular pulse of specific duration (about 60 μsec.) is thus produced at the anode of V_4.

Fig. 1.



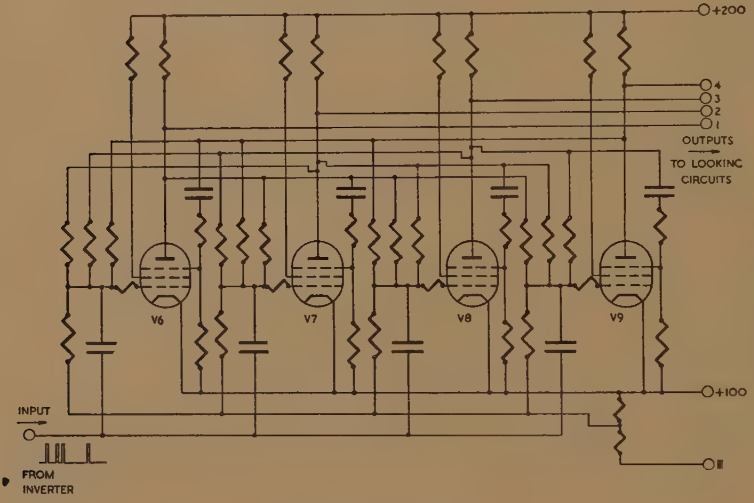
Random-time pulse generator.

The remainder of the equipment can be divided into three parts: the first determines k, the second enables the value of k to be registered, and the third provides the timing pulses.

5.2. *The k-determining Unit.*

The k-determining unit is a ring counter using hard valves only. Because it needs positive pulses to operate it, an inverting stage follows the single-stroke multivibrator. The ring counter, whose circuit is shown in fig. 2, has four stable states and would be described by some writers as a scale-of-four unit ; the type of circuit can be used for scales of higher or smaller numbers. The values of the resistors have been so proportioned that any one of these stable states is maintained in the absence of incoming pulses. In stable state 1, V_6 does not conduct but V_7, V_8 and V_9 do. A positive pulse from the inverting stage applied at the input causes V_6 to conduct and its anode voltage to fall. This fall produces a negative step which is transmitted as a pulse to the second control grid (here the suppressor grid) of V_7, thereby cutting off the anode current of V_7 and causing the anode potential of V_7 to rise rapidly. The first control grids thus receive the following signals in addition to the initial pulse ; those of V_7, V_8 and V_9 a negative step from the anode of V_6 and those of V_6, V_8 and V_9 a positive step from the anode of V_7. As a result stable state 2 is set up, V_6, V_8 and V_9 conducting and V_7 non-conducting.

Fig. 2.



Scale-of-four unit.

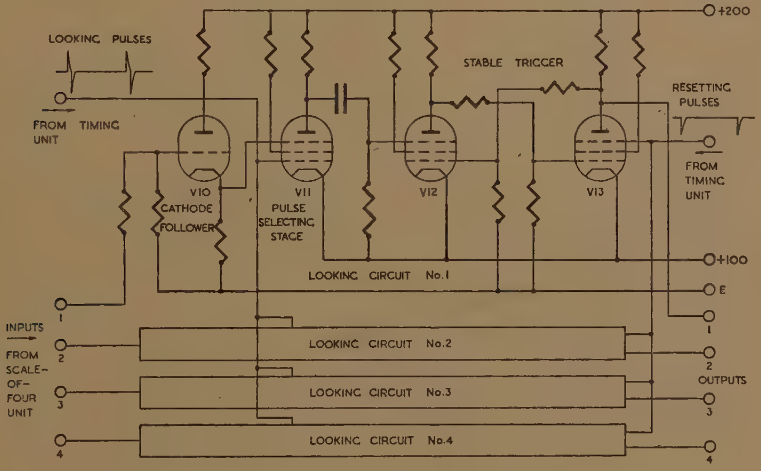
Further input pulses produce, in turn, stable state 3 (V_8 alone non-conducting), stable state 4 (V_9 alone non-conducting), etc. The Chain $V_6V_7V_8V_9$ clearly forms a ring, the passage from stable state 4 to stable state 1 being effected in the same way as any other passage. It will be seen that if $4m+k$ pulses are supplied to the ring counter, where m is any integer, the final steady state will be independent of m.

Even though the component values of the ring counter are given small tolerances the four stages cannot in practice be made identical ; the sensitivity to the incoming pulses will therefore vary from one stable state to the next. For example, the critical amplitude of input pulse to cause a change from stable state 1 to stable state 2 differs slightly from that required to make the change from stable state 2 to stable state 3. If the design permitted the pulses fed to the ring to differ in amplitude over a range including the critical values, bias would be introduced. In the present design, all the pulses are of a constant amplitude, adequate, with a liberal safety factor, to ensure infallible operation of the ring. Constancy of shape of the input pulses ensures that less important irregularities of the ring counter make no contribution to the overall performance.

5.3. *Number (k) Registering Unit.*

The design of four looking circuits which achieve the registration of the value of k, obtaining at the end of each unit time interval of duration t, is based on a pulse selecting circuit and a stable trigger circuit (see fig. 3).

Fig 3.

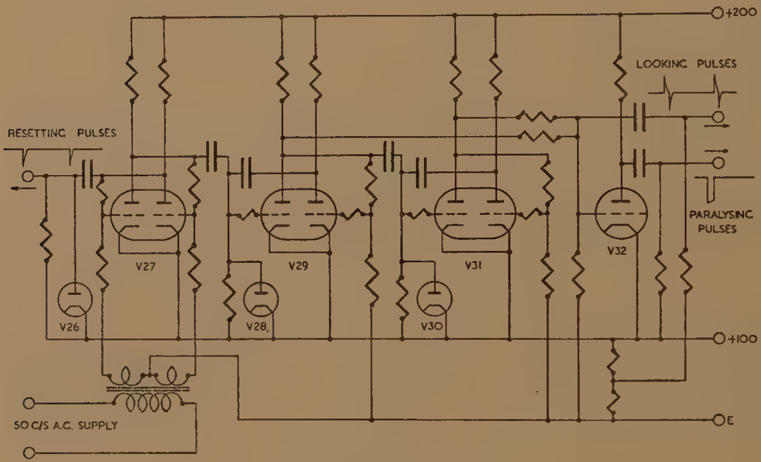

+200
LOOKING PULSES
FROM TIMING UNIT
STABLE TRIGGER
RESETTING PULSES
FROM TIMING UNIT
V10
CATHODE FOLLOWER
V11
PULSE SELECTING STAGE
V12
V13
+100
LOOKING CIRCUIT No.1
E
INPUTS
FROM SCALE-OF-FOUR UNIT
1
2
3
4
LOOKING CIRCUIT No.2
LOOKING CIRCUIT No.3
LOOKING CIRCUIT No.4
OUTPUTS
1
2
3
4

Number registering unit.

Consider looking circuit 1 which is controlled by the anode potential of V_6. In the normal condition V_{11} and V_{13} are cut off and V_{12} is conducting. The component values of the circuit of the cathode follower V_{10}, the valve coupling the anode of V_6 of the ring counter to the looking circuit 1 are so chosen that when the ring counter is in steady state 1 (*i. e.* the anode potential of V_6 is high) the potential of the suppressor grid of V_{11} (the pulse selecting valve) is equal to or just above that of the cathode of V_{11}. If, on the other hand, the ring counter is in any state other than 1, the potential of the suppressor grid of V_{11} is held sufficiently negative to cut off the anode current of V_{11} irrespective of the control grid potential. Looking circuits 2, 3 and 4 are similar to 1 and are controlled respectively by valves V_7, V_8 and V_9 of the ring counter. In the intervals between the reception of pulses (events) by the ring counter, only one of its valves is in the non-conducting state and hence for only one of the pulse-selecting valves is the suppressor grid held at a potential at or above the relevant cathode potential. At the end of each time interval a positive pulse, of duration short compared with t, is applied by the timing unit (see fig. 4) to the first control grid of V_{11} and simultaneously to the control grids of the corresponding valves in the other three looking circuits. Cathode current flows momentarily in each pulse-selecting valve, but in only one valve does the suppressor grid potential permit the flow of anode current, *i. e.* the anode potential of only one pulse-selecting valve is momentarily depressed.

The stable trigger circuit of looking circuit 1 consists of V_{12} and V_{13}, the anodes and first control grids of which are so connected as to permit of two stable conditions only of the circuit. One (condition A) is that in which V_{12} is conducting and V_{13} is cut off. If a negative impulse is produced at the anode of the pulse-selecting valve, it is transmitted to the suppressor grid of V_{12} and stops the flow of anode current in V_{12}. The anode potential of V_{12} consequently rises; V_{13} conducts, and its anode potential falls, driving the control grid of V_{12} below the cut-off point. This sequence of events produces the other condition (B), V_{12} cut off and V_{13} conducting, which is maintained after the disappearance of the negative pulse from the suppressor grid of V_{12}.

Fig. 4.



Timing unit.

At a specific time, t_r ($<t$), after the looking pulse, the timing unit applies a negative pulse (the resetting pulse) to the suppressor grid of V_{13} and to the suppressor grids of the corresponding valves of looking circuits 2, 3 and 4. By a sequence of events, similar to that which changed the stable trigger circuit from condition A to condition B, the circuit returns to condition A. Thus, during the first part of a unit time interval, one (and only one) of the four stable trigger circuits will be in

condition B, and this particular circuit determines the digit (*i. e.* 1, 2, 3 or 4) to be recorded for this unit time interval : *e. g.*, if the trigger circuit in the looking circuit 2 is tripped to condition B, digit 2 is recorded. The resetting pulse restores that circuit which has been triggered to condition B and has no effect on the remaining three circuits already in condition A ; it ensures that the four looking circuits are in condition A for the determination of k at the next looking pulse. The duration of the condition B for each digit (*i. e.* the interval t_r between the looking and resetting pulses) must be sufficient to enable the recording equipment to work satisfactorily. In the present design a choice of approximately $t/2$ for t_r simplified the design of the timing unit, while being sufficiently long to ensure satisfactory operation of a high-speed multi-pen recorder.

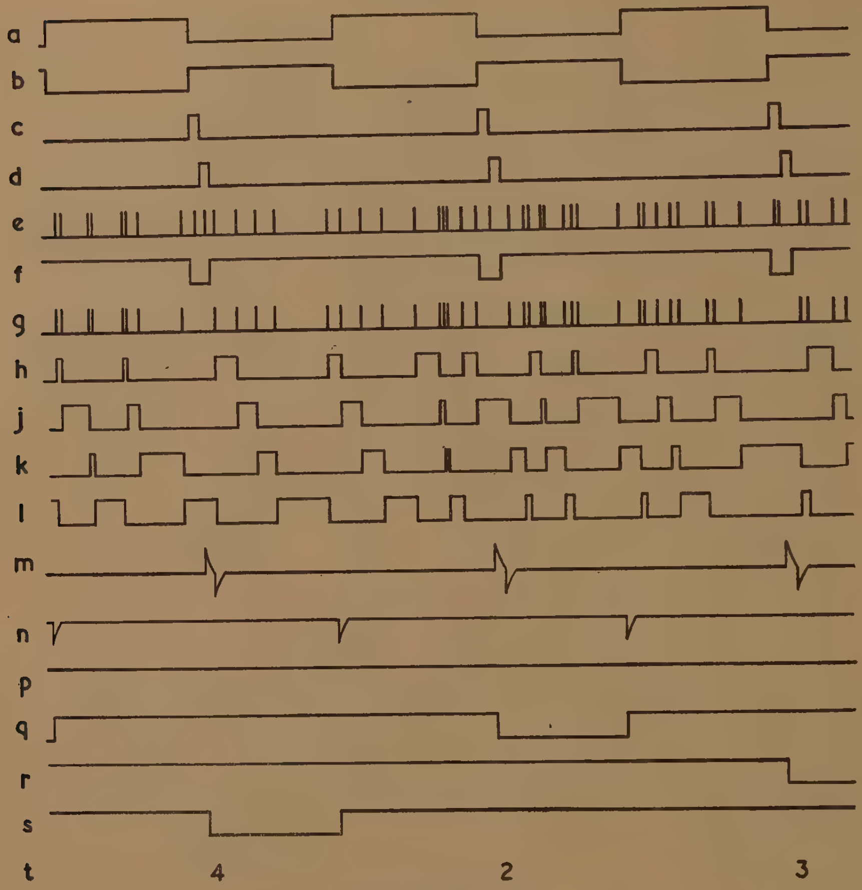
If the ring counter were in the act of changing from one stable state to the next at the instant of the looking pulse, two or possibly none or the looking circuits might respond, producing two digits simultaneously, *e. g.* 1 and 2, or none at all. In order to eliminate this undesirable possibility, the pulses incoming to the ring counter are cut off, by paralysing the single-stroke multivibrator for a time interval t_p ($\ll t$), starting before and finishing after the looking instant (see fig. 5).

5.4. *The Timing Unit.*

The requirements of the timing unit have been indicated in the previous section ; it must provide suitably timed paralysing, looking and resetting pulses. In addition some master timing unit is needed to control the unit time interval. A value of 20 mS. was chosen for t and was controlled by the 50 c/s. mains supply. Longer time intervals, derived from a supply of lower frequency, could be used without major circuit modification. Shorter intervals would be possible provided that (*a*) the necessary mean count per second, x/t, could be obtained with the available radium source, (*b*) that t/x is still great compared with the resolving time of the G.M. counter, (*c*) the speed of operation of the many switching circuits permits and (*d*) suitable recording gear is available.

The 50 c/s. supply drives a squaring circuit (V_{27}) which gives the two voltage outputs shown in fig. 4. The voltage wave-form at the right-hand anode is differentiated and the positive pulses removed by V_{26}. The remaining negative pulses are the resetting pulses. The voltage on the left-hand anode, after differentiation, triggers a single-stroke multivibrator (V_{29}) whose operation is similar to that of the single-stroke multivibrator of section 5.1, and produces a short rectangular positive pulse of a duration, determined largely by the component values of an RC combination, of 0·5 mS. at the L.H. anode of V_{29}. The diode V_{28} removes the positive pulses of the differentiated waveform and ensures that the rectangular pulses are free of overshoots. The rectangular pulse, after differentiation, triggers another single-stroke multivibrator (V_{31}) which is identical with that of V_{29}, producing another rectangular

Fig. 5.



Timing diagram.

Key.

(*a*) Waveform at the L.H. anode of V_{27} (fig. 4).
(*b*) Waveform at the R.H. anode of V_{27} (fig. 4).
(*c*) Waveform at the L.H. anode of V_{29} (fig. 4).
(*d*) Waveform of the L.H. anode of V_{31} (fig. 4).
(*e*) Pulses of Geiger-Muller counter: as at anode of V_3 (fig. 1).
(*f*) Paralysing pulse: as at the output of the timing unit (fig. 4).
(*g*) Pulses of G.M. counter after passing through the paralysed valve, *i.e.* waveform applied to the ring counter: as at the anode of V_5 (fig. 1).
(*h*) Waveform at the anode of V_6 (fig. 2).
(*j*) Waveform at the anode of V_7 (fig. 2).
(*k*) Waveform at the anode of V_8 (fig. 2).
(*l*) Waveform at the anode of V_9 (fig. 2).
(*m*) Looking pulses: as at the output of the timing unit (fig. 4).
(*n*) Resetting pulses: as at the output of the timing unit (fig. 4).
(*p*) Output No. 1 (fig. 3).
(*q*) Output No. 2 (fig. 3).
(*r*) Output No. 3 (fig. 3).
(*s*) Output No. 4 (fig. 3).
(*t*) Output digits.

positive pulse of 0·5 mS. duration at the L.H. anode of V_{31}. This pulse is also differentiated ; its leading edge produces a positive pulse which is used as the looking pulse and its trailing edge produces a negative pulse which is ineffective because the first control grids of the pulse-selecting valves are normally biased below cut-off. The two 0·5 mS. rectangular pulses are thus consecutive. They are applied to the resistive network in the grid circuit of V_{32}, which is so proportioned that V_{32} is normally cut off. During the periods when one or other of the 0·5 mS. positive pulses are applied to the network, the valve conducts. A rectangular negative pulse of 1 mS. duration is therefore produced at the anode of V_{32} ; this pulse is used as the paralysing pulse.

Fig. 5 shows the sequence of timing pulses and illustrates the counting of a typical set of pulses produced by the G.M. counter and the digit recorded.

It will be seen that the ring counter is not restored to steady state 1 after the looking pulse ; instead it is left in the state (say k_1) obtaining at the end of the previous period. Hence at the end of the next unit time interval during which $4m+k_2$ events are recorded, the state reached is k_1+k_2 or k_1+k_2-4, according as $k_1+k_2<5$ or not respectively. The digits recorded correspond therefore to k_1, k_1+k_2, $k_1+k_2+k_3$, etc., and not to k_1, k_2, k_3, etc. If k_1, k_2, k_3, etc., are a random series however, the series produced by any process, systematic with respect to the suffixes of the k's and maintained in the same scale, must also be random. For example, the series corresponding to k_1+k_2, k_2+k_3, k_3+k_4, etc., is random ; so also is that corresponding to k_1, k_1+k_2, $k_1+k_2+k_3$, etc. (multiples of 4 being ignored).

6. SETTING-UP PROCEDURE.

Two important adjustments must be made : (1) the h.t. voltage must ensure that the Geiger-Muller counter is operated in the middle of its plateau and (2) the distance of the radium from the counter must be adjusted to give a suitable mean count. After the h.t. voltage has been adjusted to the value appropriate to the G.M. counter, the output of the pulse generator (fig. 1) is connected to a pulse meter, which may be either a condenser integrator or an electronic multiple-scale counter. (If it is the latter, it should have a resolving time not greater than 10^{-4} sec.). The position of the radium is then adjusted to give a mean count of about 1000 per sec.

7. THE ANALYSIS OF SERIES.

Equipments designed on the lines described have produced many series, containing in all many hundred thousands of digits, in the form of traces made on charts by a multi-pen high-speed siphon recorder. The choice of a relatively long unit time interval, *e. g.* 150 mS., would have made possible direct recording, as a series of digits, on a teleprinter tape, a relay set and a teleprinter replacing the siphon recorder ; but speed took precedence over presentation in the application for which the equipment was originally designed.

It was necessary to make tests for local randomness on sample sections of the series produced and, in order to do so, translation of the traces to series of digits (or letters) was at first undertaken. A set of keys labelled 1, 2, . . . was connected to the electromagnetically operated keyboard of a teleprinter, functioning as a typewriter ; with this aid, an operator translated at a rate of about 100 digits a minute.

Kendall and Babington Smith have set useful precedents for the testing of the randomness of a series of numbers, although neither their tests nor any other series of tests can be claimed to be exhaustive. K. and B. S. considered four tests adequate in practice :—the frequency, serial, poker and gap tests. The frequency test determines whether any one digit occurs, in aggregate, significantly more often than would be expected at random ; the serial test determines whether any sequence of two digits (*e. g.* 16, 23, 82, 44, . . . in the scale of 10) so occurs. More lengthy sequences are dealt with in the poker test. The gap test is based on an analysis of the number of interposed digits between successive 1's, 2's, 3's, etc. They decided significance by the χ^2 test for all four tests, with rejection levels of 1 per cent. and 99 per cent.

If some reduction in the number of tests has to be made, for practical reasons, the tests remaining should " be capable of detecting any types of bias which are likely to arise in forming the tables of numbers " (a quotation from K. and B. S. used in a slightly different context).

To date, only the frequency, serial and gap tests have been applied systematically to the sample series mentioned. It has been possible to combine much of the labour of this testing with that of the translation of the traces to a series of digits on a teleprinter tape. Thus each key was connected to an electromechanical counter (of the type used in telephony to record the number of calls originated by a subscriber and often known as a subscriber's meter) and the necessary records for the frequency test at once obtained. An additional set of relays, so connected that they recorded each successive pair of digits as one of s^2 possible sequences on the appropriate meter of an additional set of subscribers' meters, reduced the labour in carrying out the serial test. If s is large the number of relays and counters required is large ; a scale of 10 requires about 35 relays and 100 electromechanical counters. The gap test was applied to the digit 1 only, use being made of a rotary switch (uniselector), a few relays and a third set of meters. Each contact of the switch was connected to a meter. The switch stepped one position whenever any key other than No. 1 was operated. A series of n digits not including a 1—following a digit 1—caused the switch to be stepped to its nth contact. If the next key depressed was No 1, (*a*) the meter connected to this nth contact operated once and (*b*) the switch returned to its home position, ready to count the next gap.

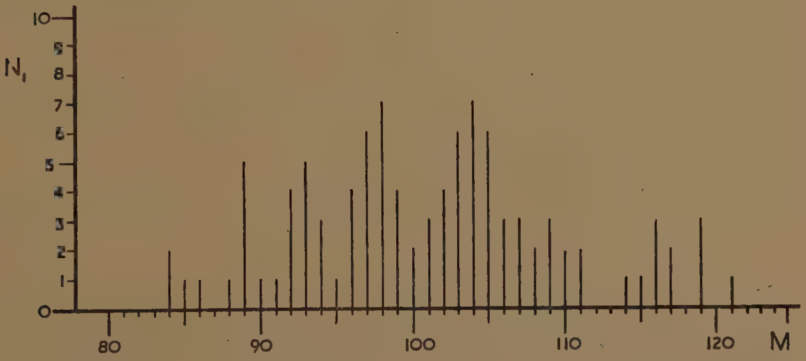
The χ^2 test with rejection limits 1 per cent. and 99 per cent. has been applied consistently; long series have been divided into blocks, each of, say, 1000 digits. The number of blocks failing the tests has not been exceptional. The t-test and the two mean t-test have also been applied; in general they have led to the rejection of more blocks than has the

TABLE.

Number of gaps of length g between successive 1's in a series of 30,000 $(=n_t)$ digits in the scale of 3, containing 10,111 $(=n_1)$ 1's.

$g=$	0	1	2	3	4	5	6	7, 8, 9	>9
Found..........	3443	2245	1504	1011	610	410	296	401	191
Expected :– $\frac{n_1}{n_t}\left(\frac{n_t-n_1}{n_t}\right)^g n_1$	3407	2260	1498	994	659	437	290	404	166
....	...	...	...	...	...	...	...	...	...
as part of a long run, without any knowledge of the number of 1's :— $\frac{1}{3}\left(\frac{2}{3}\right)^g$ 10,000.	3333	2222	1482	988	659	439	293	413	174

Fig. 6.



The 30,000 digits were divided into blocks, each of 300, and the number of blocks, N_1, having M 1's is shown.

χ^2 test. Fig. 6 shows some of the data of a frequency test applied to a series of 30,000 digits in the scale of 3. The table shows the results of a gap test applied to the same series of digits.

The results of the frequency test have justified the steps, described in section 5.1, taken to eliminate the lack of randomness which unequal sensitivities in the ring counter could otherwise introduce.

Because the ring counter is not restored to one particular stable position for the start of each unit time interval, the serial test is of much importance. The result of too low a value of x would escape the frequency test, but be evident in the serial test. Because there was no reason to expect any patterning in the series of digits, elaborate poker tests were not undertaken ; those poker tests which were made, and the gap tests, detected no patterning.

Throughout, the tests have shown that the equipments used have behaved as expected. Mean counts have been great enough to prevent systematic bias being detectable for sequences of less than about 10^7 digits.

8. Discussion.

A lengthy comparison of the method described of producing a random sequence of numbers and that of K. and B. S. is not justified. K. and B. S. aimed at producing one sequence of 100,000 numbers in the scale of 10 ; the intense concentration which their observers must have given to their work was worth while because it led to one, much wanted, reliable sequence. The present method aims at eliminating the human element, at speed and at long sequences. The form of its output is suitable for use with electrical and electronic calculating machines.

Use of the method will enable statisticians who at present make repeated calls on the K. and B. S. tables, with any detriment arising from the use of the same series over and over again, to produce, in a convenient form, new and, if necessary, lengthier series. Continuous electrical monitoring on the series produced might then be worth while ; it would involve electronic counting equipment unless the rate of production was reduced several times, when little more than the equipment mentioned in section 7 would be necessary.

The reliability of the equipment described has proved to be high ; considerable deterioration of the valves is permissible without detriment to the overall performance, except in the amplifier and the h.t. unit, where simple maintenance tests are possible.

The authors thank the Post Office Engineering Department for its permission to publish.

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XCV. *A Theorem in Operational Calculus.*

By N. N. Bose, Lucknow Christian College, Lucknow *.

[Received April 29, 1946.]

A function $h(x)$ is said to be operationally related to $f(p)$ if

$$f(p)=p\int_0^\infty e^{-px}h(x)\,dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(i)}$$

provided the integral converges. This is symbolically written as

$$f(p)\doteqdot h(x).$$

A number of theorems regarding the evaluation of integrals with the help of operational calculus has been given from time to time. The object of this paper is to establish the following theorem :—

If $f(p)\doteqdot h(x)$, then

$$\int_1^\infty f(p)\phi(p)\,dp\doteqdot\int_0^\infty h(x)\psi(x)\,dx,$$

where
$$\psi(x)=\int_0^\infty p\,.\,e^{-px}\phi(p)\,dp,$$

provided the integral converges.

Proof :—Multiplying both sides of (i) by $\phi(p)$ and integrating *with respect to p* between the limits *one* to *infinity*, we have

$$\int_1^\infty f(p)\phi(p)\,dp=\int_1^\infty \phi(p)\,p\,dp\int_0^\infty e^{-px}h(x)\,dx$$

$$=\int_0^\infty h(x)\,dx\int_1^\infty pe^{-px}\phi(p)\,dp.$$

The change in the order of integration is permissible if

$$\int_0^\infty h(x)\,dx \quad \text{and} \quad \int_0^\infty x\phi(x)\,dx$$

both exist.

Now, if $\int_1^\infty pe^{-px}\phi(p)\,dp$ be put equal to $\psi(x)$, then the above may be written as

$$\int_1^\infty f(p)\phi(p)\,dp=\int_0^\infty h(x)\psi(x)\,dx.$$

This simple theorem connects an integral with limits from one to infinity with another with limits from zero to infinity.

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* Communicated by R. S. Varma, D.Sc.

Since integrals with limits from zero to infinity are well known, the above theorem is extremely useful in the evaluation of integrals with limits from one to infinity.

As a special case of the general theorem we have the corollary that, if

$$f(p) \doteqdot h(x),$$

then

$$\int_1^\infty f(p) \cdot p^{-1} \left(\frac{p+1}{p-1}\right)^{\frac{1}{2}k} \cdot P^k_{m-\frac{1}{2}}(p)\, dp$$

$$= \int_0^\infty \frac{1}{x} W_{k,m}(2x) h(x)\, dx \quad R(k)<1 \quad \text{and} \quad |\arg z| < \frac{\pi}{2}. \qquad \text{. . . (ii)}$$

This follows by considering the following integral due to Meijer (1)

$$W_{k,m}(z) = \tfrac{1}{2} z \int_1^\infty e^{-zt} \left(\frac{t+1}{t-1}\right)^{\frac{1}{2}k} P^k_{m-\frac{1}{2}}(t)\, dt,$$

where $R(k)<1$ and $|\arg z| < \frac{\pi}{2}$.

Example 1.—If we take

$$f(p) = \left(\frac{p}{y^2}\right)^{-\nu} e^{-\frac{y^2}{p}}$$

and $h(x)=(xy)^{\frac{1}{2}\nu} J_\nu(2y\sqrt{x})$ in (ii), then we have

$$\int_0^\infty \left(\frac{p}{y^2}\right)^{-\nu} p^{-1} \left(\frac{p+1}{p-1}\right)^{\frac{1}{2}k} P^k_{m-\frac{1}{2}}(p) e^{-\frac{y^2}{p}}\, dp$$

$$= \int_0^\infty \frac{1}{x} W_{k,m}(2x) \,.\, (xy^2)^{\frac{1}{2}\nu} J_\nu(2y\sqrt{x})\, dx, \qquad \text{. . . (iii)}$$

where $R(k)<1$ and $R(\nu \pm m + \frac{1}{2}) > 0$.

Using the identity

$$J_\nu(yt) = \frac{(\frac{1}{2}ty)^\nu}{\Gamma(\nu+1)}\, {}_0F_1(\nu+1\,;\, -\tfrac{1}{4}y^2t^2) \qquad \text{. . . . (iv)}$$

and effecting a slight transformation, the R.H.S. of (iii) reduces to

$$\frac{y^{2\nu}}{2^\nu \Gamma(\nu+1)} \int_0^\infty x^{\nu-1} W_{k,m}(x)\, {}_0F_1(\nu+1\,;\, -\tfrac{1}{2}y^2x)\, dx.$$

The last integral can be evaluated by the help of Pasricha's integral (2)

$$\int_0^\infty x^{\mu-1} e^{-(\alpha^2+\frac{1}{2})x} W_{k,\nu}\, {}_rF_s\left(\begin{matrix} l_1, l_2 \ldots l_r \\ m_1, m_2 \ldots m_s \end{matrix}\,;\, yx^p\right) dx$$

$$= \sum_{m=0}^\infty \frac{(l_1, m) \ldots (l_r, m)}{(m_1, m) \ldots (m_s, m)} \cdot \frac{\Gamma(a+pm)\Gamma(b+pm)}{m\,!\,\Gamma(c+pm)}$$

$$\times {}_2F_1\left(\begin{matrix} a+pm, b+pm \\ c+pm \end{matrix}\,;\, -\alpha^2\right),$$

where $a=\mu+\nu+\frac{1}{2}$, $b=\mu-\nu+\frac{1}{2}$, $c=\mu-k+1$ true under certain conditions.

Doing so we easily see that

$$\int_1^\infty p^{2\nu-1}\left(\frac{p+1}{p-1}\right)^{\frac{1}{2}k} \mathrm{P}^k_{m-\frac{1}{2}}(p)e^{-\frac{y^2}{p}}\,dp$$

$$=\frac{y^{2\nu}}{2^\nu}\sum_{r=0}^{\infty}\frac{(-\frac{1}{2}y^2)^r\Gamma(\nu+m+\frac{1}{2}+r)\Gamma(\nu-m+\frac{1}{2}+r)}{\Gamma(\nu+1+r)r\,!\,\Gamma(\nu-k+\frac{1}{2}+r)}$$

$$\times{}_2\mathrm{F}_1\left(\begin{matrix}\nu+m+\frac{1}{2}+r,\ \nu-m+\frac{1}{2}+r\\ \nu-k+1+r\end{matrix};\ \frac{1}{2}\right).$$

This result is true under the conditions

$$\mathrm{R}(k)<1 \quad\text{and}\quad \mathrm{R}(\nu\pm m+\tfrac{1}{2})>0.$$

Example 2.—If we take

$$f(p)=\Gamma(\nu+\tfrac{1}{2})p\mathrm{D}_{-\nu-\frac{1}{2}}(pe^{\frac{\pi}{4}i})\,\mathrm{D}_{-\nu-\frac{1}{2}}(pe^{-\frac{\pi}{4}i})$$

and

$$h(x)=\sqrt{\pi}\mathrm{J}_\nu(\tfrac{1}{2}x^2)$$

in (ii), then

$$\Gamma(\nu+\tfrac{1}{2})\int_1^\infty \mathrm{D}_{-\nu-\frac{1}{2}}(pe^{\frac{\pi}{4}i})\,\mathrm{D}_{-\nu-\frac{1}{2}}(pe^{-\frac{\pi}{4}i})\left(\frac{p+1}{p-1}\right)^{\frac{1}{2}k}\mathrm{P}^k_m(p)\,dp$$

$$=\sqrt{\pi}\int_0^\infty \frac{1}{x}\mathrm{W}_{k,\,m}(2x)\mathrm{J}_\nu(\tfrac{1}{2}x^2)\,dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{v})$$

Using (iv) in the R.H.S. of (v) and changing the variable we see that it is equal to

$$\frac{\sqrt{\pi}}{4^{2\nu}\Gamma(\nu+1)}\int_0^\infty x^{2\nu-1}\mathrm{W}_{k,\,m}(x)\ {}_0\mathrm{F}_1\left(\nu+1;\ -\frac{1}{256}x^4\right)dx.$$

On using the integral due to Pasricha given above we have, when

$$\mathrm{R}(k)<1 \quad\text{and}\quad \mathrm{R}(2\nu\pm m+\tfrac{1}{2})>0,$$

$$\int_1^\infty \mathrm{D}_{-\nu-\frac{1}{2}}(pe^{\frac{\pi}{4}i})\,\mathrm{D}_{-\nu-\frac{1}{2}}(pe^{-\frac{\pi}{4}i})\left(\frac{p+1}{p-1}\right)^{\frac{1}{2}k}\mathrm{P}^k_{m-\frac{1}{2}}(p)\,dp$$

$$=\frac{\sqrt{\pi}}{4^{2\nu}}\sum_{r=0}^{\infty}\frac{(-)^r(\frac{1}{4})^{4r}\Gamma(2\nu+m+\frac{1}{2}+4r)\Gamma(2\nu-m+\frac{1}{2}+4r)}{\Gamma(\nu+1+r)\Gamma(r+1)\Gamma(2\nu-k+1+4r)}$$

$$\times{}_2\mathrm{F}_1\left(\begin{matrix}2\nu+m+\frac{1}{2}+4r,\ 2\nu-m+\frac{1}{2}+4r\\ 2\nu-k+1+4r\end{matrix};\ \frac{1}{2}\right).$$

I am indebted to Dr. R. S. Varma for his kind suggestions and guidance in the preparation of this paper.

References.

(1) C. S. Meijer, *Math. Ann.*, cxii. p. 480 (1936).

(2) Pasricha, "Some Infinite Integrals Involving Whittaker Functions," *Jour. Ind. Math. Soc.*, vii. pp. 46–50 (1943).

XCVI. *On some Integrals involving E-functions.*

By N. N. Bose,

Lucknow Christian College, Lucknow *.

[Received February 28, 1947.]

§ 1. MacRobert's function $E(\alpha, \beta :: x)$ can be defined by the equation⁽¹⁾

$$E(\alpha, \beta :: x) = \sum_{\alpha, \beta} \Gamma(\beta-\alpha)\Gamma(\alpha)x^{\alpha}F(\alpha ; \alpha-\beta+1 ; x)\,\dagger. \quad . \quad . \quad . \quad (1)$$

It satisfies the relation⁽²⁾

$$\int_0^{\infty} e^{-t}t^{\gamma-1}E(\alpha, \beta :: xt)dt$$
$$= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+\gamma)} F\left(\begin{matrix}\alpha, \beta \\ \alpha+\beta+\gamma\end{matrix} ; \frac{x-1}{x}\right), \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $R(\alpha+\gamma)>0$, $R(\beta+\gamma)>0$, $R(x)>\frac{1}{2}$.

The connection between this function and Whittaker's function $W_{k, m}(x)$ is given by the equation

$$E(\tfrac{1}{2}-k-m, \tfrac{1}{2}-k+m :: x)$$
$$= \Gamma(\tfrac{1}{2}-k-m)\Gamma(\tfrac{1}{2}-k+m)x^{-k}e^{\frac{1}{2}x}W_{k, m}(x). \quad . \quad . \quad (3)$$

The object ‡ of this paper is to establish some infinite integral formulæ involving $E(\alpha, \beta :: x)$. This is done by applying the operational calculus as developed by Van der Pol.

Two functions $f(p)$ and $h(x)$ are said to be operationally related to each other if

$$f(p) = p\int_0^{\infty} e^{-px}h(x)dx, \; p>0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

provided that the integral converges.

Symbolically, we write this in the form

$$f(p) \doteqdot h(x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The functions $f(p)$ and $h(x)$ are called the *image* and the *original* respectively.

Use will be made of Lerch's Theorem⁽³⁾, stated in the following form.

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* Communicated by the Author.

† The notation $\sum_{\alpha, \beta}$ indicates that to the expression following the symbol is to be added a similar expression with α and β interchanged.

‡ I am indebted to Dr. R. S. Varma for suggesting this paper and for guidance in its preparation.

If $\phi(p, y) \doteqdot f(x, y)$, where we regard y as a constant, if $\psi(p) \doteqdot g(x)$ and if

$$\int_0^\infty \phi(p, y)dy = \psi(p),$$

then

$$\int_0^8 f(x, y)dy = g(x).$$

This can be deduced by taking originals on either side.

§ 2. Now, in equation (2), replace t and x by py and $1/p$ respectively: then

$$\int_0^\infty e^{-py}y^{\gamma-1}\mathrm{E}(\alpha, \beta : : y)dy$$
$$= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+\gamma)p^{\gamma}} \mathrm{F}\left(\begin{matrix}\alpha, \beta\\ \alpha+\beta+\gamma\end{matrix}; 1-p\right), \quad . \; . \; . \; . \; . \; . \quad (6)$$

where $\mathrm{R}(\alpha+\gamma)>0$, $\mathrm{R}(\beta+\gamma)>0$, $0<p<2$.

Next, replace p by $1/p$ and multiply by $p^{1-\alpha+\gamma}$; then (6) becomes

$$\int_0^\infty e^{-y/p}\left(\frac{p}{y}\right)^{1-\alpha+\gamma} y^{-\alpha+2\gamma}\mathrm{E}(\alpha, \beta : : y)dy$$
$$= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+\gamma)} p^{-\alpha+1} \mathrm{F}\left(\begin{matrix}\alpha, \beta\\ \alpha+\beta+\gamma\end{matrix}; 1-\frac{1}{p}\right), \quad . \; . \; . \; . \quad (7)$$

where $\mathrm{R}(\alpha+\gamma)>0$, $\mathrm{R}(\beta+\gamma)>0$, $p>\frac{1}{2}$.

But(4)

$$\left(\frac{p}{y}\right)^{1-\alpha+\gamma} e^{-y/p} \doteqdot (xy)^{\frac{1}{2}(\alpha-\gamma-1)}\mathrm{J}_{\alpha-\gamma-1}(2\sqrt{xy}), \quad . \; . \; . \; . \quad (8)$$

where $\mathrm{R}(\alpha-\gamma)>0$; and the R.H.S. of (7) is equal to(5)

$$\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+\gamma)} p^{\beta-\alpha+1} \mathrm{F}\left(\begin{matrix}\beta, \; \beta+\gamma\\ \alpha+\beta+\gamma\end{matrix}; 1-p\right)$$
$$\doteqdot x^{\alpha-\beta-1}\,\mathrm{E}(\beta, \beta+\gamma : : x); \quad . \; . \; . \; . \; . \quad (9)$$

where $\mathrm{R}(\alpha)>0$, $\mathrm{R}(\alpha+\lambda)>0$, by (6).

Now multiply (8) by $y^{-\alpha+2\gamma}\,\mathrm{E}(\alpha, \beta : : y)$, and take the L.H.S. to be $\phi(p, y)$ in Lerch's Theorem, as stated above: then

$$\int_0^\infty (xy)^{\frac{1}{2}(\alpha-\gamma-1}\mathrm{J}_{\alpha-\gamma-1}(2\sqrt{xy})y^{-\alpha+2\lambda}\mathrm{E}(\alpha, \beta : : y)\,dy$$
$$= x^{\alpha-\beta-1}\mathrm{E}(\beta, \beta+\gamma : : x), \quad . \; . \; . \; . \; . \; . \; . \quad (10)$$

provided that $\mathrm{R}(\alpha+\gamma)>0$, $\mathrm{R}(\beta+\gamma)>0$, $\mathrm{R}(\alpha-3\gamma+\underline{3})>0$.

Thus, from (3), if $\beta=\frac{1}{2}-k+m$, $\gamma=-2m$,

$$\int_0^\infty y^{-\frac{1}{2}\alpha-3m-\frac{1}{2}}\mathrm{J}_{\alpha+2m-1}(2\sqrt{xy})\mathrm{E}(\alpha, \tfrac{1}{2}-k+m : : y)\,dy$$
$$= \Gamma(\tfrac{1}{2}-k+m)\Gamma(\tfrac{1}{2}-k-m)x^{\frac{1}{2}\alpha-2m-1}e^{\frac{1}{2}x}\mathrm{W}_{k,\,m}(x), \quad . \; . \; . \; . \; . \; . \quad (11)$$

where $\mathrm{R}(\alpha-2m)>0$, $\mathrm{R}(\frac{1}{2}-k-m)>0$, $\mathrm{R}(\alpha+6m+\frac{3}{2})>0$.

Again, in (6) replace p by $1+1/p$ and multiply by $p^{-\nu}$; then, if $p>0$,

$$\int_0^\infty e^{-y}e^{-y/p}(p/y)^{-\nu}y^{\gamma-\nu-1}\mathrm{E}(\alpha,\beta::y)\,dy$$

$$=\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+\gamma)}\sum_{r=0}^{\infty}(-1)^r\frac{(\gamma\,;\,r)^*}{r\,!}p^{-\nu-r}\,\mathrm{F}\left(\begin{matrix}\alpha,\beta\\ \alpha+\beta+\gamma\end{matrix};-\frac{1}{p}\right) \quad (12)$$

where $\mathrm{R}(\alpha+\gamma)>0$, $\mathrm{R}(\beta+\gamma)>0$, $\mathrm{p}>1$.

Now

$$\left(\frac{p}{y}\right)^{-\nu}e^{-y/p}\doteqdot(xy)^{\frac{1}{2}\nu}\mathrm{J}_\nu(2\sqrt{xy}), \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where $\mathrm{R}(\nu)>-1$, and

$$p^{-\nu-r}\mathrm{F}\left(\begin{matrix}\alpha,\beta\\ \alpha+\beta+\gamma\end{matrix};\frac{1}{p}\right)\doteqdot\frac{x^{\nu+r}}{\Gamma(\nu+r+1)}\mathrm{F}\left(\begin{matrix}\alpha,\beta\\ \alpha+\beta+\gamma,\nu+r+1\end{matrix};-x\right). \quad (14)$$

where $\mathrm{R}(\nu+r)>-1$, $p>1$.

Hence, on taking originals of both sides of (12), we have

$$\int_0^\infty e^{-y}(xy)^{-\frac{1}{2}\nu}\mathrm{J}_\nu(2\sqrt{xy})y^{\gamma-1}\mathrm{E}(\alpha,\beta::y)\,dy$$

$$=\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+\gamma)\Gamma(\nu+1)}\sum_{r=0}^{\infty}(-1)^r\frac{(\gamma\,;\,r)x^r}{r\,!\,(\nu+1\,;\,r)}\mathrm{F}\left(\begin{matrix}\alpha,\beta\\ \alpha+\beta+\gamma,\nu+r+1\end{matrix};-x\right)$$

$$. \quad . \quad . \quad . \quad (15)$$

where $\mathrm{R}(\alpha+\gamma)>0$, $\mathrm{R}(\beta+\gamma)>0$.

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(7) Erdelyi, J. Lond. Math. Soc. xiii. pp. 146–148 (1938).

XCVII. *Notices respecting New Books.*

Cosmic Rays. By L. JÁNOSSY. [Pp. xiii+424.] (The Clarendon Press, Oxford, 1948.) Price 35*s*. 0*d*.

THIS comprehensive and up-to-date account of both the experimental and theoretical aspects of the study of Cosmic Rays is a timely addition to the Clarendon Press International Series of Monographs. It is authoritatively written by an investigator who has himself made numerous contributions to the subject. The book is addressed primarily to the research workers in the field of cosmic rays, and perhaps also to those in nuclear physics. To such readers it is a useful critical survey and compendium. The first 74 pages give a concise historical survey and a review of experimental techniques. The remaining pages of text are devoted to an analysis of all the more important aspects of cosmic rays, including considerations of the theory of fast collisions, the nature of cosmic rays, the instability of the meson, the cascade theory, geomagnetic effects, extensive air showers and the nature of the meson. This section is that which will be of value to the active investigator. It contains a good deal of mathematical theory clearly expressed, and treats of a considerable amount of experimental data. It is followed by a comprehensive bibliography of more than 500 references. There is a useful 25-page appendix, giving a brief summary of the theory of the statistical treatment of observations with worked examples. The book evidently does not appear to be intended as an undergraduate's text-book, yet Chapter I (Historical), Chapter II (Experimental), most of Chapter III (Collision Theory), and the whole of Appendix I (Statistics), in all some 165 pages, could be read with profit by all students of physics in their final honours year. Included are numerous excellent illustrations, but perhaps the 24 reproductions of cloud chamber photographs might have been increased in number since such pictures are strikingly effective. The quality of printing and the production is of the best. S. TOLANSKY.

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* Here $(\gamma\,;0)=1$, $(\gamma\,;r)=\gamma(\gamma+1)(\gamma+2)\ldots(\gamma+r-1)$, where r is a positive integer.

Modern Physics. By G. E. M. JAUNCEY. 3rd Edition (1948.) [Pp. xiii+561.] D. van Nostrand. Price 33*s*. 0*d*.

THIS book was written to meet the needs of the U.S.A. physics student who has already had a grounding in classical physics. It is clear from the frequency of reprinting since the 1937 edition that this book is popular in rhe U.S.A. It has many attractive features, but also a number which depart somewhat from usual British practice. The first chapter gives an excellent review of the historical evolution of physics. There follow Chapters II (Calculus), III (Wave Motion), IV (Alternating Currents), V (Radiation Theory), VI (Movement of Charged Bodies), in all 100 pages of classical material which certainly cannot be described as " modern physics." In Great Britain this would already have been covered in the earlier classical course to which it naturally belongs, and one feels that these 100 pages overburden the book. The remaining 400 pages largely cover what one usually describes here as atomic physics. They include, too, a good chapter on kinetic theory of gases. A departure from usual British practice in a book mainly devoted to atomic physics is the insertion of a 45-page chapter on electronics, valve circuits and radio. The sections on radio-activity and on isotopes are much too elementary, but that on cosmic rays is good. Attractive sections on astrophysics are included. There is, however, some unevenness as to level running throughout the whole book. Included are six pictures of leading contemporary physicists, and it is regrettable that these are only of men resident in the U.S.A.—surely a broader view might have been taken. The book is beautifully illustrated with a profusion of good diagrams, and will be useful in our libraries. S. TOLANSKY.

Modern Operational Calculus: with Applications in Technical Mathematics. By N. W. MCLACHLAN. [Pp. xiv+218.] (Macmillan and Co., Ltd., 1948.) Price 21*s*. 0*d*.

THIS book is another contribution by Dr McLachlan to the list of texts dealing with those branches of mathematics which are important in engineering and technology. It is a well-written book and the development of the subject-matter is sufficiently rigorous to satisfy all but the most exacting mathematicians. At the same time the needs of those primarily interested in the applications of operational methods have been kept in mind and many of the examples given relate to problems in electrical engineering, theory of heat conduction, etc.

In chapter I the Laplace transform is defined and some of its properties are discussed. The rules of the operational calculus are given in chapter II and in chapters III and IV application is made to the solution of ordinary and partial differential equations with constant coefficients. Various definite integrals which may be evaluated by the use of the Laplace transform are discussed in chapter V, and methods for deriving Laplace transforms are treated in chapter VI. Techniques involving pulses of various kinds are now in common use in several branches of technology, and the mathematical treatment of a pulse involves the theory of the Laplace transform for a finite interval. This is given in chapter VII. Various mathematical questions are discussed in a set of appendices and a short list of Laplace transforms is given.

The reader will need to have a good knowledge of analysis up to the stage of an honours undergraduate. Complex integration is necessary in chapter VII, and at least an elementary knowledge of the properties of Bessel functions is assumed. The book will be of value to the mathematician just moving into the realms of technology, and to the engineer who seriously wishes to understand one of the most powerful tools in technical mathematics. L. S. G.

A Text-book of Mathematical Analysis: the Uniform Calculus and its Applications. By R. L GOODSTEIN. [Pp. xii+475.] (Oxford University Press, 1948.) Price 30*s*. 0*d*.

IN this book a bold attempt has been made to solve one of the outstanding problems in the presentation of the differential and integral calculus to undergraduates. This problem is to develop the subject along lines which neither depend upon geometrical intuition nor upon the Dedekind number theory. The basic notion in the present development is that of a convergent sequence of terminating decimals. This is presented in a practical way in the first chapter, and continuity is discussed in the second chapter. Then, the differential and integral calculus is built up to the extent covered by the syllabus for the general honours degree at London University.

Restriction is made to uniform limiting processes, but the resulting " uniform calculus " is no different in content from the classical calculus; only this restriction enables a rigorous development without the abstractions involved in the classical theory. It is claimed by the author that the present approach is more *practical*, that is, may be more readily understood by the average undergraduate, since it is based on practical constructions rather than the developments according to Dedekind and Russell.

It will be interesting to see whether the methods of the present book appeal very widely to those engaged in the teaching of the calculus. If it does, Professor Goodstein will have rendered a great service to mathematics. But if the majority of teachers persist in the traditional presentation, it is unlikely that the book will appeal to undergraduates as a suitable text-book. It would seem that what is needed now to help to bring about the adoption of the author's methods is further publicity, whereby the existing teachers of the calculus would see the advantages of the present methods over those commonly used.

L. S. G.

Analytic Mechanics. By Professor A. E. CURRIER. [Pp. 306.] (U.S.A.: The United States Naval Institute, Annapolis, Maryland, 1948.) Price $4.75.

THOSE of us old enough to have used the formidable, and very original, *Notes on Dynamics*, by the late Sir George Greenhill, expect to see a high standard set (and attained) by a treatise on mechanics written for the benefit of naval cadets. Professor Currier has not disappointed us. He has developed a course on the mechanics of particles and of rigid bodies moving in two dimensions which is logical, vividly clear and in touch with physical reality. His treatment of the mass concept is condensed, and would be improved were it discussed in greater detail, but that is a very minor matter.

The principal topics discussed are—systems of forces in equilibrium, stresses, centroids, moments of inertia, friction, strength of materials, beams, particle dynamics, work, energy and potential functions, motion of a system of particles, impulses. Illustrative examples range widely from the period of a rocking chair to the flight of a rocket. The treatment of these subjects is such as to make the book helpful to students in the final schools of physics or of engineering.

The late Professor L. N. G. Filon was wont to insist on the essentially experimental nature of applied mathematics and initiated a laboratory course in the subject at University College London, as did, *longo intervallo*, the writer of this review in the University College of North Wales, Bangor. It is to be hoped that Professor Currier will produce a companion laboratory manual on the subject. The number of illustrative experiments, static and dynamic, which can be devised is legion, and a selection would form the basis of a book which, coming from Professor Currier, would certainly be helpful, original and stimulating. A. F.

Chambers's Four Figure Mathematical Tables. By Dr. L. J. COMRIE. [Pp. 64.] (Edinburgh and London: W. & R. Chambers, 1947.) Price 6*s*.

THIS book of tables is different—and better. It contains all the usual tables together with a uumber of tables not so usual in a four-figure compendium—primes, factors, proportional parts, binomial coefficients, the Gaussian or normal curve, factorials, interpolation coefficients. The type and spacing have been carefully considered and the spacing chosen does away with the necessity for rules. The text abounds in original (and helpful) footnotes, and the table of physical constants is refreshingly useful. The change of a leading figure in the middle of a line is indicated by a change of font which is impossible to overlook. A. F.

General Properties of Matter. By NEWMAN and SEARLE. [Pp. xi+431.] (London: Edward Arnold, 1948.) Price 21*s*.

THE appearance of a fourth edition of this book, first published in 1929, is sufficient testimony to its well-deserved popularity. It handles, on classical lines, those topics—elasticity, gravitation, wave-motion, capillarity, viscosity, kinetic theory, osmosis, diffusion, Fourier analysis and dimensional theory—which are commonly treated under this title. The subjects are developed at a level suited to first-year honours university students, and pointers for further reading are given. The authors show special skill in dealing with border-line subjects in a manner likely to evoke the interest of students of engineering and of chemistry.

The book is to be numbered among the few which discuss the problem of the proportionality of weight to mass and describe the experiments of Newton, Bessel and Eötvös on this important problem. A. F.

De Revolutionibus. By NICOLAUS COPERNICUS. Preface and Book I. Translated by J. F. Dobson, assisted by S. Brodetsky. [Pp. iv+32.] (Royal Astronomical Society, 1947.) Price 3*s*. 6*d*.

John Couch Adams and the Discovery of Neptune. By W. M. SMART. [Pp. iv+56.] (Royal Astronomical Society, 1947.) Price 5*s*.

THESE two substantial booklets, printed originally as 'Occasional Notes of the Royal Astronomical Society', will be welcomed by many readers outside the circle of professed astronomers. 'The Revolutions of the Celestial Spheres', first printed just over four hundred years ago, in 1543, marks a turning point in human outlook. The prediction of the location of a planet beyond Uranus as a result of mathematical analysis of discrepancies between its observed and calculated motion, a prediction confirmed by the optical discovery of the new planet, just over a hundred years ago, in 1846, will always rank as an outstanding achievement of the human intellect.

In the translator's preface no reference is made to any other translations of Copernicus. It seems, however, that hitherto fragments only of *De Revolutionibus* have been translated into English, though Edward Rosen, in the preface to his translation of the *Commentariolus* in 1939, proposes to undertake a full translation, "if this project is not pre-empted". With the relative inaccessibility of the Latin versions, and the decreasing familiarity with Latin, comparatively few even of those who speak and write of the Copernican revolution can have better than a second-hand knowledge of Copernicus's actual work. Whether the present translation is to be followed by translations of the other five Books is not stated, but the main requirement of providing a clear impression of the approach and outlook of Copernicus is well met by this easily-flowing translation of the first Book; and the judicious notes deal concisely and usefully with obscurities and vague references in the original. In view of the later history it is surprising to find how casually Copernicus puts forward his views. There is surely a touch of light-hearted humour in his remark in the Preface addressed to Pope Paul III that certain people, when they hear that he ascribes movement to the Earth, will cry out that he should at once be "hissed off the stage" ("*statim me explodendum .... clamitent*"). The presentation is clear, straightforward, and sublimely reasonable, as may be illustrated by a quotation from Book I, which should perhaps be regarded as a general survey, preliminary to the subsequent detailed treatment. After giving the famous diagram, with the central sun, and the planets in their order, "I think it easier to believe this", Copernicus says, "than to confuse the issue by assuming a vast number of Spheres, which those who keep Earth at the centre must do. We thus rather follow Nature, who producing nothing vain or superfluous often prefers to endow one cause with many effects. Though these views are difficult, contrary to expectation, and certainly unusual, yet in the sequel we shall, God willing, make them abundantly clear at least to mathematicians".

No full-length biography of Adams has yet been written. His letters, papers and diaries have, however, now passed into the custody of Professor Smart; and this long memoir, full of understanding of Adams as a man, and of critical admiration of his work, shows that they could not be in better keeping. A full sketch is given of Adams's earlier and later life, but the core of the memoir is the fascinating story of the discovery of Neptune. While still an undergraduate, Adams in 1841, at the age of 22, formed the design of investigating whether the irregularities in the motion of Uranus might be attributed to the action of an undiscovered planet beyond it. His prediction of the approximate location of a new planet was communicated in 1845 to others, including Airy, before Le Verrier's publication of his entirely independent work, which led directly to the optical discovery of the planet by Galle and D'Arrest. Now that more information is available, and the fierce fires of the early controversies on the priority question have burned out, a more objective examination of the sequence of events can be made, and it is apparent how, through the very greatness of the occasion, regrettable but very human failings of such men as Airy and Challis assumed the proportions of grave moral delinquencies. It is noteworthy that at first the young Englishman was in England regarded as a very minor figure compared with the great Frenchman; as he was in France, but there the evidence for any other view was not available. Now, in history, Adams and Le Verrier rightly share the honour of the great achievement, and it is pleasant to know that after the Neptune episode the two were on terms of the warmest friendship. This wholly admirable memoir brings out vividly that science is an affair of persons as well as of things.

E. C. S.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

XCVIII. *Behaviour of Stressed Aluminium at Room Temperature.*

By Prof. E. A. Owen, Y. H. Liu, Ph.D. and D. P. Morris, B.Sc.,
University College of North Wales, Bangor *.

[Received January 8, 1948.]

[Plates III.–V.]

The main purpose of the work described in this paper was to study, with the aid of X rays, the behaviour of aluminium—in a state of high purity, or containing pure elements, either soluble or insoluble—after the removal of the stresses to which it had been subjected. The material was mainly in plate form, but the investigation was extended to include powder and powder foils which supplied valuable information to supplement that obtained with lump material. Observations were also made on filed and polished surfaces with a view to finding out if the particular phenomena exhibited by these surfaces could be explained in the light of the results obtained with stressed material.

Calvet, Trillat and Paic (1935) showed that aluminium of high purity recrystallizes at 0° C. so that the material would be eminently suitable for the study of recrystallization occurring at room temperature, after the removal of the applied stress. Accurate measurements of crystal parameters could be made because Dehlinger (1930) had shown that the amount of line-broadening with stressed aluminium was almost negligible, from which it was concluded that very little lattice distortion was produced on deformation. Wood (1939) interpreted the absence of line-broadening to mean that the crystallite size was not reduced below about 10^{-4} mm. Smith and Wood (1940, 1941, 1942, 1944) showed that aluminium which had been strained beyond its elastic limit, gave an expanded lattice after the removal of the tensile stress to which it had been subjected.

In an attempt to measure accurately the lattice parameter of pure aluminium in plate form, preliminary work showed that after the plate had been compressed to half its original thickness, the lattice parameter value, which had been considerably increased as the result of the application of the stress, gradually returned, at room temperature, to a value which remained constant, but which was higher than the accepted value of the parameter for fully annealed material. This result, being at the time contrary to expectation, started the extensive investigation of the phenomenon which is about to be described.

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* Communicated by the Authors.

APPARATUS.

In addition to the fibre and focusing X-ray cameras generally employed for accurate measurement of lattice parameters, a third camera was constructed which proved most useful in the investigation. It was a back reflection camera suitable for use with lumps of material as well as with powder, having mechanisms incorporated in its design for the rotation as well as the oscillation of the specimen during irradiation with X rays. A detailed description of this camera will be published elsewhere.

APPLICATION OF STRESS.

It is important, in a given series of observations, that the stress is always applied in the same manner and a note taken of the direction in which the X-ray beam falls on the specimen in relation to the direction in which the stress has been applied. It was decided not to reduce the thickness of the specimen by rolling, in the main, because of uncertainty concerning the direction of the stress, but to compress it between two sheets of polished steel, the edges of the plate specimen being free to expand laterally, at right angles to the direction of application of the stress. Experiments were also made with a plate specimen which had been placed between two sheets of aluminium and the whole compressed between steel sheets; this was done with a view to diminishing tangential surface stresses, which might have had an effect on the recovery of the specimen after removing the stress normal to the surface.

ISOTROPIC PROPERTIES OF ALUMINIUM.

It is important as a first step to check that the behaviour of cold-worked aluminium is isotropic. Accordingly the new camera, which is capable of yielding accurate values of interplanar spacings, was employed to obtain X-ray photographs with an aluminium plate, which had been reduced 50 per cent in thickness, when different targets were mounted in the X-ray tube. The radiations employed were those of copper, cobalt, iron and chromium. The specimen, after compression, had been kept at room temperature for 155 days before the X-ray photographs were taken, and had reached a steady state. The specimen was mounted in position on the camera and remained undisturbed throughout the series of photographs taken with different targets, so that the distance between the mid-plane of the film and the surface of the specimen remained constant and the same spot was irradiated with every radiation used. Reflections were obtained from the following sets of planes: (115) and (333), (024), (133), (004), and (222), and the value of the crystal parameters calculated from the interplanar spacings agreed with the mean value within one part in 20,000, and with the parameter determined by the ordinary extrapolation method for which two sets of planes are necessary, to within the same accuracy. It was concluded from these observations that aluminium exhibits isotropic deformation and that the extrapolation method may be employed with this metal to calculate values of the lattice parameter from measurements made on structure-spectrum photographs.

RESULTS.

Pure Aluminium.

Plate Specimens.

The first part of the investigation was devoted to the recovery at room temperature of aluminium of 99·998 per cent* purity after it had been subjected to compressional stress. The material had been supplied in ingot form and had remained at room temperature in the laboratory for a number of years. The mean value of the lattice parameter obtained from annealed powder samples containing a full range of particle size, was 4·0406KX at 18° C.†

Small plates were carefully sawn off the ingot; one face of some of these plates consisted of the undisturbed surface of the ingot. The mean parameter value found by reflecting X rays from this face in three different specimens was $4{\cdot}0411_3$KX, a value considerably higher than the powder value. After annealing at 500° C. for three days, and cooling in air, the lattice parameter fell to the equilibrium value of 4·0406KX. In the course of the investigation it was observed that the initial lattice parameter had no effect on the recovery process, when the amount of cold-work done on the specimen was fairly severe as in the present case. If, however, the amount of cold-work done were very small, the initial parameter of the specimen might then be of importance.

The plates were all given a uniform initial heat-treatment—1 hour at 500° C.—and were compressed in the manner already indicated until they were reduced 9·9, 27·0, 50·5, 58·8 and 93 per cent respectively in thickness, all in one stage except the last, which was done in two steps with an interval of about 15 minutes between them.

Stationary back-reflection and structure-spectrum photographs were taken alternatively at different periods of time after the release of stress, the samples being kept at room temperature. The X-ray beam fell on each specimen in the direction the stress was applied, namely, normal to the surface, and it is to be emphasized that the results herein described obtain strictly only under these conditions. Cobalt radiation was employed.

A survey of the stationary back-reflection photographs will be made at a later stage when crystal growth is considered.

The measurements of the structure-spectra supplied accurate information concerning the changes that occur in the lattice dimensions. Samples of these spectra are reproduced in fig. 1 (Pl. III.). The results, plotted in fig. 2, show that in each specimen the expanded lattice after compression decreases at room temperature and finally reaches a constant parameter value of about 4·0412KX. Observations taken over a long period of time after the compression, showed that once this state was reached there was no tendency in the parameter value to change when the material was kept at room temperature. The results also show that the final parameter value reached is independent of the degree of compression to which the material was subjected within the range represented between 10 and 93 per cent reduction in thickness. For clarity the 93 per cent curve has been omitted in fig. 2. The rate of recovery, however, depended greatly upon the amount of cold-work done on the specimen ; the heavier the cold-work the greater the rate of recovery. Fig. 2 contains a curve giving the results obtained with a specimen that had been cold-rolled to produce 98 per cent reduction, the final thickness being 0·09 mm. In this case there was

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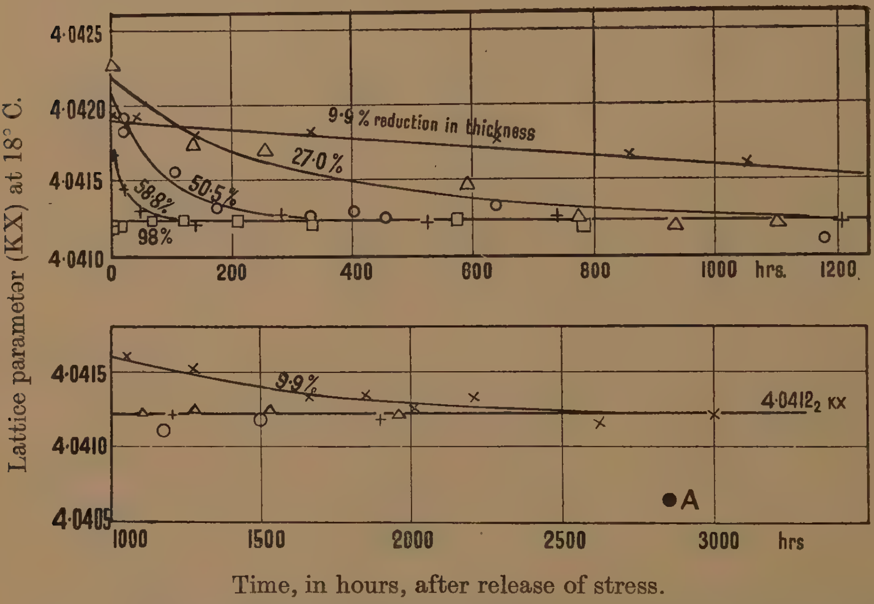
* The aluminium used throughout the investigation was of this purity.

† All parameter values in the text are given at 18° C.

Fig. 2.



Curves showing the lattice parameter of aluminium (purity 99·998%) in plate form at different times after subjecting the material to compressional stress.

immediate recovery after cold-working, but it was recovery to what may be termed the *metastable state*, the constant parameter value being the same as the final parameter value reached by the compressed specimens which had undergone less reduction in thickness.

Closer examination of the results summarized in fig. 2 leads to the conclusion that the curves can be satisfactorily represented by a relation of the form $a = a_1(1 + be^{-ct})$, where a is the parameter at any given time t, and a_1 is the final constant value which it attains ; b and c are constants for any given curve, but they assume different values as the compression is varied ; for example, c increases numerically as the amount of cold-work done on the material increases. The curves were the same whether the specimens were compressed between steel or aluminium plates.

The next step was to examine further the compressed specimens which had reached the steady state at room temperature. Taking the 50·5 per cent specimen as a representative sample of the group, this had reached the metastable state in about 350 hours after being cold-worked. During the subsequent 15 weeks the lattice parameter was found to remain constant at the value 4·0412KX. Seventeen weeks after the removal of the stress, the specimen was annealed at 500° C. for three days. This treatment brought the lattice parameter value to $4{\cdot}0406_6$KX—the point A in fig. 2—in agreement with the parameter value obtained with powder containing a full range of particle size. Hence after a compressed specimen had reached the metastable state at room temperature, it required a definite amount of energy to activate the material so as to cause the lattice to contract further to its final equilibrium value.

Grain growth in stressed material.

A close examination of the stationary back-reflection photographs (see fig. 3, Pl. IV.) taken with cold-worked material and extending over a long period of time throughout which the camera and tube assembly had remained undisturbed—the specimen mounted in position and exposures made at frequent intervals of one spot on the surface of the specimen—supplied information concerning grain growth, at room temperature, in the material as it recovered from the effect of the stress to which it had been subjected.

A back-reflection photograph, taken with an annealed plate, showed Laue spots from large crystals, and, in addition, faint broken rings consisting of numerous small diffraction spots indicating the presence of small crystals, the size of which was greater than 10^{-4} mm. A photograph taken immediately after compressing the plate (50 per cent reduction) showed continuous rings which were not of uniform intensity, indicating a certain amount of preferred orientation amongst the crystallites (fig. 3*a*, Pl. IV.). The doublets were well resolved ; the lines were slightly diffuse in outline but they could be accurately measured. There were no definite spots to be seen on the lines immediately after compression. The lattice parameter at this stage was found to be 4·0420KX. About 7½ hours after cold-working the material, several spots had made their appearance. They were very small and fairly uniformly distributed over the diffraction rings. As time went on, the spots became more numerous and those that were already formed increased in size and intensity. The spots did not appear to move relative to the diffraction rings in a radial direction. This was the state of the material deduced from photographs taken up to 76 hours after the compression. After 172 hours, the size and relative intensity of the most intense spots seemed to have become almost stationary, but a few new spots were still appearing. Photographs taken after 840, 1157 and 1344 hours showed exactly the same pattern and characteristics as that taken after 455 hours, there being no apparent change in the number, relative intensity or size of the spots in these photographs. Throughout this series of photographs, in addition to sharp spots due to new crystal growth, there were present continuous rings. The background in the photographs was always exceptionally clear; there was no indication that the scattering in the photograph taken immediately after compression was greater than it was in photographs taken subsequently. Also the continuous diffraction rings showed only slight broadening even in photographs taken immediately after the compression.

These back-reflection photographs reveal, as do the parameter measurements, the changing condition of the material after stress; they show that a steady state is reached, and the parameter measurements supply the further information that this is not the final equilibrium state.

Photographs taken long after the stressed material had reached the metastable condition at room temperature showed some interesting features. The 9·9 per cent specimen had reached the metastable state in about 15 weeks. A photograph taken ten weeks later showed very few diffraction spots—about half a dozen at the usual exposure—and these were small and somewhat faint. The rings were continuous and showed variation of intensity along their circumferences, but the doublets were well resolved. The 27 per cent specimen reached the metastable state in about seven weeks; photographs taken about 14 weeks later showed the same characteristics as the previous photographs, the only difference being that there were a few more spots, but even now the number visible was less than a dozen. A 58·8 per cent specimen photographed about 20 weeks after it had reached the metastable state, yielded a picture with numerous spots dotted irregularly about the diffraction rings, but even in this case the crystallites were not profuse. In a back-reflection photograph taken with pure aluminium after 93 per cent reduction by compression, the crystallites, 14 days after the stress was removed, were numerous. The heavier the cold-work the more pronounced was the recrystallization taking place at room temperature.

Continuous diffraction rings were a feature of all the photographs. There existed in the specimen during these long periods of time either a large number of crystallites or unreleased stresses; or both conditions may have co-existed.

It was of interest to examine material even more severely worked. For this purpose a specimen, after 98·5 per cent reduction *by rolling*, was examined. Ten minutes after cold-working the material, the rings observed in a back-reflection photograph were covered with diffraction spots, and there were only two very short lengths of continuous lines, but these had many spots superposed upon them; 51 hours after rolling no continuous portions remained, the diffraction rings consisting entirely of isolated spots. These photographs differed from those taken with material less severely worked in that now the continuous lines had disappeared and only spots, which were numerous, remained; these spots were reflections from crystals which had grown immediately after or during cold-working and the lattice parameter was that of the metastable state (4·0412KX).

If the deformation of pure aluminium is sufficiently vigorous, recrystallization takes place during or immediately after the deformation. Much of the energy is expended as heat, some of it is stored in the material—momentarily in heavily worked material—and is released during the recrystallization, but a small fraction is always retained and accounts for the metastable state in which the material exists when it has reached a steady stage at room temperature. In this state the crystals possess a slightly enlarged lattice parameter ; to reduce this to its final equilibrium value, heat must be applied. The interesting fact is that the residual energy retained by the material when it is in the metastable state causes the same increase in parameter (for pure aluminium) whatever the amount of cold-work done on the material, or in other words, the increase in parameter corresponding to the metastable state is the same whether a small or a large amount of recrystallization has taken place.

Powder.

A preliminary investigation of pure aluminium powders had revealed a slight increase in the lattice parameter as the size of particle decreased. Before proceeding with the main problem it was important to ascertain the relation between lattice parameter and particle size accurately. Three grades of annealed powders were examined with the results shown in Table I. Over the whole range of particle size investigated, there was a difference of parameter of about 0·0005KX.

TABLE I.

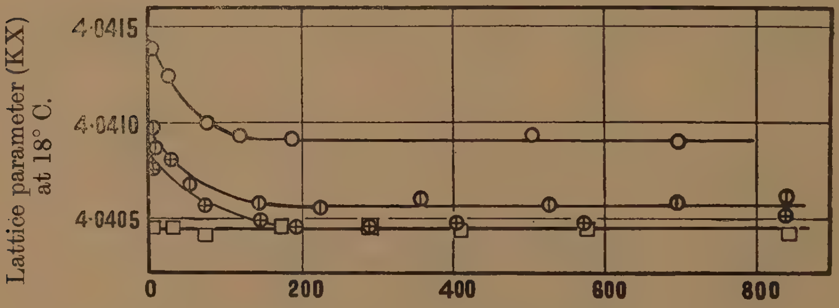
Grade of powder (Al. purity 99·998 per cent)	Lattice parameter
below 350 mesh	$4{\cdot}0409_1$KX
250–350 mesh	$4{\cdot}0405_4$
120–140 mesh	$4{\cdot}0404_4$

Further measurements were made with powders which had not undergone any heat treatment. These unannealed powders yielded structure-spectra, the lines in which were well defined and could be accurately measured, even though the photographs were taken immediately after the powders were prepared. The results obtained with one grade of powder only will be cited as a typical case. With powder of 120–140 mesh grade, the value of the lattice parameter was found to remain constant at room temperature over the whole period of observation, which extended over 843 hours. The mean value was $4{\cdot}0404_5$KX, which is identical with the parameter value of annealed powder of the same grade. Thus aluminium of this degree of purity recovered so rapidly from the severe stresses produced in the filing operation, that no change in the parameter value from that of fully annealed powder could be detected even in photographs taken immediately after the powder was prepared. There was no indication of two stages in the process of recovery to the final equilibrium condition such as was found with stressed plates of pure aluminium.

Powder-foils.

The powders, the lattice parameters of which are given in Table I., were rolled into thin foils in order to investigate how the lattice parameter of the distorted powder grains compared with that of the compressed sheet prepared from a plate of the material. The foils were produced by rolling quantities of powders between two clean sheets of aluminium. By this method the individual grains would be subjected to compressional forces very similar to those used to compress the plates in that they would have freedom to expand laterally. The lattice parameters were measured at different times after compression. The results of these measurements are shown in fig. 4, which also shows the parameter values of unannealed 120–140 mesh powder at different times after the production of the powder by filing. Immediately after compression, the foils exhibit expanded lattices, but unlike compressed sheets from solid plates, the lattice parameter decreases in course of time until it reaches the equilibrium value for the grade of powder concerned. There was no metastable condition observed with any of the foils, in spite of the fact that the initial expanded lattice of the 350 powder-foil had a parameter which exceeded the constant parameter value corresponding to the metastable state found with compressed sheet.

Fig. 4.



Time, in hours, after release in stress.

Recovery curves of foils of compressed aluminium powders.

○—Powder passed through 350 mesh.
⦶— ,, ,, ,, 250 ,, and held by 350 mesh.
⊕— ,, ,, ,, 120 ,, ,, ,, 140 mesh.
□—Unannealed powder of 120–140 mesh grade.

The difference between powder-foil and sheet compressed from a lump of material is that, in the former, the particles are sufficiently compressed to hold together but the particle boundaries still exist, there being no real fusion of one particle with the other, else the curves in fig. 4 would not reach final parameter values characteristic of the grade of powder composing the foil. In compressed sheet, however, the crystal boundaries still exist even though broken up and extended. In the foil the particles are free to recover, as in the powder freshly prepared by filing, but in compressed sheet the deformed crystals do not enjoy the same freedom and the energy supplied by the compression is not so readily released. There is an indication here that the crystal boundaries newly formed by slipping in addition to the original grain boundaries may play an important part in the phenomenon under investigation.

Aluminium Alloys.

To study the influence of impurities on the recovery process of aluminium, it is essential to differentiate between the effects brought about by soluble and insoluble atoms. For this purpose it was decided to prepare different aluminium alloys, some containing atoms that entered into solid solution and others, atoms that were insoluble and which, therefore, would be distributed along the grain boundaries.

Two alloys were made with gallium and lead respectively, the gallium being spectroscopically pure and the lead 99·999 per cent pure. Gallium is soluble and lead insoluble in aluminium. The alloy ingots were subjected to alternate cold-working and annealing several times in order to secure homogeneity and to attain the equilibrium condition.

Aluminium-gallium alloy.

The solid solubility of gallium in aluminium is about 2·4 atomic per cent at 400° C. and about 5·0 atomic per cent at room temperature (Hansen, 1936). To avoid complication which might arise from the effect of cold-work on the solid solubility of gallium in aluminium, it was decided to use a dilute solution for the present purpose ; an alloy containing 0·53 atomic per cent of gallium was accordingly made. After alternate forging and annealing at 400° C. extending over a period of 32 days, the alloy ingot yielded a lattice parameter of $4{\cdot}0414_8$KX. Powders taken from both ends of the ingot and annealed at 400° C. for 17 hours, gave parameter values of $4{\cdot}0414_8$ and $4{\cdot}0415_9$KX. All the values are in close agreement showing that the ingot is homogeneous and in a state of equilibrium.

The ingot was then reduced 53·1 per cent in thickness by compression, and parameter measurements were made at different times after cold-working. The results are plotted in fig. 5. The characteristics of the curve are similar to those of the curves obtained with pure aluminium, an initial high parameter value ($4{\cdot}0426_6$KX) immediately after the compression, falling in course of time to a steady value of about 4·0421KX, which is higher than the equilibrium value (4·0415KX). Similar results were obtained with an alloy containing 3 atomic per cent of gallium.

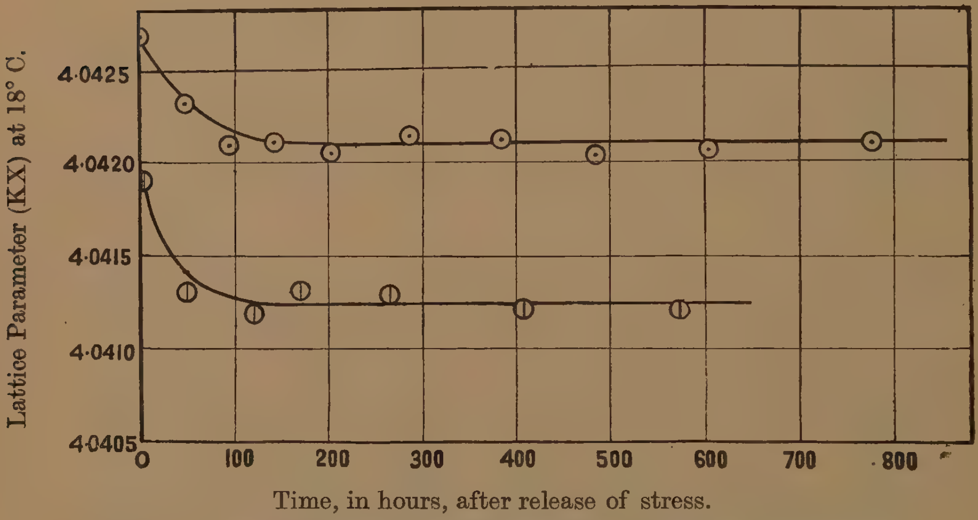
As with pure aluminium, a foil of the alloy made from 250 mesh powder, showed no indication of the metastable state, the final value of the parameter reached being that of the material in the equilibrium state, but the compression of the powder has the effect of immediately raising its parameter value.

Aluminium-lead alloy.

An alloy containing 0·13 atomic per cent of lead yielded a parameter of $4{\cdot}0406_9$KX after being treated over a period of 62 days to alternate cold-working and heating at 320° C. It was then reduced 63·8 per cent in thickness by compression and the lattice parameter measured. The results are shown in fig. 5.

An examination of the curve obtained with the lead alloy, and a comparison with those obtained with pure aluminium in fig. 2, reveals the fact that the insoluble atoms exert no marked influence on the recovery process after compression. The curves are all very similar, the metastable state reached yielding in each case a parameter value of about 4·0412KX.

Fig. 5.



Curves showing the change in lattice parameter of alloys of aluminium with gallium, and lead, after release of stress.

⦶—Al-Pb alloy plate. ⊙—Al-Ga alloy plate.

When, however, the foreign atom enters the lattice, there is a change in the curve, the final metastable value of the parameter being, with gallium, much higher than with pure aluminium or its alloys with insoluble atoms, but the general characteristics of the recovery process were the same as before.

During the investigation of the behaviour of aluminium alloys after release of stress, stationary back-reflection photographs were taken at frequent intervals. In all cases the diffraction doublets were well resolved, but the lines were not so sharply defined as with pure aluminium. Several weeks after cold-working, only a few spots had appeared on the diffraction rings. This indicated that foreign atoms present in the materials, whether in the lattice or along the grain boundaries, hindered recrystallization and grain growth to a very remarkable degree. The residual strain, after the materials have reached the metastable state, is slightly greater on the average in alloys containing soluble atoms than it is in alloys containing insoluble atoms.

DISCUSSION OF THE RESULTS.

When the metallic plate is compressed and the crystals undergo fragmentation, the increased lattice parameter found in the compressed plate might be explained on the ground that the lattice parameter increases as the grain size decreases. But certain observations made in the course of the investigation are opposed to this view. For instance, the grains in material after 98·5 per cent reduction appeared to be larger than those in a 350 grade powder, and yet the parameter of the former is larger than that of the latter, which is contrary to expectation after the results obtained with different particle sizes. The lattice distortion produced by cold-work must be due to some other cause.

It is significant that when powders are compressed into foils, lattice distortion is produced, but after annealing at room temperature for some time, the increased parameter value returns to its final equilibrium value, whereas in the plate it returns to the metastable value, which is higher than the final equilibrium value. The difference between plate and powder foil is that in the latter the grains of which it is composed are independent of each other as in the powder, but in the plate the grains are firmly held together by crystal boundaries which form a network throughout the material. The grains in the powder foil have more freedom to change their shape and size after deformation than those in solid sheet. From these observations it would appear that the crystal boundaries in the solid plate are an important factor in the problem.

When atoms insoluble in aluminium are introduced into the body of the material, they reside in the grain boundaries, and experiment shows that the impure material after compression behaves like pure aluminium. The foreign atoms have little influence on the rate of recovery, and the material finally reaches the same parameter as pure aluminium. But the rate of recrystallization during recovery is slowed up by the presence of these atoms. Powder foils of an alloy of aluminium with an insoluble metal behave like powders of pure aluminium; the lattice parameter after compression is increased, but it returns finally, at room temperature, to the equilibrium value of the powder.

Atoms that are soluble in aluminium enter the crystal lattice, and their effect on the behaviour of the stressed alloy is quite marked in one respect, that the parameter value, which initially assumed a high value after the material was stressed, gradually returned to a constant value—with gallium in solution—which was higher than the metastable value found with pure aluminium or its alloys with insoluble metals. The residual strain in the material was slightly bigger with soluble than with insoluble atoms, but the rate of recovery was unaltered. The rate of recrystallization during the period of recovery showed a marked slowing up as with the insoluble elements.

The lines in X-ray structure-spectra arise from diffraction by a series of atomic planes, so that crystal units must be preserved under the severest conditions of deformation. This is brought about by the slipping of planes over one another, and to preserve crystal units of sufficient size to diffract the X-ray beam several atomic planes must move together. If these planes maintained their relative positions, then the lattice parameter would remain unaltered. But it was found that the lattice parameter, under the conditions of the present experiments, increased after compression, and that the diffraction lines showed slight broadening. This broadening might be due to at least two causes ; firstly, the linear dimensions of the crystallites may be just below about 1000 Å., and secondly, there may be slight variation in the distances between atomic planes. Both causes may be operating together. With pure aluminium it was found that line broadening did not appear to be modified whilst the material was recovering after stress at room temperature. During recovery the lattice parameter changed from a higher to a lower value which remained constant but was not the equilibrium value ; recrystallization also occurred. But in spite of these changes, slight line broadening persisted throughout, so that it would seem reasonable to assume that the line broadening is due mainly to the variation in the distances between the atomic planes in the crystals that diffract the radiation, the mean distance being greater than it is in material in the equilibrium state. On this view, the crystals between consecutive slip planes undergo a shearing strain, that is, between layers of material which experience plastic flow, there exist crystalline layers which experience elastic strain.

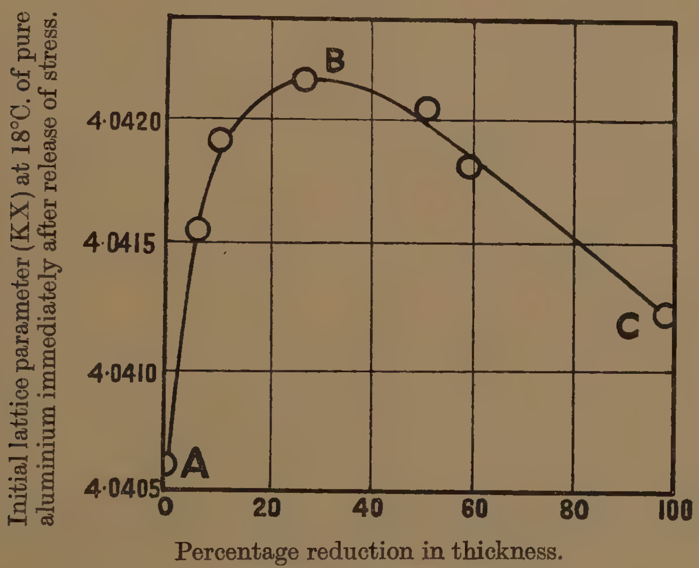
Experiment shows that after the release of stress, the elastic strain diminishes in accordance with an exponential law ; the tangential stresses in the plastic boundaries give way under the influence of the elastic stresses in the crystalline material contained between them. Experiment also shows that the boundaries created by slip during compression play the most important part in the recovery process ; the natural grain boundaries do not seem to have an appreciable effect on the recovery process.

But why do the crystalline layers display a residual strain after they have reached the steady state at room temperature ? This may be explained if it is assumed that the plastic layers will give way only to a certain extent, and that they will be able to support tangential stresses of the magnitude of the residual stresses observed in the material when it has reached the metastable state. The residual stresses are practically the same in all the cases investigated, although there is an indication that they are slightly greater in alloys containing soluble elements.

When the material is severely cold-worked, it returns immediately to the metastable state. In this case the material has been made to flow so freely that the tangential stresses in the plastic layers are very small and the crystalline layers are not strained as they were when the amount of cold-work done on the material was less severe, but even so, residual stresses remain in the recrystallized material which is formed during or immediately after the compression.

The parameter values immediately after the specimens were cold-worked appear to vary irregularly; for instance, the initial value of the parameter after 27 per cent reduction is greater and that after 58·8 per cent reduction smaller than the initial parameter value after 9·9 per cent reduction, an expansion followed by a contraction. This result is somewhat similar to that obtained with copper by Wood (1939). After extrapolating the curves in fig. 2 to zero time, the initial parameter, after different reductions in thickness, yielded the values which are plotted against the percentage reduction in thickness in fig. 6. This curve may be interpreted as follows: along the limb AB the tangential stresses in the surface of the plastic layers bordering the crystalline layers, increase as the deformation of the material increases causing increasing strain of the crystal lattice. This strain reaches a maximum value at B. After passing this maximum the material becomes more plastic, and flows more and more freely as we proceed along the limb BC of the curve. The tangential stresses in the surfaces of the plastic layers thus continuously decrease until, as the reduction in thickness approaches 100 per cent the elastic strain in the crystalline layers reaches the minimum value possible at room temperature, the aluminium then being in a metastable condition. It remains in this metastable state at room temperature until it is again heat treated when the residual stresses are released and the parameter returns to its final equilibrium value.

Fig. 6.



Curve showing the relation between the percentage reduction in thickness of a pure aluminium plate and the initial lattice parameter immediately after the release of stress.

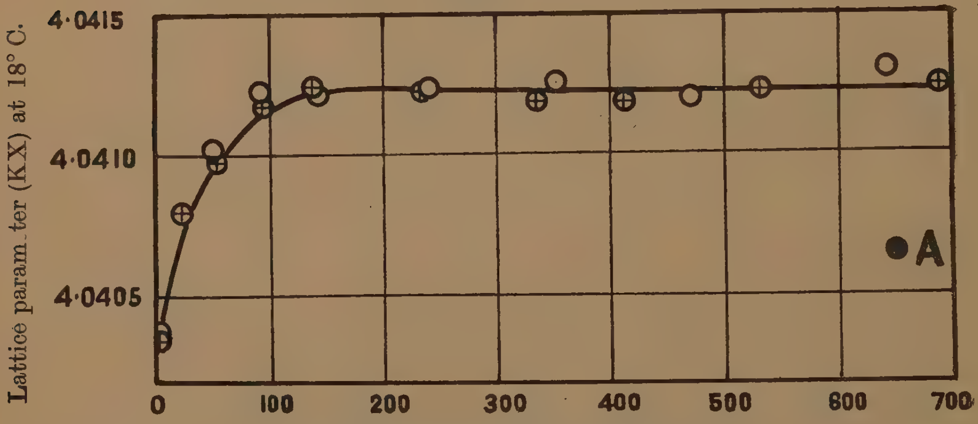
Some interesting phenomena observed with filed and polished surfaces were examined in the light of the foregoing results.

Filed Surfaces.

The filing operation breaks the surface of a plate into small crystals and underneath there must be a layer of material which was subjected to severe cold-work when the filings were torn off. In the first instance two specimens of aluminium in the metastable state were taken, and one face of each was gently filed by rubbing the specimens lightly a few times in two directions at right angles to each other, on the surface of a smooth file. Very little heat was developed and no heat treatment was given after the filing operation.

In a series of photographs taken after different periods of time, the two specimens yielded exactly the same results (see fig. 7). They both gave, after filing, a contracted lattice parameter of 4·0404KX, a value in agreement with that for coarse powder. In course of time the lattice expanded, and in about 150 hours after the filing operation was completed, it reached the steady value of 4·0412KX, the characteristic parameter value of a lump specimen of pure aluminium in the metastable state. One of the specimens which had remained at room temperature for 650 hours was annealed at 500° C. for seven days. This heat treatment brought the parameter value to $4{\cdot}0406_4$KX, in agreement with the mean powder value (point A in fig. 7).

Fig. 7.



Time (hrs.) elapsed after cold working.

Recovery curve of the filed surface of a plate of aluminium (purity 99·998 %).

It appeared from these observations that the crystals produced in the surface of the plate by filing, which had initially a low parameter value, finally assumed the parameter value of the crystals in the body of the material—the metastable value in this case. But this conclusion had to be further examined. This was done with a plate which had been annealed to produce a lattice parameter of 4·0406KX, the final equilibrium value. The filing operation was carried out as before, and the lattice parameter was again found to change from a low initial value to the value 4·0412KX.

Hence whether the crystals in the body of the material were initally in the metastable or in the final equilibrium condition, the crystals in the filed surface grew so that the lattice parameter changed from about 4·0404KX to 4·0412KX. This behaviour can be explained if it is supposed that in the filing operation enough cold-work is done to produce a layer of material in the metastable state, and that the broken strained surface crystals which have initially a low parameter value, change until they eventually attain the lattice parameter of the material to which they are anchored.

Polished Surfaces.

Back-reflection photographs showed that the texture of the lines was different for polished and for filed surfaces. Two typical stationary back-reflection photographs taken with filed and with polished surfaces are shown in figs. 8 (*a*) and 8 (*b*), (Pl. V.). The filed surface yielded diffraction rings consisting of numerous well-defined spots, whereas the polished surface yielded smooth continuous diffraction rings with hardly any spots. Also the lattice parameter of the filed surface started at the low value of 4·0404KX, gradually increasing in course of time to 4·0412KX, but that of the polished surface—the surface was rubbed on fine emery paper—yielded a parameter of $4{\cdot}0412_4$KX immediately after polishing, and remained at this value until the material was heat treated. After heat treatment the parameter was 4·0406KX, the equilibrium value.

The polished surface therefore assumes the metastable state immediately after polishing, and the crystallites in the surface are in a strained condition. The residual stresses remain until the material is suitably heat treated. Polishing and filing produce the same metastable state in the surface as severe cold-work produces throughout the body of the material.

We (E. A. Owen and Y. H. Liu) take this opportunity of acknowledging our indebtedness and thanks to the Department of Scientific and Industrial Research for a grant which enabled us to carry out the work.

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18th December, 1947.

XCIX. *Note on the Field of a Magnet immersed in a Permeable Medium.*

By LEIGH PAGE *.

[Received April 5, 1948.]

IN a footnote to an article published in this journal †, Prof. G. H. Livens states, without further specification, that the proof given in my 'Introduction to Theoretical Physics' for the magnetic field H due to one pole of a long needle-like permanent magnet immersed in an isotropic medium of constant permeability μ is wrong "as it omits terms of the same magnitude as some which are retained." In subsequent correspondence Prof. Livens has made clear to me his objection, which I would now like to answer.

The proof which Livens criticizes depends essentially on Gauss's law, which states that the flux of magnetic induction B out of any closed surface vanishes. To obtain the field at a distance r from the north pole of the needle (the south pole being at such a great distance that its effect can be neglected), a sphere of radius r is described about the north pole as centre. Then, if a is the cross-sectional area of the needle, the zero flux of B out through this spherical surface is

$$(4\pi r^2-a)B_0+aB_i=0,$$

where $B_0=\mu H$ is the magnetic induction outside the needle in the permeable medium and B_i is the magnetic induction inside the needle. If I is the uniform intensity of magnetization inside the magnet and m the strength of the north pole, $B_i=-4\pi I$ and $aB_i=-4\pi aI=-4\pi m$. Hence, neglecting a as compared with $4\pi r^2$,

$$H=\frac{m}{\mu r^2}.$$

Livens's objection seems to rest on his contention that Gauss's law for the flux of B is not valid when the surface or a portion of it lies in a magnet. Therefore we shall cut a slot in the needle and fill the slot with the same permeable medium which surrounds the magnet. Then we have in the slot the magnetic analog of a parallel-plate condenser, with a pole-strength $-I$ per unit area on the one surface and I per unit area on the other, where I is the intensity of magnetization *inside* the needle. According to Livens, this gives rise to a magnetic force $-4\pi I$ in the slot and a magnetic induction $-4\pi\mu I$ instead of the magnetic induction $-4\pi I$ calculated above. But he has forgotten that the permeable medium in the slot has become polarized by the field existing there, with the result that the sum of the pole-strength on the first surface of the slot, plus that on the adjacent surface of the permeable medium lying in the slot, is $-I+(1-1/\mu)I=-I/\mu$. Hence the magnetic force in the slot is $-4\pi I/\mu$ and the magnetic induction is $-4\pi I$.

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* Communicated by the Author.

† Phil. Mag. xxxviii. p. 455 (1947).

Livens's contention that Gauss's law for the flux of B is not valid when the surface passes through a magnet is equivalent to the assertion that tubes of magnetic induction are not continuous when passing through the surface of a magnetic material. This would lead to the absurd conclusion that the induced electromotive force due to change of magnetic flux through a turn of wire containing an iron core would depend upon whether the flux was calculated by integrating over a surface passing through the core or over one which bulged so as to pass around it !

The derivation of the expression for H due to a long needle-like magnet immersed in a magnetic medium of constant μ, which is outlined above, does not pretend to represent more than an approximation which approaches exactitude as the cross-section of the needle is reduced. The exact solution, where the needle is taken to be a very narrow uniformly magnetized prolate spheroid, was published * several years ago.

Sloane Physics Laboratory,
Yale University, New Haven,
Connecticut.
March 25, 1948.

C. *Distribution of Reaction Times for Laminar Flow in Cylindrical Reactors.*

By R. C. L. Bosworth, Ph.D., D.Sc.†

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§ 1. *Introduction.*

In judging the efficiency and performance of a vessel in which a given chemical reaction takes place in a continuous manner, *i. e.* in which the reactants flow in at one end and the resultants out at the other continuously, two factors are involved : the reaction constant or constants, and the reaction time or the time taken by a molecule in passing through the reaction zone. In many cases of calculations based on this quantity (see for example Wenner (1941)) the reaction time is taken as the length of the reaction vessel in the direction of flow divided by the mean stream velocity. Such a procedure may be nearly correct when the reactant flows turbulently through long reaction vessels (as for example in the cracking of petroleum), but in most industrial reactors the reaction velocities are so slow that the flow in vessels of reasonable size is necessarily laminar in nature. In laminar flow it is well known that the different filaments of the stream travel at different speeds. The different sections of the fluid will therefore have different reaction times. At any given rate of flow and given reaction vessel we are concerned not with a single reaction time but with a population of reaction times exhibiting a certain frequency distribution curve.

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* L. Page, Phys. Rev. xliv. p. 112 (1933).

† Communicated by the Author.

Two similar reaction vessels built on different scales will give comparable results only when they show the same distribution curve for the population of reaction times. In general it is not sufficient that the mean reaction time should be the same in each case. As both Forster and Geib (1934) and Damköhler (1936, 1937) have shown that the processes of diffusion and convection are both concerned in determining the composition of the resultant stream.

If the reaction takes place in a single phase, that is, if the reaction is homogeneous, the molecules passing down the centre of the reactor will travel faster than all the others and consequently have the lowest reaction time. If the mean reaction time is just sufficient, under the physical conditions obtaining, to complete the reaction, the central core will emerge with the reaction incomplete. On the other hand, a molecule travelling near the wall will flow slowly and have a long reaction time. For the part of the stream near the wall therefore the reaction may proceed too far.

The problem of the distribution of reaction times involves both the distribution of the stream line (or statistical stream line in the case of turbulent flow) velocity and the problem of the diffusion of the individual molecules from one stream line to another during the transit of the reaction zone. Reactors having the same mean reaction time but different distribution curves would give different efficiencies, and if tested under such different conditions of operation as temperature, flow rate, etc. would appear to give different optimum working conditions. In the development of a chemical project from the laboratory to the plant scale it is usual to commence the study of the reaction in small vessels and to increase the scale in a series of steps. Unless the model reactors give the same distribution curve for reaction times they will give misleading information as to the design of the next larger scale. This subject is therefore worthy of close study.

This paper will be concerned with the distribution of reaction times in unpacked cylindrical reactors, the reaction being either homogeneously catalysed or uncatalysed and the flow being laminar in nature. The paper will deal first with the distribution under the conditions of negligible diffusion and will later take into account the effect of diffusion between the streamlines and also of diffusion along the streamlines.

§ 2. *Laminar Flow without Diffusion.*

Let φ be the pressure gradient acting down a cylindrical reaction vessel of length L and radius R, and let η be the viscosity of the fluid.

Then the velocity v of a filament distance r from the axis is given by the well-known expression due originally to Poiseuille :

$$v=\frac{\phi}{4\eta}(R^2-r^2). \quad \ldots\ldots \quad (1)$$

Similarly the volume Q flowing through the reactor in unit time is given by

$$Q=\int_0^R 2\pi rv\,dr=\frac{\pi\phi}{8\eta}R^4. \quad \ldots\ldots \quad (2)$$

The time (τ) taken by the filament distance r from the axis to travel the length (L) of the reaction vessel is given by the expression

$$\tau=\frac{L}{v}=\frac{4\eta L}{\phi}\frac{1}{R^2-r^2}. \quad \ldots\ldots \quad (3)$$

On differentiation equation (3) becomes

$$\frac{d\tau}{dr}=\frac{8\eta Lr}{\phi}\frac{1}{(R^2-r^2)^2}. \quad \ldots\ldots \quad (4)$$

Consider now a hollow cylinder of fluid of internal radius r and external radius $r+dr$. The time taken by the fluid in this cylinder to pass the reaction zone is given by equation (3) above. The volume of fluid entering this hollow cylinder in unit time is equal to

$$2\pi rv\,dr.$$

The fraction of the total volume of fluid which in any time enters the hollow cylinder we shall write as $F_r dr$. Clearly :

$$\begin{aligned}F_r\,dr&=\frac{2\pi r\phi(R^2-r^2)8\eta}{4\eta\pi\phi R^4}dr\\&=4(R^2-r^2)\frac{r\,dr}{R^4}. \quad \ldots\ldots \quad (5)\end{aligned}$$

The fraction $F_\tau d\tau$ of the molecules which pass through the reaction zone with transit time lying between τ and $\tau+d\tau$ is given by

$$\begin{aligned}F_\tau\,d\tau&=F_r\frac{1}{\frac{d\tau}{dr}}d\tau\\&=4(R^2-r^2)^3\frac{r}{R^4}\frac{\phi}{8\eta Lr}d\tau\\&=\frac{32\eta^2L^2}{\phi^2R^4}\cdot\frac{1}{\tau^3}d\tau. \quad \ldots\ldots \quad (6)\end{aligned}$$

This expression holds for all large values of τ down to a certain minimum value τ_0 representing the time of transit for the central core of fluid.

For all values of τ less than τ_0 $F_\tau=0$. Thus equation (6) may be more completely written

$$F_\tau d\tau = H(\tau-\tau_0)\frac{32\eta^2L^4}{\phi^2R^4}\frac{1}{\tau^3}d\tau, \quad \dots\dots \quad (7)$$

where $H(x)$ is the Heaviside function defined by

$$H(x)=0 \quad x<0,$$
$$H(x)=1 \quad x\geqslant 0.$$

The minimal value τ_0 may be obtained from equation (3) by putting $r=0$. Thus

$$\tau_0=\frac{4\eta L}{\phi}\frac{1}{R^2}. \quad \dots\dots\dots \quad (8)$$

Equation (7) may thus further be simplified to

$$F_\tau=H(\tau-\tau_0)\frac{2\tau_0^2}{\tau^3}. \quad \dots\dots\dots \quad (9)$$

Thus the frequency distribution in a cylindrical reactor (with negligible diffusion) depends only on the minimum frequency and not on any geometrical parameter of the vessel. Two different cylindrical vessels, one long and thin and with a high flow rate and the other short and wide and associated with a low flow rate, will give the same distribution curves if the minimal transit times are adjusted to be the same.

We shall use the symbol τ^* to denote the dimensionless ratio of the reaction time to that of the central core, thus:

$$\tau^*=\tau/\tau_0. \quad \dots\dots\dots\dots \quad (10)$$

Then $F_{\tau^*}d\tau^*$ denoting the fraction of the stream exhibiting a relative reaction time lying between τ^* and $\tau^*+d\tau^*$ is given by

$$F_{\tau^*}=H(\tau^*-1)\frac{2}{\tau^{*3}}. \quad \dots\dots\dots \quad (11)$$

The distribution function is that of an inverse cube with a sharp cut-off at $\tau^*=1$. The mean relative reaction time τ^*_m is given by

$$\tau^*_m=\frac{\int_1^\infty \tau^*F_{\tau^*}d\tau^*}{\int_1^\infty F_{\tau^*}d\tau^*}$$
$$=2.$$

The mean reaction time is $2\tau_0$, or twice the minimum reaction time.

§ 3. *The Effect of Interlaminar or Radial Diffusion.*

In the discussion above we have supposed that the individual molecules travel only along the fluid stream lines. We will now take into account the fact that there is a certain amount of exchange of molecules between the different lamellæ and that this interlaminar diffusion modifies the nature of the distribution function.

As above, we consider a hollow cylinder of fluid with internal radius r and external radius $r+dr$ entering the reaction zone. The fraction $F_r dr$ of the total stream entering this hollow cylinder is given by equation (5) above. During passage of the reaction zone a certain subfraction diffuses from the hollow cylinder. The stream velocity and time of transit of this subfraction will thus be modified. We will assume that the effective time of transit for a subfraction $f(x)dx$ is that appropriate to the distance x from the centre of the tube attained by the diffusing molecule in half the time of transit of the vessel. The time of transit is now given by

$$\tau=\frac{4\eta L}{\phi}\frac{1}{R^2-x^2}. \qquad \ldots \ldots \quad (12)$$

The probability (p) that any particular molecule will in a time $\frac{1}{2}\tau$ diffuse a distance $\zeta=x-r$ is proportional to

$$e^{-\zeta^2/2D\tau},$$

where D is the appropriate diffusion coefficient. Diffusion a distance ζ from a hollow cylinder of radius r can result in the particular molecule moving outwards to a cylindrical surface of radius $r+\zeta$, or inwards to a cylindrical surface of radius $r-\zeta$. If we associate an equal probability with each unit area of these two cylindrical surfaces the relative amounts of material diffusing outwards and inwards are respectively proportional to

$$2\pi x e^{-\zeta^2/2D\tau} \quad \text{and} \quad 2\pi(2r-x)e^{-\zeta^2/2D\tau}.$$

Accordingly the subfraction $f(x)dx$ is given by

$$f(x)=Axe^{-\zeta^2/2D\tau}. \qquad \ldots \ldots \quad (13)$$

The constant A is to be evaluated from the condition that diffusion does not alter the total quantity of material, thus :

$$\int_{-8}^{+8} f(x)dx=1$$

or

$$A=r\sqrt{2\pi D\tau}. \qquad \ldots \ldots \quad (14)$$

In our specific problem the fluid is restricted by the walls to values of x between $\pm$R. When the cylindrical wave of diffusing particles moving outwards strikes the wall, we will suppose that it is reflected and converges again towards the centre. Likewise we will suppose the particles diffusing inwards (negative values of ζ) can pass right through the singularity at the centre and spread out as a divergent wave towards the walls. Any particular value of x may therefore be attained by a whole series of different values of ζ, namely at $x-r$, $-x-r$, $-2R-r+x$, $2R-x-r$, $2R+x-r$, $-2R-r-x$, $4R-x-r$, $-4R-r+x$, $4R+x-r$, etc.

We may thus rewrite equation (13) taking into account multiple reflections from the walls as

$$f(x)=\frac{1}{\sqrt{2\pi D\tau}}\frac{x}{r}\{e^{-(x-r)^2/2D\tau}+e^{-(x+r)^2/2D\tau}+e^{-(2R-r+x)^2/2D\tau}$$

$$+e^{-(2R-r-x)^2/2D\tau}+e^{-(2R+r-x)^2/2D\tau}+e^{-(2R+r+x)^2/2D\tau}+\ldots\} \quad . \quad (13a)$$

From equation (12) we have the following relationship between x and τ:

$$xdx=\frac{2\eta L}{\phi\tau^2}d\tau \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and

$$x^2=R^2-\frac{4\eta L}{\phi\tau}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Equation (8) may be rewritten in the form

$$R=\frac{4\eta L}{\phi\tau_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8a)$$

And equation (16) becomes

$$x=R\sqrt{1-\frac{\tau_0}{\tau}}=R\sqrt{1-\frac{1}{\tau^*}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$=Ra \text{ (say)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17a)$$

The subfraction $f_\tau d\tau$ arising from $F_r dr$ and having a transit time lying between τ and $\tau+d\tau$ is thus given by

$$f_\tau=\frac{1}{\sqrt{2\pi D\tau}}\frac{1}{r}\frac{2\eta L}{\phi\tau^2}\{e^{-(Ra-r)^2/2D\tau}+e^{-(Ra+r)^3/2D\tau}+e^{-(R\overline{2-a}-r)^2/2D\tau}$$

$$+e^{-(R\overline{2+a}-r)^2/2D\tau}+e^{-(R\overline{2-a}+r)^2/2D\tau}+e^{-(R\overline{2+a}+r)^2/2D\tau}$$

$$+e^{-(R\overline{4-a}-r)^2/2D\tau}\ldots\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The fraction $F_\tau d\tau$ of the total molecular stream which passes through the reaction zone with transit time lying between τ and $\tau+d\tau$ is given by the integral

$$F_\tau=\int_0^R f_\tau F_r dr$$

$$=H(\tau-\tau_0)\frac{2\eta L}{\phi\tau^2}\frac{1}{\sqrt{2\pi D\tau}}\sum_{i=0}^{\infty}\int_0^R(R^2-r^2)\{e^{-(Rb_i-r)^2/2D\tau}+e^{-(Rb_i+r)^2/2D\tau}\}.$$

$$. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The integral is to be summed over all values of b_i:

$$b_0=a,$$
$$b_1=2-a,$$
$$b_2=2+a,$$
$$b_3=4-a, \text{ etc.}$$

These quantities are related by the set of equations :

$$b_0-1=-(b_1-1)$$

and

$$b_i+1=b_{i+2}-1. \tag{20}$$

We have two integrals to solve, namely,

$$\int_0^R (R^2-r^2)e^{-(Rb-r)^2/2D\tau}\,dr \quad \text{and} \quad \int_0^R (R^2-r^2)e^{-(Rb+r)^2/2D\tau}\,dr.$$

In the first integral we put

$$r=Rb+\sqrt{2D\tau}\,y$$

and the integral then becomes

$$\sqrt{2D\tau}\int_{-Rb/\sqrt{2D\tau}}^{R(1-b)/\sqrt{2D\tau}} \{R^2(1-b^2)-2Rb\sqrt{2D\tau}\,y-2D\tau y^2\}e^{-y^2}\,dy,$$

which may readily be shown to be

$$\sqrt{2\pi D\tau}\left[\tfrac{1}{2}\{R^2(1-b^2)-D\tau\}\left\{\operatorname{erf}\frac{R(1-b)}{\sqrt{2D\tau}}+\operatorname{erf}\frac{Rb}{\sqrt{2D\tau}}\right\}+\frac{1}{2\sqrt{\pi}}R\sqrt{2D\tau}\right.$$
$$\left.\times\left\{(b+1)\exp\left(\frac{-R^2(1-b)^2}{2D\tau}\right)-b\exp\left(\frac{-R^2b^2}{2D\tau}\right)\right\}\right]$$

where exp (x) is the exponential function e^x and erf x the Gaussian error function defined by

$$\operatorname{erf} x=\frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\,dy \qquad \text{(Jeffreys, p. 111, 1931).}$$

In the second integral we put

$$r=Rb-\sqrt{2D\tau}\,y$$

and the integral then becomes

$$\sqrt{2D\tau}\int_{+Rb/\sqrt{2D\tau}}^{R(b+1)/\sqrt{2D\tau}} \{R^2(1-b^2)+2Rb\sqrt{2D\tau}\,y-2D\tau y^2\}e^{-y^2}\,dy$$

which becomes

$$\sqrt{2\pi D\tau}\left[\tfrac{1}{2}\{R^2(1-b^2)-D\tau\}\left\{\operatorname{erf}\frac{R(b+1)}{\sqrt{2D\tau}}-\operatorname{erf}\frac{Rb}{\sqrt{2D\tau}}\right\}-\frac{1}{2\sqrt{\pi}}R\sqrt{2D\tau}\right.$$
$$\left.\times\left\{(b-1)\exp\left(\frac{-R^2(1+b)^2}{2D\tau}\right)-b\exp\left(\frac{-R^2b^2}{2D\tau}\right)\right\}\right].$$

Equation (19) may thus be written in the form

$$F_\tau=H(\tau-\tau_0)\frac{8\eta L}{R^2\phi\tau^2}\sum_{i=0}^{\infty}\left[\left\{\left(\tfrac{1}{2}-\tfrac{1}{2}b_i^2-\frac{D\tau}{2R^2}\right)\right\}\left\{\operatorname{erf}\frac{R(b+1)}{\sqrt{2D\tau}}-\operatorname{erf}\frac{R(b-1)}{\sqrt{2D\tau}}\right\}\right.$$
$$+\frac{1}{2\sqrt{\pi}}\frac{\sqrt{2D\tau}}{R}\left\{(b+1)\exp\left(\frac{-R^2(1-b)^2}{2D\tau}\right)\right.$$
$$\left.\left.-(b-1)\exp\left(\frac{-R^2(1+b)^2}{2D\tau}\right)\right\}\right].$$

We now make the substitution

$$B=\frac{R}{\sqrt{2D\tau_0}}=R^2\sqrt{\frac{\phi}{8D\eta L}}, \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where B is a dimensionless quantity involving the radius, the diffusion coefficient and the minimal reaction time. In terms of B and τ^*, the (dimensionless) relative reaction time, the distribution function may be expressed as

$$\begin{aligned} F_{\tau^*}=H(\tau^*-1)\frac{2}{\tau^{*3}}\sum_{i=0}^{\infty}\Bigg[&\tfrac{1}{2}\tau^*\left(1-b_i^2-\frac{\tau^{*2}}{2B^3}\right)\left(\operatorname{erf}\frac{B(b_i+1)}{\sqrt{\tau^*}}-\operatorname{erf}\frac{B(b_i-1)}{\sqrt{\tau^*}}\right)\\ &+\frac{1}{2\sqrt{\pi}}\frac{\sqrt{\tau^{*3}}}{B}\left\{(b_i+1)\exp\left(\frac{-B^2(1-b_i)^2}{\tau^*}\right)\right.\\ &\left.-(b_i-1)\exp\frac{-B^2(1+b_i)^2}{\tau^*}\right\}\Bigg] \quad . \quad . \quad . \quad (22) \end{aligned}$$

$$=H(\tau^*-1)\frac{2}{\tau^{*3}}\Psi(B,\tau^*). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The function Ψ which takes into account the effect of interlaminar diffusion is a function only of B and τ^*. All reaction vessels having the same mean reaction time and the same values for the dimensionless parameters B will exhibit the same reaction time distribution functions.

All the b_i's which occur in the series for $\Psi(B,\tau^*)$ are greater than unity with the single exception of b_0 and

$$b_0 \leqslant 1.$$

Accordingly all the error functions except $\operatorname{erf}\dfrac{B(b_0-1)}{\sqrt{\tau^*}}$ are positive and we may express Ψ in the form

$$\begin{aligned} \Psi=\tfrac{1}{2}\tau^*\Bigg[&\left\{\left(1-b_0^2-\frac{\tau^*}{2B^2}\right)\left(\operatorname{erf}\frac{B}{\sqrt{\tau^*}}(b_0+1)+\operatorname{erf}\frac{B}{\sqrt{\tau^*}}(1-b_0)\right)\right\}+\sum_{i=0}^{\infty}\\ &\times\left\{\left(1-b_i^2-\frac{\tau^*}{2B^2}\right)\left(\operatorname{erf}\frac{B}{\sqrt{\tau^*}}(b+1)-\operatorname{erf}\frac{B}{\sqrt{\tau^*}}(b-1)\right)\right\}\Bigg]\\ &+\frac{1}{2\sqrt{\pi}}\frac{\sqrt{\tau^{*3}}}{B}\left\{\sum_{i=0}^{\infty}(b_i+1)e^{-\frac{B^2}{\tau^*}(b_i-1)^2}-\sum_{i=0}^{\infty}(b_i-1)e^{-\frac{B^2}{\tau^*}(b_i+1)^2}\right\} \\ &\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24) \end{aligned}$$

A simplified approximate value of Ψ valid for large values of B may be obtained by expanding the error functions as a series in the argument thus :

$$\operatorname{erf} x=1-\frac{1}{\sqrt{\pi}x}e^{-x^2}\left(1-\frac{1}{2x^2}+\frac{3}{4x^4}-\ldots.\right).$$

To this approximation equation (24) becomes

$$\begin{aligned}\Psi&=1-\frac{\tau^{*2}}{2\mathrm{B}^2}+\frac{\tau^{*5/2}}{2\sqrt{\pi}\mathrm{B}^3}\sum_{i=0}^{\infty}\left\{\frac{1}{(b_i-1)^2}e^{-\frac{\mathrm{B}^2}{\tau^*}(b_i-1)^2}+\frac{1}{(b_i+1)^2}e^{-\frac{\mathrm{B}^2}{\tau^*}(b_i+1)^2}\right\}\\&=1-\frac{\tau^{*2}}{2\mathrm{B}^3}+\frac{\tau^{*3/2}}{\sqrt{\pi}\mathrm{B}^4}\left\{\frac{1}{(\sqrt{\tau^*}-\sqrt{\tau^*-1})^2}e^{-\frac{\mathrm{B}^2}{\tau^{*2}}(\sqrt{\tau^*}\ \sqrt{\tau^*-1})^2}\right.\\&\qquad+\frac{1}{(\sqrt{\tau^*}+\sqrt{\tau^*-1})^2}e^{-\frac{\mathrm{B}^2}{\tau^*}(\sqrt{\tau^*}+\sqrt{\tau^*-1})^2}+\frac{1}{(3\sqrt{\tau^*}-\sqrt{\tau^*-1})^2}\\&\qquad\left.\times e^{-\frac{\mathrm{B}^2}{\tau^{*2}}(3\sqrt{\tau^*}-\sqrt{\tau^*-1})^2}+\ldots\right\}.\end{aligned}$$

Of this series of exponential terms all but the first two are negligibly small, provided the quantity B is large. Accordingly we have for the function Ψ:

$$\begin{aligned}\Psi=1-\frac{\tau^*}{2\mathrm{B}^2}+\frac{\tau^{*3/2}}{\sqrt{\pi}\mathrm{B}^3}&\left\{\frac{1}{(\sqrt{\tau^*}-\sqrt{\tau^*-1})^2}e^{-\frac{\mathrm{B}^2}{\tau^{*2}}(\sqrt{\tau^*}-\sqrt{\tau^*-1})^2}\right.\\&\left.+\frac{1}{(\sqrt{\tau^*}+\sqrt{\tau^*-1})^2}e^{-\frac{\mathrm{B}^2}{\tau^{*2}}(\sqrt{\tau^*}+\sqrt{\tau^*-1})^2}\right\}.\quad .\quad .\quad .\quad (25)\end{aligned}$$

Except when we are dealing with a system in which $\tau^*\leqslant 2$, equation (25) may be further simplified by omitting the second exponential in braces.

§ 4. *The Effect of Intralaminar or Longitudinal Diffusion.*

The time of transit of any particular molecule through the reaction zone will also be modified if its molecular motion carries it either ahead or behind its initial position in the fluid stream line. Thus if it diffuses ahead of the stream velocity its transit time will be reduced, and conversely if it diffuses backwards against the streaming motion its transit time will be increased.

We will suppose that the length of the reaction vessel is L. Let again τ be the time taken for a particular point A on the stream line to travel the distance L. The probability ($p_x dx$) that a molecule occupying the point A when the stream enters the reaction zone will have moved a distance not less than x or more than $x+dx$ ahead (or behind) the stream after a time τ is given by

$$p_x dx=\frac{1}{\sqrt{4\pi\mathrm{D}\tau}}e^{-x^2/4\mathrm{D}\tau}dx.$$

The molecules which move behind by a distance x will have a transit time τ' given by

$$\tau'=\tau\frac{\mathrm{L}}{\mathrm{L}+x}.$$

We have therefore for the probability $p(\tau, \tau')d\tau'$ that a molecule travelling along a stream line of transit time τ, will itself show a transit time lying between τ' and $\tau'+d\tau'$, the expression

$$p(\tau, \tau')d\tau' = \frac{1}{2\sqrt{\pi D\tau'}} e^{-L^2\left(\frac{\tau'}{\tau}-1\right)^2/4D\tau'} \cdot \frac{\tau L}{\tau'^2} d\tau'. \quad . \quad . \quad . \quad (26)$$

The probability that the stream line point has a transit time lying between τ and $\tau+d\tau$ is, as above given by

$$F_\tau d\tau = H(\tau-\tau_0)\frac{2\tau_0}{\tau^3}\Psi(B, \tau)d\tau. \quad . \quad . \quad . \quad . \quad . \quad (23\ a)$$

Consider now the whole stream entering the reaction zone. The fraction with a value of the molecular transit time lying between τ' and $\tau'+d\tau'$ is given by $\Phi_{\tau'}d\tau'$, where

$$\Phi_{\tau'} = \int_0^\infty H(\tau-\tau_0)\frac{\tau_0^2}{\tau^2}\frac{L}{\sqrt{\pi D\tau'^5}} e^{-\frac{L^2}{4D\tau'}\left(\frac{\tau}{\tau'}-1\right)^2} \Psi(B, \tau)d\tau. \quad . \quad (27)$$

$\Psi(B, \tau)$ under conditions of comparatively mild diffusion, as we have seen above, does not vary appreciably from unity. Accordingly we may write it as $\Psi(B, \tau')$ and take it outside the integral sign. Equation (27) now becomes

$$\Phi_{\tau'} = \Psi(B, \tau')\frac{\tau_0^2 L}{\sqrt{\pi D\tau'^5}}\int_{\tau_0}^\infty \frac{1}{\tau^2} e^{-\frac{L^2}{4D\tau'}\left(\frac{\tau}{\tau'}-1\right)} d\tau \quad . \quad . \quad . \quad (28)$$

$$= \Psi(B, \tau')\frac{\tau_0^2 L}{\sqrt{\pi D\tau'^5}}\int_{\frac{\tau_0}{\tau_1}-1}^\infty \frac{1}{1+u^2} e^{-L^2u^2/4D'} du.$$

The expression $1/1+u^2$ is approximately e^{-u^2} when u is small, and when u is large the contribution to the whole integral is negligible anyhow. Thus we may write the equation above in the approximate form

$$\Phi_{\tau'} \simeq \Psi(B, \tau')\frac{\tau_0^2 L}{\sqrt{\pi D\tau'^5}}\int_{\frac{\tau_0}{\tau'}-1}^\infty e^{-(L^2/4D\tau'+1)} du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

$$= \Psi(B, \tau')\frac{\tau_0^2 L}{\sqrt{\pi D\tau'^5}}\frac{1}{2\tau'}\sqrt{\frac{4\pi D\tau'}{L^2+4D\tau'}}\left(1+\operatorname{erf}\sqrt{\frac{L^2+4D\tau'}{4D\tau'}}\left(1-\frac{\tau_0}{\tau_1}\right)\right)$$

$$= \Psi(B, \tau')\frac{\tau_0^2}{\tau'^3}\left\{1+\operatorname{erf}\frac{L}{\sqrt{4D\tau'}}\left(1-\frac{\tau_0}{\tau'}\right)\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

the last expression being obtained using the fact that for all problems involving only mild diffusion the length of the reaction zone L will be very large in comparison with $\sqrt{4D\tau'}$. Equation (30) may be written in an alternative form involving only dimensionless quantities by using the relative reaction time τ^* defined here as

$$\tau^* = \tau'/\tau_0.$$

This alternative form reads

$$\Phi_{\tau^*}=\Psi(\mathrm{B},\tau^*)\frac{1}{\tau^{*3}}\left\{1+\operatorname{erf}\frac{\mathrm{L}}{\sqrt{4\mathrm{D}\tau_0}}\left(\frac{1}{\sqrt{\tau^*}}-\frac{1}{\sqrt{\tau^{*3}}}\right)\right\}. \quad . \quad . \quad (31)$$

Since $\mathrm{L}/\sqrt{4\mathrm{D}\tau_0}$ is a large quantity the error function will be sensibly equal to unity for all values of τ^* except those in the immediate vicinity of unity. For $\tau^*=1$ the error function takes the value zero and for values of τ^* less than unity the error function becomes negative, rapidly tending to the values -1 as τ^* falls below unity.

The effect of intralaminar diffusion is thus to replace the sharp cut-off at the lower frequencies, represented by the Heaviside function, $\mathrm{H}(\tau-\tau_0)$, by the more gradual change represented by

$$1+\operatorname{erf}\frac{\mathrm{L}}{\sqrt{4\mathrm{D}\tau_0}}\frac{1}{\sqrt{\tau^*}}\left(1-\frac{1}{\tau^*}\right).$$

The drop to zero frequencies in the population curve is thus the sharper the larger the value of the dimensionless quantity $\mathrm{L}/\sqrt{4\mathrm{D}\tau_0}$. A given reaction (given value of τ_0) will give a reaction time population curve which will tend the more nearly to the curve of inverse cubes the longer the vessel or the larger the diameter.

§ 5. *Discussion.*

In the absence of the modifications produced by diffusion the distribution function for the reaction times of a fluid moving in stream lines through a cylindrical reactor consists as we have seen of an inverse cube-law curve with a sharp cut-off at the minimal reaction time. As equations (23), (24), (25) and (31) show diffusion of the reacting molecules effects two modifications in this distribution curve. In the first place the sharp cut-off at the lower end of the curve is replaced by one which is the more gradual the lower the value of the dimensionless quantity

$$\frac{\mathrm{L}}{\sqrt{4\mathrm{D}\tau_0}},$$

where τ_0 is the reaction time at which the frequency curve would cut off in the absence of all diffusion. Due to the operation of diffusion some molecules have a reaction time less than τ_0.

In the second place the operation of diffusion replaces the inverse cube law distribution curve by one in which the frequencies at higher reaction times are somewhat higher (equation (22)). This flattening of the distribution curve at the higher reaction times is the more pronounced the lower the value of the dimensionless quantity

$$\frac{\mathrm{R}}{\sqrt{2\mathrm{D}\tau_0}}.$$

For any given chemical system the quantity τ_0 is determined by the rate constant of the controlling reaction. In consequence, if the same chemical system is studied in a number of vessels of different sizes the values of the applied pressure gradients (ϕ) must of necessity be adjusted to give the same values of τ_0 in all the vessels concerned. Both types of departure from the inverse cube power law is therefore less pronounced in the larger vessels, provided the flow remains laminar in nature. With this one restriction (laminar flow), in vessels sufficiently large the distribution curve becomes completely independent of the ratio of length to radius, so that we are at liberty to choose any value for this ratio without upsetting the relative proportions of the molecular assembly exposed to reaction conditions for stated times.

In an inverse cube law distribution 99 per cent of the population lies between relative reaction times (τ^*) of 1 and 10. We may regard the effect of radial diffusion as negligible provided that nowhere within this range is the value of F_τ changed by diffusion by more than 1 per cent. In other words the effect of radial diffusion will be regarded as negligible provided,

$$\Psi(B, \tau^*) < 0{\cdot}01 \quad \ldots\ldots\ldots\ldots \quad (32)$$

for $1 \leqslant \tau^* \leqslant 10$.

Reference to the approximate equation (25) shows that Ψ is the largest at the largest values of τ^*. At $\tau^*=10$, Ψ attains the value 1·01 (*i. e.* is 1 per cent. greater than unity) for a value of B (or $R/\sqrt{2D\tau_0}$) equal to 13. The effect of radial diffusion may therefore be neglected provided that

$$R > 13\sqrt{2D\tau_0}$$

or

$$R > 18\sqrt{D\tau_0}. \quad \ldots\ldots\ldots\ldots \quad (33)$$

In the absence of diffusion the *mode* of the distribution, or the point at which the reaction time population curve attains the maximum frequency, occurs at $\tau=\tau_0$. The operation of longitudinal diffusion shifts the mode to values of τ greater than τ_0, by replacing the sharp cut-off, $H(\tau-\tau_0)$ by

$$\tfrac{1}{2}\left[1+\operatorname{erf}\frac{L}{\sqrt{4D\tau_0}}\frac{1}{\sqrt{\tau^*}}\left(1-\frac{1}{\tau^*}\right)\right].$$

We will assume that the effect of longitudinal diffusion is negligible when the shift in the position of the mode is less than $\delta\tau^*=0{\cdot}01$, or approximately

$$\frac{L}{\sqrt{4D\tau_0}}(1-0{\cdot}99) > 1{\cdot}8\,\dagger$$

or

$$L > 360\sqrt{D\tau_0}. \quad \ldots\ldots\ldots\ldots \quad (34)$$

Since it is, in chemical engineering circles, regarded as the normal practice to make the length of a reaction vessel at least ten times the diameter, *i. e.* twenty times the radius, condition (34) is automatically satisfied when condition (33) holds. For all lengths and radii satisfying the above inequalities the reaction time distribution function may be regarded as independent of the relative shape of the reaction vessel.

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† The number 1·8 has been selected because (to two places of decimals erf 1·8=0·99.

§ 6. *Examples.*

Example (a). *The Oxidation of Nitrosyl Chloride.*

Wenner (1941) pages 243–53 discusses the kinetics of this reaction from the point of view of industrial design. At 300° C. and 5 atmospheres pressure the stoichiometrical mixture of nitrosyl chloride and air requires a mean reaction time ($2\tau_0$) of 8·5 seconds. τ_0 is thus 4·25 seconds. The diffusion coefficient for NOCl in air may be estimated by means of a modified form of the Sutherland equation given by Sherwood (1937) page 78, namely,

$$D=0{\cdot}0043\frac{T^{3/2}}{P(V_A^{1/3}+V_B^{1/3})^2}\sqrt{\frac{1}{M_A}+\frac{1}{M_B}}, \quad . \quad . \quad . \quad . \quad (35)$$

where

D is the diffusion coefficient in cm.^{2} per sec.,

P the total pressure in atmospheres,

T the absolute temperature,

M_A and M_B are the molecular weights of the two constituents, and

V_A and V_B the corresponding molecular volumes.

On inserting the appropriate numerical values the diffusivity of NOCl in air at 300° C. and 5 atmospheres pressure becomes 0·056 cm.^{2}/sec.

Accordingly we have

$$\sqrt{D\tau_0}=0{\cdot}48 \text{ cm.}$$

For studies in laboratory equipment where R is of the order of 1 cm. or less the value of B is of the order unity and diffusion produces a very profound modification of the reaction time distribution curve, and not until $R=8{\cdot}5$ cm. (or 3·5 inches) may the effect of diffusion be regarded as negligible. Similarly longitudinal diffusion may only be neglected in reaction vessels longer than 170 cm. (6 feet).

The application for the whole of the conditions given in this paper for the vanishing of the effect due to diffusion is dependent on a laminar nature for the flow. Turbulent flow gives a radial velocity distribution very different from parabolic and further gives an augmented effective diffusion coefficient. It is accordingly desirable to examine the Reynolds's numbers for the flowing reaction mixtures. We will assume that the length of the reactor is twenty times the radius (R). The mean velocity of flow is therefore

$$\frac{20R}{8{\cdot}5} \text{ cm. per sec.}$$

The kinematic viscosity is taken as that of air at 300° C. and 5 atmospheres, namely 0·062 Stokes. The Reynolds's number is therefore

$$75R^2.$$

Conditions of turbulent flow (Reynolds's number=2100) are therefore attained at a radius of

$$5{\cdot}3 \text{ cm.}$$

Turbulent flow is therefore attained before the radius has become sufficiently large to permit the effect of diffusion to be neglected.

Examples (b). *The Thermal Cracking of Acetic Acid.*

When acetic acid vapour containing small amounts of a catalyst such as phosphoric acid is passed through a reaction vessel at 750° C. and atmospheric pressure, the acid is dehydrated with the formation of acetic anhydride as the major product, according to the chemical equation

$$2CH_3COOH \rightarrow (CH_3CO)_2O + H_2O.$$

Experiments have shown that the mean reaction time ($2\tau_0$) required for this process is 2·2 seconds. We thus have

$$\tau_0 = 1{\cdot}1 \text{ sec.}$$

The value of D estimated by means of equation (35) amounts to

$$0{\cdot}39 \text{ cm.}^2 \text{ per sec.}$$

We accordingly have

$$\sqrt{D\tau_0} = 0{\cdot}65 \text{ cm.}$$

In this system therefore effects of radial diffusion are negligible provided R is greater than 12 cm., and the effects of longitudinal diffusion negligible provided L is greater than 240 cm.

On assuming that the kinematic viscosity is equal to 2/3 the diffusivity †, namely

$$0{\cdot}26 \text{ Stokes,}$$

and that the length of the reactor is twenty times the radius we find that the critical Reynolds's number is reached at a radius of

$$5{\cdot}5 \text{ cm.}$$

In other words the flow becomes turbulent before the reactor can be made sufficiently large to render the effect of diffusion on the distribution of reaction times negligible. It should be mentioned of course that conditions for the attainment of a large radius with laminar flow are less favourable if extremely elongated reaction vessels are used, as the use of such vessels would entail faster velocities.

We may conclude therefore that, although the effect of diffusion in modifying the reaction time distribution curve can be made negligible by using sufficiently large reaction vessels, it frequently happens that conditions of turbulent flow set in before the vessels are made sufficiently large.

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† The inverse fifth power law of attraction between molecules gives :
Diffusivity=1·5 × kinematic viscosity.

This conclusion may be regarded as a direct consequence of the fact that in gaseous systems the diffusivity (D) is of the same order as the kinematic viscosity (ν). Let ν be taken as SmD, where Sm is the Schmidt number, a dimensionless quantity, usually lying between 0·6 and 0·9 in gaseous systems. Let τ_0 be arbitrary and let us consider the condition most favourable for simultaneous laminar flow and negligible diffusion, namely

$$L=20R.$$

We have for the condition of laminar flow

$$\frac{2vR}{\nu}<2100,$$

where v the velocity of flow is $20R/2\tau_0$ or $10R/\tau_0$.

Thus the condition for laminar flow becomes

$$\frac{20R^2}{SmD\tau_0}<2100$$

or

$$R<10{\cdot}2\sqrt{SmD\tau_0}. \qquad (36)$$

While the condition for negligible diffusion reads

$$R>18\sqrt{D\tau_0}. \qquad (37)$$

In gaseous systems where Sm is of the order unity these conditions are mutually exclusive. In liquid systems, however, the Schmidt number is a large quantity ranging from 10^3 for limpid liquids to figures larger than 10^9 for highly viscous liquids, so that equations (36) and (37) are no longer incompatable and there may be a considerable range of reactor sizes in which the flow is laminar and yet the reaction time distribution function effectively uninfluenced by diffusion.

Summary.

The molecules of a stream of fluid passing through a cylindrical reactor do not all spend the same time in the reaction zone but exhibit a reaction time distribution curve. For laminar flow and in the absence of molecular diffusion, the distribution curve is that of an inverse cube with a sharp cut-off at the minimal reaction time. Further, the distribution curve is independent of the shape of the reactor (or the ratio of length to radius).

The operation of molecular diffusion modifies the distribution curve by rounding off the cut-off at low reaction times and by increasing the frequency of occurrence at high reaction times. If R and L are the radius and length respectively of the reactor, D is the diffusion coefficient and τ_0 the miminum time of passage through the reaction zone in the absence of diffusion, then the modifications produced by diffusion may be regarded as negligible provided

$$R>18\sqrt{D\tau_0}$$

and

$$L>360\sqrt{D\tau_0}.$$

Thus the modifying effect of diffusion on the reaction time distribution curve is most pronounced in the smallest vessels.

Consideration of two examples of gaseous reactions shows, in each case, that the critical Reynolds's numbers are exceeded and the state of turbulent flow attained before the reaction vessels are made sufficiently large to fulfil the conditions for negligible diffusion. In liquid systems, however, there is a possible wide range of reactor sizes for which the flow is laminar yet the effect of diffusion negligible.

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CI. *On the Analysis of a Small Arbitrary Disturbance in a Homogeneous Isotropic Elastic Solid.*

By C. G. Pendse, M.A., Ph.D.*

[Received October 1, 1945.]

There is a well-known result in the theory of elasticity that there are two basic types of motion, classified as equivoluminal and irrotational, when a homogeneous isotropic elastic solid under no external forces executes small oscillations about its position of equilibrium. For each type there is a characteristic wave equation and the wave velocities for the two types are different. It does not appear, however, to have been pointed out clearly that when the motion is arbitrary, that is, neither wholly irrotational nor wholly equivoluminal, it consists of two distinct parts, depending on equations similar to those for the basic types but involving physical quantities not necessarily the same as those in the equations for the basic types.

The object of the present note is to clear up this point. In § 1 a brief account of the classical theory is given. In § 2 it is shown how far that theory is incomplete; and in § 3 the analysis is completed for an arbitrary motion. In § 4 a physical discussion of the results is given.

§ 1. The Equations of Motion.

Let $Oxyz$ be a frame of reference, and let

$\overrightarrow{\mathrm{r}}=(x, y, z)$ be the position vector for a particle of the solid in the unstrained state;

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* Communicated by the Author.

$\vec{r}+\vec{d}=(x, y, z)+(u, v, w)$ be the position vector for the same particle in the strained state, at the instant t, $\vec{d}=(u, v, w)$ being assumed to be a function of $\vec{r}$ and t;

$\theta=\operatorname{div} \vec{d}$,

$\vec{\omega}=\frac{1}{2} \operatorname{curl} \vec{d}$.

Let λ and μ be Lamé's constants for the solid, ρ its density, and $c_1^2=\frac{(\lambda+2\mu)}{\rho}$ and $c_2^2=\frac{\mu}{\rho}$.

When $\vec{d}$ is small the equation of motion is

$$\rho \frac{\partial^2 \vec{d}}{\partial t^2}=(\lambda+\mu) \operatorname{grad} \theta+\mu \nabla^2 \vec{d}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

when there are no external body forces *.

Equation (1) involves as dependent variables the vector function $\vec{d}$ and the scalar function θ. In terms of rectangular Cartesian components, the equation (1) is equivalent to three interdependent scalar equations containing four dependent variables, viz. u, v, w, θ.

Equations, each of which contains only one dependent variable, can be obtained by taking the divergence and the curl of both sides of equation (1). The derived equations are :—

$$\rho \operatorname{div}\left(\frac{\partial^2 \vec{d}}{\partial t^2}\right)=(\lambda+\mu) \operatorname{div} \operatorname{grad} \theta+\mu \operatorname{div} \nabla^2 \vec{d},$$

i. e.

$$\rho \frac{\partial^2 \theta}{\partial t^2}=(\lambda+\mu)\nabla^2\theta+\mu\nabla^2\theta=(\lambda+2\mu)\nabla^2\theta,$$

and

$$\rho \operatorname{curl}\left(\frac{\partial^2 \vec{d}}{\partial t^2}\right)=(\lambda+\mu) \operatorname{curl} \operatorname{grad} \theta+\mu \operatorname{curl} \nabla^2 \vec{d},$$

i. e.

$$\rho \frac{\partial^2 \vec{\omega}}{\partial t^2}=\mu \nabla^2 \vec{\omega}.$$

On dividing by ρ, the above equations become

$$\frac{\partial^2 \theta}{\partial t^2}=c_1^2 \nabla^2 \theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{\partial^2 \vec{\omega}}{\partial t^2}=c_2^2 \nabla^2 \vec{\omega}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

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* *Cf.* A. E. H. Love, 'The Mathematical of Theory Elasticity,' 4th ed. 2nd imp. chap. XIII. (1934).

Equations (2) and (3) are independent of each other and can co-exist for any small motion; equation (3) is equivalent to three independent scalar equations.

It is, however, by no means possible to assert that the motion of the solid will always be governed by a single equation like either (2) or (3) or, indeed, an equation involving one dependent variable (either scalar or vector). In the general case the motion will be complex.

§ 2. THE CHIEF SPECIAL TYPES OF MOTION.

Equations (2) and (3) can co-exist and consequently it is possible to have "waves" in dilatation and rotation simultaneously. One is liable to get the misleading impression from text-books that the dilatational and the rotational waves are mutually exclusive. It is, however, more correct to say that, if the actual motion consists of an "unmixed" wave, the motion is one in which either the rotation or the dilatation does not have the characteristics of a wave. This can be shown as follows.

Since
$$\text{curl curl } \vec{d} = \text{grad div } \vec{d} - \nabla^2 \vec{d},$$
the equation (1) can be written in two alternative but equivalent forms, viz.

$$\rho \frac{\partial^2 \vec{d}}{\partial t^2} = (\lambda+\mu) \text{ grad div } \vec{d} + \mu \nabla^2 \vec{d},$$

or
$$\rho \frac{\partial^2 \vec{d}}{\partial t^2} = (\lambda+\mu) \text{ curl curl } \vec{d} + (\lambda+2\mu) \nabla^2 \vec{d}.$$

Hence the equation (1) reduces to an equation for one dependent variable if, and only if, either

$$\text{curl curl } \vec{d} = 0,$$

i. e.
$$\text{curl } \vec{\omega} = 0, \quad \ldots \ldots \ldots \ldots \quad (4)$$

or
$$\text{grad div } \vec{d} = 0,$$

i. e.
$$\text{grad } \theta = 0. \quad \ldots \ldots \ldots \ldots \quad (5)$$

$\vec{\omega}=0$ and $\theta=0$ are *particular* solutions only of equations (4) and (5) respectively, and these are the ones generally referred to in the literature on the subject.

We now consider separately the cases in which curl $\vec{\omega}=0$ and grad $\theta=0$.

Case I.—curl $\vec{\omega}=0$.

Let the suffix 1 be applied to the various quantities, so that we have the dependent variables $\vec{d}_1$, θ_1 and $\vec{\omega}_1$. The equations become

$$\frac{\partial^2 \vec{d}_1}{\partial t^2} = c_1^2 \nabla^2 \vec{d}_1, \quad \ldots \ldots \ldots \ldots \quad (6)$$

$$\frac{\partial^2\theta_1}{\partial t^2}=c_1^2\nabla^2\theta_1, \quad \ldots\ldots\ldots \quad (7)$$

$$\frac{\partial^2\vec{\omega}_1}{\partial t^2}=c_2^2\nabla^2\vec{\omega}_1. \quad \ldots\ldots\ldots \quad (8)$$

Since

$$\text{curl curl }\vec{\omega}_1\equiv\text{grad div }\vec{\omega}_1-\nabla^2\vec{\omega}_1,$$

and as curl $\vec{\omega}_1=0$ and div $\vec{\omega}_1=0$, $\nabla^2\vec{\omega}_1=0$; and the equation (8) becomes

$$\frac{\partial^2\vec{\omega}_1}{\partial t^2}=0. \quad \ldots\ldots\ldots \quad (9)$$

That is to say,

$$\vec{\omega}_1=(a_1+b_1t)\text{ grad }\psi_1, \quad \ldots\ldots\ldots \quad (10)$$

where ψ_1 is a harmonic function of position.

The implication of (10) is that there can be rotation while $\vec{d}_1$ satisfies an "unmixed" equation. As regards the secular term in (10), one can only say that, after the elastic limits are exceeded, the equations of motion in the original form would not be valid.

If either a_1 and $b_1=0$ or ψ_1 is constant, $\vec{\omega}_1=0$ for all values of $\vec{r}$ and t, and we have the usual case of irrotational motion.

In irrotational motion there exists a scalar function ϕ_1 of $\vec{r}$ and t such that $\vec{d}_1=\text{grad }\phi_1$, which would satisfy the equation

$$\frac{\partial^2\text{ grad }\phi_1}{\partial t^2}=c_1^2\nabla^2\text{grad }\phi_1,$$

i. e.

$$\text{grad}\left(\frac{\partial^2\phi_1}{\partial t^2}-c_1^2\nabla^2\phi_1\right)=0,$$

i. e.

$$\frac{1}{c_1^2}\frac{\partial^2\phi_1}{\partial t^2}-\nabla^2\phi_1=F(t),$$

a function of t.

Without any loss of generality $F(t)$ can be absorbed in ϕ_1 and we can write

$$\frac{1}{c_1^2}\frac{\partial^2\phi_1}{\partial t^2}-\nabla^2\phi_1=0. \quad \ldots\ldots\ldots \quad (11)$$

The equations (6) and (11) are the same as equation (2), which is valid in the most general case. From the relation $\vec{d}_1=\text{grad }\phi_1$, it is easy to see that in the case of a $\left\{\begin{matrix}\text{plane}\\ \text{spherical}\end{matrix}\right\}$ wave the direction of $\vec{d}_1$ is parallel to the $\left\{\begin{matrix}\text{line of propagation}\\ \text{radius vector}\end{matrix}\right\}$.

Case II.—grad $\theta=0$.

Let the suffix 2 be applied to the various quantities, so that we have the dependent variables $\vec{d}_2$, θ_2 and $\vec{\omega}_2$.

Since grad $\theta_2=0$, θ_2 is independent of $\vec{r}$ and the equation (2) becomes $\frac{\partial^2\theta_2}{\partial t^2}=0$; and we have

$$\theta_2=b_0+b_1t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where b_0 and b_1 are constants. If $b_1=0$, it would mean that there is a permanent dilatation. If $b_1\neq 0$, it would mean that the dilatation would go on increasing and eventually the approximations subject to which the equation (1) holds good are no longer valid.

Equations (1) and (2) become

$$\frac{\partial^2\vec{d}_2}{\partial t^2}=c_2{}^2\nabla^2\vec{d}_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\frac{\partial^2\vec{\omega}_2}{\partial t^2}=c_2{}^2\nabla^2\vec{\omega}_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Equation (13) is the same as equation (3). It is easy to verify that, in the case of a plane sine wave, $\vec{d}_2$ is perpendicular to the direction of propagation if div $\vec{d}_2\equiv\theta_2=0$, *i. e.* if the wave is equivoluminal.

§ 3. MOTION UNDER ARBITRARY CONDITIONS.

§ 2 above contains the only two special types of motion which are possible. In each of them the equation (1) reduces to an equation involving only one dependent variable. When, however, the initial conditions are such that neither case *I* nor case *II* discussed above is possible, the solution of (1) is very complicated. Love * has given a solution in the most general case. He has made use of the scalar and vector potential functions to split up the vector $\vec{d}$. It may be mentioned, however, that the use of scalar and vector potentials is not free from ambiguity, as these quantities are not always uniquely determined. Below is sketched a method showing how $\vec{d}$ can be determined in many cases.

Since $$\text{curl curl } \vec{d}=\text{grad div } \vec{d}-\nabla^2\vec{d},$$

we can write $$2 \text{ curl } \vec{\omega}=\text{grad } \theta-\nabla^2\vec{d},$$

i. e. $$\nabla^2\vec{d}=\text{grad } \theta-2 \text{ curl } \vec{\omega}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where θ and $\vec{\omega}$ satisfy equations (2) and (3) respectively.

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* A. E. H. Love, *loc. cit.* pp. 303–304. Also Love, "The propagation of wave motion in an isotropic elastic medium," Proc. London Math. Soc. (2). i pp. 291–344 (1903).

Provided the terminal conditions are known it is possible to determine θ and $\vec{\omega}$ from (2) and (3), by using Kirchhoff's method, as functions of $\vec{r}$ and t. Assuming that this has been done, equation (15) is equivalent to Poisson's equation, and its solution is then

$$\vec{d}=\frac{1}{4\pi}\iiint\frac{2\,\mathrm{curl}\,\vec{\omega}-\mathrm{grad}\,\theta}{r'}\,d\tau+\vec{H}, \quad . \quad . \quad . \quad . \quad (16)$$

where r' is the distance from the point at which the value of $\vec{d}$ has to be found, the integration extending over the whole of the solid and $\vec{H}$ is a harmonic vector function of position * depending possibly on t. $\vec{H}$ will be determined by the conditions of the problem.

§ 4. Physical Discussion of the Results.

It may be useful to indicate the physical significance of the results of § 2. It is customary in the literature on elasticity and seismology to speak of longitudinal waves and transverse waves. In seismology the names P-waves and S-waves are used for the longitudinal waves and transverse waves respectively. The above analysis shows, however, that there can be waves, especially plane waves, which are neither longitudinal nor transverse, but which have the same velocities as those waves have, under conditions in which elastic limits are liable to be exceeded. After a catastrophic earthquake, a permanent or semipermanent change on the surface of the earth is produced ; and in this case we have conditions in which the above additional types of waves may possibly exist.

CII. *On Development of Turbulent Liquid Motion over an Infinite Plate.*

By S. D. Nigam, M.Sc.†

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1. *Introduction.*

Dr. Ray⁽¹⁾ has worked out the problem of the growth of turbulence on a very large plate. The flow is supposed to be two-dimensional, and though depending on time, is taken to be uniform in every section across the layer of turbulence. The results obtained by him relate to the spread

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* $\vec{H}$ is used instead of the surface integral, over the boundary of the solid, involving $(\vec{n}.\nabla)\vec{d}$, where $\vec{n}$ is the unit vector along the outward normal to the surface of the solid. The actual formula would be

$$4\pi\vec{d}=\iiint\frac{2\,\mathrm{curl}\,\vec{\omega}-\mathrm{grad}\,\theta}{r'}\,d\tau+\iint\left\{\frac{1}{r'}(\vec{n}.\nabla)\vec{d}-\vec{d}(\vec{n}.\nabla)\frac{1}{r'}\right\}dS,$$

where the surface integral is taken over the boundary of the solid.

† Communicated by the Author.

of a two-dimensional turbulent motion perpendicular to the plate, while the motion at any instant in every section perpendicular to the direction of flow remains identical. The solutions are not valid in the immediate vicinity of the plate, because the viscous terms have been utterly neglected from the equations of motion. Nevertheless, the viscous effect has been to some extent taken into consideration by calculating the total depth of the turbulent layers according to the one-seventh law of Blasius.

The present note is an attempt to solve the same problem after including the terms corresponding to viscous stresses in the equations of motion and taking full account of the viscous sub-layer adjacent to the plate. In this layer the flow is laminar and the eddy-viscosity is zero. Next to this layer is the region where turbulence and viscosity are both effective, but at very great distance from the plate the viscous-turbulent flow asymptotically merges into the constant flow U_0.

2. *Equations of Motion.*

The equation of motion based on the vorticity transfer theory of Taylor (2), for a fully developed turbulent flow is

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=\rho\overline{l'v'}\frac{\partial^2 U}{\partial y^2}, \quad \ldots \ldots \quad (1)$$

where U is the component of mean velocity along the plate, V the component perpendicular to it, ρ the density of the fluid and $\overline{l'v'}$ is the eddy kinemetic viscosity of the fluid. K. Takahasi (3) proposes that both the stresses due to turbulence and viscosity are additive, and then (1) becomes

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=(\nu+\rho\overline{l'v'})\frac{\partial^2 U}{\partial y^2}. \quad \ldots \ldots \quad (2)$$

Making Prandtl's assumption

$$\overline{l'v'}=l\left(\frac{\partial U}{\partial y}\right), \quad \ldots \ldots \quad (3)$$

the equation (2) becomes

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=\rho l^2\frac{\partial U}{\partial y}\cdot\frac{\partial^2 U}{\partial y^2}+\nu\frac{\partial^2 U}{\partial y^2}. \quad \ldots \quad (4)$$

Assuming that U is a function of y and t only, and that V is zero everywhere, we obtain

$$\frac{\partial U}{\partial t}=\rho l^2\frac{\partial U}{\partial y}\cdot\frac{\partial^2 U}{\partial y^2}+\nu\frac{\partial^2 U}{\partial y^2}. \quad \ldots \ldots \quad (5)$$

In the laminar sub-layer where the viscosity is predominant and the eddy-viscosity vanishes, (5) becomes

$$\frac{\partial U}{\partial t}=\nu\frac{\partial^2 U}{\partial y^2}. \quad \ldots \ldots \quad (6)$$

Setting

$$y/2(\nu t)^{\frac{1}{2}}=\eta,$$

$$U=KU_0 f(\eta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where U_0 is the velocity of flow in the main stream, and K is the constant of integration. We get

$$U=KU_0 \operatorname{erf} \eta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\frac{\partial U}{\partial y}=KU_0 \operatorname{erf}' \eta \cdot \frac{1}{2(\nu t)^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Again, in the region where viscosity and turbulence are both comparable, the equation (5) is taken in all its entirety ; assuming a form of " l," the mixture length, slightly different from that of Kármán, viz.

$$l^2=\alpha \frac{\partial U}{\partial y} \Big/ \frac{\partial^2 U}{\partial y^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where α is some unknown constant.

Equation (5) becomes

$$\frac{\partial U}{\partial t}=\alpha\left(\frac{\partial U}{\partial y}\right)^2+\nu\frac{\partial^2 U}{\partial y^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Again, putting

$$U=U_0 F(\eta), \quad y \cdot \phi(t)=\eta,$$

$\phi(t)$ being a function of " t " only, (11) becomes

$$y\phi'(t)F'(\eta)=\alpha U_0\phi^2(t)F'^2(\eta)+\nu\phi^2(t)F''(\eta). \quad . \quad . \quad . \quad . \quad (12)$$

Choosing $\phi(t)$ in such a way that either y or t do not occur explicitly in (12), we see that it has to satisfy the relation

$$\phi'(t)=-2\nu\phi^3(t),$$

whence we get

$$\phi(t)=\frac{1}{2(\nu t)^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Equation (12) becomes

$$F''(\eta)+\frac{\alpha U_0}{\nu}F'^2(\eta)+2\eta F'(\eta)=0,. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

the dashes denoting differentiation with respect to η. An integral of (14) is

$$F(\eta)=\frac{\nu}{\alpha U_0}\left[D+\log\{\operatorname{erf}\eta+B\}\right]. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Therefore

$$U=\frac{\nu}{\alpha}\left[D+\log\{\operatorname{erf}\eta+B\}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and

$$\frac{\partial U}{\partial y}=\cdot\frac{\nu}{\alpha}\frac{\operatorname{erf}'\eta}{\operatorname{erf}\eta+B}\cdot\frac{1}{2(\nu t)^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Taking B=0 and $\quad \frac{\nu D}{\alpha}=U_0,$

$$U=U_0+\frac{\nu}{\alpha}\log\{\operatorname{erf}\eta\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16\,a)$$

$$\frac{\partial U}{\partial y}=\frac{\nu}{\alpha}\frac{\operatorname{erf}'\eta}{\operatorname{erf}\eta}\frac{1}{2(\nu t)^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17\,a)$$

If at any time t, $\delta(t)$ be taken as the width of the laminar sub-layer, the conditions to be satisfied are: $U=0$ when $y=0$ for all values of t. The velocity and stress are continuous on $y=\delta$; and at a great distance from the plate (theoretically infinite) the viscous-turbulent flow merges into the constant flow U_0. It can be shown that (8) makes the velocity zero on $y=0$. Equating the values of U from (8) and (16 *a*), and $\partial U/\partial y$ from (9) and (17 *a*) on $y=\delta$, we get

$$\log\left\{\operatorname{erf}\frac{\delta}{2(\nu t)^{\frac{1}{2}}}\right\}=\frac{\alpha U_0}{\nu}\left[K\operatorname{erf}\frac{\delta}{2(\nu t)^{\frac{1}{2}}}-1\right], \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$\operatorname{erf}\frac{\delta}{2(\nu t)^{\frac{1}{2}}}=\frac{\nu}{K\alpha U_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Since (18) and (19) hold at all times, $\frac{\delta}{2(\nu t)^{\frac{1}{2}}}$ must be a numerical constant.

Taking
$$\delta=2M(\nu t)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

we get
$$\log\frac{\nu}{K\alpha U_0}=1-\frac{\alpha U_0}{\nu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18\,a)$$

$$\operatorname{erf} M=\frac{\nu}{K\alpha U_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19\,a)$$

These equations determine M and K in terms of α, which in turn may be known if we know the velocity in the viscous-turbulent region at some point at any given instant of time. (16 *a*) makes the velocity U_0 for $\eta=\infty$.

3. *Discussion of Solution.*

The velocity component inside the laminar sub-layer is given by

$$U=KU_0\operatorname{erf}\left(\frac{y}{2(\nu t)^{\frac{1}{2}}}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and outside this layer by

$$U=U_0+\frac{\nu}{\alpha}\log\left(\operatorname{erf}\frac{y}{2(\nu t)^{\frac{1}{2}}}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The thickness of the viscous sub-layer is

$$\delta(t)=2M(\nu t)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The initial stage of motion can be had by putting $t=0$ in (21), (22) and (23). Equation (21) then gives the velocity on the plate, which is obviously zero; and (22) gives the velocity at all points inside the fluid, which is U_0; and (23) gives the width of the laminar sub-layer which is zero initially. The velocity of the fluid at great distance from the plate may be obtained from (22), which tends to U_0 as $y\to\infty$. For the sake of definiteness and to get better insight into the mechanism of turbulence, we take definite numerical values for the constants involved. Thus taking

$$\operatorname{erf} \mathrm{M}=0{\cdot}04511=\frac{\nu}{\alpha \mathrm{KU}_0},$$

$$\mathrm{M}=0{\cdot}04,$$

the equation (18*a*) gives $$\frac{\alpha \mathrm{U}_0}{\nu}=4{\cdot}0987$$

and $\delta(t)=0{\cdot}08(\nu t)^{\frac{1}{2}}$; $$\mathrm{K}=5{\cdot}408.$$

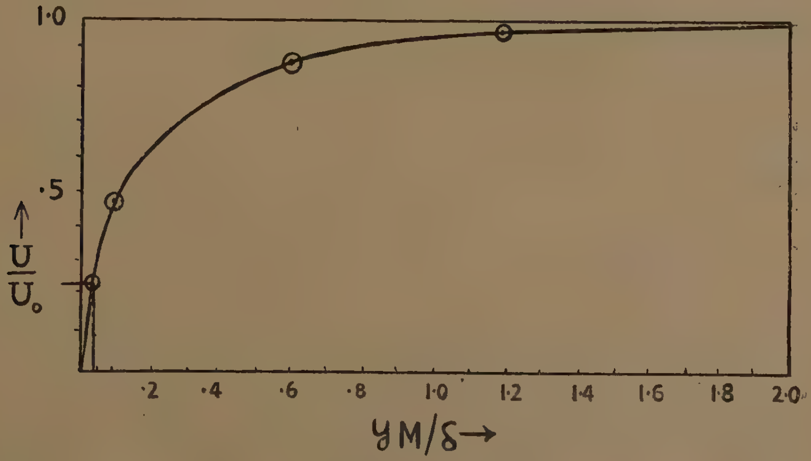
For points defined by $y \leqslant \delta(t)$,

$$\frac{\mathrm{U}}{\mathrm{U}_0}=5{\cdot}408 \times \operatorname{erf}\left(\frac{y\mathrm{M}}{\delta}\right)$$

and, for $y \geqslant \delta(t)$, $$\frac{\mathrm{U}}{\mathrm{U}_0}=1+\frac{1}{1{\cdot}78} \log_{10} \operatorname{erf}\left(\frac{y\mathrm{M}}{\delta}\right).$$

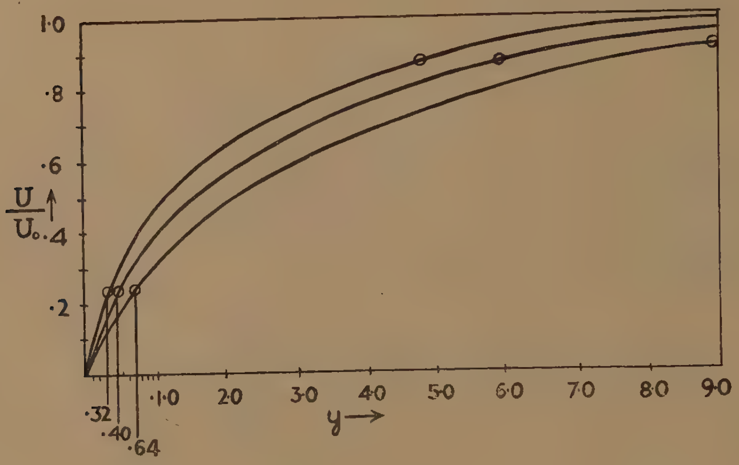
ym/δ	$\operatorname{erf}(ym/\delta)$	U/U_0	$[y]t=\frac{16}{\nu}$	$[y]t=\frac{25}{\nu}$	$[y]t=\frac{64}{\nu}$
0·0125	0·01360	0·0735	0·1	0·125	0·200
0·025	0·0282	0·1525	0·2	0·25	0·400
0·04	0·04511	0·2440	0·32	0·40	0·640
0·05	0·05637	0·2998	0·40	0·5	0·8
0·06	0·06762	0·3427	0·48	0·6	0·96
0·07	0·07886	0·3803	0·56	0·7	1·12
0·08	0·09008	0·4127	0·64	0·8	1·28
0·1	0·11246	0·4668	0·8	1·0	1·6
0·6	0·60386	0·8770	4·8	6·0	9·6
0·9	0·74210	0·9447	7·2	9·0	
1·2	0·91031	0·9779			
2·0	0·99532	0·9989			

Fig. 1.



Shows the velocity rise in the viscous layer and the turbulent region. Vel. on $y=\delta$ is $0{\cdot}244\ u_0$.

Fig. 2.



Shows the velocity profile at $t=\frac{16}{\nu}, \frac{25}{\nu}, \frac{64}{\nu}$; the width of viscous sub-layer is 0·32, 0·40, 0·64 respectively.

In conclusion, I express my gratefulness to Prof. M. Ray, D.Sc., for his kind help and guidance throughout this work.

Supplementary Note.

The physical conditions of the problem have been satisfied by suitably combining the solution of (14) for the viscous-turbulent motion with the solution of (6). Equation (6) has been obtained under the assumption that the effect of eddy-viscosity near the plate in the viscous sub-layer is negligible and hence the first term on the right-hand side of (5) may be neglected in comparison with the second term.

It should be noted that the solution of (14) makes the velocity negatively infinite at $\eta=0$; which shows that it breaks down in the immediate vicinity of the plate, and is therefore taken to represent the motion for values of η greater than or equal to M $\left(\text{where } M=\frac{2(\nu t)^{\frac{1}{2}}}{\delta}\right)$. In order to complete the solution, it is joined smoothly with the viscous solution as explained in the above paragraph.

The solutions can be extrapolated to give the initial motion; but it is not possible to predict the time after which the steady state is reached. The solutions are therefore expected to be valid only for the period preceding the steady state and, once this state of affairs is established, the solutions break down. In fact they represent the stage of development of motion, because obviously they give absurd results by putting $t=\infty$.

References.

(1) Dr. Ray, Phil. Mag. ser. 7, xxvii. p. 240 (1939).
(2) 'Modern Developments in Fluid Dynamics,' ii. p. 369 (1938).
(3) *Proc. Physico. Mth. Soc. of Japan*, xxi. (Nov. 1939).
(4) Numerical value of the error function has been taken from David Brunt's 'The Combination of Observations' (Camb. 1931).

Agra College, Agra.
10 January, 1947.

CIII. *On Turbulent Liquid Motion Outside a Circular Boundary.*

By S. D. Nigam, M.Sc.*

[Received July 2, 1947.]

1. *Introduction.*

Dr. Ray has discussed the problem of turbulent motion of uniform stream in the neighbourhood of a circular boundary from the point of view of "vorticity transport" theory of Taylor, starting with Taylor-Goldstein equations, transforming them into forms suitable for treatment of motion near a circular boundary and then integrating them numerically by the method of successive approximations. His solutions reveal a peculiar motion. The motion in the neighbourhood of the boundary is retrograde and changes into direct motion at a distance from the circular boundary equal to about one-third of the radius of the circle. It has been possible to construct an analytical solution by taking a different form for "l," the mixture length. The solutions obtained on this new assumption give the necessary fit at the boundary and a better insight into the turbulence pattern. The solutions do not take into account the viscous layer touching the wall, but have been adjusted to make $\partial u/\partial y$ infinite at the wall; our solutions break down in the region where the wake is formed. To avoid repetition I have not given the deduction of the equation of motion, for which reference may be made to the paper cited in the beginning.

2. *Equations of Motion.*

The equation(1) of motion is

$$\frac{r}{r+y}u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{uv}{r+y}=\overline{L_2\nu'}\left(\frac{\partial^2 u}{\partial y^2}+\frac{1}{r+y}\frac{\partial u}{\partial y}-\frac{u}{(r+y)^2}\right), \quad . \quad (1)$$

and the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial}{\partial y}\left(\frac{r+y}{r}\nu\right)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where r is the radius of the circle, u is the velocity parallel to the boundary, and v the velocity along the radius vector. Making the usual assumption of Prandtl for the mixture length, we set

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* Communicated by the Author.

$$\overline{L_2 v'} = -l^2 \frac{\partial u}{\partial y} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equation (1) becomes

$$\frac{r}{r+y} u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{uv}{r+y} = -l^2 \frac{\partial u}{\partial y} \left(\frac{\partial^2 u}{\partial y^2} + \frac{1}{r+y} \frac{\partial u}{\partial y} - \frac{u}{(r+y)^2} \right) . \quad . \quad (4)$$

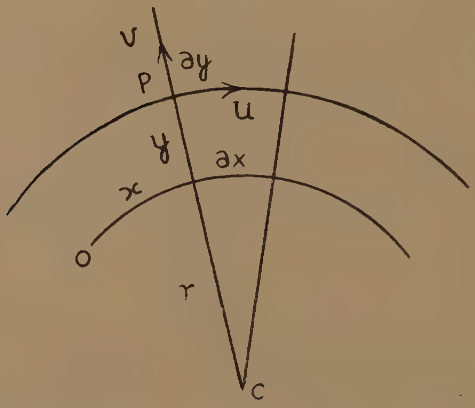
Again setting

$$\frac{x}{r} = \xi, \qquad \frac{y}{r} = \eta, \qquad \frac{l}{r} = l',$$

the equations (2) and (4) now become

$$\frac{\partial u}{\partial \xi} + \frac{\partial}{\partial \eta} [(1+\eta) v] = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Fig. 1.



$$\frac{1}{1+\eta} u \frac{\partial u}{\partial \xi} + v \frac{\partial u}{\partial \eta} + \frac{uv}{1+\eta} = -l'^2 \frac{\partial u}{\partial \eta} \left[\frac{\partial^2 u}{\partial \eta^2} + \frac{1}{1+\eta} \frac{\partial u}{\partial \eta} - \frac{u}{(1+\eta)^2} \right] . \quad (6)$$

To solve these equations we introduce the stream function ψ, such that

$$u = \frac{\partial \psi}{\partial \eta}, \qquad v(1+\eta) = -\frac{\partial \psi}{\partial \xi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

putting

$$l'^2 = f(\eta) \cot \xi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\psi = \mathrm{A}\, \mathrm{F}(\eta) \sin \xi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\left. \begin{aligned} u &= \mathrm{A}\, \mathrm{F}'(\eta) \sin \xi, \\ v &= -\mathrm{A} \frac{\mathrm{F}(\eta)}{1+\eta} \cos \xi. \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Where a dash denotes differentiation with respect to η. Equation (6) now becomes

$$(1+\eta)(\mathrm{F}'^2 - \mathrm{FF}'') - \mathrm{FF}' = -f(\eta) \, . \, \mathrm{F}''[(1+\eta)^2 \mathrm{F}''' + (1+\eta)\mathrm{F}'' - \mathrm{F}']. \quad (11)$$

The conditions to be satisfied on the circular boundary are

$$\frac{\partial u}{\partial y}=\infty \quad \text{and} \quad v=u=0,$$

i. e.

$$F''(0)=\infty\,; \quad F'(0)=F(0)=0.$$

Solution of the Differential Equation.

Let

$$F(\eta)=\eta^p \quad \text{where} \quad 1<p<9/5.$$

By choosing p in this way it is obvious that all the boundary conditions are satisfied. Equation (11) becomes

$$\eta^p=-f(\eta)p(p-1)[\eta^{p-1}p(p-2)+\eta^{p-2}(p-1)(2p-3)+\eta^{p-3}(p-1)(p-2)].$$

Therefore

$$f(\eta)=\frac{-\eta^3}{p(p-1)[(p-1)(p-2)+(p-1)(2p-3)\eta+p(p-2)\eta^2]}. \quad . \quad (12)$$

This equation makes $f(0)=0$, as should be the case. It should be noted that $f(\eta)$ is always positive if $\cot\xi$ is positive. Hence the expression in the denominator of (12) should be negative for small values of η and those values of p which are given by the relation $1<p<9/5$.

$$E\equiv p(p-2)\eta^2+(p-1)(2p-3)\eta+(p-1)(p-2).$$

The roots of the quadratic equation obtained by putting $E=0$ are given by

$$\frac{-(p-1)(2p-3)\pm\sqrt{(p-1)(5p-9)}}{2p(p-2)}.$$

The roots are complex for $1<p<9/5$ and therefore the expression has the same sign as $p(p-2)$. The roots are real for $9/5\leqslant p<2$. The expression will not have a consistent sign for small values of η, when $9/5\leqslant p<2$, although the values of p satisfying this relation satisfy all the boundary conditions. Hence p must satisfy the inequality $1<p<9/5$.

4. *Discussion of Velocity.*

The velocities in the turbulent field are given by

$$u=Ap\eta^{p-1}\sin\xi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

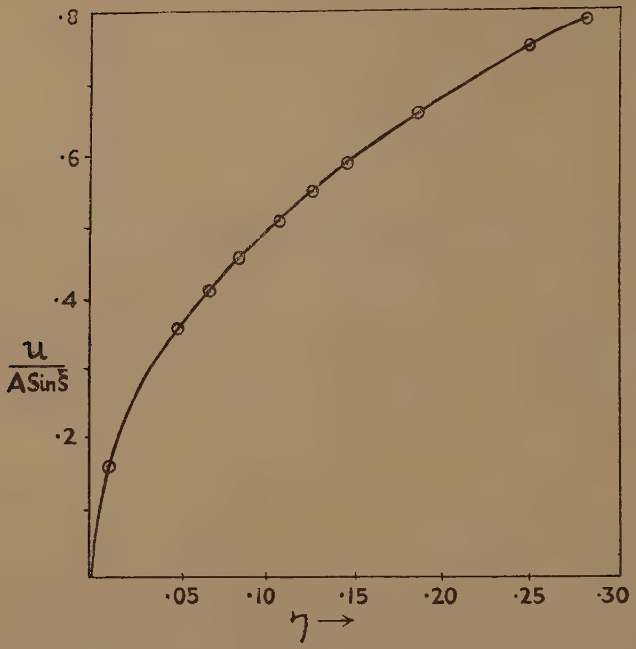
$$v=-A\frac{\eta^p}{1+\eta}\cos\xi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and

$$\psi=A\eta^p\sin\xi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

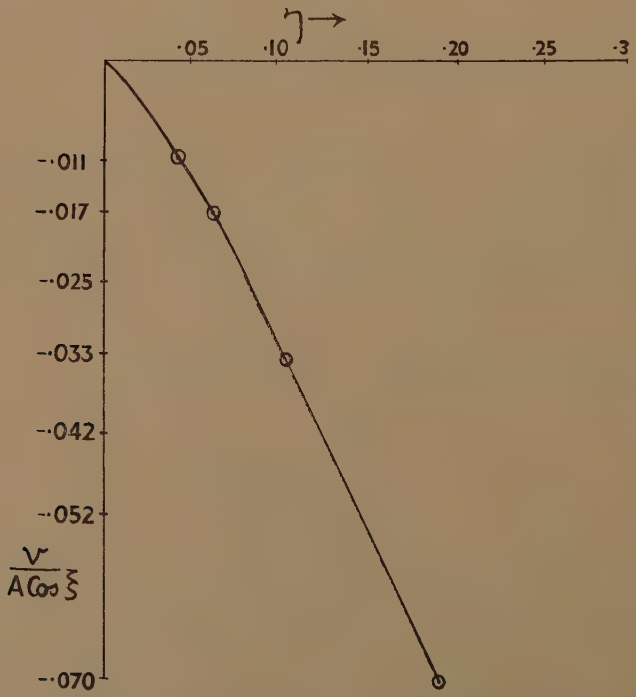
The motion no more keeps the retrograde character. Both u and v vanish on the circular wall. The points $x/r=0$ and $x/r=\pi$ are outside the region under consideration. The velocity distribution is given by (13) and (14). The formulæ contain an unknown constant which may be found experimentally by observing the velocity at a definite point in the turbulent region near the wall.

Fig. 2.



Shows the profile of the velocity parallel to the circular wall along a radius.

Fig. 3.



Shows the profile of the radial velocity along a radius in the turbulent layer. The curve shows that for large values of the fall is nearly straight.

TABLE I.

$p=1{\cdot}5$.

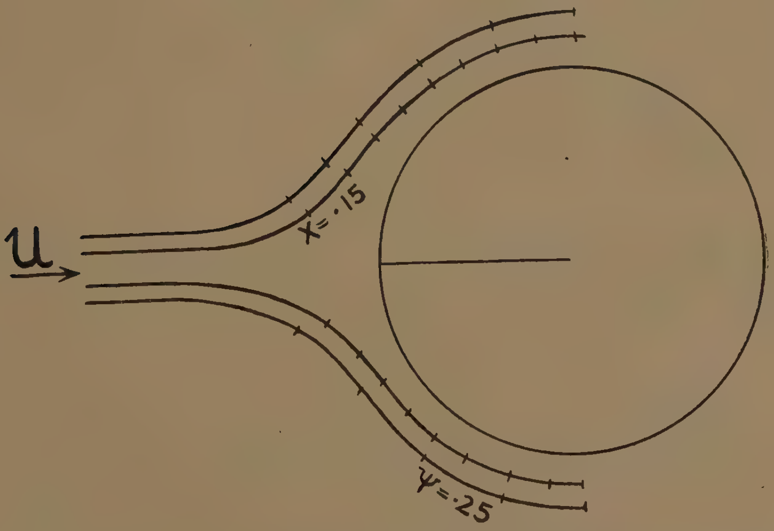
η	0·05	0·07	0·09	0·11	0·13	0·15	0·19	0·25	0·28	0·30
$u/\mathrm{A}\sin\xi$	0·3354	0·3969	0·4500	0·4975	0·5408	0·5809	0·6538	0·7500	0·7938	0·8216
$-v/\mathrm{A}\cos\xi$	0·0106	0·0173	0·0247	0·0328	0·0414	0·0505	0·0695	0·1000	0·1157	0·1267

The stream lines can be traced from the equation (15) by taking different values of ψ. They are shown below.

TABLE II.

ξ.	10	20	30	40	50	60	70	80	90
				$\psi=0{\cdot}15$.					
η.	0·9070	0·5772	0·4481	0·3791	0·3373	0·3107	0·2943	0·2843	0·2830
				$\psi=0{\cdot}25$.					
η.	1·175	0·8115	0·630		0·474		0·4136		0·397

Fig. 4.



In conclusion I want to express my thanks to Prof. M. Ray, D.Sc., for his kind help in this work.

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(1) Dr. M. Ray, Phil. Mag. ser. 7, xxviii. p. 231 (1939).

Agra College, Agra.

CIV. *Why not discard the Spinor Calculus ?*

By Otto F. Fischer[*],
Consulting Engineer, Norra Kungsvägen 8, Lidingö I,
Stockholm, Sweden.

[Received February 14, 1946.]

Abstract.

The 2-dimensional Pauli matrices and the 4-dimensional Dirac matrices introduced in order to extend the Schrödinger equation to consider firstly a magnetic moment (spin) and secondly also an electric moment (elementary dipole) have inspired the development of a rather extensive Spinor Calculus by many ingenious mathematicians. Hereby the relativistic " world " structure for Maxwell's and d'Alembert's systems of equations has been replaced by a number of new hypercomplex " unit " matrices. But was this really an advantage ? Perhaps we are cultivating a habit of treatment which makes things more complicated than necessary. The author here suggests a return to the " world " system of the relativity theory, expressed synthetically in Hamilton's quaternions as indicated in two pre-war papers[†] in the Philosophical Magazine, and arrives at an apparently more natural and very simple synthetic form for the generalized Schrödinger equation.

1. *Introduction.*

The most general description of electron waves demands wave functions ψ of 2 or better 4 dimensions instead of the scalar functions of the original Schrödinger equation. To manipulate the case of 2-dimensional functions the Pauli-matrices

$$\sigma_x=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \sigma_y=\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix} \quad \sigma_z=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} \quad \delta=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad . \quad . \quad (1)$$

were introduced, indicating two additional quantum numbers. So far as this goes the Spinor Calculus admittedly has had its advantages as a 2-dimensional approximation. But when the same road is extended by Dirac towards the more exact 4-dimensional wave expressions, a number of complicated " unit " matrices become involved, such as

$$\alpha_x=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \quad \sigma_y=\begin{bmatrix}-i&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&i\end{bmatrix} \quad \alpha_z=\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix} \quad \alpha_t=\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}$$

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[*] Communicated by the Author.

[†] Phil. Mag. xxvii. p. 375 (1939) and xxx. p. 135 (1940).

$$\alpha_0=\begin{bmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{bmatrix} \qquad \ldots \ldots \quad (2)$$

or a set $\beta_x\,\beta_y\,\beta_z\,\beta_t\,\beta_0$ with β_0 on the α_x form, or a set $\gamma_x\,\gamma_y\,\gamma_z\,\gamma_t\,\gamma_0$ with γ_t on the α_x form. These different sets are used *ad libitum*. Sometimes one is more convenient, sometimes another.

The corresponding wave-function components are of the form

$$\psi^1=\mathrm{H}_x+i\mathrm{H}_y,\quad \psi^2=\mathrm{H}_z-i\mathrm{H}_t,\quad \psi^3=-i(\mathrm{E}_x+i\mathrm{E}_y),\quad \psi^4=-i(\mathrm{E}_z-i\mathrm{E}_t), \qquad \ldots \ldots \quad (3)$$

where **H** and **E** are analogues to the magnetic and electric field strengths, completed by "time" components. In the 2-dimensional Pauli-case the "electrical" wave components ψ^3 and ψ^4 are neglected.

The point is that the four "world" components have been combined "crosswise" into the complex wave functions of the Spinor Calculus, combining H_x with H_y, H_z with H_t a.s.f. without apparent physical meaning, instead of the classical combination-wise of the Maxwell and Einstein theories, where all the magnetic components are real and all the electrical components are imaginary. This introduces a break between the methodics of relativity theory and that of advanced wave mechanics. Is that really justified? Are the advantages to quantum mechanics large enough to balance the disadvantages of separating more and more the technics of two so important regions of modern science, which ought to support and inspire each other to a more complete united understanding of nature?

In the following investigation I shall begin by presenting in Part A some of the well-known formulas in present use, which we shall try to simplify. Hereby I shall follow the text book by Frenkel: 'Wave Mechanics, Part II., Advanced General Theory', Oxford 1934, referred to as "Fr. II". Then in Part B I shall try to show how the technics of relativity theory may be followed up synthetically to a concentrated expression of the most general Schrödinger equation. Hereby I shall make use of Hamilton's quaternions, continuing the start I made before the war in the two above mentioned papers, which I shall refer to as "Phil. Mag. I." and "Phil. Mag. II."

PART A.—SOME WELL-KNOWN FORMULAS TO BE SIMPLIFIED.

2. *The Generalized Maxwell Equations.*

According to Fr. II., p. 264, the Maxwell equations

$$\operatorname{curl}\mathbf{H}-\frac{1}{c}\frac{\delta\mathbf{E}}{\delta t}=\frac{1}{c}\rho\mathbf{V} \qquad \operatorname{curl}\mathbf{E}+\frac{1}{c}\frac{\delta\mathbf{H}}{\delta t}=0 \qquad \ldots \quad (4)$$

$$\operatorname{div} \mathbf{E}=\rho \qquad \operatorname{div} \mathbf{H}=0 \quad . \quad . \quad . \quad . \quad (5)$$

may be written on the following generalized form :

$$\left.\begin{array}{ll} u_y M_z - u_z M_y - u_t N_x = u_x M_t, & u_y N_z - u_z N_y + u_t M_x = u_x N_t \\ u_z M_x - u_x M_z - u_t N_y = u_y M_t, & u_z N_x - u_x N_z + u_t M_y = u_y N_t \\ u_x M_y - u_y M_x - u_t N_z = u_z M_t, & u_x N_y - u_y N_x + u_t M_z = u_z N_t \end{array}\right\}, \quad . \quad (4\,a)$$

$$u_x N_x + u_y N_y + u_z N_z = -u_t M_t, \qquad u_x M_x + u_y M_y + u_z M_z = u_t N_t \quad . \quad (5\,a)$$

where **M** and **N** may be any vectors, not necessarily **H** and **E** ; but say components of the 4-dimensional wave function. Likewise u may represent any 4-dimensional differential operator.

Fr. II. p. 245, writes

$$u_x = \frac{\hbar}{i}\frac{\delta}{\delta x} - \frac{e}{c}A_x, \quad u_y = \frac{\hbar}{i}\frac{\delta}{\delta y} - \frac{e}{c}A_y, \quad u_z = \frac{\hbar}{i}\frac{\delta}{\delta z} - \frac{e}{c}A_z,$$

$$u_t = \frac{1}{c}\left(\frac{\hbar}{i}\frac{\delta}{\delta t} + e\phi\right), \quad \hbar = \frac{h}{2\pi}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where **A** is the vector potential of **H** and ϕ the scalar potential of **E**.

With (4) and (5) we have a system of partial differential equations of the first order. But the wave equation is of the second order of the d'Alembert equation type :

$$\left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2} - \frac{1}{c^2}\frac{\delta^2}{\delta t^2}\right)\psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

(*cf.* Fr. II. p. 260). In Fr. II. p. 267, an attempt to generalize these into wave equations on the classical line has actually been made as follows :

$$\left.\begin{array}{l} DM_t + \frac{\hbar e}{ic}(\mathbf{HM} - \mathbf{EN}) = 0, \\ DN_t + \frac{\hbar e}{ic}(\mathbf{HN} + \mathbf{EM}) = 0, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\left.\begin{array}{l} D\mathbf{N} + \frac{\hbar e}{ic}\{(\mathbf{E}\times\mathbf{M} - \mathbf{E}M_t) + (\mathbf{H}\times\mathbf{N} - \mathbf{H}N_t)\} = 0, \\ D\mathbf{M} + \frac{\hbar e}{ic}\{(\mathbf{H}\times\mathbf{M} - \mathbf{H}M_t) - (\mathbf{E}\times\mathbf{N} - \mathbf{E}N_t)\} = 0, \end{array}\right\} \quad . \quad . \quad (9)$$

where

$$D = u_x^2 + u_y^2 + u_z^2 - u_t^2 + m_0^2 c^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

corresponds to the operator (7) to which the energy of the rest mass has been added. But there the matter has been given up. Presumably because the equations (8) and (9) have appeared to be " too complicated " as representatives of the Schrödinger equation.

3. *The Dirac Equations.*

The Pauli generalization of Schrödingers theory, representing the electrons magnetic moment by $\mu=\mu\sigma$, the wave function by $\psi^{\dagger}=\{\psi^1, \psi^2\}$ and adding $-\mu\mathbf{H}$ with

$$\mathrm{P}\psi=(\mathrm{K}+\mathrm{P}_t)\psi=0, \qquad \mathrm{P}_t=\frac{\hbar}{i}\frac{\delta}{\delta t}\delta,$$

$$\mathrm{K}=\left(\frac{1}{2m_0}\mathbf{u}^2+\mathrm{V}\right)\delta-\mu\mathbf{H} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

may be considered an approximate simplification of Dirac's exact theory of 4-dimensional wave functions, to which we therefore shall confine ourselves here. The first order "Maxwell" equations are alternatively written $\mathrm{A}\psi=0$, $\mathrm{B}\psi=0$ or $\Gamma\psi=0$ where, for instance, the latter is specified:

$$\left.\begin{aligned} &\Gamma\psi=(\gamma_x u_x+\gamma_y u_y+\gamma_z u_z+\gamma_t u_t+\gamma_0 m_0 c)\psi=0, \\ &\qquad\qquad\qquad\qquad \psi=\begin{Bmatrix}\psi^1\\ \psi^2\\ \psi^3\\ \psi^4.\end{Bmatrix} \end{aligned}\right\} \quad . \quad . \quad (12)$$

The second order "d'Alembert" or generalized Schrödinger equation according to Fr. II. p. 314 is written

$$\mathrm{Q}\psi=0, \qquad \mathrm{Q}=\mathrm{D}\delta-\frac{\hbar e}{c}(\mathbf{H}\xi+\mathbf{E}\eta)=\mathrm{D}\delta-\frac{\hbar e}{c}(\mathbf{H}-i\rho\mathbf{E})\xi,$$

$$\xi_x=i\beta_y\beta_z, \qquad \xi_y=i\beta_z\beta_x, \qquad \xi_z=i\beta_x\beta_y, \qquad \eta_x=i\beta_x\beta_t,$$

$$\eta_y=i\beta_y\beta_t, \qquad \eta_z=i\beta_z\beta_t \qquad \rho=\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix} \quad . \quad . \quad (13)$$

representing 4 equations of complex components.

In the energy expression

$$\mathrm{E}=\mathrm{V}+v_x g_x+v_y g_y+v_z g_z+m_0c^2\sqrt{1-\frac{v^2}{c^2}}, \quad . \quad . \quad . \quad (14)$$

the theory according to Fr. II. p. 319 puts

$$\mathbf{g}=\mathbf{u}, \qquad \mathbf{v}=c\gamma \qquad \sqrt{1-\frac{v^2}{c^2}}=\gamma_0. \quad . \quad . \quad . \quad . \quad (15)$$

PART B.—APPLICATION OF HAMILTON'S QUATERNIONS.

4. *Introduction.*

The perfection of the mathematical instrument, which the calculus of Hamilton's quaternions has supplied us with many years ago, has been testified by many authors (*cf.* for instance, J. A. Schouten: Grundlagen der Vekter und Affineranalysis, Teubner, 1914), and if it could supplant the Spinor Calculus it would be a worthy successor indeed. The quaternion consists, as we know, of the assembly of a usual 3-dimensional vector and a scalar with the multiplication rule

$$O \underset{\cdot}{\times} F = \mathbf{O} \times \mathbf{F} - \mathbf{O} \,.\, \mathbf{F} + o \,.\, f + \sigma \,.\, \mathbf{F} + f \,.\, \mathbf{O}, \qquad O = \mathbf{O} + \sigma, \qquad F = \mathbf{F} + f, \qquad \ldots \ldots \quad (16)$$

i.e. a combination of vector multiplication and scalar multiplication with the peculiarity that the latter, carried out between two vectors, changes the sign. In Phil. Mag. I. (2) this procedure is detailed at length, referring to the coordinate unit axes $\frac{\mathbf{i}}{1}, \frac{\mathbf{i}}{2}, \frac{\mathbf{i}}{3}$ of a Cartesian system, completed by a scalar as 4th coordinate.

By application to Maxwell's and Einstein's theories this 4th coordinate is supposed to belong to the time axis corresponding to the radius vector

$$L = x\frac{\mathbf{i}}{1} + y\frac{\mathbf{i}}{2} + z\frac{\mathbf{i}}{3} + ict, \quad l = ict \quad \ldots \quad (17)$$

with

$$\Diamond = \frac{\mathbf{i}}{1}\frac{\delta}{\delta x} + \frac{\mathbf{i}}{2}\frac{\delta}{\delta y} + \frac{\mathbf{i}}{3}\frac{\delta}{\delta z} + \frac{\delta}{\delta l} = \nabla - \frac{i}{c}\frac{\delta}{\delta t} \quad \ldots \quad (18)$$

for 4-dimensional differential operator and Laplace operator

$$\square = \widetilde{\Diamond} \underset{\cdot}{\times} \Diamond = \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2} - \frac{1}{c^2}\frac{\delta^2}{\delta t^2} = \Delta + \frac{\delta^2}{\delta l^2}, \quad \ldots \quad (19)$$

(*cf.* d'Alembert) corresponding to the "norm."

$$\mathrm{N}(O) = \tilde{O} \underset{\cdot}{\times} O = O_x{}^2 + O_y{}^2 + O_z{}^2 + \sigma^2 \quad \ldots \ldots \quad (20)$$

of any quaternion with $\tilde{O} = \sigma - \mathbf{O}$ for "adjungate quaternion." We shall here only deal with the case of the special relativity theory, although transition to the general theory can be made rather fluently for many of the formulas.

The potentials **A** and ϕ shall here be represented by the quaternion

$$F = \mathbf{F} + if = \mathbf{A} + i\phi, \quad \ldots \ldots \quad (21)$$

to be substituted in (16) with

$$O = \mathbf{O} + i\sigma = -\widetilde{\Diamond} = \nabla + \frac{i}{c}\frac{\delta}{\delta t} \quad \ldots \ldots \quad (22)$$

leading to the new quaternion

$$C^\dagger = O \underset{\cdot}{\times} F = \mathbf{O} \times \mathbf{F} + i(\sigma\mathbf{F} \times f\mathbf{O}) - (\mathbf{OF} + \sigma f) = \mathbf{C} + \mathrm{H} + i(\mathbf{C}^* - i\nu), \quad (23)$$

(*cf.* Phil. Mag. II. (51)) where

$$\left.\begin{aligned} \mathbf{C} &= \mathbf{O} \times \mathbf{F} = \nabla \times \mathbf{A} = \operatorname{curl} \mathbf{A} = \mathbf{H} \\ \mathbf{C}^* &= \sigma\mathbf{F} + f\mathbf{O} = \frac{1}{c}\frac{\delta \mathbf{A}}{\delta t} + \nabla\phi = -\mathbf{E} \end{aligned}\right\}, \quad \ldots \quad (24)$$

while

$$\left.\begin{aligned} \mathscr{H} + \nu &= -(\mathbf{OF} + \sigma f) \\ &= -\left(\nabla\mathbf{A} + \frac{1}{c}\frac{\delta\phi}{\delta t}\right) = 0 \end{aligned}\right\}, \quad \ldots \quad (25)$$

so that
$$C^{\dagger}=\mathbf{H}-i\mathbf{E}, \qquad \qquad (26)$$
in the case that we adopt the " Minkowski assumption " (*cf.* Phil. Mag. I. (8)).

The Maxwell equations coincide simply with :
$$\tilde{O}\underset{\cdot}{\times}C^{\dagger}=0, \qquad -\tilde{O}=\Diamond, \qquad \qquad (27)$$
(*cf.* Phil Mag. I. (22)) and the generalized form (4 *a*) (5 *a*) is obtained directly if we put :
$$\left.\begin{array}{c} H=\mathbf{H}+\mathscr{H}=\mathbf{M}-\mathrm{M}_t=M \qquad E=\mathbf{E}+iv=\mathbf{N}-\mathrm{N}_t=N \\ u=-\tilde{O}=\mathbf{u}-iu_t=\frac{\hbar}{i}\Diamond-\frac{e}{c}F. \end{array}\right\} \qquad (28)$$

5. *The Generalized Schrödinger Equation.*

We now shall reserve the term $C^{\dagger}$ for having only the electromagnetic meaning (26), while for the generalized form (28) we define *the 4-dimensional wave function* :
$$\mathit{\Psi}=M-iN=\mathit{\Psi}-\Psi_t, \qquad \mathit{\Psi}=\mathbf{M}-i\mathbf{N}, \qquad \Psi_t=\mathrm{M}_t-i\mathrm{N}_t=-(\mathscr{H}+v) \qquad (29)$$
while the differential operator (10) becomes
$$\mathrm{D}=\mathrm{N}(u)+m_0{}^2c^2=\tilde{u}\underset{\cdot}{\times}u+m_0{}^2c^2, \qquad \qquad (30)$$
after which the d'Alembert second order expressions (8), (9) can be synthetically represented in the following simple *generalized Schrödinger equation* :
$$\mathrm{D}\mathit{\Psi}+\frac{\hbar e}{ic}C^{\dagger}\underset{\cdot}{\times}\mathit{\Psi}=\boxed{\left(\mathrm{D}+\frac{\hbar e}{ic}(\mathbf{H}-i\mathbf{E})\right)\underset{\cdot}{\times}\mathit{\Psi}=0.} \qquad (31)$$
In the case that $C^{\dagger}=0$ all the component equations become alike corresponding to the usual Schrödinger equation ($u=\frac{\hbar}{i}\Diamond$ by $F=0$). If **E** is neglected the number of component equations is halved (as the imaginary members fall out) and we may substitute the Pauli matrices if we so prefer. But then again we might " freeze " to the road leading to the 4-dimensional Dirac equation (13) with all its complicated unit matrices and peculiar interpretations (15). Is it justified to discard good century old Hamiltonian tools for new ones of this kind ? Is (31) more complicated or less fit for future inspiration ? Would it not be better to readopt the Quaternion Calculus for more widespread use and diminish the use of the Spinor Calculus instead or discard it altogether ? ‡

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‡ *Note :* A considerable amount of further detailing of course is required before we seriously approach the possibility indicated in the title of this paper. But so much preparatory work as a matter of fact has been carried out that examples of application could already be given when sufficient space is available.

As a matter of form I shall finish by adding that if we write the Dirac wave function components as follows:

$$\psi^1=\mathrm{M}_x+i\mathrm{M}_y, \qquad \psi^2=\mathrm{M}_z-i\mathrm{M}_t, \qquad \psi^3=-i(\mathrm{N}_x+i\mathrm{N}_y), \qquad \psi^4=-i(\mathrm{N}_z-i\mathrm{N}_t) \quad . \; . \; . \; . \; . \; . \quad (32)$$

then the relations to the above wave-function components become

$$\left.\begin{array}{ll}\psi^1=\frac{1}{2}(\Psi_1+\tilde{\Psi}_1+i(\Psi_2+\tilde{\Psi}_2)), & \psi^2=\frac{1}{2}(\Psi_3+\tilde{\Psi}_3-i(\Psi_4+\Psi_4)), \\ \psi^3=\frac{1}{2}(\Psi_1-\tilde{\Psi}_1+i(\Psi_2-\tilde{\Psi}_2)), & \psi^4=\frac{1}{2}(\Psi_3-\tilde{\Psi}_3-i(\Psi_4-\tilde{\Psi}_4)),\end{array}\right\} \quad . \quad (33)$$

if we here let $\tilde{\Psi}$ indicate the conjugate complex values. Conversely we have

$$\left.\begin{array}{ll}\Psi_1=\frac{1}{2}(\psi^1+\tilde{\psi}^1+\psi^3-\tilde{\psi}^3), & \Psi_2=\frac{1}{2i}(\psi^1-\tilde{\psi}^1+\psi^3+\tilde{\psi}^3), \\ \Psi_3=\frac{1}{2}(\psi^2+\tilde{\psi}^2+\psi^4-\tilde{\psi}^4), & \Psi_4=\frac{i}{2}(\psi^2-\tilde{\psi}^2+\psi^4+\tilde{\psi}^4).\end{array}\right\} \quad . \quad (34)$$

Also it follows as a matter of course that any adjustments of energy values (cf. $m_0{}^2c^2$) to fit particular cases may be made in (31) as well as in (8) (9).

CV. *A Theory of Viscosity and Thermal Conductance of Liquids and Highly Compressed Gases.*

By Balbhadra Prasad*.

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A. *Viscosity of Liquids.*

In developing his theory of viscosity of liquids Andrade⁽¹⁾ considered the liquid state of matter to be more akin to the solid state in order to explain the negative temperature coefficient of viscosity. Gases too under high pressures have a negative temperature coefficient of viscosity⁽²⁾, like liquids. In the following picture Andrade's concept of the mechanism of equalization of momentum of liquid molecules of different layers having different velocities has been used.

A rough idea of the space actually occupied per square centimetre can be had by determining (*a*) the number of molecules per square centimetre from the knowledge of molecular weight, density and Avogadro's number and (*b*) the cross section of the molecule from the knowledge of viscosity of the substance in gaseous state. Table I. gives data on gaseous nitrogen and a number of liquids. It is evident from this table that the various layers of a liquid are so much packed that if the mechanism of viscosity in case of liquids was the same as in case of gases, the viscosity of liquids would have been extremely small. But the reverse is the case.

Liquids are marked by cohesive forces. Molecules in the various layers of a liquid do not have the same freedom of movement as the molecules of a gas have. Accepting Andrade's picture of the transfer of momentum—only molecules whose potential energy is less by " E " are capable of leaving their layers and colliding with molecules of the neighbouring layers. Such collisions result in equalization of momentum. The number of such molecules per cubic centimetre from Boltzmann's law is $ne^{\mathrm{E}/k\mathrm{T}}$ when n is the number of molecules per cubic centimetre—we come to the conclusion that the number of molecules per layer which are capable of leaving the layer and colliding with the molecules of the neighbouring layers is $n^{\frac{2}{3}}e^{2\mathrm{E}/3k\mathrm{T}}$. The number of collisions which a molecule moving up and down from one layer to another will make per second $c/3d$, where "c" is the velocity of the molecule and "d" is the distance between the layers. As all the collisions may not be effective this number may be put as $fc/3d$. The factor "f" may have something to do with the shape and length of the molecule. The following equation can therefore be written :

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* Communicated by the Author.

$$n^{\frac{2}{3}}e^{2\mathrm{E}/3k\mathrm{T}} \cdot \frac{fc}{3d} \cdot m\frac{dv}{dx}d=\eta\frac{dv}{dx},$$

substituting $\sqrt{3k\mathrm{T}/m}$ for "c" and ρ/m for n we get

$$\eta=a'\rho^{\frac{2}{3}}\mathrm{T}^{\frac{1}{2}}e^{b'/\mathrm{T}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where constant $a'=\dfrac{1}{\sqrt{3}}fm^{-\frac{1}{2}}k^{\frac{1}{2}}$ and constant $b'=\dfrac{2\mathrm{E}}{3k}$. Further, as Table II. shows,

$$\log\rho^{\frac{2}{3}}\mathrm{T}^{\frac{1}{2}}=\mathrm{L}-\frac{\beta}{\mathrm{T}} \quad \text{or} \quad \rho^{\frac{2}{3}}\mathrm{T}^{\frac{1}{2}}=a''e^{-b''/\mathrm{T}}$$

in the case of liquids. This reduces equation (1) to

$$\eta=a'a''e^{b'-b''/\mathrm{T}}=\mathrm{A}e^{b/\mathrm{T}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In deriving this equation changes in volume as well as the changes in the fruitful momentum equalizing collisions due to changes in temperature have been taken into account. Hence the modification made by Andrade which was shown (3) to be unnecessary on grounds of experiment is shown to be unnecessary on theoretical grounds also.

B. *Thermal Conductance of Liquids.*

Passage of molecules from one isothermal layer to another is not likely to play a prominent part in thermal conductance of liquids. Assuming that the mechanism of conductance is the same as that of viscosity it follows :

$$\frac{f}{3} \cdot n^{\frac{2}{3}} \cdot e^{2\mathrm{E}/3k\mathrm{T}} \cdot \frac{c}{d} \cdot \mathrm{C}_p \cdot m \cdot \frac{d\mathrm{T}}{dx} \cdot d=\mathrm{K}\frac{d\mathrm{T}}{dx} \quad \text{or} \quad \mathrm{K}=\mathrm{C}_p\eta,$$

where "K" is thermal conductance and C_p is specific heat at constant pressure. Therefore in the case of simple unassociated liquids thermal conductance should decrease with temperature and should roughly be an exponential function of temperature. Results given in Table III. show that the thermal conductance of liquids decreases with temperature. Whether this decrease is exponential or not cannot be decided. Table IV. shows that $\mathrm{C}_p\times\eta$ is two to six times "K" instead of being equal to it, but this difference has its parallel in the case of gases also. From the available data it cannot be said whether $\mathrm{C}_p\eta/\mathrm{K}$ is independent of temperature or not.

C. *Viscosity of Compressed Gases.*

The viscosity of a gas at ordinary pressure is governed by the transfer of molecules from one layer to another having different velocities. But as the pressure of a gas is increased the occupied space per square centimetre goes on increasing. An idea of the occupied space per square centimetre can be had from the following figures obtained, from Michels and Gibson's data⁽²⁾ on nitrogen at 25° C., in the same way as figures for Table I. :—

Pressure atmos./(cm.)^2	15·37	104·5	212·4	320·4	430·2	630·4	742·0	854·1
Occupied space	0·041	0·141	0·224	0·280	0·313	0·362	0·385	0·400

At very high pressures the mechanism of viscosity of a gas is bound to be the same as that of a liquid. Hence it is expected that as the pressure is increased the temperature coefficient of viscosity will go on diminishing, and beyond a certain pressure it will become negative as in the case of liquids and at still higher pressures where the occupied space per square centimetre becomes comparable to that of the liquids the variation of viscosity with temperature will be governed by the equation $\eta=a'\rho^{\frac{2}{3}}\mathrm{T}^{\frac{1}{2}}e^{b'/\mathrm{T}}$. The expectation is partly justified inasmuch as viscosity of gases decreases with temperatures at high pressures as may be seen in Table V. The data are insufficient to test any equation. There is need of viscosity data at high pressures, say at a pressure of a thousand atmospheres and above, at a number of temperatures to decide the point.

D. *Thermal Conductance of Compressed Gases.*

From what has been discussed under sections (B) and (C) it is evident that the mechanism of thermal conductance of liquids and highly compressed gases will be the same . As the pressure is increased the temperature coefficient of conductance of a gas is expected to decrease and finally become negative. From the meagre data, which are available and which are given in Table VI., on the thermal conductance of gases it seems that at high pressures thermal conductance decreases with rise in temperature.

TABLE I.

Substance	t° C.	Cross-section $\times 10^{14}$	No. of mols. per sq. cm. $\times 10^{-14}$	Occupied surface /sq. cm.
Gaseous nitrogen at pressure of 15·37 atmos.	25	0·077	0·527	0·041
Liquid pentane	36	0·36	3·00	1·00
Liquid benzene	80	0·248	3·47	0·86
Liquid mercury	300	0·0945	11·03	1·00
Liquid bromine	25	0·206	4·76	0·98

TABLE II $\left(\text{for equation } \log_{10} \rho^{\frac{2}{3}}\mathrm{T}^{\frac{1}{2}}=\mathrm{L}-\frac{\beta}{\mathrm{T}}\right)$.

Liquid	Temp. range 0° C.	L	β	Maximum % error
n-hexane	0–80	1·197	24·6	0·1
n-heptane	0–110	1·222	29·0	0·2
n-octane	0–140	1·235	30·4	0·2
cyclo-hexane	0–110	1·266	30·8	0·1
Benzene	0–80	1·327	39·0	0·4
Methyl acetate	0–100	1·296	24·3	0·2
Methyl chloride	—40–0	1·290	23·2	0·1
Carbon tetrachloride	0–100	1·483	33·5	0·4
Bromo benzene	0–160	1·490	41·0	0·2
Tin	300–1200	2·229	170·0	0·6
Sodium	100–650	1·510	92·3	0·4
Potassium	100–650	1·478	95·0	0·6
Gallium	42·9–1100	2·130	123·0	1·4

TABLE III.

Thermal Conductance of Liquids.

Substance	Temp.	$\mathrm{K}\times10^4$	Substance	Temp.	$\mathrm{K}\times10^4$
Acetone	30	4·29	*n*-Hexyl alcohol	30	3·86
Acetone	75	4·03	*n*-Hexyl alcohol	75	3·74
Ether	30	3·29	*n*-Hexyl alcohol	100	3·68
Ether	75	3·22	*n*-Heptyl alcohol	30	3·89
Ethyl bromide	30	2·86	*n*-Heptyl alcohol	75	3·76
Ethyl bromide	75	2·73	*n*-Heptyl alcohol	100	3·72
Ethyl iodide	30	2·65	*n*-Octyl alcohol	30	3·98
Ethyl iodide	75	2·61	*n*-Octyl alcohol	75	3·87
n-Pentane	30	3·22	*n*-Octyl alcohol	100	3·75
n-Pentane	75	3·07	*n*-Nonyl alcohol	30	4·01
Carbon disulphide	30	3·82	*n*-Nonyl alcohol	75	3·91
Carbon disulphide	75	3·62	*n*-Nonyl alcohol	100	3·82
Toluene	30	3·64	Methyl chloride	—10	4·49
	(Bridgeman)				
Toluene	75	3·39	Methyl chloride	0	4·28
Toluene	30	3·57	Methyl chloride	10	4·08
	(Smith)				
Toluene	75	3·38	Methyl chloride	20	3·87
n-Amyl alcohol	30	3·88	Methyl chloride	30	3·67
n-Amyl alcohol	75	3·75			
n-Amyl alcohol	100	3·68			

TABLE IV.

Substance	Temperature	$C_p \times \eta \times 10^4$	$K \times 10^4$
Acetone	30	16·0	4·4
Ether	30	11·0	3·3
Ethyl iodide	30	9·0	2·7
n-Pentane	30	11·0	3·2
Carbon disulphide	30	7·7	3·8
Toluene	30	22·0	3·6

TABLE V.

Viscosity of nitrogen[2] in C.G.S. units ×10⁶.

Temp. in Centigrade degrees Pressure in atmos./(cm.)^2	20	50	75
15·37	180·6 *	191·3	202·2
104·5	200·5	208·8	217·5
212·4	236·7	237·3	240·8
320·4	276·4	273·7	274·6 *
430·2	319·4	312·9	307·9
630·4	399·7	378·6	370·1 *
742·0	440·8	416·5	401·6
854·1	480·2	455·0	436·2
965·7	520·5	491·3	470·3

* These figures are slightly inaccurate.

Viscosity of carbon dioxide[2] in C.G.S. units ×10⁶.

Temp. in Centigrade degrees Pressure in atmos./(cm.)^2	32	35	40
20	162	163	167 **
40	175	174	176
60	187	178	187
80	528	361	218
100	663 **	612 **	483

** Interpolated data.

TABLE VI.

Thermal Conductance of compressed carbon dioxide.

Press. Kg./(cm.)^2	Temp.	K×10^7	Press. Kg./(cm.)^2	Temp.	K×10^7
46·4	12·0	563	65·4	29·8	736
45·9	27·3	511	65·9	33·8	688
48·1	26·9	525	90·0	36·9	2584
48·2	33·9	521	90·1	50·0	1010

All data or the materials for the data of all the tables except Table V. have been taken from the main and supplementary volumes of Physico-Chemical tables of Landolt, Bornstein, Roth and Scheel.

The author is thankful to Mr. S. S. Panda for helping him in some of the calculations.

Summary.

1. Assuming that the picture of the mechanism of viscosity of liquids as given by Andrade is correct it has been shown that the viscosity of a liquid is given by the equation $\eta=a'\rho^{\frac{2}{3}}\mathrm{T}^{\frac{1}{2}}e^{b'/\mathrm{T}}$.

2. It has been further shown that in case of liquids $\rho^{\frac{2}{3}}\mathrm{T}^{\frac{1}{2}}=a''e^{-b''/\mathrm{T}}$. Hence viscosity equation reduces itself to Andrade's simple equation $\eta=\mathrm{A}e^{b/\mathrm{T}}$. The corrections made to this simple equation of Andrade are therefore unnecessary.

3. The mechanism of thermal conductance of liquids is bound to be the same as that of viscosity. This view is not in contradiction with the existing data. The derived equation for conductance is $\mathrm{K}=\mathrm{C}_p\eta$.

4. Viscosity and thermal conductance of highly compressed gases are bound to have the same mechanism as those of liquids as the space per layer will get so crowded under very high pressures that large scale of transfer of molecules from one layer to another will not be possible. The existing data in the literature seem to support this view.

References.

(1) Andrade, Phil. Mag. (7) xvii. p. 497 (1934) ; p. 698 (1934).
(2) Philips, Proc. Roy. Soc. A, lxxxvii. p. 48 (1912) ; Michels and Gibson, *ibid.*, A, cxxxiv. p. 288 (1932).
(3) Srinivasan and Prasad, Phil. Mag. (7) xxxiii. p. 258 (1942).

Mayurbhary Chemical Laboratory,
Ravenshaw College, Cuttark, India.

CVI. *Generalized Coordinates in Substitutive Networks.*

By D. W. C. Shen, Ph.D., Electrical Engineering Department, Queen Mary College, London University*.

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In a previous paper (1) the writer has shown that a general star network with mutual impedances between all branches can be substituted by an equivalent mesh network without coupling between the links. In polyphase networks, the use of substituted variables such as Fortescue's symmetrical coordinates (2), (3), (4), (5) to eliminate the coupling between sequence networks is very helpful in practical computations as well as in theoretical discussions of the performance of completely symmetrical systems. It is true that the very skilful application of the single-phase static network analyser to take care of unbalanced faults of a three-phase system has widened the utility of the method. Still it must be recognized that the solution using symmetrical coordinates is not necessarily the most direct and the simplest. If the network itself does not exhibit perfect symmetry, the sequence networks derived by symmetrical coordinates may be coupled together in a far more complicated manner than the original circuit, and such transformations back and forth represent lost motion to a great extent.

Blondel and Park (6) proposed two-reaction coordinates in the analysis of partly symmetrical synchronous machines. Later Clarke (7) introduced modified symmetrical coordinates and Kimbark (8) two-phase coordinates in their analysis of partially-symmetrical circuits such as the open-△ and Scott-connected transformer and auto-transformer banks, in which the impedances of two of the phases are equal and the positive- and negative-sequence impedances of the rest of the system could be assumed equal. These sets of coordinates reduce the number of mutual connections between the transformed networks but do not entirely eliminate the coupling. A review of the literature reveals the fact that all these sets of coordinates were discovered more or less from a hit-and-miss approach. Recognizing the physical nature of the problem, we expect that the generalized coordinates which will completely eliminate the coupling between the substitutive networks of the most general three-phase circuit would necessarily be functions of the impedance elements of the circuit and also that there might be some simple geometrical relations among the various known coordinates in three-dimensional space.

In this paper a general transformation is presented on the basis of Hamilton-Cayley identity of a hyper-number network impedance. The generalized coordinates thus found are functions of the impedance elements as expected. Moreover, in the case of symmetrical rotating machines the coordinates will be reduced to symmetrical coordinates and in the case of symmetrical bilateral networks the coordinates may be reduced to either symmetrical coordinates, modified symmetrical coordinates or two-phase coordinates. In the light of Hamilton-Cayley Identity, we will note that there is a very simple interpretation of some of these transformations geometrically, namely, that of unitary and orthogonal transformations or rotation of space. Besides, in the case of symmetrical bilateral networks, the number of degrees of freedom (or the number of independent real parameters available for adjustment) will be seen from simple geometrical properties in the generalized coordinates which completely separate the component networks along the three principal axes of Gibb's ellipsoid (11). This transformation, unlike that for partial elimination of mutual coupling, is unique.

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* Communicated by Professor W. J. John.

Transformation of a general three-phase Network Impedance to a Tonic or Right Tensor.

Let the impedance dyadic or matrix of a general three-phase network be given as Ƶ, which in nonion form with base vectors omitted is

$$\text{Ƶ}=\begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix}, \quad . \; . \; . \; . \; . \; . \; . \; . \qquad (1)$$

where the diagonal elements are self-impedances and the other elements the mutual impedances of the network. The three scalar invariants of Ƶ are Ƶ_s (the scalar of Ƶ being the sum of the nonion coefficients of the leading diagonal)

$$\text{Ƶ}_s=z_{11}+z_{22}+z_{33} \quad . \; . \; . \; . \; . \; . \; . \; . \qquad (2a)$$

$(\text{2nd Ƶ})_s$ (the scalar of the second of Ƶ being the sum of the first minors of the leading diagonal),

$$(\text{2nd Ƶ})_s=\begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}+\begin{vmatrix} z_{22} & z_{23} \\ z_{32} & z_{33} \end{vmatrix}+\begin{vmatrix} z_{33} & z_{31} \\ z_{13} & z_{11} \end{vmatrix}, \quad . \; . \qquad (2b)$$

and 3rd Ƶ (the third of Ƶ being the determinant of the coefficients)

$$\text{3rd Ƶ}=\begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix}. \quad . \; . \; . \; . \; . \; . \; . \qquad (2c)$$

The cubic

$$x^3-\text{Ƶ}_s x^2+(\text{2nd Ƶ})_s x-\text{3rd Ƶ}=0 \quad . \; . \; . \; . \; . \; . \qquad (3)$$

is called the characteristic equation of the dyadic. The Hamilton-Cayley theorem states that a dyadic or in general a hyper-number satisfies its characteristic equation identically, *i. e.*

$$\text{Ƶ}^3-\text{Ƶ}_s\text{Ƶ}^2+(\text{2nd Ƶ})_s\text{Ƶ}-\text{3rd Ƶ}\equiv 0. \quad . \; . \; . \; . \; . \; . \qquad (4)$$

It should be noted, in special cases, a dyadic may satisfy identically an equation of the second or first degree. Suppose the three roots r, s, t of the characteristic equation are distinct, then Hamilton-Cayley identity can be factored as the product of three distinct factors, *i. e.*

$$(\mathfrak{Z}-r\mathfrak{I})\,.\,(\mathfrak{Z}-s\mathfrak{I})\,.\,(\mathfrak{Z}-t\mathfrak{I})=0. \qquad \ldots\ldots \qquad (5)$$

Here $\mathfrak{I}$ is an idemfactor. It is obvious that each of these factors is a planar dyadic and hence there is a direction in space normal to the plane of the consequents so that the planar dyadic when used as a prefactor is an annihilator. Thus we may find three vectors $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ from the following equations

$$\left.\begin{array}{l} (\mathfrak{Z}-r\mathfrak{I})\,.\,\mathbf{A}_1=0 \\ (\mathfrak{Z}-s\mathfrak{I})\,.\,\mathbf{A}_2=0 \\ (\mathfrak{Z}-t\mathfrak{I})\,.\,\mathbf{A}_3=0 \end{array}\right\}. \qquad \ldots\ldots \qquad (6)$$

It can be shown that $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ are noncoplanar if r, s, t are distinct. If we express $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ in terms of the orthogonal unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ such as

$$\left.\begin{array}{l} \mathbf{A}_1=a_{11}\mathbf{i}+a_{21}\mathbf{j}+a_{31}\mathbf{k} \\ \mathbf{A}_2=a_{12}\mathbf{i}+a_{22}\mathbf{j}+a_{32}\mathbf{k} \\ \mathbf{A}_3=a_{13}\mathbf{i}+a_{23}\mathbf{j}+a_{33}\mathbf{k} \end{array}\right\}, \qquad \ldots\ldots \qquad (7)$$

then (6) may be written

$$\left.\begin{array}{l} \begin{bmatrix} z_{11}-r & z_{12} & z_{13} \\ z_{21} & z_{22}-r & z_{23} \\ z_{31} & z_{32} & z_{33}-r \end{bmatrix}\,.\,(a_{11}\mathbf{i}+a_{21}\mathbf{j}+a_{31}\mathbf{k})=0, \\ \begin{bmatrix} z_{11}-s & z_{12} & z_{13} \\ z_{21} & z_{22}-s & z_{23} \\ z_{31} & z_{32} & z_{33}-s \end{bmatrix}\,.\,(a_{12}\mathbf{i}+a_{22}\mathbf{j}+a_{32}\mathbf{k})=0, \\ \begin{bmatrix} z_{11}-t & z_{12} & z_{13} \\ z_{21} & z_{22}-t & z_{23} \\ z_{31} & z_{32} & z_{33}-t \end{bmatrix}\,.\,(a_{13}\mathbf{i}+a_{23}\mathbf{j}+a_{33}\mathbf{k})=0. \end{array}\right\}. \qquad . \quad (8)$$

And it is possible to obtain the directions of $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ from the following determinants if the second and third rows are not proportional :—

$$\left.\begin{array}{l} \mathbf{A}_1=\alpha\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ z_{21} & z_{22}-r & z_{23} \\ z_{31} & z_{32} & z_{33}-r \end{vmatrix}, \quad \mathbf{A}_2=\beta\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ z_{21} & z_{22}-s & z_{23} \\ z_{31} & z_{32} & z_{33}-s \end{vmatrix} \\ \qquad\qquad\qquad\qquad\qquad\quad \mathbf{A}_3=\gamma\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ z_{21} & z_{22}-t & z_{23} \\ z_{31} & z_{32} & z_{33}-t \end{vmatrix} \end{array}\right\}. \qquad (9)$$

α, β, γ are independent parameters.

Using these three non-coplanar vectors $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ as base vectors, we may write the impedance dyadic as

$$\mathfrak{Z}=\mathbf{A}_1\mathbf{B}_1+\mathbf{A}_2\mathbf{B}_2+\mathbf{A}_3\mathbf{B}_3 \quad . \; . \; . \; . \; . \; . \; . \quad (10)$$

and also the idem factor as

$$\mathfrak{I}=\mathbf{A}_1\mathbf{A}_1^\dagger+\mathbf{A}_2\mathbf{A}_2^\dagger+\mathbf{A}_3\mathbf{A}_3^\dagger. \quad . \; . \; . \; . \; . \; . \; . \quad (11)$$

Here $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$ are vectors yet to be determined and $\mathbf{A}_1^\dagger$, $\mathbf{A}_2^\dagger$, $\mathbf{A}_3^\dagger$ form a system reciprocal to $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$. $\mathbf{A}_1^\dagger$, $\mathbf{A}_2^\dagger$, $\mathbf{A}_3^\dagger$ can be found from $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ by dint of the following definitions :—

$$\mathbf{A}_1^\dagger=\frac{\mathbf{A}_2\times\mathbf{A}_3}{[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3]}, \quad \mathbf{A}_2^\dagger=\frac{\mathbf{A}_3\times\mathbf{A}_1}{[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3]}, \quad \mathbf{A}_3^\dagger=\frac{\mathbf{A}_1\times\mathbf{A}_2}{[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3]}, \quad . \; . \; . \quad (12)$$

where $[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3]$ represents the scalar triple product.

Obviously, with the aid of (7), (12) may be expressed in the following form :

$$\left.\begin{aligned}\mathbf{A}_1^\dagger&=b_{11}\mathbf{i}+b_{12}\mathbf{j}+b_{13}\mathbf{k},\\ \mathbf{A}_2^\dagger&=b_{21}\mathbf{i}+b_{22}\mathbf{j}+b_{23}\mathbf{k},\\ \mathbf{A}_3^\dagger&=b_{31}\mathbf{i}+b_{32}\mathbf{j}+b_{33}\mathbf{k}.\end{aligned}\right\}. \quad . \; . \; . \; . \; . \; . \quad (13)$$

It is not difficult to show from (6), (10), (11) that

$$\mathbf{B}_1=r\mathbf{A}_1^\dagger, \quad \mathbf{B}_2=s\mathbf{A}_2^\dagger, \quad \mathbf{B}_3=t\mathbf{A}_3^\dagger. \quad . \; . \; . \; . \; . \quad (14)$$

and hence the impedance dyadic $\mathfrak{Z}$ is reduced to the trinomial form

$$\mathfrak{Z}=r\mathbf{A}_1\mathbf{A}_1^\dagger+s\mathbf{A}_2\mathbf{A}_2^\dagger+t\mathbf{A}_3\mathbf{A}_3^\dagger, \quad . \; . \; . \; . \; . \; . \; . \quad (15)$$

which will be called a tonic.

Now, if we transform $\mathfrak{Z}$ to $\mathfrak{Z}_{\text{seq}}$ by multiplying (15) on the left by $(\mathbf{i}\mathbf{A}_1^\dagger+\mathbf{j}\mathbf{A}_2^\dagger+\mathbf{k}\mathbf{A}_3^\dagger)$ and on the right by $(\mathbf{A}_1\mathbf{i}+\mathbf{A}_2\mathbf{j}+\mathbf{A}_3\mathbf{k})$, we get

$$\begin{aligned}\mathfrak{Z}_{\text{seq}}&=(\mathbf{i}\mathbf{A}_1^\dagger+\mathbf{j}\mathbf{A}_2^\dagger+\mathbf{k}\mathbf{A}_3^\dagger)\,.\,(r\mathbf{A}_1\mathbf{A}_1^\dagger+s\mathbf{A}_2\mathbf{A}_2^\dagger+t\mathbf{A}_3\mathbf{A}_3^\dagger)\,.\,(\mathbf{A}_1\mathbf{i}+\mathbf{A}_2\mathbf{j}+\mathbf{A}_3\mathbf{k})\\ &=r\mathbf{ii}+s\mathbf{jj}+t\mathbf{kk}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (16)\end{aligned}$$

which has been called by Gibbs a right tensor.

It may perhaps be more easily recognized by the matrician if we use (7) and (13) so that the transformation in (16) can be rewritten in the following form :

$$\mathfrak{Z}_{\text{seq}}=\begin{bmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{bmatrix}\cdot\begin{bmatrix} z_{11} & z_{12} & z_{13}\\ z_{21} & z_{22} & z_{23}\\ z_{31} & z_{32} & z_{33}\end{bmatrix}\cdot\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=\begin{bmatrix} r & 0 & 0\\ 0 & s & 0\\ 0 & 0 & t\end{bmatrix} \quad . \; . \; . \quad (17)$$

Note that the base vectors given in (9) are the generalized coordinates which, being functions of the impedance elements in the general case, have three independent parameters and will transform the general network to completely independent sequence networks.

In the case of a symmetrical rotating machine, either synchronous or induction type, the nine distinct scalar coefficients (complex or real) given in the impedance dyadic (1) merge into three in number because

$z_{11}=z_{22}=z_{33}=$ say z_0, $z_{21}=z_{13}=z_{32}=$ say z_1, and $z_{31}=z_{12}=z_{23}=$ say z_2. Hence the impedance dyadic takes the following circulant form :

$$\not{\mathbf{Z}}=\begin{bmatrix} z_0 & z_2 & z_1 \\ z_1 & z_0 & z_2 \\ z_2 & z_1 & z_0 \end{bmatrix}. \quad \ldots \ldots \quad (18)$$

Physically it is quite evident that in a symmetrical rotating machine the self-impedances of the different phases are all equal to each other and that the second phase is related to the first phase not in the same way as the third phase to the first phase, but rather as the third phase to the second phase or the first to the third phase. Viewing from another angle, we note that exchanging two-line leads will exchange the phase rotation of a machine.

The three scalar invariants of (18) are

$$\not{\mathbf{Z}}_s=3z_0,\ (\text{2nd }\not{\mathbf{Z}})_s=3(z_0^2-z_1z_2),\ \text{3rd }\not{\mathbf{Z}}=z_0^3+z_1^3+z_2^3-3z_0z_1z_2. \quad . \quad (19)$$

Solving the characteristic cubic, we find the three roots r, s, t to be

$$r=z_0+z_1+z_2,\quad s=z_0+az_1+a^2z_2,\quad t=z_0+a^2z_1+az_2, \quad . \quad . \quad (20)$$

when a and a^2 are the two complex cube roots of unity.

Our next step is to determine the three base vectors $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ from (9). We have

$$\left.\begin{aligned} \mathbf{A}_1&=\alpha\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ z_1 & -z_1-z_2 & z_2 \\ z_2 & z_1 & -z_1-z_2 \end{vmatrix}=\alpha(z_1^2+z_2^2+z_1z_2)(\mathbf{i}+\mathbf{j}+\mathbf{k}), \\ \mathbf{A}_2&=\beta\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ z_1 & -az_1-a^2z_1 & z_2 \\ z_2 & z_1 & -az_1-z^2z_2 \end{vmatrix}=\beta(a^2z_1^2+az_2^2+z_1z_2)(\mathbf{i}+a^2\mathbf{j}+a\mathbf{k}), \\ \mathbf{A}_3&=\gamma\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ z_1 & -a^2z_1-az_2 & z_2 \\ z_2 & z_1 & -a^2z_1-az_2 \end{vmatrix}=\gamma(az_1^2+a^2z_2^2+z_1z_2)(\mathbf{i}+a\mathbf{j}+a^2\mathbf{k}). \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad . \quad (21)$$

Let us adjust α, β, γ to make $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ each of unit length. We get

$$\mathbf{A}_1=\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}),\quad \mathbf{A}_2=\frac{1}{\sqrt{3}}(\mathbf{i}+a^2\mathbf{j}+a\mathbf{k}),\quad \mathbf{A}_3=\frac{1}{\sqrt{3}}(\mathbf{i}+a\mathbf{j}+a^2\mathbf{k}). \quad (22)$$

The reciprocal system is easily evaluated as

$$\mathbf{A}_1^\dagger=\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})=\mathbf{A}_1,\quad \mathbf{A}_2^\dagger=\frac{1}{\sqrt{3}}(\mathbf{i}+a\mathbf{j}+a^2\mathbf{k})=\mathbf{A}_3,$$

$$\mathbf{A}_3^\dagger=\frac{1}{\sqrt{3}}(\mathbf{i}+a^2\mathbf{j}+a\mathbf{k})=\mathbf{A}_2 \quad \ldots \ldots \ldots \quad (23)$$

and the impedance dyadic of (18) is transformed by the following operation :

$$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \cdot \begin{bmatrix} z_0 & z_2 & z_1 \\ z_1 & z_0 & z_2 \\ z_2 & z_1 & z_0 \end{bmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}. \qquad (24)$$

to the well-known Fortescue's symmetrical components

$$\begin{bmatrix} z_0+z_1+z_2 & 0 & 0 \\ 0 & z_0+az_1+a^2z_2 & 0 \\ 0 & 0 & z_0+a^2z_1+az_2 \end{bmatrix} \quad . \quad . \quad . \quad . \quad (25)$$

We should note that this transformation is effected by two unitary dyadics or matrices characterized by the property that the reciprocal of the dyadic (or matrix or tensor) is equal to the transpose of the conjugate (or the conjugate of the transpose). Thus, let $\mathfrak{U}$ and $\mathfrak{V}$ be any two unitary dyadics, then we have

$$\mathfrak{U}=\mathfrak{U}_t^{-1}, \quad \mathfrak{U}_t=\mathfrak{U}^{-1}, \quad \mathfrak{U}^* . \mathfrak{U}_t=\mathfrak{U}_t . \mathfrak{U}=\mathfrak{I}. \quad . \quad . \quad . \quad (26)$$

$$(\mathfrak{U} . \mathfrak{V})_t^*=\mathfrak{V}_t^* . \mathfrak{U}_t^*=\mathfrak{V}^{-1} . \mathfrak{U}^{-1}=(\mathfrak{U} . \mathfrak{V})^{-1}. \quad . \quad . \quad . \quad (27)$$

In geometry, such an operation as (24) represents an affine transformation of space with origin fixed.

If, instead of transforming the impedance $\mathfrak{Z}$ in (18) to a right tensor by the operation of (24), we transform it to a tonic with $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ found in (23) as base vectors, then according to (15) we obtain

$$\mathfrak{Z}=(z_0+z_1+z_2)\mathbf{A}_1\mathbf{A}_1+(z_0+az_1+a^2z_2)\mathbf{A}_2\mathbf{A}_3+(z_0+a^2z_1+az_2)\mathbf{A}_3\mathbf{A}_2. \quad (28)$$

It is interesting to note that though in this case we have the same scalar coefficients yet the nature of transformation is quite different. Here the sequence components are used as the rectangular coordinates of new vectors, while in the right tensor transformation they are used as nonion coefficients of new dyadics. (28) can also be obtained by splitting up the circulant impedance dyadic $\mathfrak{Z}$ of (18) into three orthogonal dyadics or matrices such as

$$\mathfrak{Z}=\begin{bmatrix} z_1 & z_2 & z_1 \\ z_1 & z_1 & z_2 \\ z_2 & z_1 & z_1 \end{bmatrix}=z_1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}+z_1\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}+z_2\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad (29)$$

and each of these orthogonal dyadics or matrices can again be written as the sum of three Hermitians characterized by the property that the conjugate of the dyadic or matrix is equal to its transpose.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}+\frac{1}{3}\begin{bmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{bmatrix}=\frac{1}{3}\begin{bmatrix} 1 & a^2 & a \\ a & 1 & a^2 \\ a^2 & a & 1 \end{bmatrix}. \quad (30a)$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \frac{a}{3}\begin{bmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{bmatrix} + \frac{a^2}{3}\begin{bmatrix} 1 & a^2 & a \\ a^2 & 1 & a^2 \\ a^2 & a & 1 \end{bmatrix} \qquad (30b)$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \frac{a^2}{3}\begin{bmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{bmatrix} + \frac{a}{3}\begin{bmatrix} 1 & a^2 & a \\ a & 1 & a^2 \\ a^2 & a & 1 \end{bmatrix} \qquad (30c)$$

Therefore (29) becomes

$$\mathbf{Z} = (z_0+z_1+z_2)\tfrac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + (z_0+az_1+a^2z_2)\tfrac{1}{3}\begin{bmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{bmatrix}$$

$$+(z_0+a^2z_1+az_2)\tfrac{1}{3}\begin{bmatrix} 1 & a^2 & a \\ a & 1 & a^2 \\ a^2 & a & 1 \end{bmatrix}$$

$$=(z_0+z_1+z_2)\mathbf{A}_1\mathbf{A}_1+(z_0+az_1+a^2z_2)\mathbf{A}_2\mathbf{A}_3+(z_0+a^2z_1+az_2)\mathbf{A}_3\mathbf{A}_2. \quad (28)$$

Suppose z_0, z_1, z_2 of (18) are real, then the instantaneous impedance dyadic is so-called a cyclotonic. The two roots s, t in (20) are conjugate

$$s=z_0+az_1+a^2z_2=\text{say } z\epsilon^{j\theta}, \quad t=z_0+a^2z_1+az_2=\text{say } z\epsilon^{-j\theta}. \quad . \quad (31)$$

and we will choose α, β and γ of (21) such that $\mathbf{A}_1$ is still of unit length, but $\mathbf{A}_2$ and $\mathbf{A}_3$ become rotating vectors $\epsilon^{+jp}\mathbf{A}_2$ and $\epsilon^{-jp}\mathbf{A}_3$ respectively, or call the new vectors $\mathbf{A}_1'$, $\mathbf{A}_2'$, $\mathbf{A}_3'$, we have

$$\mathbf{A}_1=\mathbf{A}_1', \qquad \mathbf{A}_2=\epsilon^{-jp}\mathbf{A}_2', \qquad \mathbf{A}_3=\epsilon^{jp}\mathbf{A}_3' \qquad . \quad . \quad . \quad (32)$$

Substituting (31) and (32) into (28) ,we get

$$\mathbf{Z}=\begin{bmatrix} z_0 & z_2 & z_1 \\ z_1 & z_0 & z_2 \\ z_2 & z_1 & z_0 \end{bmatrix}=(z_0+z_1+z_2)\mathbf{A}_1\mathbf{A}_1+z\epsilon^{j\theta}\mathbf{A}_2'\mathbf{A}_3'+z\epsilon^{-j\theta}\mathbf{A}_3'\mathbf{A}_2'$$

$$=(z_0+z_1+z_2)\mathbf{A}_1\mathbf{A}_1+\tfrac{1}{2}z\cos\theta\{(\mathbf{A}_2'+\mathbf{A}_3')(\mathbf{A}_3'+\mathbf{A}_2')+(\mathbf{A}_2'-\mathbf{A}_3')(\mathbf{A}_3'+\mathbf{A}_2')\}$$

$$+\tfrac{1}{2}\mathbf{j}z\sin\theta\{(\mathbf{A}_2'+\mathbf{A}_3')(\mathbf{A}_3'-\mathbf{A}_2')+(\mathbf{A}_2'-\mathbf{A}_3')(\mathbf{A}_3'+\mathbf{A}_2')\}.$$

Now let us define

$$\mathbf{d}=\frac{\mathbf{A}_2'+\mathbf{A}_3'}{\sqrt{2}}, \qquad \mathbf{q}=\mathbf{j}\frac{(\mathbf{A}_2'-\mathbf{A}_3')}{\sqrt{2}}, \quad . \quad . \quad . \quad . \quad . \quad (33)$$

then we have

$$\mathbf{Z}=(z_0+z_1+z_2)\mathbf{A}_1\mathbf{A}_1+z\cos\theta(\mathbf{dd}-\mathbf{qq})-z\sin\theta(\mathbf{dq}-\mathbf{qd}). \quad . \quad (34)$$

which is expressed in terms of the real base vectors $\mathbf{A}_1$, $\mathbf{d}$, $\mathbf{q}$, the so-called Blondel's or Park's two-reaction coordinates.

Transformation of a Bilateral Symmetrical three-phase Network Impedance to a Special Tonic or Right Tensor.

In practice, a three-phase network usually contains many branches, and most branches of most three-phase networks are symmetrical and bilateral, or at least they may be considered so with negligible error. The impedance dyadic of a bilateral symmetrical three-phase network is given as

$$\mathfrak{Z}=\begin{bmatrix} z_0 & z & z \\ z & z_0 & z \\ z & z & z_0 \end{bmatrix}. \qquad (35)$$

Obviously, it may be considered as a special case of eq. (18) when $z_1=z_2=$ say z.

The roots of the characteristic cubic of (35) are

$$r=z_1+2z, \qquad s=t=z_2-z. \qquad (36)$$

It should be noted that in the case the three roots r, s, t are not distinct, the reduction by the method given above to the trinomial form of (15) is not unique. For two equal roots, there are two forms:

one giving a special tonic:

$$\mathfrak{Z}=r\mathbf{A}_1\mathbf{A}_1^\dagger+s(\mathbf{A}_2\mathbf{A}_2^\dagger+\mathbf{A}_3\mathbf{A}_3^\dagger), \qquad (37)$$

and the other a so-called shear:

$$\mathfrak{Z}=r\mathbf{A}_1\mathbf{A}_1^\dagger+s[\mathbf{A}_2\mathbf{A}_2^\dagger+(\mathbf{A}_2+\mathbf{A}_3)\mathbf{A}_3^\dagger] \qquad (38)$$

dependent whether the dyadic satisfies a quadratic identically or not. In the form (37), the dyadic would satisfy a quadratic in $\mathfrak{Z}$ namely $(\mathfrak{Z}-r\mathfrak{I})\,.\,(\mathfrak{Z}-s\mathfrak{I})=0$ identically. While for the shear (38), the lowest degree equation that it satisfies identically is the cubic (4).

As the dyadic given in (35) satisfies a quadratic identically, it is a special tonic. In fact, since it is self-transposed, it must be a right tensor and our method as above given is valid.

To determine $\mathbf{A}_1$, we have

$$\mathbf{A}_1=\alpha\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ z & -2z & z \\ z & z & -2z \end{vmatrix}=\alpha z^2(\mathbf{i}+\mathbf{j}+\mathbf{k}), \qquad (39)$$

which again becomes

$$\mathbf{A}_1=\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$$

after adjusting α to make the vector to have unit length. To determine $\mathbf{A}_2$ or $\mathbf{A}_3$, we assume $\mathbf{A}_2=a_{12}\mathbf{i}+a_{22}\mathbf{j}+a_{32}\mathbf{k}$ and $\mathbf{A}_3=a_{13}\mathbf{i}+a_{23}\mathbf{j}+a_{33}\mathbf{k}$.

Then we have from

$$\begin{bmatrix} z_0-(z_0-z) & z & z \\ z & z_0-(z_0-z) & z \\ z & z & z_0-(z_0-z) \end{bmatrix} . (a_{12}\mathbf{i}+a_{22}\mathbf{j}+a_{32}\mathbf{k})$$

$$=2(a_{12}+a_{22}+a_{32})(\mathbf{i}+\mathbf{j}+\mathbf{k})=0.$$

namely $\mathbf{A}_2 . \mathbf{A}_1=0$ or $a_{12}+a_{22}+a_{32}=0,$

hence $\mathbf{A}_2=a_{12}\mathbf{i}+a_{33}\mathbf{j}-(a_{12}+a_{22})\mathbf{k}$ . . . . (40)

Likewise $\mathbf{A}_3 . \mathbf{A}_1=0$ or $a_{13}+a_{23}+a_{33}=0,$

hence $\mathbf{A}_3=a_{13}\mathbf{i}+a_{23}\mathbf{j}-(a_{13}+a_{23})\mathbf{k}$ . . . . (41)

Thus there are an infinite number of ways of choosing $\mathbf{A}_2$ or $\mathbf{A}_3$, the only restriction being $[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3]\neq 0$, since the dyadic is assumed to be complete. Geometrically, $\mathbf{A}_2$ or $\mathbf{A}_3$ may take on any values and assume any directions on the plane perpendicular to $\mathbf{A}_1$ except that $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ should be non-coplanar.

For instance, let us take

$$a_{12}=\frac{1}{\sqrt{3}},\quad a_{22}=\frac{a^2}{\sqrt{3}},\quad a_{13}=\frac{1}{\sqrt{3}},\quad a_{23}=\frac{a}{\sqrt{3}},$$

and get

$$\mathbf{A}_2=\frac{1}{\sqrt{3}}(\mathbf{i}+a^2\mathbf{j}+a\mathbf{k}),\quad \mathbf{A}_3=\frac{1}{\sqrt{3}}(\mathbf{i}+a\mathbf{j}+a^2\mathbf{k})$$

which, together with the value of

$$\mathbf{A}_1=\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$$

as already found, form again the symmetrical components.

Furthermore, taking $\frac{2}{3}, -\frac{1}{3}$ for a_{12} and a_{22}, and $0, \frac{1}{\sqrt{3}}$ for a_{13}, a_{23}, we find

$$\mathbf{A}_2=\frac{1}{3}(2\mathbf{i}-\mathbf{j}-\mathbf{k}) \quad\text{and}\quad \mathbf{A}_3=\frac{1}{\sqrt{3}}(\mathbf{j}-\mathbf{k}). \quad . \; . \; . \quad (42)$$

These together with the value of $\mathbf{A}_1=\frac{2}{3}(\mathbf{i}+\mathbf{j}+\mathbf{k})$ are Clarke's "Modified Symmetrical Components."

Again, if we take $\mathbf{A}_1=\frac{2}{3}(\mathbf{i}+\mathbf{j}+\mathbf{k})$ together with the $\mathbf{A}_2$ and $\mathbf{A}_3$ of (42), we get Kimbark's "Two-phase Coordinates."

Finally, with all the vectors having unit length, viz.

$$\mathbf{A}_1=\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}},\quad \mathbf{A}_2=\sqrt{\frac{2}{3}}\mathbf{i}-\sqrt{\frac{1}{6}}\mathbf{j}-\sqrt{\frac{1}{6}}\mathbf{k},\quad \mathbf{A}_3=\frac{1}{\sqrt{2}}(\mathbf{j}-\mathbf{k}). \quad (43a)$$

or in matrix form

$$\begin{array}{cccc} & \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathbf{A}_1 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \mathbf{A}_2 & \sqrt{\frac{2}{6}} & -\sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{6}} \\ \mathbf{A}_3 & 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \end{array} \quad . \quad . \quad . \quad . \quad . \quad . \quad (43b)$$

we easily recognize them to be those listed by Boyajian in his discussion of Kimbark's paper, multiplied by different factors for each row. In other words, Boyajian deals with numerics so the coefficients given by him are direction numbers while the writer deals with base vectors so the coefficients given in (43*b*) are direction cosines—a difference common to Wills' Vector and Tensor Analysis and other treatises on the subject. From the very interesting geometrical interpretation of three-phase quantities by Sah (9, 10), we note from (43*a*) or (43*b*) $\mathbf{A}_1$ the so-called isoclinic unit vector has direction cosines all equal to $\frac{1}{\sqrt{3}}$ while $\mathbf{A}_3$, being the line of interesction $(\mathbf{j}-\mathbf{k})$ between the coordinate plane $\mathbf{j}\,\mathbf{k}$ and the so-called fundamental plane, has direction cosines $0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$. $\mathbf{A}_2$ is simply equal to $\mathbf{A}_3 \times \mathbf{A}_1$. Thus $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ of (43*a*) are real orthogonal unit vectors ; they are related to the stationary symmetrical coordinates of (22) in the same way as the two-reaction coordinates $\mathbf{A}_1$, $\mathbf{d}$, $\mathbf{q}$ of (33) are related to the rotational symmetrical coordinates $\mathbf{A}_1$, $\mathbf{A}_2'$, $\mathbf{A}_3'$ given in (32). From the phase axes $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, the stationary and rotational symmetrical coordinates are obtained by unitary transformation while (43*a*) and two-reaction coordinates are obtained by orthogonal transformation.

Using the undetermined values of a_{12}, a_{22}, a_{13}, a_{23} of (40) and (41), we evaluate the reciprocal system as follows :

$$\left.\begin{aligned} \mathbf{A}_1^{\dagger} &= \frac{a_{12}a_{23}-a_{13}a_{22}}{D}(\mathbf{i}+\mathbf{j}+\mathbf{k}), \\ \mathbf{A}_2^{\dagger} &= \frac{1}{\sqrt{3D}}[(a_{13}+2a_{23})\mathbf{i}-(2a_{13}+a_{23})\mathbf{j}+(a_{13}-a_{23})\mathbf{k}], \\ \mathbf{A}_3^{\dagger} &= \frac{1}{\sqrt{3D}}[-(a_{12}-2a_{22})\mathbf{i}+(2a_{12}+a_{22})\mathbf{j}+(a_{22}-a_{12})\mathbf{k}]. \end{aligned}\right\} . \quad (44)$$

When D is the determinant

$$\begin{vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ a_{12} & a_{22} & -a_{12}-a_{22} \\ a_{13} & a_{23} & -a_{13}-a_{23} \end{vmatrix} .$$

After transformation the impedance dyadic (35) takes the form :—

$$\begin{bmatrix} z_0+2z & 0 & 0 \\ 0 & z_0-z & 0 \\ 0 & 0 & z_0-z \end{bmatrix}. \qquad (45)$$

Number of Degrees of Freedom.

When the three roots of the characteristic cubic are distinct, the directions of $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ are uniquely determined though their magnitudes may take on any values. Therefore the most general transformation which will completely eliminate the coupling of this type of impedance network would contain three adjustable parameters. In mathematical symbols, we have :—

$$\mathbf{A}_1=\alpha(a_{11}\mathbf{i}+a_{21}\mathbf{j}+a_{31}\mathbf{k}),$$
$$\mathbf{A}_2=\beta(a_{12}\mathbf{i}+a_{22}\mathbf{j}+a_{32}\mathbf{k}),$$
$$\mathbf{A}_3=\gamma(a_{13}\mathbf{i}+a_{23}\mathbf{j}+a_{33}\mathbf{k}),$$

in which α, β, γ are scalars whose values may be adjusted at will and a_{11}, a_{12}, a_{13}, a_{21} . . . a_{33} are direction cosines of $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$.

When two of the roots of the characteristic cubic are equal, the direction of $\mathbf{A}_1$ is uniquely determined but the direction as well as the magnitude of the other two vectors $\mathbf{A}_2$ and $\mathbf{A}_3$ are arbitrary, the only restriction being that $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ should be non-coplanar. In this case the most general transformation would be

$$\mathbf{A}_1=\alpha(a_{11}\mathbf{i}+a_{21}\mathbf{j}+a_{31}\mathbf{k}),$$
$$\mathbf{A}_2=\beta[a_{12}\mathbf{i}+a_{22}\mathbf{j}-(a_{12}+a_{22})\mathbf{k}],$$
$$\mathbf{A}_3=\gamma[a_{13}\mathbf{i}+a_{23}\mathbf{j}-(a_{13}+a_{23})\mathbf{k}],$$

in which the constants a_{12} or a_{22} and a_{13} or a_{23} are also adjustable in addition to α, β, and γ. Thus in this case, we have all-fold five adjustable parameters.

Finally, for the case of three equal roots $r=s=t$, the impedance dyadic may be transformed to three different forms : one being simply the idemfactor multiplied by r, namely

$$\mathfrak{Z}=r\mathfrak{I}, \qquad (46)$$

and the other two special shears of either the form

$$\mathfrak{Z}=r[\mathbf{A}_1\mathbf{A}_1^{\dagger}+\mathbf{A}_2\mathbf{A}_2^{\dagger}+(\mathbf{A}_2+\mathbf{A}_3)\mathbf{A}_3^{\dagger}] \qquad (47)$$

or

$$\mathfrak{Z}=r[\mathbf{A}_1\mathbf{A}_1^{\dagger}+(\mathbf{A}_1+\mathbf{A}_2)\mathbf{A}_2^{\dagger}+(\mathbf{A}_2+\mathbf{A}_3)\mathbf{A}_3^{\dagger}], \qquad (48)$$

according to whether the dyadic satisfies an equation of the first degree, namely $\mathfrak{Z}-r\mathfrak{I}=0$, or the quadratic $(\mathfrak{Z}-r\mathfrak{I})^2=0$, or only the Hamilton-Cayley Identity. The directions as well as the magnitudes of $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ are all arbitrary and there would be nine adjustable parameters. In bilateral three-phase network, the last case is rather trivial as the impedance without mutual effects in each phase.

From the above considerations, we see that to avoid mutual coupling by properly choosing the axes is from a dyadician's point of view to write the dyadic in a trinomial form useful to the problem. If the dyadic is reduced to a right tensor characterized by an ellipsoid with three principal axes, no coupling exists, when referred to these axes. In the case of general bilateral network typified by a self-transposed dyadic known as a right tensor, the transformation is unique. In the case of partial elimination, when the network is symmetrical with respect to one phase, there is usually more than one possibility. As for the most general impedance network, the coordinates to completely separate the networks are dependent on the impedance elements because each part of the network has its own ellipsoid with different sets of principal axes. It seems rather impracticable if not impossible to work out a scheme by which different axes may be used in the same problem at the same time.

The writer wishes to thank Professor W. J. John and Dr. H. Tropper for their encouragement and helpful criticism of the manuscript.

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CVII. *The Efficiency of the Barrier Layer Photo-cell.*

By R. A. HOUSTOUN, D.Sc., University of Glasgow *.

[Received February 26, 1948.]

IN a leaflet issued in connection with one of the selenium barrier layer photo-cells occurs the sentence : "It is interesting to note that the conversion of light energy into electrical energy takes place at the remarkably high efficiency of approximately 50 per cent." It is the purpose of this paper to describe some experiments made to test the truth of this statement.

§ 1.

It is obvious that the efficiency must vary with the wavelength and the first step was to measure this variation for a series of five cells. There were :—

A=A Weston Photronic.

B=Another Weston Photronic. The sensitive area of these cells is circular, 37 mm. in diameter, and it is protected by glass.

C=A cell made by Renny and Wender, Ltd. Area circular, 38 mm. in diameter.

D=Cell supplied by Evans Electro-selenium, Ltd. Rectangular, 30×13 mm.^{2}

E=Cell of German origin. Rectangular, 30×15 mm.^{2}

C, D and E have no glass cover.

The spectrometer used had as collimator and telescope lenses a Ross Wide Angle Xpres, working at F 4, and a Pentac, working at F 2·9. Their focal lengths were respectively $8\frac{1}{4}$ and 8 inches. The prism was of dense flint, its refracting angle was 60° and the surface of its faces measured $3 \times 1{\cdot}85$ square inches. The spectrum fell on a screen with a slit in it, which moved along a scale, and the photo-cell was placed behind this slit. The slit was about $\frac{1}{2}$ mm. wide and the spectrum was 1 cm. high. The source of light was a 250-watt projector lamp, focused on the slit of the collimator by a lens and turned edgeways, so that the images of the different filaments were in contact. This lamp was run off the 250-volt lighting circuit. The voltage was read after every observation and a correction of the intensity made when necessary ; this correction was usually of the order of one per cent. The galvanometer was a moving coil one of 143-ohms resistance and a deflection of 1 mm. on the scale was equivalent to $2{\cdot}43 \times 10^{-9}$ amp. There was no other resistance in the circuit but the galvanometer and the size of the deflections was regulated by changing the collimator slit width.

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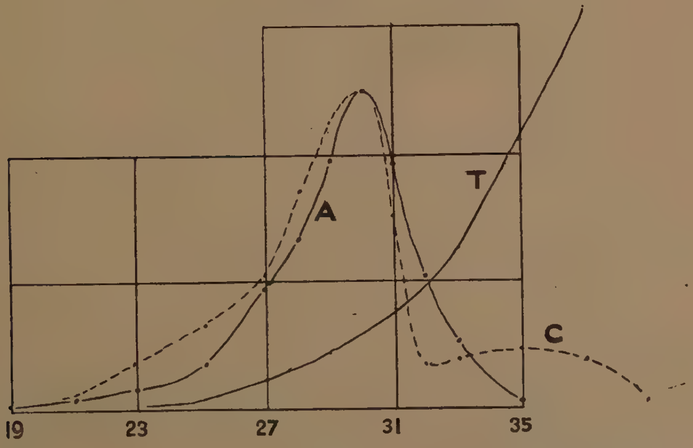
* Communicated by the Author.

The results are in the following table :

Scale reading	mμ	A	B	C	D	E	Thermopile
1·5	370	0·7	1·3	0·5	1·3	1·3	—
1·9	409	1·1	2·3	2·2	3·0	3·2	—
2·1	431	3·4	4·2	4·3	4·6	5·2	0·5
2·3	455	6·1	7·6	14·6	10·3	8·7	1·5
2·5	489	14·1	18·8	26·6	25·1	19·7	3·8
2·7	530	37·9	45·5	46·2	46·0	44·8	8·2
2·8	556	52·8	66·2	69·0	67·9	62·9	12·4
2·9	588	79·0	90·0	90·8	87·0	91·4	16·4
3·0	636	100·0	100·0	100·0	100·0	100·0	23·4
3·1	685	77·2	63·4	61·0	50·3	63·5	31·1
3·2	736	41·0	25·4	13·6	14·9	12·8	40·2
3·3	786	20·3	15·2	15·2	6·3	3·2	51·2
3·5	889	2·3	2·6	17·6	1·0	1·3	84·0
3·7	990	0·7	1·3	11·4	0·3	1·0	126·0
3·9	—	0·7	1·0	1·1	0·0	0·6	84·0

Fig. 1.

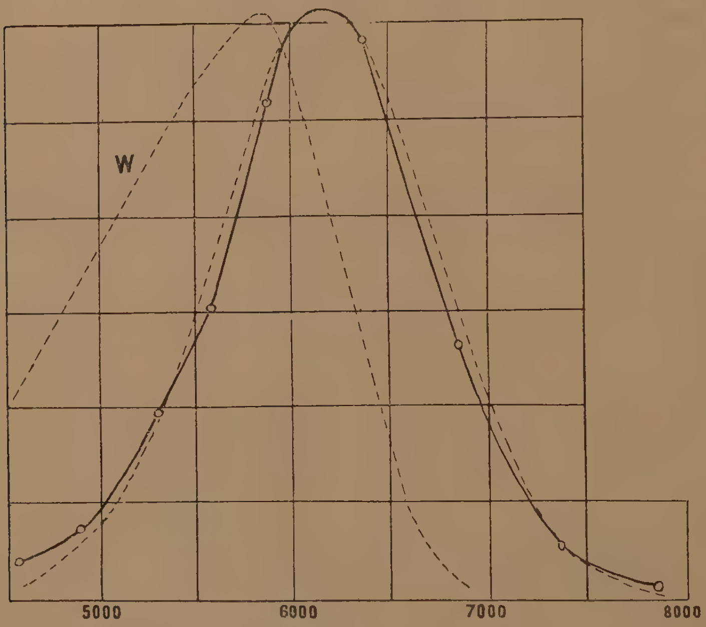


The first column gives the scale reading, the second the wavelength, and the next five are galvanometer deflections for the five cells respectively; the scales being adjusted to make the maximum value 100 in each case. Each value is the mean of three settings. Each cell has the maximum at the same wavelength and there is little difference between the cells except for one point ; cell C shows a second maximum in the red. Fig. 1 shows the results for cell A as a full line and for cell C as a broken line. The dip between the maxima for C occurs at λ=736 mμ.

§ 2.

Curves A and C in fig. 1 represent galvanometer deflections against deviation for a prismatic spectrum. The efficiency is the ratio of the electrical energy produced by the cell per second to the energy of radiation incident on it per second. The electrical energy is proportional to the square of the current, *i. e.* to the square of the numbers in columns A, B, C, D and E. To measure the incident energy, it is necessary to go through the spectrum with a non-selective absorber. A Moll vacuum thermopile, made by Kipp and Zonen, was borrowed from the Chemistry Department for the purpose, and used with the same galvanometer and slit widths, as were employed for the photo-cells. The result is given in arbitrary units in the last column of the table and is shown as curve T in fig. 1. By dividing the squares of columns A, B, C, D and E by column T, numbers can be obtained proportional to the efficiency as a function of the wavelength, and the full curve in fig. 2 was obtained in this way ; it represents the efficiency of cell A as a function of the wavelength. It was not necessary to apply a slit-width correction, as the same slit width was used both for photo-cell and for thermopile.

Fig. 2.



The curve has almost the shape of a probability curve. The broken curve superposed on it is a probability curve calculated to fit the same data. The photronic book of data gives a " spectral sensitivity " curve in which the ordinates are proportional to current. To make it an efficiency curve, it is necessary to square the ordinates *. This has been done and the result is shown as the curve W in fig. 2. It differs considerably from mine ; so also do the curves given by other makers of selenium cells and they also differ among themselves.

§ 3.

The curve in fig. 2 gives only relative values of the efficiency. The latter is greatest in the orange part of the spectrum, in the region transmitted by the Kodak γ spectral filter ; I next determined its absolute value for each of the five photo-cells in question for this region. According to W. Gerlach, the Hefner lamp radiates $2{\cdot}25\times10^{-5}$ calorie per sq. cm. per sec. at a distance of 1 metre in a horizontal direction. The glass window of the thermopile was removed and it was exposed to a Hefner lamp at this distance. Thus the combination, thermopile and galvanometer, was standardised : a deflection of 1 mm. on the scale was found to indicate a radiation of $2{\cdot}32\times10^{-7}$ cal. per sq. cm. per sec. or 0·974 microwatt per sq. cm.

Each of the five photo-cells was then exposed to the radiation from a 15-watt lamp through the γ filter at a distance of one metre from the lamp and I, the current through the galvanometer, noted. The rate at which the cell is doing work is RI^2, where R is the resistance of the galvanometer, 143 ohms. The thermopile was then exposed through the same filter at the same distance and the reading recorded ; it showed that energy was being received at the rate of 86·7 microwatts per cm.^{2} The efficiency is RI^2 divided by the product of this quantity and the area of the surface of the photo-cell. The following table gives the results :

Cell	RI^2	Efficiency
A	$9{\cdot}06\times10^{-7}$ microwatt	$9{\cdot}7\times10^{-10}$
B	$1{\cdot}53\times10^{-6}$,,	$1{\cdot}65\times10^{-9}$
C	$7{\cdot}19\times10^{-8}$,,	$7{\cdot}34\times10^{-11}$
D	$3{\cdot}28\times10^{-7}$,,	$9{\cdot}7\times10^{-10}$
E	$1{\cdot}54\times10^{-9}$,,	$3{\cdot}9\times10^{-12}$

It should be stated that cell E has deteriorated very much in the past two years. Formerly it was as good as D.

§ 4.

The results in the table are all very small. But it must be remembered that the efficiency increases with the illumination, and the illumination of the cells during the above measurements was only about one-tenth foot candle. The efficiency also varies with the resistance in the circuit. The variation with these factors was next investigated for cell A by placing it on a photometer bench and connecting it in series with a resistance box

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* Probably also the ordinates should have been multiplied by the ordinates of curve T in fig. 1. This would diminish the discrepancy.

and microammeter. The source of light was a 100-watt clear bulb run off the D.C. lighting circuit. The voltage was checked by a voltmeter and the intensity corrected for its variation, as before. No colour filter was used.

The results are shown in figs. 3 and 4. Fig. 3 shows the variation of efficiency with resistance in the circuit for illuminations of 12·8, 28·9, 52·0, 85·5 and 144·5 foot candles respectively. In order to make a compact diagram, each curve is plotted on a vertical scale which is two-thirds of the vertical scale of the curve immediately below it. It is obvious from the diagram that the efficiency increases with illumination, also that there is a maximum value for each illumination which shifts towards the left as the illumination increases.

Fig. 3.

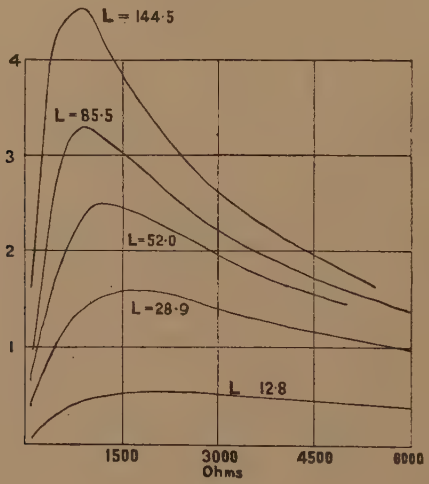


Fig. 4 shows the variation of efficiency with illumination, when the resistance in the circuits is 20,000, 10,000, 5000 and 1000 ohms respectively. The two lower curves are plotted on the same vertical scale, but in order to make the diagram compact, the 5000 ohm curve is plotted on two-thirds and the 1000 ohm curve on one-third this scale. It is obvious from the diagram that there is a maximum value of the efficiency for each resistance, which shifts towards the right as the resistance diminishes.

The maximum efficiencies recorded in figs. 3 and 4 are $2{\cdot}3\times10^{-8}$ and $2{\cdot}35\times10^{-8}$ watts per foot-candle. The area of the cell was

$$\pi\left(\frac{3{\cdot}7}{2}\right)^2 \text{ cm.}^2 = 10{\cdot}7 \text{ cm.}^2$$

The higher of these two values is consequently $2{\cdot}2\times10^{-9}$ watts per cm.^{2} per foot-candle. If we use Gerlach's value, since the Hefner unit is nine-tenths of a candle, one foot-candle is

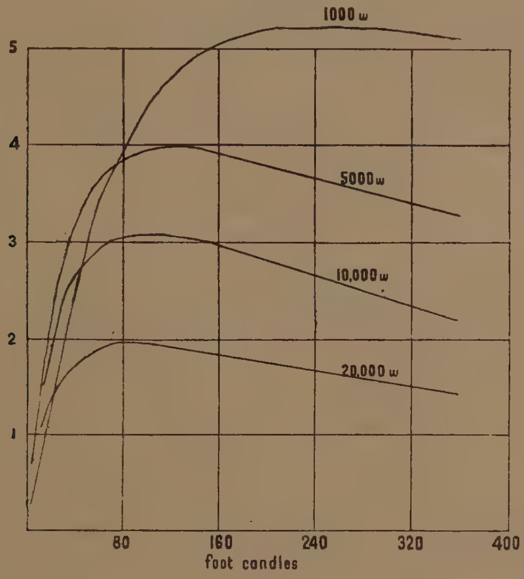
$$\frac{10}{9}\times\frac{2{\cdot}25\times10^{-5}\times4{\cdot}2\times10^{7}}{(0{\cdot}3048)^2} \text{ ergs per cm.}^2 \text{ sec.,}$$

or $1 \cdot 13 \times 10^4$ ergs per cm.^{2} sec. The efficiency is consequently

$$\frac{2 \cdot 2 \times 10^{-9} \times 10^7}{1 \cdot 13 \times 10^4} = 2 \times 10^{-6}.$$

This is for the whole spectrum, but only about one-thirtieth of the radiated energy falls into the visible spectrum and the efficiency varies considerably throughout the visible spectrum. We can conclude, therefore, from the above calculation, that the results in figs. 3 and 4 show that the efficiency may be as high as 2×10^{-4}, which is a considerable advance on the values for orange light given in the table.

Fig. 4.



§ 5.

The question can be treated mathematically. I have shown in a former paper *, that the action of the selenium barrier layer photo-cell is represented very well by the equation

$$\frac{1}{\mathrm{I}} = \frac{k}{\mathrm{E}}\left(1 + \frac{\mathrm{R}}{r}\right)\frac{1}{\mathrm{L}} + \frac{\mathrm{R}}{\mathrm{E}},$$

where I is the external current, k a quantity that varies with wavelength, E a constant, r the internal resistance, R the external resistance and L the intensity of the incident light. The efficiency is RI^2/L. It is a maximum when $\sqrt{\mathrm{L}}/\mathrm{I}\sqrt{\mathrm{R}}$ is a minimum.

From the equation above, we obtain

$$\frac{\sqrt{\mathrm{L}}}{\mathrm{I}\sqrt{\mathrm{R}}} = \frac{k}{\mathrm{E}}\left(1 + \frac{\mathrm{R}}{r}\right)\frac{1}{\sqrt{\mathrm{R}}\sqrt{\mathrm{L}}} + \frac{\sqrt{\mathrm{R}}\sqrt{\mathrm{L}}}{\mathrm{E}}.$$

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* R. A. Houstoun, Phil. Mag. xxxiii. p. 226 (1942).

This may be written in the two following forms

$$\frac{\sqrt{\mathrm{L}}}{\mathrm{I}\sqrt{\mathrm{R}}}=\left[\left(\frac{k}{\mathrm{E}\sqrt{\mathrm{L}}}\right)^{\frac{1}{2}}\frac{1}{\mathrm{R}^{\frac{1}{4}}}-\left(\frac{k}{\mathrm{E}r\sqrt{\mathrm{L}}}+\frac{\sqrt{\mathrm{L}}}{\mathrm{E}}\right)^{\frac{1}{2}}\mathrm{R}^{\frac{1}{4}}\right]^2$$
$$+2\sqrt{\frac{k}{\mathrm{E}\sqrt{\mathrm{L}}}\left(\frac{k}{\mathrm{E}r\sqrt{\mathrm{L}}}+\frac{\sqrt{\mathrm{L}}}{\mathrm{E}}\right)}, \quad . \quad . \quad . \quad \mathrm{I}$$

$$\frac{\sqrt{\mathrm{L}}}{\mathrm{I}\sqrt{\mathrm{R}}}=\left[\left(\frac{k}{\mathrm{E}}\right)^{\frac{1}{2}}\left(\frac{1}{\sqrt{\mathrm{R}}}+\frac{\sqrt{\mathrm{R}}}{r}\right)^{\frac{1}{2}}\frac{1}{\mathrm{L}^{\frac{1}{4}}}-\left(\frac{\sqrt{\mathrm{R}}}{\mathrm{E}}\right)^{\frac{1}{2}}\mathrm{L}^{\frac{1}{4}}\right]^2$$
$$+2\sqrt{\frac{k}{\mathrm{E}}\left(\frac{1}{\sqrt{\mathrm{R}}}+\frac{\sqrt{\mathrm{R}}}{r}\right)\frac{\sqrt{\mathrm{R}}}{\mathrm{E}}}. \quad . \quad . \quad . \quad \mathrm{II}$$

In the first form R is the independent variable and L is constant. The expression is a minimum when the square is zero, *i. e.* when

$$\mathrm{L}=k\left(\frac{1}{\mathrm{R}}-\frac{1}{r}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathrm{III}$$

In the second form L is the independent variable and R is constant. The expression is a minimum when the square is zero, *i. e.* when

$$\mathrm{L}=k\left(\frac{1}{\mathrm{R}}+\frac{1}{r}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathrm{IV}$$

If L and 1/R are taken as coordinates, III and IV represent parallel straight lines on opposite sides from the origin and at equal distances from it.

The coordinates of the maxima can be read off from figs. 3 and 4. They are

R varying (fig. 3)		L varying (fig. 4)	
R	L	R	L
ohms	foot candles	ohms	foot candles
2100	12·8	20,000	90
1600	28·9	10,000	113
1400	52·0	5000	130
800	85·5	1000	230
750	144·5	—	—

They are graphed in fig. 5. If we make allowances for error of observation, the points lie on the parallel lines required by theory and, by measuring the intercepts, r is found to be 2350 ohms.

The existence of the maxima depends on the relative variation of RI^2 and L, which naturally cannot be measured to the same degree of accuracy as they can. As the cell A is subject to fatigue, greater accuracy could be obtained by using other cells and also by heaping observations. The values of R and L above are those obtained from a first inspection of the curves; they were not "improved" after comparison with one another.

If we make the square zero in I

$$\frac{\mathrm{L}}{\mathrm{RI}^2}=\frac{4k}{E^2}\left(\frac{k}{r\mathrm{L}}+1\right),$$

and if we increase L indefinitely, the efficiency is given in the limit by

$$\frac{\mathrm{RI}^2}{\mathrm{L}}=\frac{E^2}{4k}.$$

If we make the square zero in II

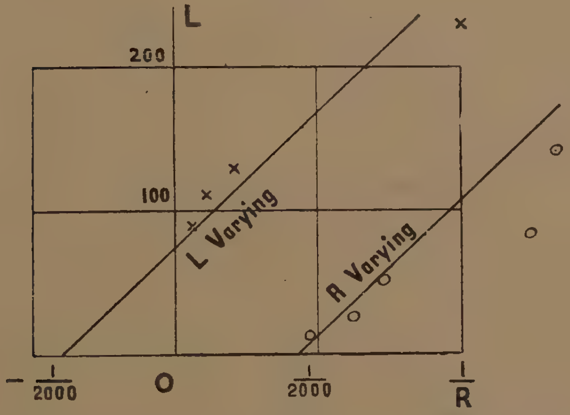
$$\frac{\mathrm{L}}{\mathrm{RI}^2}=\frac{4k}{\mathrm{E}^2}\left(1+\frac{\mathrm{R}}{r}\right),$$

and if we make R zero, the efficiency is given again by the same expression.

We find from fig. 5 that k is $1\cdot8\times10^5$ foot-candle ohms and from previous work * on this cell it was known that its value of E was 0·2 volt. Hence $\mathrm{E}^2/4k=5\cdot5\times10^{-8}$ watts per foot-candle. We have already found that one foot-candle is $1\cdot13\times10^4$ ergs per cm.^{2} sec. Thus the efficiency is

$$\frac{5\cdot5\times10^{-8}\times10^7}{1\cdot13\times10^4\times10\cdot7}\doteqdot5\times10^{-6}.$$

Fig. 5.



This is an average for the spectrum ; for the most favourable wavelength it might be 100 times this. So we arrive at the value 5×10^{-4} which is $2\frac{1}{2}$ times as great as the maximum actually recorded in figs. 3 and 4, but we do not known that the theory holds, when L becomes infinite and R becomes zero.

§ 6.

An E.E.L. photo-cell, the surface of which was circular and of 5·8 cm. diameter was placed at a distance of 5 cm. from the filament of a clear 100-watt lamp. The current through a milliammeter of 5-ohms resistance was 10 mA. Thus the rate of production of electrical energy was 500 microwatts. By measuring the solid angle subtended by the sensitive surface, it was found that the rate at which energy fell on it was 8 watts.

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* R. A. Houstoun and A. F. Howatson, Phil. Mag. xxxvi. p. 279 (1945).

Thus the average efficiency for the whole spectrum was

$$5\times10^{-4}/8=6{\cdot}2\times10^{-5}.$$

When cell B was placed at a distance of 8 cm. from the same lamp and the radiation received through the γ filter, the current was 44 μA., measured on a microammeter of resistance 39·4 ohms. When the radiation was received by the thermopile through the same filter but at a distance of 12 cm., the current was 47 μA., the same microammeter being used. This makes the efficiency $6{\cdot}4\times10^{-8}$.

To sum up, the highest efficiency found in this paper for approximately monochromatic light is $6{\cdot}4\times10^{-8}$; the highest efficiencies recorded for white light by two different arrangements were 2×10^{-6} and $6{\cdot}2\times10^{-5}$ respectively. Photo-cells vary greatly in their efficiency. But if any barrier layer cell has an efficiency at all approaching 50 per cent, it must be a very exceptional cell and it will retain this efficiency only for a short time.

When the large E.E.L. photo-cell was connected to the milliammeter of 5-ohms resistance and pointed towards the sky, the highest current I have been able to obtain on a January afternoon is 4 mA. This means a rate of working of 8×10^{-5} watt. If the slates on the roof of the Natural Philosophy Department in which I write were all replaced by photo-cells of this efficiency, the total activity under similar conditions would be 32 watts, sufficient to run one electric lamp. So the photo-cell does not promise well as a source of energy.

CVIII. *Notices respecting New Books.*

The Chemical Society, 1841–1941: *A Historical Review.* By TOM SIDNEY MOORE and JAMES CHARLES PHILIP. [Pp. 236, 10 plates.] (London: Chemical Society, 1947.) Price 12*s*. 6*d*.

British Chemists. Edited by ALEXANDER FINDLAY and WILLIAM HOBSON MILLS. [Pp. 432, 16 plates.] (London: Chemical Society, 1947.) Price 18*s*.

THE history of a science may be written as a logical development of ideas, or in terms of the lives of its eminent men and the growth of the societies which foster it. The latter method is illustrated by these two volumes published by the Chemical Society to mark its centenary which, but for the war, would have been celebrated in 1941. The first gives a historical review of the Society together with extracts from its records, the second short biographies of distinguished chemists who have died since the outbreak of the first world war. The names of the editors and authors are a guarantee of the high quality of the contents which form a valuable contribution to the history of British chemistry in the last century.

The Chemical Society, founded for the advancement of chemical science, has more than justified the faith of its pioneers, one of whom was Richard Phillips, an Editor of the *Philosophical Magazine* at the time. It was the first national chemical body, and it is of interest to find that A. W. Hofmann, the first Director of the Royal College of Chemistry and President of the Society in 1863, took a leading part when he moved to Berlin in 1865 in the formation of the German Chemical Society, and became its first President.

The membership of the Society, originally seventy-seven, exceeded six thousand in 1945; but this progress was not maintained without difficulties. The growth of professionalism in chemistry in the latter half of the nineteenth century led to division of opinion on the functions of the Society, and the controversies on the admission of women to the Fellowship, first broached in 1880 but not effected until 1920, make interesting reading. Accommodation has been a difficulty. The Society moved into the present Burlington House in 1873, but now "the position is that after a century of public-spirited activity the Society is housed in premises which in no single respect are even tolerably adequate for its nationally important work."

The first meeting of the Society decided that "discoveries and observations" should be communicated and published, and that a library should be formed. This library is now one of the finest chemical libraries in existence, and through a co-operation scheme with other chemical organizations has served an increasingly wide circle of readers. The problem of housing the vast collection of books is, however, not yet solved. The publication of the Journal has been supplemented by the well-known Abstracts, first published on a comprehensive scale in 1871, and later extended by arrangement with other bodies, by the volumes of memorial lectures, and since 1904 by the valuable annual reports on the progress of chemistry.

The history of the Society cannot be told apart from the men who took a leading part in its affairs, and the second of these volumes is thus in some ways complementary to the first, in that it gives an account of the lives, together with photographs, of sixteen eminent British chemists, ten of them presidents and four vice-presidents, whose work was done mainly in the second half-century of the Society's existence. This distinguished company includes W. Crookes, who was already nine years old when the Society was founded, J. Dewar, H. E. Armstrong, R. Meldola, H. B. Dixon, W. Ramsay, W. H. Perkin (jun.), A. G. Perkin, A. G. Green, A. Harden, W. J. Pope, G. T. Morgan, A. Lapworth, J. F. Thorpe, T. M. Lowry and G. Barger. Some account is given of the scientific achievements of each, and, what is as interesting, something, frequently at first-hand, of their personal history, their interests and recreations and what manner of men they were. The task of selection must have been difficult and one can naturally think of names that might have been included. The dominance of organic chemistry in this period is striking, especially when one thinks of the European physical chemists who have died since 1914—W. Ostwald, W. Nernst, S. Arrhenius, to mention only three.

Chemists all over the world owe a debt of gratitude to the Chemical Society for its work, and all who are interested in the history of chemistry in the last hundred years are further indebted for these two attractively produced and well-printed centenary volumes.

J. W. B.

Research: *A Journal of Science and its Applications*. Vol. I. Nos. 6, 7. [Pp. xii+48 each.] (London: Butterworths Scientific Publications. March, April, 1948.) Price 3*s*. 6*d*. each.

The journal 'Research' was launched in October, 1947, with the purpose of forging a link between pure science and its applications in industry and elsewhere, under the guidance of a strong scientific advisory broad, with Sir John Anderson as Chairman. It was proposed to cover the broad front of science purposefully and practically, with a limitation in view-point rather than in subject-matter; 'Research' was to be "functional." There is undoubtedly scope for such a journal, and it is of interest to see how the programme is working out.

In the two issues under notice, about two-thirds of the space is devoted to articles of a general kind, with well-defined themes, but usually, in the scientific articles, involving reference to many branches or sub-branches of science. An article on "The Nutrient Balance in the Sea", for example, is chemical, physical and biological; under "Recent Advances in Electron Microscopy" applications to metallurgical and biological problems are dealt with as well as the instrument itself and the techniques developed. The reviewer has found the articles on themes with which he is unfamiliar clearly presented and interesting (often much more so than he expected from the titles); and those on more familiar themes could be read without that discomfort which is sometimes produced by so-called "popular" writing. This suggests that the journal is fulfilling well one of its purposes in making more easily accessible to the scientist or industrialist authoritative information about interesting and exciting work in fields with which he is unfamiliar, but which may have bearing on his own work, or to which his own work might contribute. Among articles of other types, one on "German Scientific and Technical Publications of To-day" may be mentioned as likely to be widely useful.

The notes on the Research Associations and the D.S.I.R. are a valuable feature, for knowledge of the activities of these bodies is much too narrowly confined. The book reviews are sufficiently full to give a reasonable indication of the content and character of the books, though the problem of selecting books appropriate for review in the journal can hardly be an easy one.

It is as yet too early for useful comment to be made on the Research Supplement, which is at present rather thin. As the journal becomes more widely known, it may well become generally accepted as most appropriate for certain types of paper, and the research section may have to be enlarged.

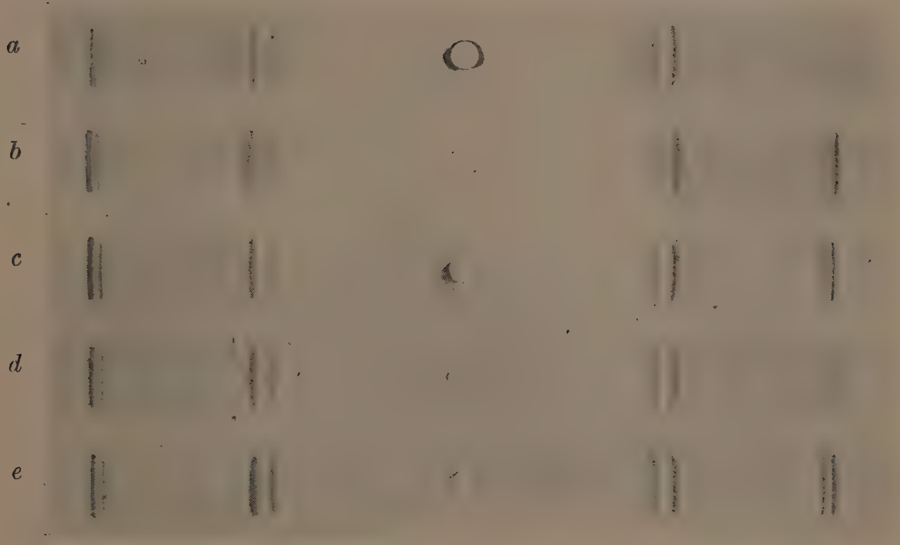
In short, the journal has made an excellent start, and there is no doubt that it forms a most valuable complement to existing journals. The title, however, is deplorable, and its supersession by the present sub-title, or something similar, is much to be desired. E. C. S.

The Half-Yearly Journal of the Mysore University. New series, Section B, 1944, 1945, 1946. (Bangalore.) 1 *Re* each issue.

At intervals of six months, the University of Mysore publishes a Journal of which Section B contains articles on science, engineering and medicine. The six parts to which this review relates contain a notable collection of serious original contributions covering a wide range of subjects. The general high quality appears to be well maintained, with very few noticeable lapses. The reader may feel a little surprise to find, among such titles as "Potential Problems with Axial Symmetry," "The Drainage Pattern of Mysore State," "The Skeletal System of *Calotes Versicolor*," an article on the calibration of a bridge wire, and it is true that this article does not seem to add much to the methods laid down in text-books of practical physics 40 years ago. On the other hand, this isolated shadow does serve the useful purpose of bringing out the lights in the picture, which includes—to quote a few papers among many which impress the reader—articles on "Titanium Minerals, their Occurrences, Metallurgy and Uses," by C. S. Pichamuthu; "The Principal Magnetic Susceptibilities of Some Metal Crystals," by S. R. Rao; "Inequalities relating to Arithmetic and Geometric Means," by T. S. Nanjundiah; and "Recent advances in Blood and Blood-substitute Therapy," by V. R. Naidu. Some of the work is naturally of local interest, as in "Enumeration of Flowering Plants round about Bangalore," by C. S. Venkatesh and H. C. Govinda. This title perhaps shows a slightly local quality in the English, too; but it must be said that the journal is remarkably free from local idiom. Indeed, in literary quality the Journal bears comparison with the best of our own scientific publications. There can be no doubt that this journal will have to be read by all serious workers, and—particularly in view of its remarkably low price—could well be added to the library list of institutions who do not already take it. J. P. A.

[The Editors do not hold themselves responsible for the views expressed by their correspondents.]

Fig. 1.



High angled lines in structure-spectra taken with a plate of pure aluminium and with cobalt radiation.

(*a*) Initial state; equilibrium condition.
(*b*) Immediately after release of stress (58·8% reduction in thickness).
(*c*) 403 hrs. ,, ,, ,,
(*d*) 525 hrs ,, ,, ,,
(*e*) 1896 hrs. ,, ,, ,,

CIX. *A Theory of Arbitrary Aerofoils in Cascade* *.

By A. R. Howell, M.A., Research Engineer with
Messrs. Power Jets (R. & D.), Ltd. †

[Received January 14, 1946.]

I. Introduction.

The design of elements employing cascades of aerofoils requires a knowledge of the flow of the fluid through such cascades. The solution of this flow is usually obtained from experimental tests (1), (2), (3), (4), (5), (6), (7), (8), (9), but a theoretical solution would be of use in order to supplement these tests. Again, the suitability of any cascade could be estimated by a theoretical method before any tests were actually made.

Theoretical solutions have been evolved for the flow of an ideal fluid past thin aerofoils and flat plates in cascade. These solutions are obtained by the transformation of known flows through cascades (10), (11), (12), (13) or by transforming known flows past isolated bodies (14), (15), (16), (17) or by using approximate methods (8), (9), (18), (19). Recently Collar (20) and Merchant and Collar (21), by transforming cascades of ovals, have obtained solutions for certain series of aerofoils in cascade similar to the Joukowski type.

The purpose of this note is to give a solution of the theoretical flow of an ideal fluid past arbitrary aerofoils in cascade.

II. Method of Solution.

1. *General Description.*

The method is general and direct, that is, it begins with an arbitrary aerofoil in cascade. Four conformal transformations are then used in order to reduce the flow past the aerofoil in cascade to that of a known flow past a circle.

The basis of the method is the fairly simple transformation $\zeta_1 = \tanh Z_1$, where Z_1 is the plane of the aerofoil in cascade. The ζ_1 plane then consists of a single aerofoil. An ordinary Joukowski transformation now changes this aerofoil into an irregular oval, and another Joukowski transformation changes this into an approximate circle, the axes of reference being chosen to suit the transformations. The approximate circle is then transformed into an accurate circle by using a very generalized transformation such as that of Theodorsen (22) and von Kàrmàn (23).

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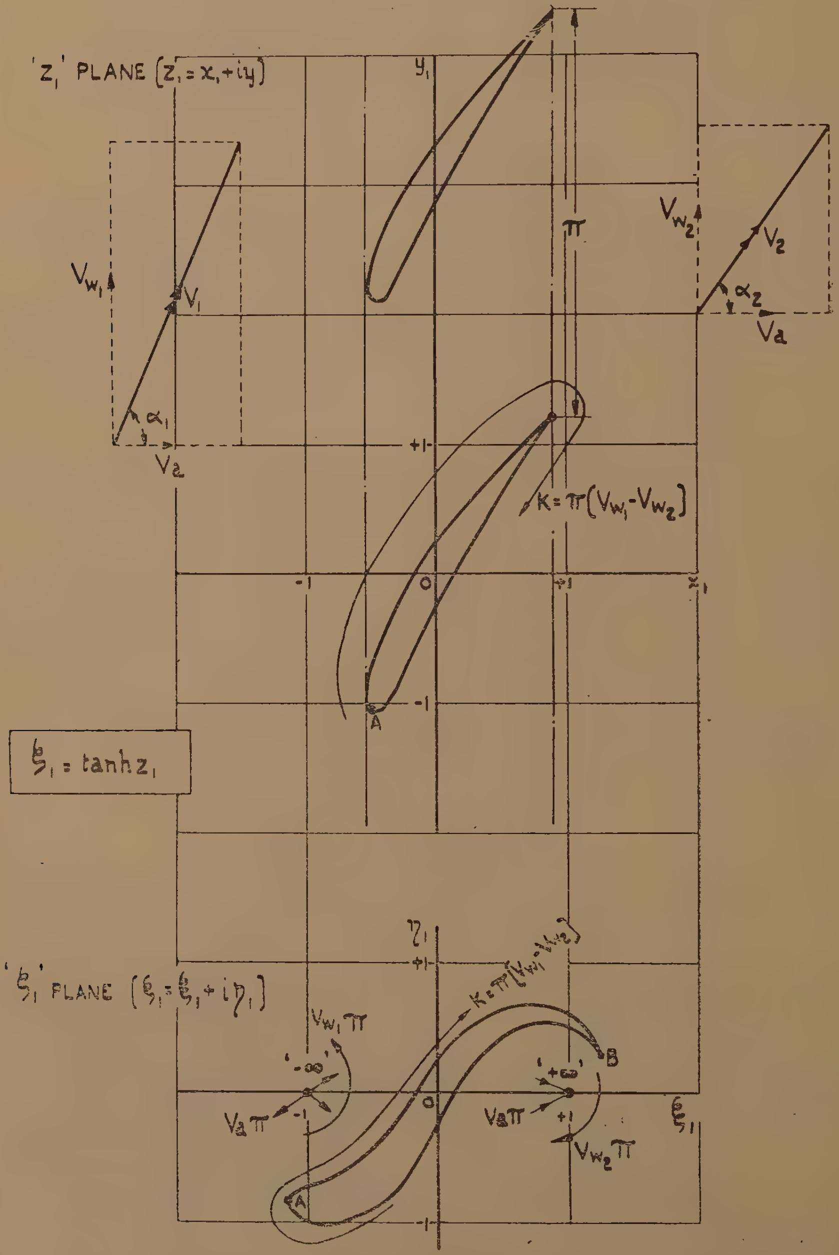
* This paper was first issued as " R. A. E. Note No. E. 3859 " in March 1941.

† Communicated by the Author.

2. *First Transformation.*

In the Z_1 plane the aerofoils are placed along the y_1 axis at a distance π apart and the chords taken to give the appropriate pitch/chord ratio.

Fig. 1.

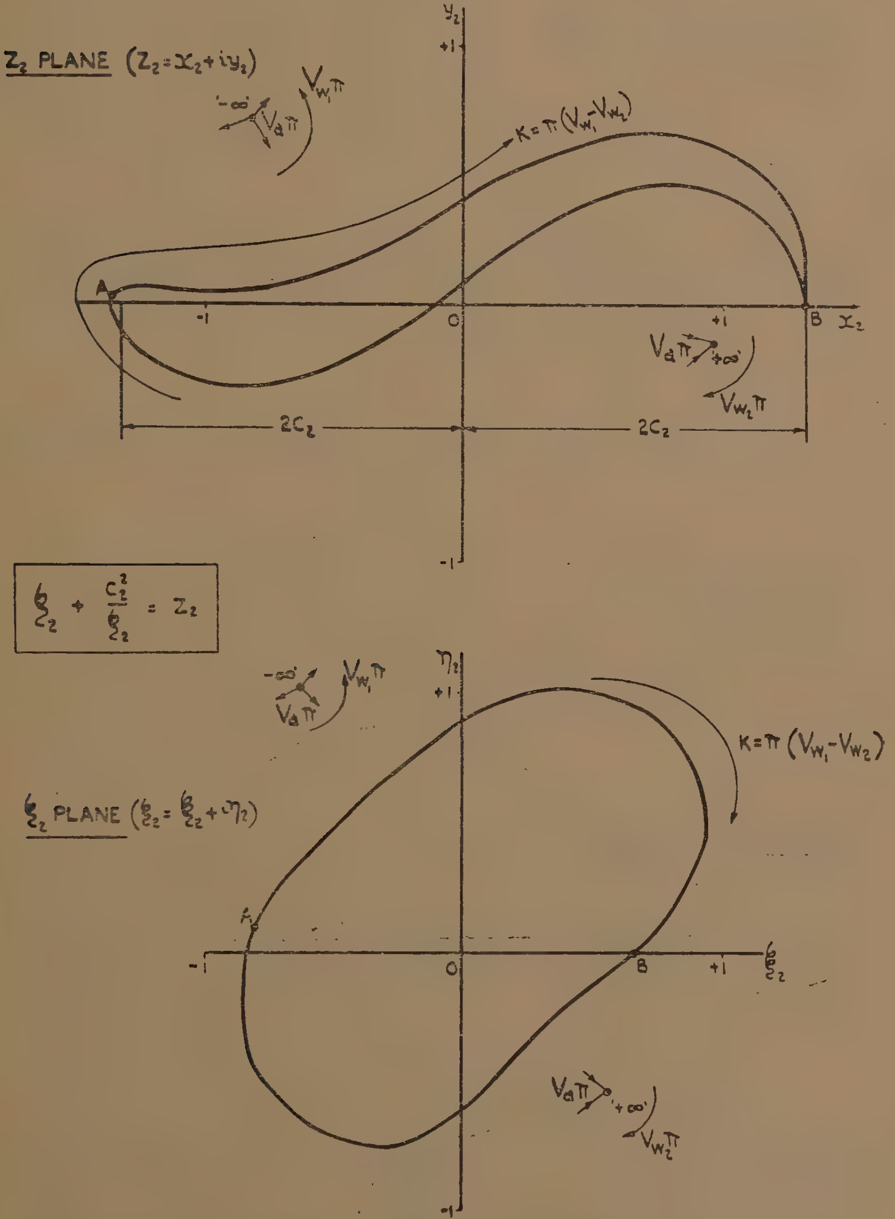

'Z₁' PLANE (Z₁ = x₁ + iy₁)
y₁
V_W₁
V₁
α₁
Va
π
V_W₂
V₂
α₂
Va
+1
K = π(V_W₁ − V_W₂)
−1
O
+1
x₁
−1
A
ζ₁ = tanh z₁
η₁
+1
'ζ₁' PLANE (ζ₁ = ξ₁ + iη₁)
K = π(V_W₁ − V_W₂)
V_W₁ π
'−∞'
Va π
−1
O
'+∞'
B
Va π
+1
V_W₂ π
ξ₁
A
−1

The origin should be taken at about the centre of one of the aerofoils. The transformation used, and illustrated in fig. 1, is

$$\zeta_1 = \tanh Z_1 .$$

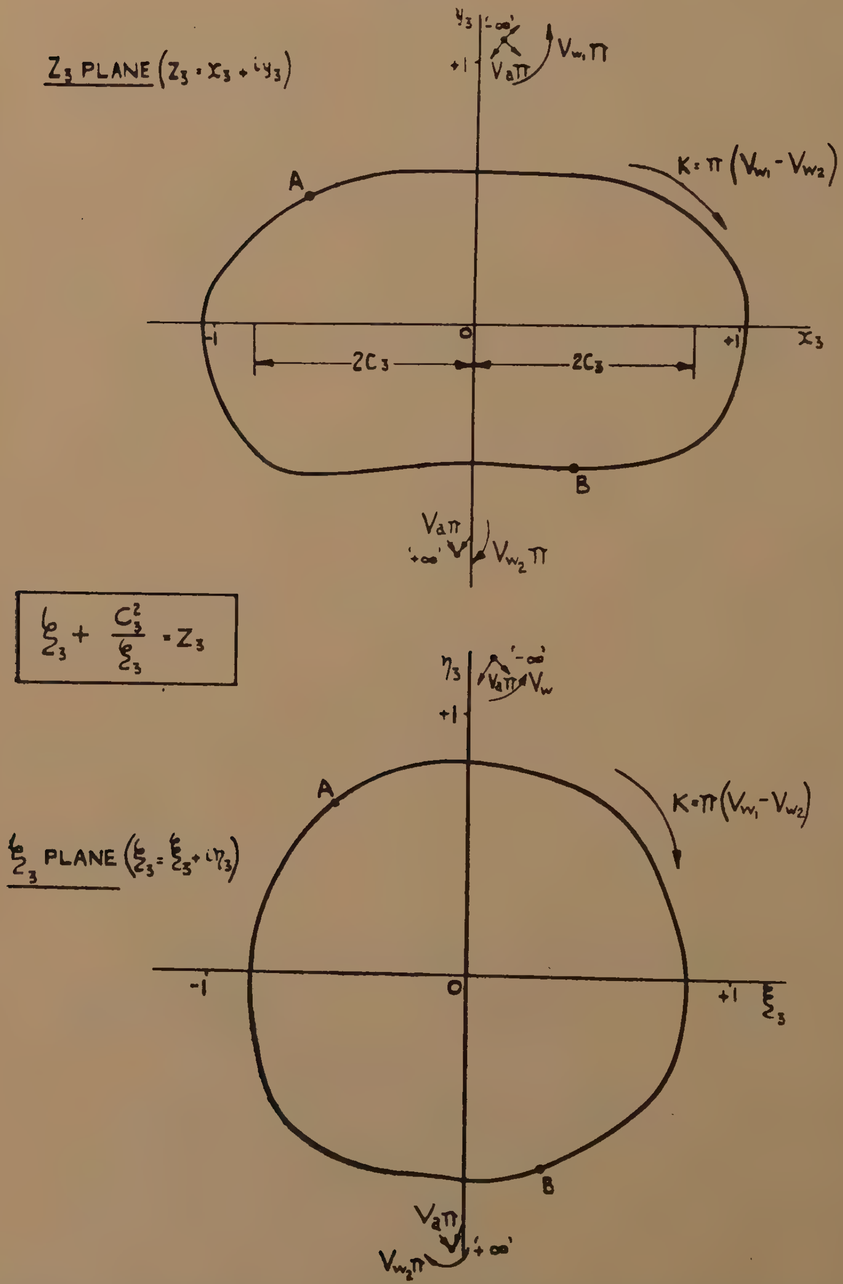
The values of ζ_1 can be obtained directly from an atlas of complex functions such as Ref. (25). The ζ_1 plane consists of a single aerofoil with a circulation $\pi(V_{w_1}-V_{w_2})$ around it, a source $V_a\pi$ and vortex V_{w_1} at the " $-\infty$ " point $(-1, 0)$ and a sink $V_a\pi$ and vortex $V_{w_2}\pi$ at the " $+\infty$ " point $(+1, 0)$—these two points corresponding to the infinities in the Z_1 plane. V_a is the (axial) component of the inlet velocity (V_1) or of

Fig. 2.


Z_2 PLANE $(Z_2 = x_2 + iy_2)$
y_2
+1
$V_{w_1}\pi$
'$-\infty$'
$V_a\pi$
$K = \pi(V_{w_1} - V_{w_2})$
A
-1
O
+1
B
x_2
$V_a\pi$
'$+\infty$'
$V_{w_2}\pi$
$2C_2$
$2C_2$
-1
$\zeta_2 + \frac{C_2^2}{\zeta_2} = Z_2$
'$-\infty$'
$V_{w_1}\pi$
$V_a\pi$
η_2
+1
$K = \pi(V_{w_1} - V_{w_2})$
ζ_2 PLANE $(\zeta_2 = \xi_2 + i\eta_2)$
A
-1
O
B
+1
ξ_2
$V_a\pi$
'$+\infty$'
$V_{w_2}\pi$
-1

the outlet velocity (V_2) perpendicular to the cascade. V_{w_1} and V_{w_2} are the (whirl) components of the velocities V_1 and V_2 respectively along the cascade. A brief discussion of the singularities of this first transformation is given in Appendix I.

Fig. 3.


Z3 PLANE (z3 = x3 + iy3)
y3
'-∞'
Vw1 π
Va π
+1
K = π (Vw1 - Vw2)
A
-1
0
+1
x3
2C3
2C3
B
Va π
'+∞'
Vw2 π
ζ3 + C3²/ζ3 = Z3
η3
'-∞'
Va π
Vw
+1
A
K = π (Vw1 - Vw2)
ζ3 PLANE (ζ3 = ξ3 + iη3)
-1
0
+1
ξ3
B
Va π
'+∞'
Vw2 π

3. *Second Transformation.*

New axes of reference are now taken for the single aerofoil of the ζ_1 plane, this new plane being designated by Z_2. Referring to fig. 2, the axes are chosen so that a point roughly midway between the trailing edge and its centre of curvature is defined by $(+2C_2, 0)$ and a point roughly midway between the leading edge and its centre of curvature defined by $(-2C_2, 0)$. The ordinary Joukowski transformation now applied and illustrated in fig. 2 is

$$\zeta_2+\frac{C_2^{\,2}}{\zeta_2}=Z_2\,.$$

The value of ζ_2 can be found by the method described in Appendix II., but any other method could be used. In general, the aerofoil in the Z_2 plane transforms into an irregular oval in the ζ_2 plane, the " $-\infty$ " and " $+\infty$ " points taking up their corresponding positions.

4. *Third Transformation.*

Using the same procedure as in section 3, new axes of reference for the Z_3 plane are found and the transformation

$$\zeta_3+\frac{C_3^{\,2}}{\zeta_3}=Z_3$$

(similar to the second transformation) is used. The irregular oval in the Z_3 plane is transformed into an approximate circle in the ζ_3 plane, as shown in fig. 3.

For some cascades, such as those with large pitch/chord ratios, this third transformation is unnecessary as the irregular oval may be taken to be an approximate circle.

5. *Fourth Transformation.*

Before proceeding with the fourth transformation the centroid and area of the approximate circle in the ζ_3 plane should be estimated. New axes are then taken through the centroid to give the Z_4 plane, as shown in fig. 4. The circle with its centre at the origin of the Z_4 plane and with area equal to the approximate circle can be taken as the base circle.

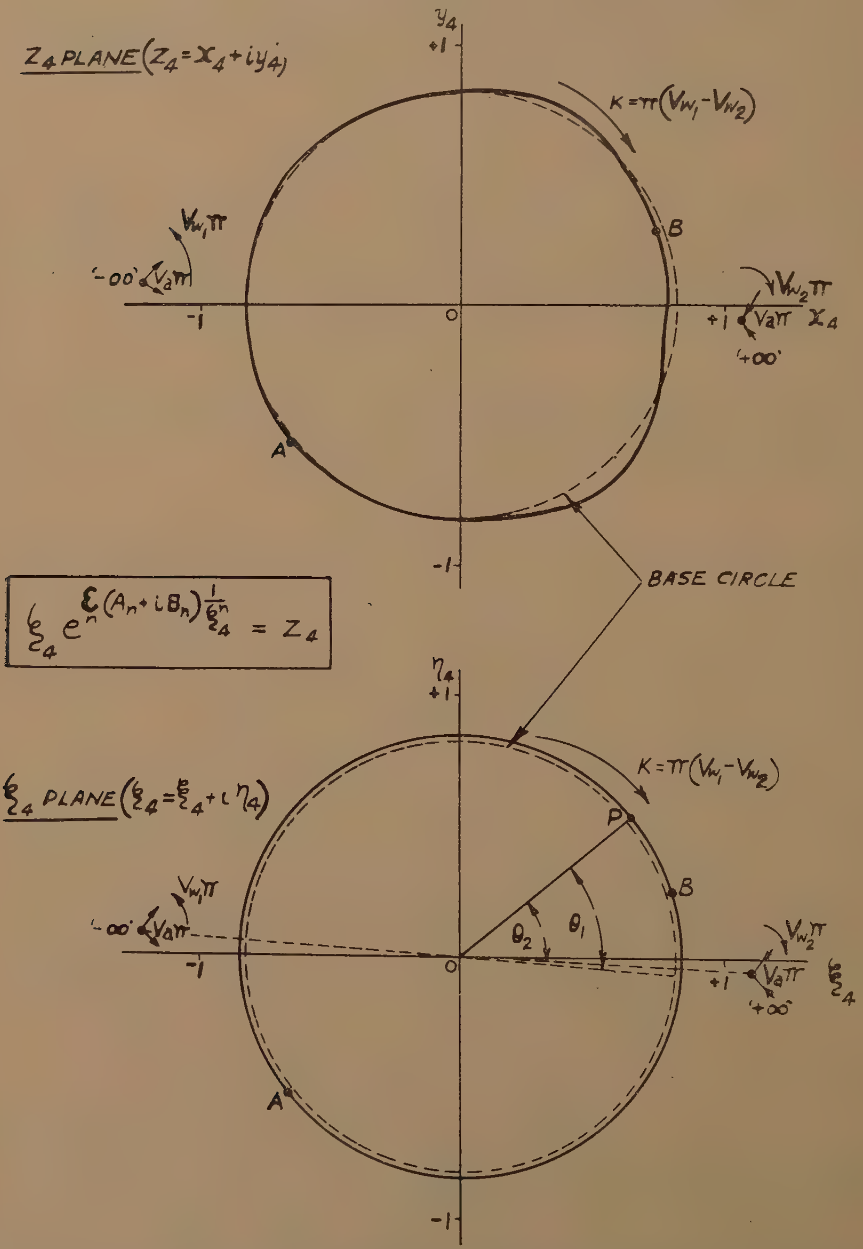
The generalized transformation used to transform the approximate circle in the Z_4 plane is

$$\zeta_4 e^{\sum_n (A_n+iB_n)\frac{1}{\zeta_4^{\,n}}}=Z_4,$$

where A_n and B_n are constants, the summation being taken to n terms depending on the accuracy required.

This transformation is illustrated in fig. 4. If the centre of the ζ_4 plane circle is taken at the origin of the axes, the values of A_n and B_n can be fairly easily calculated by means of a Fourier series. It might be mentioned that, if the area referred to in the previous paragraph had been found accurately, the ζ_4 plane circle would coincide with the base circle. Actually the requirements of the method used are that the ζ_4 plane circle should be approximately equal to the base circle.

Fig. 4.

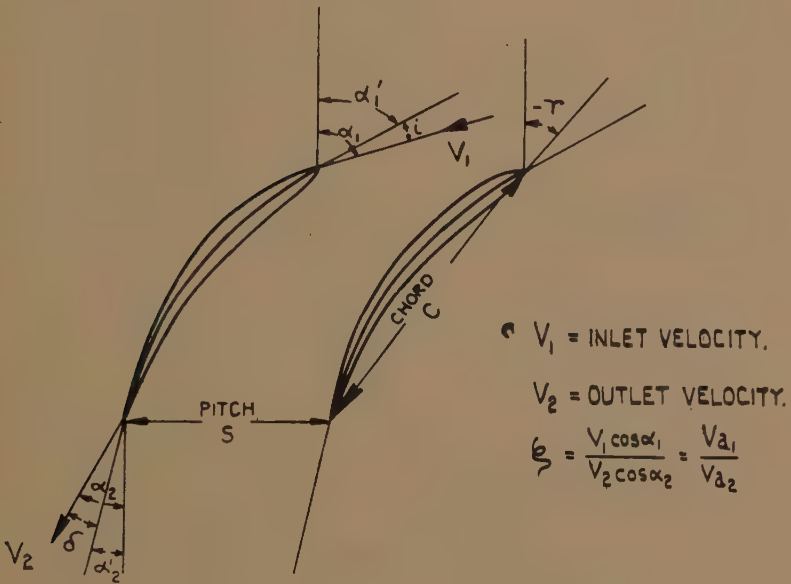

y_4
+1
Z_4 PLANE ($Z_4 = x_4 + iy_4$)
$K = \pi(V_{W_1} - V_{W_2})$
B
$V_{W_1}\pi$
'−∞'
$V_a\pi$
−1
0
+1
$V_{W_2}\pi$
$V_a\pi$
x_4
'+∞'
A
−1
BASE CIRCLE
$\zeta_4 e^{\sum_n (A_n + iB_n)\frac{1}{\zeta_4^n}} = Z_4$
η_4
+1
ζ_4 PLANE ($\zeta_4 = \xi_4 + i\eta_4$)
$K = \pi(V_{W_1} - V_{W_2})$
P
B
θ_2
θ_1
$V_{W_1}\pi$
'−∞'
$V_a\pi$
−1
0
+1
$V_{W_2}\pi$
$V_a\pi$
ξ_4
'+∞'
A
−1

The methods used in the evaluation of this transformation are indicated in Appendix III. Further information about the transformation can be obtained from refs. (22), (23), (25) and (26).

6. *Air Angles and Lift Coefficient.*

The velocities on the ζ_4 plane circle must be known before the air angles and lift coefficient in the cascade or Z_1 plane can be determined. The ζ_4 plane consists of a circle with a circulation $\pi(V_{w_1}-V_{w_2})$ around it, a source $V_a\pi$ and vortex $V_{w_1}\pi$ at the "$-\infty$" point and a sink $V_a\pi$ and vortex $V_{w_1}\pi$ at the "$+\infty$" point. The velocities on the circle due to the source, sink and vortices can be easily found. Appendix IV. gives the appropriate formulæ for the velocities. The velocities due to V_a, V_{w_1} and V_{w_2} at any point are defined by fV_a, gV_{w_1} and hV_{w_2} respectively.

Fig. 5.



α_1'
α_1
i
V_1
$-\tau$
CHORD C
PITCH S
α_2
δ
α_2'
V_2
V_1 = INLET VELOCITY.
V_2 = OUTLET VELOCITY.
$\xi = \frac{V_1 \cos\alpha_1}{V_2 \cos\alpha_2} = \frac{V_{a_1}}{V_{a_2}}$

CAMBER $\theta = \alpha_1' - \alpha_2'$

DEVIATION $\delta = \alpha_2 - \alpha_2'$

DEFLECTION $\epsilon = \alpha_1 - \alpha_2$

INCIDENCE $i = \alpha_1 - \alpha_1'$

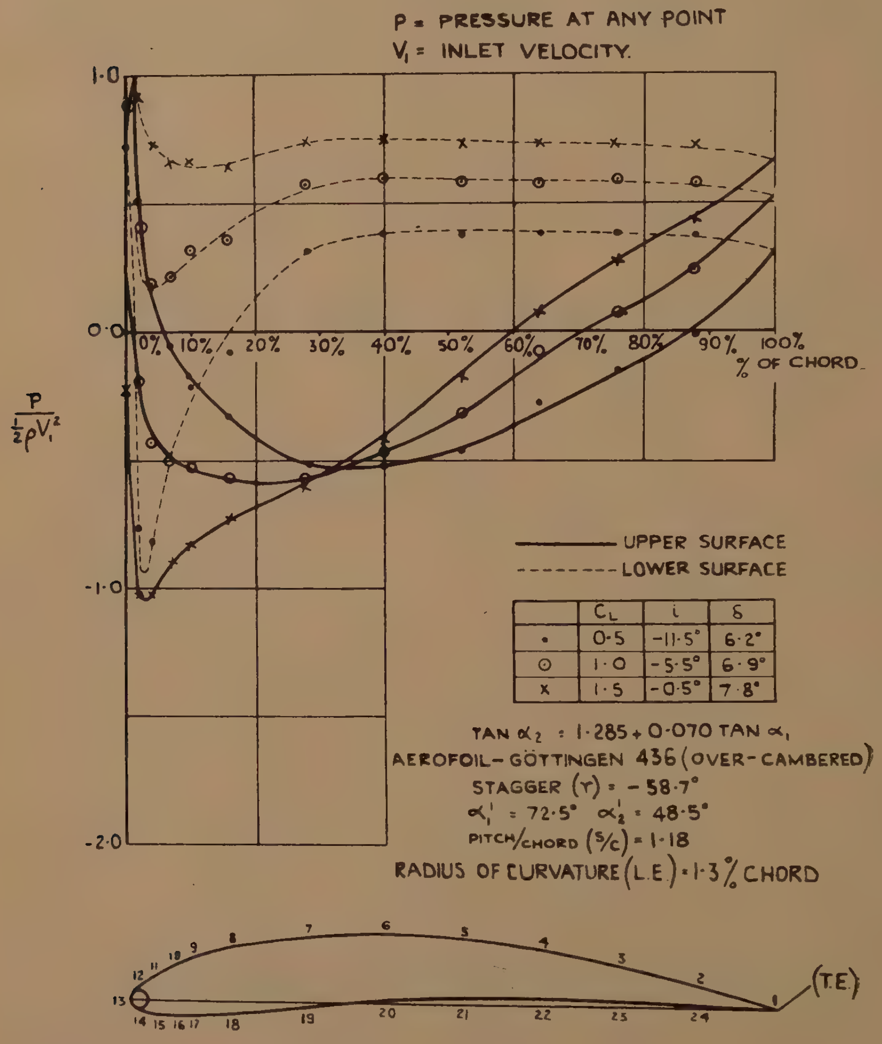
STAGGER τ $\left[= \frac{\theta}{2} - \alpha_1' \text{ FOR CIRCULAR ARC CAMBERS}\right]$

The characteristics of the flow through the cascade are found by the usual Joukowski condition, that is, one of the stagnation points on the circle must occur at the point corresponding to the trailing edge of the aerofoil in cascade. If f_t g_t and h_t are the values of f, g and h respectively at the trailing edge, then

$$f_tV_a+g_tV_{w_1}+h_tV_{w_2}=0,$$

or
$$f_t+g_t \tan \alpha_1+h_t \tan \alpha_2=0,$$
where α_1 is the inlet air angle and α_2 the outlet air angle to the cascade, both angles being measured from a normal to the cascade. The lift coefficient (C_L) at any α_1 can now be obtained by the formula

Fig. 6.



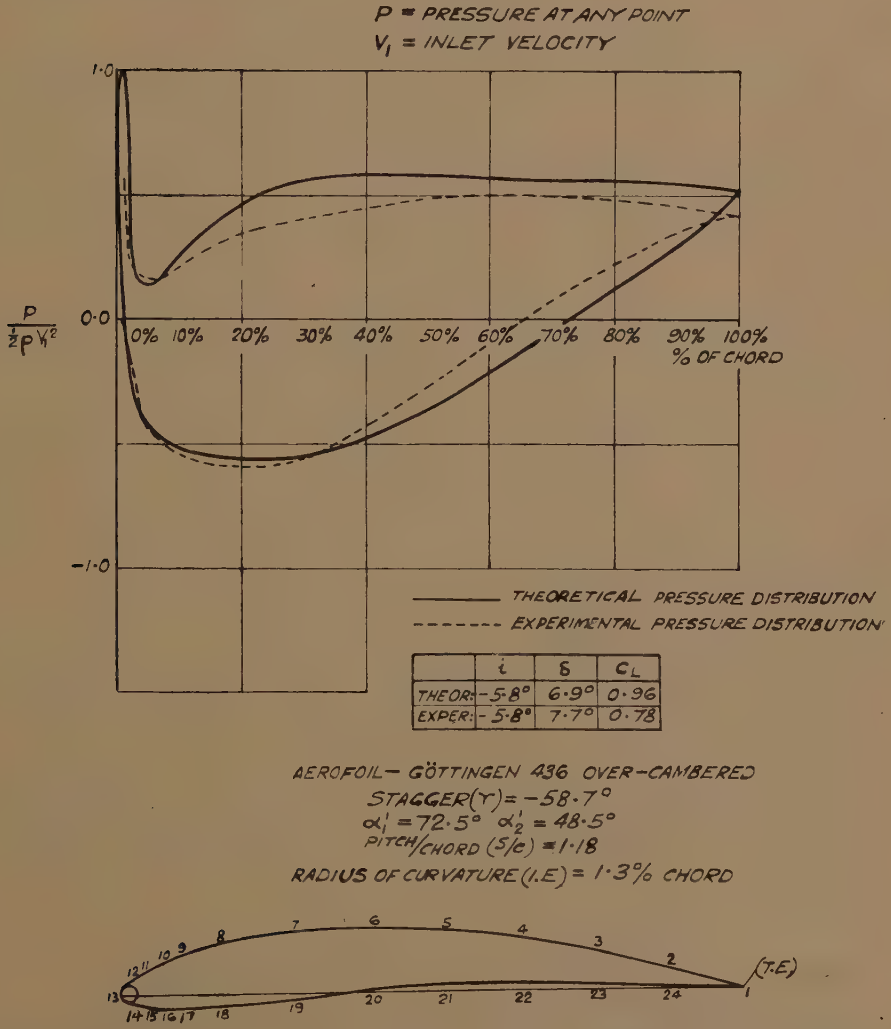
$$C_L=2S/C(\tan \alpha_1-\tan \alpha_2) \cos \alpha_m,$$
where S/C is the pitch/chord ratio and $\tan \alpha_m$ equals $(\tan \alpha_1+\tan \alpha_2)/2$.

7. *Velocity and Pressure Distributions on the Aerofoil in Cascade.*

The calculation of the velocities on the ζ_4 plane circle has been described in the previous section. The velocities on the Z_1 plane aerofoil, that is, the aerofoil in cascade, are obtained by multiplying the corresponding velocities on the ζ_4 plane circle by the velocity coefficients

$$\left|\frac{d\zeta_4}{dZ_4}\right| \left|\frac{d\zeta_3}{dZ_3}\right| \left|\frac{d\zeta_2}{dZ_2}\right| \text{ and } \left|\frac{d\zeta_1}{dZ_1}\right|.$$

Fig. 7.



	i	δ	C_L
THEOR:	−5·8°	6·9°	0·96
EXPER:	−5·8°	7·7°	0·78

The methods of obtaining these velocity coefficients are given in Appendix V. The pressures on the surface of the aerofoil are obtained from the velocities by means of Bernoulli's equation.

III. Illustrative Example.

The transformations for this example are actually those illustrated in figs. 1 to 4. Fig. 5 gives the notation used for the results plotted in fig. 6 and fig. 7. Fig. 6 shows the theoretical pressure distributions at different lift coefficients for an over-cambered Göttingen 436 aerofoil in cascade. The pitch/chord ratio (S/C) is 1·18 and the stagger (Y) is —58·7° for this cascade. Fig. 7 gives a comparison between theoretical and experimental results for the same cascade, the experimental results being taken from ref. (6). The very rapid increase of pressure to a stagnation point at the trailing edge which theoretically occurs in the immediate neighbourhood of the trailing edge is not plotted, as such an increase does not occur in experimental results.

The theoretical results are in reasonable agreement with the experimental values. The theoretical and experimental deviations (δ) in fig. 7 are respectively 6·9° and 7·7°. The theoretical lift coefficient is 0·96. The experimental lift coefficient (C_2), calculated from the air angles and drag coefficient and assuming constant axial velocity (V_a) through the cascade, is 0·88 ; the appropriate formula is

$$C=2\,S/C\,(\tan\alpha_1-\tan\alpha_2)\cos\alpha_m-C_D\tan\alpha_m,$$

where $\tan\alpha_m=\frac{1}{2}(\tan\alpha_1+\tan\alpha_2)$, as in section 6. It will be observed, however, that the lift coefficient obtained from the experimental pressure distribution is 0·78. This difference of 0·10 in the experimental lift coefficients is mainly due to the contraction of the air-stream in passing through the cascade, thereby departing from the two-dimensional criterion of constant V_a. This contraction of the air-stream is caused by the boundary layer thickening on the walls of the tunnel. The contraction coefficient (ζ) for the experimental results given was about 0·95.

IV. Limitations of Method.

The aerofoil in the ζ_1 plane after the first transformation from the Z_1 or cascade plane will usually be of an " S " shape for small pitch/chord ratios (S/C), as in fig. 1. If the S/C is very small the aerofoil becomes more of an " § " shape ; this shape is rather awkward to transform into a circle and a different procedure would have to be adopted. The practical working limit of S/C is given approximately by the condition that the projection of the aerofoil on the y_1 axis is not greater than one and a half times the pitch. For the aerofoils usually used in cascade the approximate limits are given in the following table :—

Stagger (Y)......	0°	±30°	±60°
Minimum S/C....	0·5	0·6	0·7

Pitch/chord ratios below the above values are very seldom used, so that the theoretical method can be used for most practical examples of aerofoils in cascade.

V. Conclusions.

A solution of the theoretical flow of an ideal fluid past arbitrary aerofoils in cascade can be obtained by the method described in the context. The method can be used for most practical examples of aerofoils in cascade and the results obtained are in reasonable agreement with the corresponding experimental values.

APPENDIX I.

Singularities of First Transformation.

The singular points of the first transformation

$$\zeta = \tanh Z_1$$

given by the conditions

$$\frac{\propto \zeta_1}{dZ_1} = 1 - \zeta_1{}^2 = 0 \text{ or } \infty$$

are $(\pm 1, 0)$, $(\pm\infty, \eta_1)$ and $(\xi_1, \pm\infty)$ in the ζ_1 plane, the corresponding points in the cascade of Z_1 plane being $(\pm\infty, y_1)$ and

$$\left(0, \pm \{2n-1\}\frac{\pi}{2}\right)$$

respectively. Both sets of singular points are outside the aerofoils. The velocities (V_{Z_1}) in the Z_1 plane are obtained from the velocities (V_{ζ_1}) in the ζ_1 plane by the equation

$$V_{Z_1} = V_{\zeta_1} \times \left|\frac{d\zeta_1}{dZ_1}\right|,$$

where $\left|\frac{d\zeta_1}{dZ_1}\right|$ is the modulus of $\frac{d\zeta_1}{dZ_1}$. It can be fairly easily proved that $V_{\zeta_1} \to C/|-\zeta_1^2|$ when ζ_1 approaches either of the above two sets of singular points and where C is a finite quantity other than zero. V_{Z_1} will therefore be a finite quantity other than zero at $(\pm\infty, y_1)$ and

$$\left(0, \pm\{2n-1\}\frac{\pi}{2}\right)$$

in the cascade or Z_1 plane, so that the transformation is conformal for all points.

APPENDIX II.

Second and Third Transformations.

The second and third transformations are similar and are of the form

$$\zeta_2 + \frac{C_2^2}{\zeta_2} = Z_2.$$

The evaluation of ζ_2, when $Z_2 = x_2 + iy_2$ is known, can be performed by the method given in ref. (23). The formulæ used are given below :

Let $\zeta_2 = \xi_2 + i\eta_2 = C_2 e^{\sigma 2}(\cos\tau_2 + i\sin\tau_2)$,

if $P_2 = 1 - \left(\frac{x_2}{2C_2}\right)^2 - \left(\frac{y_2}{2C_2}\right)^2.$

Then the parameters σ_2 and τ_2 are defined by the equations

$$2\sin^2\tau_2 = P_2 + \sqrt{P_2^2 + \left(\frac{y_2}{C_2}\right)^2},$$

$$\sinh\sigma_2 = \frac{y_2}{2C_2\sin\tau_2},$$

$$\cosh\sigma_2 = \frac{x_2}{2C_2\cos\tau_2}$$

and

$$e^{\sigma_2} = \sinh\sigma_1 + \sqrt{1+\sinh^2\sigma_2}.$$

APPENDIX III.

Fourth Transformation.

The approximate circle in the Z_4 plane is transformed into a circle in the final or ζ_4 plane by the equation

$$\zeta_4 e^{\sum\limits_n (A_n+iB_n)\frac{1}{\zeta_4^n}} = Z_4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Points in the Z_4 plane can be expressed in the form

$$Z_4 = ae^{\lambda+i\phi} = ae^{\lambda}(\cos\phi + i\sin\phi) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and in the ζ_4 plane as

$$\zeta_4 = ae^{\psi+i\theta} = ae^{\psi}(\cos\theta + i\sin\theta), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where a is the radius of the base circle, whose area is roughly equal to that of the approximate circle (or to the ζ_4 plane circle).

Equations (1), (2) and (3) give

$$(\lambda-\psi)+i(\phi-\theta) = \sum_n (A_n+iB_4)\frac{1}{(ae^{\psi})}(\cos n\theta - i\sin n\theta), \quad . \quad . \quad (4)$$

and therefore

$$\Delta = \lambda-\psi = \sum_n\left[\frac{A_n}{(ae^{\psi})^n}\cos n\theta + \frac{B_n}{(ae^{\psi})^n}\sin n\theta\right] \quad . \quad . \quad . \quad (5)$$

and

$$\epsilon = \phi-\theta = \sum_n\left[\frac{B_n}{(ae^{\psi})^n}\cos n\theta - \frac{A_n}{(ae^{\psi})}\sin n\theta\right]. \quad . \quad . \quad . \quad (6)$$

Let λ_0, ϕ_0 and ψ_0, θ_0 denote the values of λ, ϕ and ψ, θ for the approximate circle and the ζ_4 plane circle respectively. Both λ_0 and ψ_0 will generally be small. λ_0 will vary with ϕ_0, and ψ_0 will be a constant as $r=ae^{\psi_0}$ is the radius of the ζ_4 plane circle. Then

$$\Delta_0 = \lambda_0-\psi_0 = \sum_n\left[\frac{A_n}{(ae^{\psi_0})^n}\cos n\theta_0 + \frac{B_n}{(ae^{\psi_0})^n}\sin n\theta_0\right], \quad . \quad . \quad . \quad (7)$$

$$\epsilon_0 = \phi_0-\theta_0 = \sum_n\left[\frac{B_n}{(ae^{\psi_0})^n}\cos n\theta_0 - \frac{A_n}{(ae^{\psi_0})^n}\sin n\theta_0\right], \quad . \quad . \quad . \quad (8)$$

giving by the usual Fourier analysis

$$\psi_0 = \frac{1}{2\pi}\int_0^{2\pi} \lambda_0 d\theta_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\frac{A_n}{(ae^{\psi_0})^n} = \frac{1}{\pi}\int_0^{2\pi} \lambda_0 \cos n\theta_0 d\theta_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\frac{B_n}{(ae^{\psi_0})^n} = \frac{1}{\pi}\int_0^{2\pi} \lambda_0 \sin n\theta_0 d\theta_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

λ_0 is actually known as a function of ϕ_0 and not θ_0, but for most cascades $\epsilon_0 = \phi_0 - \theta_0$ will be small and ϕ_0 may be used instead of θ_0 for the evaluation of ψ_0, A_n and B_n. The evaluation of A_n and B_n for $n=1$ to $n=6$ is usually sufficient. From equations (2) and (9) it will be observed that, if the area referred to in connection with the base circle had been found accurately, ψ_0 would be zero, or in other words the ζ_4 plane circle would coincide with the base circle. Points outside the circle in the final or ζ_4 plane, such as the " $-\infty$ " or " $+\infty$," can be found from equations (5) and (6) by making use of the fact that ψ and θ are approximately equal to λ and ϕ respectively.

APPENDIX IV.

Velocities on ζ_4 Plane Circle.

The flow past the ζ_4 plane circle can be obtained by introducing an image system inside the circle of the source ($V_a\pi$), sink ($V_a\pi$) and vortices ($V_{w_1}\pi$ and $V_{w_2}\pi$) outside the circle. Denote the radius of the circle by r and the radius vectors of the " $-\infty$ " and " $+\infty$ " by $m_1 r$ and $m_2 r$ respectively. If the velocity V, measured in a clockwise direction, at any point P is given by the expression

$$V = fV_a + gV_{w_1} + hV_{w_2},$$

then

$$f = \frac{1}{2r}\left[\frac{\sin\theta_1}{\frac{1}{2}\left(m_1 + \frac{1}{m_1}\right) + \cos\theta_1} + \frac{\sin\theta_2}{\frac{1}{2}\left(m_2 + \frac{1}{m_2}\right) - \cos\theta_2}\right],$$

$$g = \frac{1}{4r}\left[\frac{m_1 - \frac{1}{m_1}}{\frac{1}{2}\left(m_1 + \frac{1}{m_1}\right) + \cos\theta_1}\right],$$

$$h = -\frac{1}{4r}\left[\frac{m_2 - \frac{1}{m_2}}{\frac{1}{2}\left(m_2 + \frac{1}{m_2}\right) - \cos\theta_2}\right],$$

where θ_1 and θ_2 are the angles of the point P measured from the radii through the " $-\infty$ " and " $+\infty$ " in the manner shown in fig. 4.

APPENDIX V.

Velocity Coefficients.

The formulæ for obtaining

$$\left|\frac{d\,\zeta_1}{dZ_1}\right|,\ \left|\frac{d\,\zeta_2}{dZ_2}\right| \quad \text{and} \quad \left|\frac{d\,\zeta_3}{dZ_3}\right|$$

are

$$\left|\frac{d\,\zeta_1}{dZ_1}\right|=\left|1-\zeta_1^2\right|,$$

$$\left|\frac{d\,\zeta_2}{dZ_2}\right|=\left|\frac{\zeta_2^2}{\zeta_2^2-C_2^2}\right|=\frac{e^{\sigma^2}}{2\sqrt{\sinh^2\sigma_2+\sin^2\tau_2}};$$

$$\left|\frac{d\,\zeta_2}{dZ_3}\right|=\left|\frac{\zeta_3^2}{\zeta_3^2-C_3^2}\right|=\frac{e^{\sigma^3}}{2\sqrt{\sinh^2\sigma_3+\sin^2\tau_3}}.$$

The velocity coefficient $\left|\frac{d\,\zeta_4}{dZ_4}\right|$ is given by the expressions

$$\left|\frac{d\,\zeta_4}{dZ_4}\right|=\left|\frac{\zeta_4}{Z_4}\right|\frac{1-\epsilon_0^1}{\sqrt{1+\lambda_0^{12}}}=e^{\psi_0-\lambda_0}x\frac{1-\epsilon_0^1}{\sqrt{1+\lambda_0^{12}}},$$

where $$\epsilon_0{}^1=\frac{d\epsilon_0}{d\phi_0} \text{ and } \lambda_0{}^1=\frac{d\epsilon_0}{d\phi_0}.$$

Now, $\Delta_0=\lambda_0-\psi_0$ and $\epsilon_0=\phi_0-\theta_0$, ϵ_0^1 and λ_0^1 are generally small so that the squares and higher powers of these quantities can be neglected.

Then

$$\left|\frac{d\,\zeta_4}{dZ_4}\right|=1-\Delta_0-\epsilon_0{}^1$$

$$=1+\sum_n\left[\frac{(n-1)A_n}{(ae^{\psi_0})^n}\cos n\theta_0+\frac{(n-1)B_n}{(ae^{\psi_0})^n}\sin n\theta_0\right].$$

The notation used above will be defined in either the context of this note or in the preceding appendices.

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CX. *Five-figure Table of the Function* $\int_0^\infty e^{-zy} \,.\, \mathrm{Ai}^2(y-j_1)dy$ *in the Complex Plane.*

By H. G. Hay,

Telecommunications Research Establishment, Ministry of Supply, Malvern *.

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1. *Introduction.*

The need for these tables has arisen in connection with the determination of the eigenvalues of the wave equation for centimetric wave propagation in the atmosphere. The wave equation is

$$\frac{d^2\mathrm{U}}{ds^2}+\left[s+\sum_{n=1}^{r} A_n{}^{-}e^{\alpha_n s}+\mathrm{D}_1\right]\mathrm{U}=0,$$

in which the constants A_n and α_n are known. The equations from which the eigenvalue D_1 is found involve the function $\mathrm{F}(z)=\int_0^\infty e^{-zy}.\ \mathrm{Ai}^2(y-j_1)dy$, and its first derivative, where $-j_1$ is the first zero of the Airy function. Values of $\mathrm{F}(z)$ and $\mathrm{F}'(z)$ are required for complex values of z. A full description of the method, which is due to Macfarlane, is given in reference (1). A similar function has also been discussed in the same connection by C. L. Pekeris (2).

2. *Definition of Functions Tabulated.*

The function $\mathrm{F}(z)$ is defined by the integral

$$\int_0^\infty e^{-zy}.\ \mathrm{Ai}^2(y-j_1)dy, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $-j_1=-2{\cdot}3381$ is the first zero of the Airy function $Ai(y)$, as defined by Miller (3) and tabulated in the complex plane by P. M. Woodward and Mrs. A. M. Woodward (4).

It satisfies the differential equation

$$\frac{d\mathrm{F}}{dz}=\frac{1}{2z}\left[\mathrm{A}i'(-j_1)\right]^2-\left(\frac{1}{2z}+j_1+\frac{z^2}{4}\right)\mathrm{F}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

from which the derivative can be calculated when $\mathrm{F}(z)$ is known.

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* Communicated by the Author.

Asymptotic formulæ for F and dF/dz can be found by the methods given by Pekeris (*loc. cit.*). Thus

$$F \sim 0{\cdot}491696 \frac{z(j_1+\frac{1}{4}z^2)^2+(j_1+\frac{3}{4}z)^2}{z(j_1+\frac{1}{4}z^2)^2(1+2j_1z+\frac{1}{2}z^3)}, \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

and

$$\frac{dF}{dz} \sim -0{\cdot}245848 \frac{j_1+\frac{3}{4}z^2}{z^2(j_1+\frac{1}{4}z^2)^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

where $j_1=2{\cdot}33811$.

3. *Method of Tabulation.*

The function $F(z)$ was calculated from power series obtained from the differential equation. For $|z|<0{\cdot}4$ an expansion around the origin was used. Then

$$F(z)=a_0+a_1z+a_2z^2+ \ldots., \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

where the constants a_n are obtained by substitution of this series into the differential equation for $F(z)$. For calculations beyond $|z|=0{\cdot}4$ the origin of the expansion was moved in successive steps of 0·4. If the origin is moved by a distance A put $z=z_1+A$, where z_1 is a new variable. The differential equation becomes

$$\frac{dF}{dz_1}=\frac{1}{2(z_1+A)}\Big[Ai'(-j_1)\Big]^2-\Big[\frac{1}{2(z_1+A)}+j_1+\frac{(z_1+A)^2}{4}\Big]F, \quad . \qquad (6)$$

and F is now represented by the power series

$$F(z)=b_0+b_1z_1+b_2z_1{}^2+ \ldots.. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

The constant b_0, which represents the value of F at $z=A$, therefore depends on previous calculations. The calculations at each change of origin were therefore carefully checked.

The derivative $F'(z)$ was calculated from equation (2).

4. *Contents of Tables.*

The tables, giving values of the real and imaginary parts of $F(z)$ and $F'(z)$ to five figures, cover the region

$$x=0{\cdot}0\ (0{\cdot}2)\ 4{\cdot}0,$$
$$y=0{\cdot}0\ (0{\cdot}2)\ 3{\cdot}2,$$

in the upper half of the complex plane :

Tables I. and II. : Real Part of $F(x+iy)$.
Tables III. and IV. : Imaginary Part of $F(x+iy)$.
Tables V. and VI. : Real Part of $F'(x+iy)$.
Tables VII. and VIII. : Imaginary Part of $F'(x+iy)$.

The tables may also be used to cover the corresponding region in the lower half of the complex plane by taking complex conjugates.

TABLE I. Real Part of $F(x+iy)$

$\frac{1}{2}(\Delta_{x^2}-\Delta_{y^2})$ and $\Delta_{x^2}\Delta_{y^2}$ given beneath each function value.

y \ x	0·0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0
0·0	0·49170 5735 −1189	0·36342 3771 −708	0·27344 2531 −431	0·20913 1732 −268	0·16236 1207 −170	0·12780 855 −110	0·10187 615 −73	0·08216 449 −49	0·06697 332 −33	0·05514 249 −23	0·04581 189 −16
0·2	0·46352 5156 −1013	0·34486 3425 −612	0·26096 2320 −377	0·20058 1600 −238	0·15640 1123 −152	0·12357 800 −99	0·09883 579 −66	0·07993 425 −45	0·06533 316 −30	0·05390 238 −21	0·04487 181 −15
0·4	0·38462 3586 −555	0·29256 2480 −359	0·22561 1739 −234	0·17623 1235 −155	0·13933 889 −104	0·11142 648 −70	0·09005 478 −48	0·07350 357 −33	0·06055 269 −23	0·05031 205 −17	0·04214 158 −12
0·6	0·27033 1471 14	0·21577 1182 −36	0·17305 927 −48	0·13965 718 −45	0·11346 554 −38	0·09284 427 −30	0·07652 330 −23	0·06352 256 −17	0·05309 200 −13	0·04467 157 −10	0·03783 124 −7
0·8	0·14132 −637 498	0·12718 −155 252	0·11128 66 126	0·09593 155 61	0·08209 181 28	0·07002 176 12	0·05971 159 4	0·05099 138 0	0·04365 117 −2	0·03748 98 −3	0·03230 82 −2
1·0	0·01826 −2265 763	0·03992 −1249 429	0·04874 −674 243	0·05061 −348 138	0·04888 −165 79	0·04544 −64 45	0·04131 −9 26	0·03708 19 15	0·03303 32 8	0·02930 37 4	0·02594 37 3
1·2	−0·08277 −3152 772	−0·03519 −1926 467	−0·00727 −1177 283	0·00866 −717 173	0·01726 −434 106	0·02145 −259 66	0·02298 −151 41	0·02297 −85 25	0·02208 −44 16	0·02074 −19 10	0·01920 −5 6
1·4	−0·15293 −3283 587	−0·09144 −2146 388	−0·05174 −1402 254	−0·02627 −916 166	−0·01010 −599 108	−0·00000 −391 71	0·00613 −254 46	0·00968 −164 30	0·01156 −105 20	0·01237 −66 13	0·01251 −41 9

1·6	-0·19074 -2833 315	-0·12656 -1982 245	-0·08241 -1377 180	-0·05218 -952 128	-0·03157 -657 90	-0·01760 -453 63	-0·00822 -312 43	-0·00199 -214 30	0·00208 -147 20	0·00465 -100 14	0·00621 -68 10
1·8	-0·20049 -2067 58	-0·14206 -1575 93	-0·09946 -1173 92	-0·06867 -861 78	-0·04654 -626 61	-0·03073 -453 47	-0·01949 -327 35	-0·01154 -235 25	-0·00595 -169 18	-0·00207 -121 13	0·00059 -86 9
2·0	-0·18961 -1237 -121	-0·14189 -1072 -27	-0·10486 -876 14	-0·07662 -692 28	-0·05531 -534 30	-0·03937 -407 28	-0·02752 -307 23	-0·01876 -230 18	-0·01232 -172 14	-0·00760 -128 11	-0·00417 -95 8
2·2	-0·16626 -519 -205	-0·13096 -593 -98	-0·10151 -564 -40	-0·07767 -493 -10	-0·05876 -411 4	-0·04397 -333 10	-0·03251 -264 12	-0·02370 -207 11	-0·01697 -161 10	-0·01186 -124 8	-0·00799 -95 7
2·4	-0·13755 -1 -212	-0·11402 -208 -123	-0·09248 -290 -67	-0·07378 -304 -33	-0·05810 -283 -14	-0·04524 -248 -3	-0·03486 -210 2	-0·02658 -173 4	-0·02003 -140 5	-0·01488 -112 5	-0·01086 -89 5
2·6	-0·10865 309 -173	-0·09490 56 -114	-0·08049 -81 -71	-0·06683 -146 -42	-0·05460 -168 -23	-0·04403 -167 -12	-0·03512 -153 -5	-0·02773 -134 -1	-0·02169 -114 1	-0·01679 -95 2	-0·01284 -79 3
2·8	-0·08269 447 -117	-0·07624 208 -88	-0·06763 58 -62	-0·05838 -29 -41	-0·04939 -76 -26	-0·04115 -96 -15	-0·03385 -101 -9	-0·02755 -96 -5	-0·02221 -87 -2	-0·01774 -77 0	-0·01405 -66 1
3·0	-0·06110 467 -66	-0·05959 273 -58	-0·05530 137 -46	-0·04960 47 -34	-0·04341 -9 -24	-0·03729 -41 -16	-0·03156 -57 -10	-0·02640 -63 -6	-0·02185 -62 -4	-0·01793 -58 -2	-0·01459 -52 -1
3·3	-0·04413 421 -27	-0·04563 279 -32	-0·04430 169 -30	-0·04126 89 -25	-0·03731 34 -19	-0·03300 -2 -14	-0·02869 -23 -10	-0·02461 -35 -7	-0·02087 -40 -4	-0·01753 -41 -3	-0·01461 -39 -1

TABLE II. Real Part of $F(x+iy)$

$\frac{1}{2}(\Delta_{x^2}-\Delta_{y^2})$ and $\Delta_{x^2}\Delta_{y^2}$ given beneath each function value.

y \ x	2·2	2·4	2·6	2·8	3·0	3·2	3·4	3·6	3·8	4·0
0·0	0·03839 145 —11	0·03242 112 —8	0·02758 88 —6	0·02362 69 —4	0·02035 55 —3	0·01764 44 —2	0·01538 36 —2	0·01348 29 —1	0·01186 24 —1	0·01049 20 —1
0·2	0·03767 139 —10	0·03186 108 —8	0·02714 85 —6	0·02327 67 —4	0·02008 53 —3	0·01742 43 —2	0·01520 35 —2	0·01333 28 —1	0·01174 23 —1	0·01039 19 —1
0·4	0·03557 123 —8	0·03023 97 —6	0·02586 76 —5	0·02226 61 —4	0·01927 49 —3	0·01676 40 —2	0·01468 32 —2	0·01290 26 —1	0·01139 22 —1	0·01010 18 —1
0·6	0·03224 98 —5	0·02764 79 —4	0·02382 63 —3	0·02064 51 —3	0·01798 42 —2	0·01573 34 —2	0·01383 28 —1	0·01221 23 —1	0·01082 19 —1	0·00963 16 0
0·8	0·02794 68 —2	0·02426 56 —2	0·02115 47 —2	0·01851 39 —1	0·01627 32 —1	0·01435 27 —1	0·01270 23 —1	0·01128 19 —1	0·01006 16 0	0·00900 14 0
1·0	0·02295 35 1	0·02032 32 0	0·01801 29 0	0·01600 25 0	0·01423 22 0	0·01269 19 0	0·01135 17 —1	0·01017 15 0	0·00913 12 0	0·00823 11 0
1·2	0·01761 4 4	0·01606 8 2	0·01459 10 1	0·01322 11 1	0·01198 11 1	0·01085 11 0	0·00982 10 0	0·00891 9 0	0·00808 8 0	0·00735 8 0
1·4	0·01223 —24 6	0·01171 —13 4	0·01105 —6 2	0·01033 —2 2	0·00960 1 1	0·00889 3 1	0·00820 4 0	0·00755 4 0	0·00695 4 0	0·00639 4 0

1·6	0·00708 —46 7	0·00748 —31 5	0·00757 —20 3	0·00746 —13 2	0·00721 —8 2	0·00690 —4 1	0·00653 —2 1	0·00615 0 0	0·00577 1 0	0·00539 1 0
1·8	0·00239 —62 7	0·00356 —44 5	0·00429 —31 4	0·00471 —22 2	0·00490 —15 2	0·00495 —10 1	0·00489 —7 1	0·00476 —4 1	0·00458 —2 0	0·00438 —1 0
2·0	—0·00169 —70 6	0·00007 —52 5	0·00131 —38 3	0·00217 —28 3	0·00274 —21 2	0·00310 —15 1	0·00330 —11 1	0·00340 —8 1	0·00342 —5 0	0·00338 —4 0
2·2	—0·00507 —73 5	—0·00290 —56 4	—0·00128 —42 3	—0·00009 —32 2	0·00078 —25 2	0·00140 —19 1	0·00183 —14 1	0·00212 —11 1	0·00231 —8 1	0·00241 —6 0
2·4	—0·00773 —70 4	—0·00531 —55 3	—0·00345 —43 3	—0·00202 —34 2	—0·00094 —27 2	—0·00012 —21 1	0·00050 —16 1	0·00095 —12 1	0·00128 —10 1	0·00151 —8 0
2·6	—0·00969 —64 2	—0·00718 —52 2	—0·00519 —42 2	—0·00362 —34 2	—0·00239 —27 1	—0·00142 —21 1	—0·00068 —17 1	—0·00010 —13 1	0·00034 —11 0	0·00068 —9 0
2·8	—0·01101 —56 1	0—·00853 —46 1	—0·00651 —38 1	—0·00488 —32 1	—0·00357 —26 1	—0·00252 —21 1	—0·00168 —17 1	—0·00101 —14 0	—0·00048 —11 0	—0·00007 —9 0
3·0	—0·01177 —46 0	—0·00941 —40 0	—0·00745 —34 0	—0·00583 —29 1	—0·00449 —24 1	—0·00340 —20 1	—0·00251 —17 1	—0·00179 —14 0	—0·00120 —11 0	—0·00073 —9 0
3·2	—0·01207 —36 —1	—0·00990 —32 0	—0·00805 —29 0	—0·00649 —25 0	—0·00518 —22 0	—0·00409 —18 0	—0·00318 —15 1	—0·00242 —13 0	—0·00180 —11 0	—0·00129 —9 0

TABLE III. Imaginary Part of $F(x+iy)$

$\frac{1}{2}(\Delta_{x^2}-\Delta_{y^2})$ and $\Delta_{x^2}\Delta_{y^2}$ given beneath each function value.

y \ x	0·0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0
0·0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0
0·2	—0·14923 —2364 594	—0·10399 —1483 340	—0·07387 —950 199	—0·05340 —621 119	—0·03925 —413 72	—0·02928 —280 45	—0·02216 —193 28	—0·01699 —135 18	—0·01318 —96 12	—0·01035 —69 8	—0·00820 —50 6
0·4	—0·27530 —4152 987	—0·19344 —2635 573	—0·13840 —1706 340	—0·10070 —1125 206	—0·07442 —756 127	—0·05580 —516 80	—0·04241 —358 51	—0·03264 —252 33	—0·02542 —180 22	—0·02001 —130 15	—0·01591 —95 10
0·6	—0·36068 —4978 1066	—0·25702 —3227 638	—0·18616 —2129 389	—0·13691 —1428 241	—0·10214 —973 152	—0·07723 —673 97	—0·05913 —472 63	—0·04581 —336 42	—0·03587 —242 28	—0·02839 —176 19	—0·02268 —130 13
0·8	—0·39717 —4764 844	—0·28887 —3195 534	—0·21296 —2171 341	—0·15905 —1494 220	—0·12026 —1041 144	—0·09201 —734 95	—0·07118 —524 63	—0·05564 —379 43	—0·04393 —277 29	—0·03502 —204 20	—0·02815 —152 14
1·0	—0·38673 —3720 439	—0·28921 —2639 318	—0·21834 —1878 225	—0·16643 —1343 157	—0·12809 —967 109	—0·09952 —702 76	—0·07803 —514 53	—0·06173 —379 37	—0·04925 —282 26	—0·03962 —212 19	—0·03212 —160 14
1·2	—0·33945 —2240 12	—0·26342 —1767 72	—0·20513 —1362 82	—0·16051 —1037 74	—0·12634 —786 60	—0·10007 —595 47	—0·07979 —451 36	—0·06406 —343 27	—0·05177 —262 20	—0·04212 —201 15	—0·03449 —155 11
1·4	—0·26978 —739 —307	—0·22003 —820 —130	—0·17836 —763 —45	—0·14428 —657 —6	—0·11677 —544 10	—0·09471 —441 16	—0·07707 —353 16	—0·06297 —280 14	—0·05169 —222 12	—0·04262 —175 10	—0·03532 —139 8

1·6	—0·19247 470 —459	—0·16833 5 —247	—0·14393 —205 —129	—0·12147 —281 —64	—0·10177 —291 —29	—0·08496 —271 —11	—0·07084 —238 —2	—0·05911 —203 2	—0·04940 —169 4	—0·04138 —140 5	—0·03477 —115 4
1·8	—0·11948 1232 —454	—0·11648 590 —274	—0·10734 228 —162	—0·09579 32 —94	—0·08383 —67 —54	—0·07249 —111 —29	—0·06223 —125 —15	—0·05321 —123 —7	—0·04541 —113 —3	—0·03874 —100 0	—0·03307 —87 1
2·0	—0·05843 1549 —348	—0·07030 907 —235	—0·07291 503 —154	—0·07035 254 —98	—0·06518 106 —62	—0·05889 20 —38	—0·05236 —27 —23	—0·04609 —50 —13	—0·04030 —59 —7	—0·03510 —61 —4	—0·03051 —58 —2
2·2	—0·01258 1521 —207	—0·03300 992 —162	—0·04337 626 —119	—0·04738 379 —84	—0·04753 218 —57	—0·04545 114 —38	—0·04221 49 —25	—0·03845 10 —16	—0·03459 —13 —10	—0·03085 —25 —6	—0·02736 —31 —4
2·4	0·01824 1283 —80	—0·00549 914 —86	—0·02000 630 —75	—0·02814 421 —60	—0·03202 273 —45	—0·03313 169 —33	—0·03252 99 —23	—0·03089 53 —16	—0·02873 23 —11	—0·02633 4 —7	—0·02389 —8 —5
2·6	0·03629 963 9	0·01295 750 —25	—0·00287 559 —36	—0·01306 403 —35	—0·01919 283 —31	—0·02247 192 —25	—0·02380 127 —19	—0·02385 80 —14	—0·02309 47 —10	—0·02185 25 —7	—0·02035 10 —5
2·8	0·04470 649 57	0·02391 559 15	0·00870 452 —6	—0·00199 350 —15	—0·00916 262 —17	—0·01371 191 —16	—0·01633 135 —13	—0·01759 93 —11	—0·01791 62 —8	—0·01761 39 —6	—0·01690 24 —5
3·0	0·04657 389 71	0·02928 381 34	0·01576 337 12	0·00560 281 0	—0·00175 224 —6	—0·00685 173 —8	—0·01021 130 —8	—0·01226 95 —7	—0·01335 68 —6	—0·01376 47 —5	—0·01368 32 —4
3·2	0·04450 198 66	0·03080 235 39	0·01943 234 20	0·01038 211 9	0·00343 180 1	—0·00171 147 —2	—0·00538 117 —4	—0·00787 90 —4	—0·00947 68 —5	—0·01038 50 —4	—0·01078 36 —3

TABLE IV. Imaginary Part of $F(x+iy)$

$\frac{1}{2}(\Delta_{x^2}-\Delta_{y^2})$ and $\Delta_{x^2}\ \Delta_{y^2}$ given beneath each function value.

x y	2·2	2·4	2·6	2·8	3·0	3·2	3·4	3·6	3·8	4·0
0·0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0
0·2	−0·00657 −37 4	−0·00531 −28 3	−0·00433 −21 2	−0·00358 −16 1	−0·00295 −12 1	−0·00246 −9 1	−0·00206 −7 0	−0·00174 −6 0	−0·00148 −5 0	−0·00126 −4 0
0·4	−0·01278 −70 7	−0·01035 −53 5	−0·00845 −40 4	−0·00696 −30 3	−0·00577 −23 2	−0·00482 −18 1	−0·00405 −14 1	−0·00342 −11 0	−0·00291 −9 0	−0·00249 −7 0
0·6	−0·01828 −97 9	−0·01486 −73 7	−0·01218 −56 5	−0·01006 −43 3	−0·00836 −33 2	−0·00700 −26 2	−0·00589 −20 1	−0·00499 −16 1	−0·00425 −13 0	−0·00364 −10 0
0·8	−0·02283 −115 10	−0·01865 −87 7	−0·01536 −67 5	−0·01273 −52 4	−0·01062 −40 3	−0·00892 −32 2	−0·00753 −25 1	−0·00640 −20 1	−0·00546 −16 0	−0·00469 −13 0
1·0	−0·02623 −122 10	−0·02158 −94 7	−0·01787 −73 5	−0·01489 −57 4	−0·01249 −45 3	−0·01053 −35 2	−0·00893 −28 1	−0·00761 −23 1	−0·00652 −18 1	−0·00561 −15 0
1·2	−0·02842 −120 8	−0·02357 −94 6	−0·01966 −74 5	−0·01649 −58 3	−0·01390 −46 3	−0·01179 −37 2	−0·01004 −30 1	−0·00860 −24 1	−0·00739 −20 1	−0·00639 −16 1
1·4	−0·02942 −110 6	−0·02462 −88 5	−0·02071 −70 4	−0·01750 −56 3	−0·01486 −45 2	−0·01268 −37 2	−0·01086 −30 1	−0·00934 −24 1	−0·00807 −20 1	−0·00700 −16 1

1·6	—0·02932 —94 4	—0·02480 —77 3	—0·02107 —63 3	—0·01796 —51 2	—0·01537 —42 2	—0·01320 —35 1	—0·01138 —29 1	—0·00985 —24 1	—0·00855 —20 1	—0·00745 —16 1
1·8	—0·02827 —74 1	—0·02422 —63 1	—0·02079 —53 1	—0·01790 —44 1	—0·01545 —37 1	—0·01338 —31 1	—0·01162 —26 1	—0·01012 —22 1	—0·00884 —18 1	—0·00774 —16 0
2·0	—0·02649 —53 —1	—0·02301 —47 0	—0·01999 —42 0	—0·01740 —36 0	—0·01517 —31 1	—0·01324 —27 1	—0·01159 —23 1	—0·01017 —20 0	—0·00894 —17 0	—0·00788 —14 0
2·2	—0·02417 —33 —2	—0·02132 —32 —1	—0·01878 —30 —1	—0·01654 —27 0	—0·01457 —24 0	—0·01284 —22 0	—0·01134 —19 0	—0·01002 —17 0	—0·00887 —15 0	—0·00787 —13 0
2·4	—0·02153 —14 —3	—0·01931 —18 —2	—0·01726 —19 —1	—0·01540 —19 —1	—0·01373 —18 0	—0·01223 —16 0	—0·01089 —15 0	—0·00971 —14 0	—0·00866 —12 0	—0·00774 —11 0
2·6	—0·01874 1 —4	—0·01712 —5 —3	—0·01556 —9 —2	—0·01408 —11 —1	—0·01271 —11 —1	—0·01145 —11 0	—0·01030 —11 0	—0·00927 —10 0	—0·00833 —10 0	—0·00750 —9 0
2·8	—0·01595 12 —4	—0·01488 5 —3	—0·01376 0 —2	—0·01265 —4 —1	—0·01157 —6 —1	—0·01055 —7 —1	—0·00960 —7 0	—0·00872 —7 0	—0·00791 —7 0	—0·00717 —7 0
3·0	—0·01329 20 —3	—0·01268 12 —3	—0·01196 6 —2	—0·01118 2 —1	—0·01038 —1 —1	—0·00958 —3 —1	—0·00882 —4 —1	—0·00809 —4 0	—0·00741 —5 0	—0·00677 —5 0
3·2	—0·01082 25 —3	—0·01060 17 —2	—0·01022 10 —2	—0·00973 6 —1	—0·00917 3 —1	—0·00859 1 —1	—0·00800 —1 —1	—0·00742 —2 —1	—0·00686 —3 0	—0·00633 —3 0

TABLE V. Real Part of $F'(x+iy)$

$\frac{1}{2}(\Delta_{x^2}-\Delta_{y^2})$ and $\Delta_{x^2}\Delta_{y^2}$ given beneath each function value.

$y \diagdown x$	0·0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0
0·0	−0·76643 −12343 3152	−0·53265 −7708 1794	−0·37744 −4920 1045	−0·27231 −3206 623	−0·19975 −2129 379	−0·14880 −1439 235	−0·11243 −989 149	−0·08609 −691 96	−0·06673 −490 63	−0·05231 −352 41	−0·04145 −256 28
0·2	−0·70605 −10807 2611	−0·49487 −6834 1510	−0·35328 −4410 893	−0·25654 −2902 539	−0·18926 −1944 332	−0·14170 −1324 208	−0·10755 −916 133	−0·08267 −644 86	−0·06431 −459 57	−0·05057 −331 38	−0·04018 −242 26
0·4	−0·53977 −6736 1225	−0·39000 −4488 774	−0·28575 −3029 493	−0·21220 −2070 316	−0·15962 −1433 205	−0·12153 −1005 135	−0·09361 −713 90	−0·07289 −512 60	−0·05735 −372 41	−0·04555 −273 27	−0·03652 −203 19
0·6	−0·30716 −1462 −423	−0·24089 −1383 −125	−0·18835 −1163 −9	−0·14743 −928 31	−0·11581 −721 40	−0·09143 −553 37	−0·07261 −422 31	−0·05804 −321 24	−0·04670 −245 19	−0·03782 −188 14	−0·03084 −144 10
0·8	−0·05957 3420 −1695	−0·07785 1611 −864	−0·07931 700 443	−0·07340 248 −227	−0·06483 32 −116	−0·05583 −64 −58	−0·04743 −100 −28	−0·03999 −106 −13	−0·03361 −100 −5	−0·02823 −88 −1	−0·02373 −76 1
1·0	0·15523 6670 −2210	0·06980 3771 −1226	0·02310 2135 −686	−0·00167 1205 −388	−0·01407 674 −221	−0·01955 370 −127	−0·02122 196 −73	−0·02087 97 −43	−0·01952 41 −25	−0·01775 10 −13	−0·01586 −6 −8
1·2	0·30517 7775 −1949	0·18076 4741 −1170	0·10473 2903 −704	0·05833 1785 −425	0·03012 1101 −258	0·01314 680 −158	0·00309 420 −97	−0·00268 258 −60	−0·00582 158 −37	−0·00735 95 −24	−0·00792 56 −15
1·4	0·37897 6973 −1195	0·24528 4565 −815	0·15792 2982 −542	0·10083 1948 −355	0·06353 1275 −231	0·03916 836 −150	0·02328 549 −98	0·01297 361 −64	0·00633 238 −41	0·00211 157 −27	−0·00053 103 −18

1·6	0·38405 4984 —333	0·26482 3584 —350	0·18173 2527 —293	0·12415 1762 —223	0·08438 1221 —162	0·05695 844 —115	0·03806 583 —80	0·02507 402 —56	0·01614 278 —38	0·01002 192 —26	0·00585 133 —18
1·8	0·33956 2646 331	0·24882 2247 56	0·18051 1778 —51	0·13003 1354 —81	0·09315 1007 —79	0·06640 738 —67	0·04709 537 —53	0·03319 388 —40	0·02320 280 —30	0·01604 201 —22	0·01091 145 —16
2·0	0·26833 612 667	0·21030 954 307	0·16156 974 123	0·12245 862 34	0·09192 712 —5	0·06853 565 —20	0·05079 438 —24	0·03746 334 —22	0·02748 252 —19	0·02005 189 —15	0·01453 142 —12
2·2	0·19042 —780 704	0·16199 —46 393	0·13276 285 208	0·10621 401 103	0·08358 410 46	0·06502 371 16	0·05014 315 1	0·03841 258 —6	0·02927 206 —8	0·02219 162 —8	0·01674 127 —7
2·4	0·11973 —1485 552	0·11381 —664 354	0·10095 —202 217	0·08588 39 127	0·07111 151 71	0·05779 191 37	0·04634 192 18	0·03679 175 7	0·02899 151 2	0·02271 127 —2	0·01769 104 —3
2·6	0·06344 —1643 337	0·07197 —932 254	0·07097 —476 177	0·06506 —199 117	0·05707 —39 74	0·04862 46 45	0·04060 86 26	0·03341 99 14	0·02720 98 7	0·02196 90 3	0·01761 79 1
2·8	0·02328 —1461 144	0·03924 —947 144	0·04561 —575 119	0·04612 —321 89	0·04335 —157 63	0·03896 —55 42	0·03399 4 27	0·02903 36 17	0·02443 51 10	0·02032 55 6	0·01675 54 3
3·0	—0·00238 —1130 12	0·01587 —817 55	0·02590 —555 63	0·03033 —355 57	0·03115 —212 45	0·02981 —114 34	0·02731 —50 24	0·02428 —10 16	0·02114 13 11	0·01812 26 7	0·01535 31 5
3·2	—0·01675 —782 57	0·00062 —630 —3	0·01170 —471 21	0·01804 —333 29	0·02103 —222 28	0·02178 —140 24	0·02111 —81 18	0·01961 —40 14	0·01771 —14 9	0·01566 3 7	0·01364 13 6

TABLE VI. Real Part of $F'(x+iy)$

$\frac{1}{2}(\Delta_{x^2}-\Delta_{y^2})$ and $\Delta_{x^2}\Delta_{y^2}$ given beneath each function value.

y \ x	2·2	2·4	2·6	2·8	3·0	3·2	3·4	3·6	3·8	4·0
0·0	—0·03318 —189 18	—0·02680 —141 13	—0·02183 —106 9	—0·01791 —80 7	—0·01482 —61 5	—0·01235 —48 3	—0·01036 —37 2	—0·00875 —30 1	—0·00743 —24 1	—0·00634 —19 2
0·2	—0·03224 —179 18	—0·02610 —134 12	—0·02130 —101 8	—0·01752 —77 7	—0·01452 —59 5	—0·01212 —46 3	—0·01018 —36 2	—0·00860 —29 2	—0·00731 —23 1	—0·00624 —18 1
0·4	—0·02953 —152 14	—0·02407 —115 10	—0·01997 —88 7	—0·01636 —68 5	—0·01362 —52 4	—0·01142 —41 3	—0·00963 —32 2	—0·00817 —26 1	—0·00697 —21 1	—0·00597 —17 1
0·6	—0·02531 —112 8	—0·02090 —87 6	—0·01737 —68 4	—0·01452 —53 3	—0·01220 —42 3	—0·01031 —33 2	—0·00875 —27 1	—0·00748 —22 1	—0·00642 —18 1	—0·00553 —14 1
0·8	—0·01998 —64 1	—0·01687 —53 1	—0·01429 —43 1	—0·01215 —36 1	—0·01036 —29 1	—0·00887 —24 1	—0·00763 —20 1	—0·00658 —16 1	—0·00570 —14 1	—0·00495 —11 0
1·0	—0·01402 —14 —5	—0·01232 —17 —3	—0·01078 —18 —1	—0·00942 —17 0	—0·00823 —16 0	—0·00719 —14 0	—0·00630 —12 0	—0·00552 —10 0	—0·00484 —9 0	—0·00426 —8 0
1·2	—0·00791 31 —9	—0·00759 16 —6	—0·00710 7 —4	—0·00653 1 —2	—0·00594 —2 —1	—0·00537 —3 —1	—0·00484 —4 —1	—0·00435 —4 0	—0·00389 —4 0	—0·00349 —4 0
1·4	—0·00211 67 —12	—0·00302 44 —8	—0·00348 28 —5	—0·00365 18 —4	—0·00364 11 —2	—0·00352 6 —2	—0·00334 3 —1	—0·00313 1 —1	—0·00290 0 0	—0·00267 —1 0

1·6	0·00302 92 −12	0·00112 64 −9	−0·00014 44 −6	−0·00094 30 −4	−0·00144 21 −3	−0·00173 14 −2	−0·00188 9 −1	−0·00193 6 −1	−0·00191 4 0	−0·00185 3 0
1·8	0·00724 104 −11	0·00462 75 −8	0·00277 54 −6	0·00146 39 −4	0·00055 28 −3	−0·00008 20 −2	−0·00051 14 −2	−0·00079 10 −1	−0·00096 7 −1	−0·00106 6 −1
2·0	0·01042 106 −9	0·00738 79 −7	0·00514 59 −5	0·00348 44 −4	0·00226 32 −3	0·00137 24 −2	0·00072 18 −1	0·00025 13 −1	−0·00008 9 −1	−0·00032 5 −1
2·2	0·01256 98 −6	0·00936 75 −5	0·00692 58 −4	0·00507 44 −3	0·00366 34 −2	0·00258 26 −2	0·00177 21 −1	0·00116 16 −1	0·00070 12 −1	0·00036 10 −1
2·4	0·01372 84 −3	0·01059 67 −3	0·00814 53 −2	0·00621 42 −2	0·00471 33 −2	0·00354 27 −1	0·00263 21 −1	0·00192 16 −1	0·00137 13 −1	0·00095 10 −1
2·6	0·01405 67 −1	0·01116 56 −1	0·00882 46 −1	0·00694 37 −1	0·00544 30 −1	0·00424 25 −1	0·00328 20 −1	0·00251 16 −1	0·00191 13 −1	0·00144 10 0
2·8	0·01372 49 1	0·01117 43 0	0·00905 37 0	0·00730 31 0	0·00586 26 0	0·00469 22 0	0·00373 18 −1	0·00296 15 −1	0·00233 12 −1	0·00182 10 −1
3·0	0·01289 32 3	0·01075 30 1	0·00891 28 0	·000735 25 0	0·00603 22 0	0·00492 19 0	0·00401 16 −1	0·00325 13 −1	0·00263 11 0	0·00212 9 0
3·2	0·01174 18 4	0·01003 19 1	0·00850 18 0	0·00715 17 −1	0·00597 17 0	0·00497 16 −1	0·00413 13 −2	0·00342 11 −1	0·00282 10 0	0·00231 9 1

TABLE VII. Imaginary Part of $F'(x+iy)$

$\frac{1}{2}(\Delta_{x^2}-\Delta_{y^2})$ and $\Delta_{x^2}\Delta_{y^2}$ given beneath each function value.

y \ x	0·0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0
0·0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0
0·2	0·27696 5650 —1695	0·18269 3379 — 929	0·12298 2066 —521	0·08437 1290 —299	0·05892 822 — 175	0·04182 533 —105	0·03015 352 — 64	0·02204 236 —40	0·01633 161 —25	0·01226 111 —16	0·00931 78 —11
0·4	0·49883 9660 —2736	0·33237 5856 —1526	0·22574 3625 —870	0·15609 2289 —507	0·10976 1472 —301	0·07840 964 —182	0·05683 641 —112	0·04176 433 — 70	0·03108 297 —45	0·02341 207 —29	0·01784 146 —19
0·6	0·62638 11016 —2785	0·42475 6851 —1608	0·29297 4337 —946	0·20535 2793 —566	0·14613 1829 — 345	0·10549 1216 —213	0·07719 820 —134	0·05720 561 —85	0·04289 389 —55	0·03253 274 — 36	0·02493 195 —24
0·8	0·64610 9662 —1944	0·44996 6277 —1209	0·31761 4125 —756	0·22714 2744 —476	0·16451 1848 — 303	0·12060 1259 —194	0·08945 868 — 126	0·06709 605 —83	0·05086 427 —55	0·03894 305 —37	0·03010 220 —25
1·0	0·57083 6399 —668	0·41341 4517 —537	0·30162 3172 —398	0·22188 2228 —284	0·16465 1569 — 198	0·12329 1111 —137	0·09314 792 — 95	0·07100 568 —65	0·05459 411 —45	0·04234 300 —32	0·03310 220 —22
1·2	0·43212 2461 518	0·33213 2220 138	0·25424 1823 —12	0·19458 1430 —61	0·14927 1095 —69	0·11497 828 —62	0·08900 622 —50	0·06928 466 —39	0·05426 350 —30	0·04276 263 —22	0·03391 199 —16
1·4	0·26837 —999 1237	0·22854 43 605	0·18864 454 284	0·15304 568 125	0·12301 550 47	0·09844 482 11	0·07867 401 —5	0·06293 325 —11	0·05044 259 —12	0·04056 205 —11	0·03274 160 —9

1·6	0·11358 —3268 1396	0·12401 —1554 781	0·11826 —643 431	0·10572 —177 234	0·09120 48 124	0·07707 145 64	0·06434 175 31	0·05333 173 14	0·04405 157 4	0·03635 136 0	0·02997 114 —2
1·8	—0·00970 —4177 1123	0·03438 —2391 705	0·05396 —1322 433	0·05997 —694 261	0·05881 —332 155	0·05421 —130 91	0·04823 —21 53	0·04199 34 30	0·03608 59 16	0·03074 67 8	0·02606 66 4
2·0	—0·09215 —3979 655	—0·03194 —2535 483	0·00252 —1576 335	0·02094 —955 224	0·02962 —561 146	0·03256 —316 94	0·03228 —166 59	0·03028 —76 37	0·02751 —23 22	0·02448 6 14	0·02149 21 8
2·2	—0·13535 —3125 205	—0·07330 —2199 230	—0·03344 —1498 199	—0·00873 —995 154	0·00591 —646 112	0·01400 —409 79	0·01793 —252 54	0·01930 —149 36	0·01915 —83 24	0·01814 —42 16	0·01671 —16 11
2·4	—0·14747 —2054 109	—0·09288 —1629 25	—0·05459 —1221 72	—0·02857 —882 78	—0·01143 —619 69	—0·00054 —425 55	0·00606 —285 41	0·00978 —187 30	0·01160 —119 22	0·01221 —74 15	0·01207 —43 11
2·6	—0·13897 —1078 —258	—0·09618 —1027 —99	—0·06358 —870 —19	—0·03966 —689 16	—0·02264 —523 29	—0·01088 —385 30	—0·00300 —277 26	0·00210 —195 21	0·00523 —134 17	0·00701 —91 13	0·00787 —60 10
2·8	—0·11947 —350 —280	—0·08914 —518 —148	—0·06387 —534 —68	—0·04387 —479 —24	—0·02865 —398 —1	—0·01740 —316 9	—0·00931 —242 12	—0·00365 —181 13	0·00019 —132 11	0·00270 —95 9	0·00425 —67 8
3·0	—0·09624 104 —230	—0·07679 —153 —144	—0·05876 —263 —82	—0·04328 —290 —42	—0·03067 —272 —18	—0·02076 —236 —5	—0·01321 —194 2	—0·00761 —154 5	—0·00354 —119 6	—0·00067 —90 6	0·00129 —67 5
3·2	—0·07385 311 —157	—0·06280 71 —113	—0·05095 —73 —75	—0·03976 —142 —46	—0·02995 —164 —26	—0·02177 —161 —13	—0·01518 —144 —5	—0·01002 —122 0	—0·00609 —100 2	—0·00315 —79 3	—0·00100 —62 3

TABLE VIII. Imaginary Part of $F'(x-iy)$

$\frac{1}{2}(\Delta_{x^2}-\Delta_{y^2})$ and $\Delta_{x^2}\Delta_{y^2}$ given beneath each function value.

x y	2·2	2·4	2·6	2·8	3·0	3·2	3·4	3·6	3·8	4·0
0·0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0	0·00000 0 0
0·2	0·00714 55 —7	0·00554 40 —5	0·00433 29 —3	0·00342 21 —2	0·00273 16 —2	0·00219 12 —1	0·00177 9 —1	0·00145 7 —1	0·00119 5 0	0·00098 4 0
0·4	0·01374 104 —13	0·01068 75 —9	0·00838 55 —6	0·00664 40 —4	0·00530 30 —3	0·00427 23 —2	0·00346 17 —1	0·00282 13 —1	0·00232 10 —1	0·00192 8 —1
0·6	0·01930 140 —16	0·01508 102 —11	0·01189 75 —8	0·00946 55 —5	0·00758 41 —3	0·00612 31 —2	0·00498 24 —2	0·00407 18 —1	0·00335 14 —1	0·00278 11 —1
0·8	0·02348 160 —17	0·01848 118 —12	0·01466 87 —8	0·01172 65 —5	0·00944 49 —4	0·00766 38 —3	0·00625 29 —2	0·00514 22 —2	0·00425 17 —1	0·00353 14 —1
1·0	0·02608 163 —15	0·02071 122 —11	0·01656 92 —8	0·01334 69 —5	0·01082 53 —4	0·00882 41 —3	0·00724 32 —2	0·00598 25 —2	0·00497 19 —1	0·00415 15 —1
1·2	0·02706 151 —12	0·02172 115 —9	0·01755 88 —7	0·01427 68 —5	0·01166 53 —3	0·00959 41 —2	0·00792 32 —1	0·00658 25 —1	0·00550 20 —1	0·00461 16 —1
1·4	0·02654 127 —7	0·02160 99 —6	0·01767 78 —5	0·01452 61 —4	0·01199 49 —3	0·00994 39 —2	0·00828 31 —1	0·00693 24 —1	0·00582 19 —1	0·00492 16 0

1·6	0·02476 95 —3	0·02049 78 —3	0·01701 63 —3	0·01416 51 —2	0·01183 42 —2	0·00991 34 —2	0·00833 28 —1	0·00703 22 —1	0·00596 18 —1	0·00506 15 0
1·8	0·02203 60 1	0·01861 53 0	0·01572 46 0	0·01329 39 —1	0·01125 33 —1	0·00954 28 —1	0·00811 23 —1	0·00691 19 —1	0·00590 16 —1	0·00506 15 0
2·0	0·01871 27 5	0·01619 29 3	0·01397 28 2	0·01203 26 1	0·01034 24 0	0·00890 21 0	0·00766 18 0	0·00660 15 0	0·00570 13 0	0·00492 10 0
2·2	0·01510 —1 7	0·01348 7 5	0·01193 12 3	0·01050 14 2	0·00921 14 1	0·00805 14 0	0·00703 13 0	0·00613 11 0	0·00536 10 0	0·00468 9 0
2·4	0·01150 —23 8	0·01069 —10 5	0·00977 —2 3	0·00884 3 2	0·00792 6 2	0·00707 7 1	0·00628 8 0	0·00556 8 0	0·00491 7 0	0·00435 7 0
2·6	0·00812 —38 7	0·00800 —23 5	0·00764 —13 3	0·00714 —6 2	0·00658 —2 2	0·00601 1 1	0·00544 3 1	0·00491 4 1	0·00441 4 1	0·00395 4 1
2·8	0·00512 —46 6	0·00553 —31 4	0·00562 —20 3	0·00550 —12 2	0·00526 —7 2	0·00494 —4 1	0·00459 —1 1	0·00422 1 1	0·00386 2 1	0·00351 2 1
3·0	0·00258 —49 4	0·00337 —35 3	0·00381 —25 2	0·00399 —17 2	0·00401 —11 2	0·00391 —7 1	0·00374 —4 1	0·00353 —2 1	0·00329 —1 1	0·00305 0 0
3·2	0·00052 —48 4	0·00157 —37 3	0·00224 —27 2	0·00264 —19 2	0·00286 —12 2	·000295 —10 1	0·00294 —8 —1	0·00285 —4 1	0·00273 —2 0	0·00258 —2 —1

5. *Accuracy of Tables.*

In the computation of the tables each function value was calculated to six figures and the power series expansions around two enclosing origins were used as a first check. At this stage the error did not exceed five units in the last decimal place. The values were then rounded off by the method described by P. M. and A. M. Woodward (4), which probably reduces the error to less than one unit in the sixth decimal place. The tables presented in five-figure form should be correct to the number of figures shown. An error of one unit in the fifth figure may occur where the sixth figure is in the region of five.

6. *Method of Interpolation.*

Coupled differences are given below each function value for use in P. M. Woodward's method of bivariate interpolation for a function of a complex variable. The reader is referred to the accompanying paper (5) for information on interpolation by coupled differences and for bivariate tables of the interpolation coefficients.

Acknowledgment.

The author desires to express his thanks to Miss N. Gamble for her assistance in computing the tables, and to the Ministry of Supply for permission to publish the paper, which is Crown Copyright reserved and reproduced with the permission of the Controller of H.M. Stationery Office.

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(5) P. M. Woodward, Phil. Mag. (7), xxxix. p. 594 (August, 1948).

CXI. *The Whirling of Shafts having Sections with Unequal Principal Bending Moduli.*

By J. Dick, B.Sc., Ph.D.*

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The study of the whirling of shafts may be said to date from 1869 when Rankine(1) discussed the equilibrium of a perfectly balanced rotating shaft, and gave the subject its name. Since then there have been numerour publications on the subject, one of the most notable being that of Jeffcott(2) who showed that in addition to the forced whirl at shaft speeds, due to out-of-balance effects, a uniform shaft could vibrate freely at its natural frequency, which is independent of the shaft speed.

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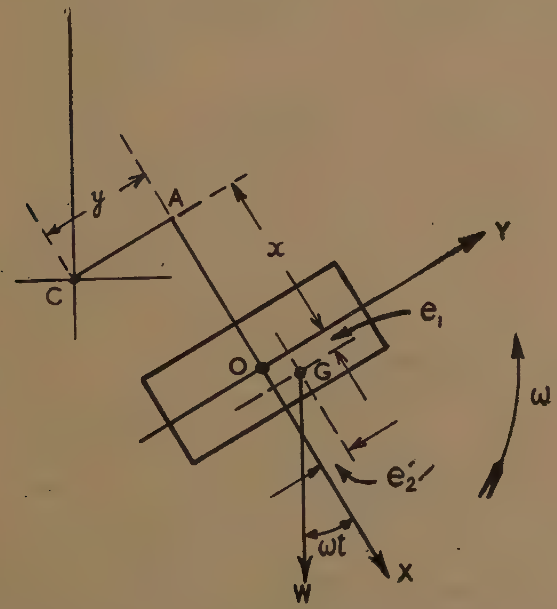
* Communicated by the Author.

In the case of a shaft having a section with unequal principal bending moduli, the free vibrations are more complex. There can be two vibrations differing in frequency, and each of the frequencies is a function of the angular velocity of the shaft. This case was analysed by Robertson(3) who applied the principle of suppression, and considered the equilibrium of the forces acting parallel to the direction of deflection and forces at right angles to these. In the present article it is proposed to approach the problem by considering the equilibrium of forces along the principal bending axes. This leads to a simpler solution than that due to Robertson, and gives a clearer picture of the various phenomena involved.

Fig. 1 shows the cross-section of a shaft having unequal principal bending moduli. O is the elastic centre of the shaft, with OX and OY as the principal axes of the section. To simplify the algebra, it is assumed that the shaft is carrying a disc of weight W, which is large compared with the weight of the shaft. An out-of-balance effect is provided by considering the centre of gravity of the disc as being at G, which is displaced from O by e_1 in the direction of OX, and e_2 in the direction of OY. The shaft is assumed to rotate at constant angular velocity ω, and due to various disturbing effects the elastic centre O moves relative to the bearing axis C. The gravitational effect is obtained by assuming that the weight W acts in the direction indicated.

Fig. 1.



Showing section with unequal principal bending moduli.

From these conditions, we can deduce the following :

Elastic displacement in direction $OX = AO = x$. . . . . . (1)

Velocity of O along $OX = \frac{dx}{dt} - \omega y$. . . . . (2)

$$\text{Acceleration of G in direction OX}=\frac{d^2x}{dt^2}-\omega^2(x+e_1)-2\omega\frac{dy}{dt}. \quad . \quad (3)$$

The corresponding expressions for the direction OY are :

$$\text{Elastic displacement in direction OY}=\text{CA}=y. \quad . \quad . \quad . \quad . \quad (4)$$

$$\text{Velocity of O along OY}=\frac{dy}{dt}+\omega x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\text{Acceleration of G in direction OY}=\frac{d^2y}{dt^2}-\omega^2(y+e_2)+2\omega\frac{dx}{dt}. \quad . \quad (6)$$

Let $F_1=\left(\frac{W}{g}\right)\omega_1^2$ be the elastic stiffness in the direction OX, and let $F_2=\left(\frac{W}{g}\right)\omega_2^2$ be the elastic stiffness in the direction OY, then if frictional effects are neglected the equation for motion in the direction OX is

$$\frac{W}{g}\frac{d^2x}{dt^2}-\frac{W}{g}\omega^2(x+e_1)-2\frac{W}{g}\omega\frac{dy}{dt}+F_1x-W\cos\omega t=0,$$

that is

$$\frac{d^2x}{dt^2}+(\omega_1^2-\omega^2)x-2\omega\frac{dy}{dt}=\omega^2e_1+g\cos\omega t. \quad . \quad . \quad . \quad . \quad (7)$$

The similar equation for motion in the direction OY is

$$\frac{d^2y}{dt^2}+(\omega_2^2-\omega^2)y+2\omega\frac{dx}{dt}=\omega^2e_2-g\sin\omega t. \quad . \quad . \quad . \quad . \quad (8)$$

Particular Integrals.

The particular integrals are

$$x=\frac{\omega^2e_1}{(\omega_1^2-\omega^2)}+\frac{g(\omega_2^2-4\omega^2)}{\{\omega_1^2\omega_2^2-2\omega^2(\omega_1^2+\omega_2^2)\}}\cos\omega t. \quad . \quad . \quad (9)$$

and

$$y=\frac{\omega^2e_2}{(\omega_2^2-\omega^2)}-\frac{g(\omega_1^2-4\omega^2)}{\{\omega_1^2\omega_2^2-2\omega\ (\omega_1\ +\omega_2^2)\}}\sin\omega t. \quad . \quad . \quad . \quad (10)$$

The first term in each case refers to the main whirl due to out of balance effects. The second term refers to the whirl induced by the gravitational pull.

Main Whirl.

The main whirl is indicated by

$$x=\frac{\omega^2e_1}{\omega_1^2-\omega^2} \quad \text{and} \quad y=\frac{\omega^2e_2}{\omega_2^2-\omega^2},$$

from which the critical speeds are obviously $\omega=\omega_1$ and $\omega=\omega_2$.

Gravitational Whirl.

The deflections due to the gravitational pull are

$$x=\frac{g(\omega_2^2-4\omega^2)}{\{\omega_1^2\omega_2^2-2\omega^2(\omega_1^2+\omega_2^2)\}}\cos\omega t$$

and

$$y=-\frac{g(\omega_1^2-4\omega^2)}{\{\omega_1^2\omega_2^2-2\omega^2(\omega_1^2+\omega_2^2)\}}\sin\omega t.$$

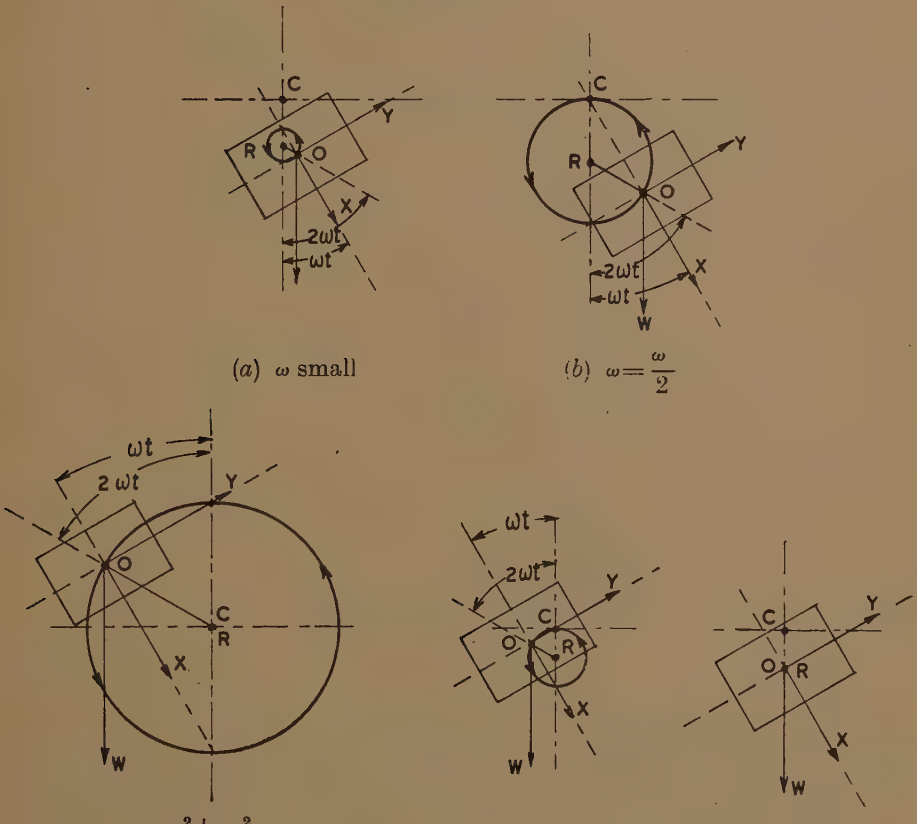
The study of these is simplified by finding the vertical and horizontal components of the deflection.

The vertical component is

$$x \cos \omega t - y \sin \omega t$$

$$= \frac{g}{2\{\omega_1^2\omega_2^2 - 2\omega^2(\omega_1^2+\omega_2^2)\}}\{(\omega_1^2+\omega_2^2-8\omega^2)+(\omega_2^2-\omega_1^2)\cos 2\omega t\}.$$

Fig. 2.



(*a*) ω small (*b*) $\omega = \frac{\omega}{2}$

(*c*) $\omega = \frac{\omega_1^2+\omega_2^2}{8}$ (*d*) $\omega = \frac{\omega_2}{2}$ (*e*) ω large

Showing locus for gravitational whirl, with $\omega_2^2 = 2\omega_1^2$.

The horizontal component is

$$x \sin \omega t + y \cos \omega t = \frac{g}{2\{\omega_1^2\omega_2^2 - 2\omega^2(\omega_1^2+\omega_2^2)\}}(\omega_2^2-\omega_1^2)\sin 2\omega t.$$

These indicate that the effects of the gravitational force is to induce the elastic centre O to travel round a circular path of radius RO

$$= \frac{g/2\,(\omega_2^2-\omega_1^2)}{\{\omega_1^2\omega_2^2 - 2\omega^2(\omega_1^2+\omega_2^2)\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

at twice the speed of rotation of the shaft.

The centre of the path is at R, which is vertically below C, and

$$\mathrm{CR}=\frac{g/2\,(\omega_1^2+\omega_2^2-8\omega^2)}{\{\omega_1^2\omega_2^2-2\omega^2(\omega_1^2+\omega_2^2)\}}. \qquad (12)$$

It is obvious from these that the value of the critical speed of the whirl induced by the gravitational pull is

$$\omega=\frac{\omega_1\omega_2}{\sqrt{2(\omega_1^2+\omega_2^2)}}. \qquad (13)$$

It is interesting to trace the variations in the radius RO, and the deflection CR, as ω increases from zero to a large value. *Fig.* 2 shows some special cases for $\omega_2^2=2\omega_1^2$. These are:

(*a*) The case for small values of ω

$$\mathrm{RO}=\frac{g}{2}\frac{\omega_2^2-\omega_1^2}{\omega_1^2\omega_2^2} \quad \text{and} \quad \mathrm{CR}=\frac{g}{2}\frac{\omega_1^2+\omega_2^2}{\omega_1^2\omega_2^2}.$$

(*b*) The case when $\omega=\omega_1/2$

$$\mathrm{RO}=\mathrm{CR}=g/\omega_1^2=\mathrm{W}/\mathrm{F}_1.$$

(*c*) A case when ω is above the critical speed, when $\omega^2=\frac{\omega_1^2+\omega_2^2}{8}$.

In this case $\mathrm{RC}=0$ and $\mathrm{RO}=\frac{2g}{\omega_2^2-\omega_1^2}$.

(*d*) The case when $\omega=\omega_2/2$

$$\mathrm{CR}=\mathrm{RO}=g/\omega_2^2=\mathrm{W}/\mathrm{F}_2.$$

(*e*) The case when ω is large

$$\mathrm{RO}=0 \quad \text{and} \quad \mathrm{CR}=\frac{2g}{\omega_1^2+\omega_2^2}.$$

The Free Vibrations.

The free vibration terms can be studied by finding the complementary functions of x and y in equations (7) and (8). If the gravitational and out of balance terms are omitted, the differential equations become

$$\frac{d^2x}{dt^2}+(\omega_1^2-\omega^2)x-2\omega\frac{dy}{dt}=0. \qquad (14)$$

and

$$\frac{d^2y}{dt^2}+(\omega_2^2-\omega^2)y+2\omega\frac{dx}{dt}=0. \qquad (15)$$

From these,

$$\frac{d^4x}{dt^4}+\{(\omega_1^2+\omega^2)+(\omega_2^2+\omega^2)\}\frac{d^2x}{dt^2}+(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)x=0. \qquad (16)$$

If the last term is positive, that is, if ω is less than ω_1 or greater than ω_2, the solution of this equation gives

$$x=\mathrm{A}_1\sin(p_1t+\alpha_1)+\mathrm{A}_2\sin(p_2t+\alpha_2) \qquad (17)$$

and

$$y=A_1\frac{p_1^2+\omega^2-\omega_1^2}{2\omega p_1}\cos(p_1t+\alpha_1)+A_2\frac{p_2^2+\omega^2-\omega_1^2}{2\omega p_2}\cos(p_2t+\alpha_2) \quad . \quad . \quad . \quad (18)$$

where

$$p_1^2=\tfrac{1}{2}[(\omega_1^2+\omega^2)+(\omega_2^2+\omega^2)-\sqrt{(\omega_1^2+\omega_2^2+2\omega^2)^2-4(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)}] \quad . \quad . \quad . \quad (19)$$

and

$$p_2^2=\tfrac{1}{2}[(\omega_1^2+\omega^2)+(\omega_2^2+\omega^2)+\sqrt{(\omega_1^2+\omega_2^2+2\omega^2)^2-4(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)}] \quad . \quad . \quad . \quad (20)$$

If ω lies between the values ω_1 and ω_2, then p_1^2 is negative, and the solution is

$$x=A_3e^{\beta t}+A_4e^{-\beta t}+A_2\sin(p_2t+\alpha_2). \quad . \quad . \quad . \quad . \quad (21)$$

and

$$y=\frac{\beta^2+\omega_1^2-\omega^2}{2\omega\beta}(A_3\epsilon^{\beta t}-A_4e^{-\beta t})+A_2\frac{p_2^2+\omega^2-\omega_1^2}{2\omega p_2}\cos(p_2t+\alpha_2) \quad . \quad (22)$$

where

$$\beta^2=\tfrac{1}{2}[-(\omega_1^2+\omega^2)-(\omega_2^2+\omega^2)+\sqrt{(\omega_1^2+\omega_2^2+2\omega^2)^2-4(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)}] \quad . \quad . \quad . \quad (23)$$

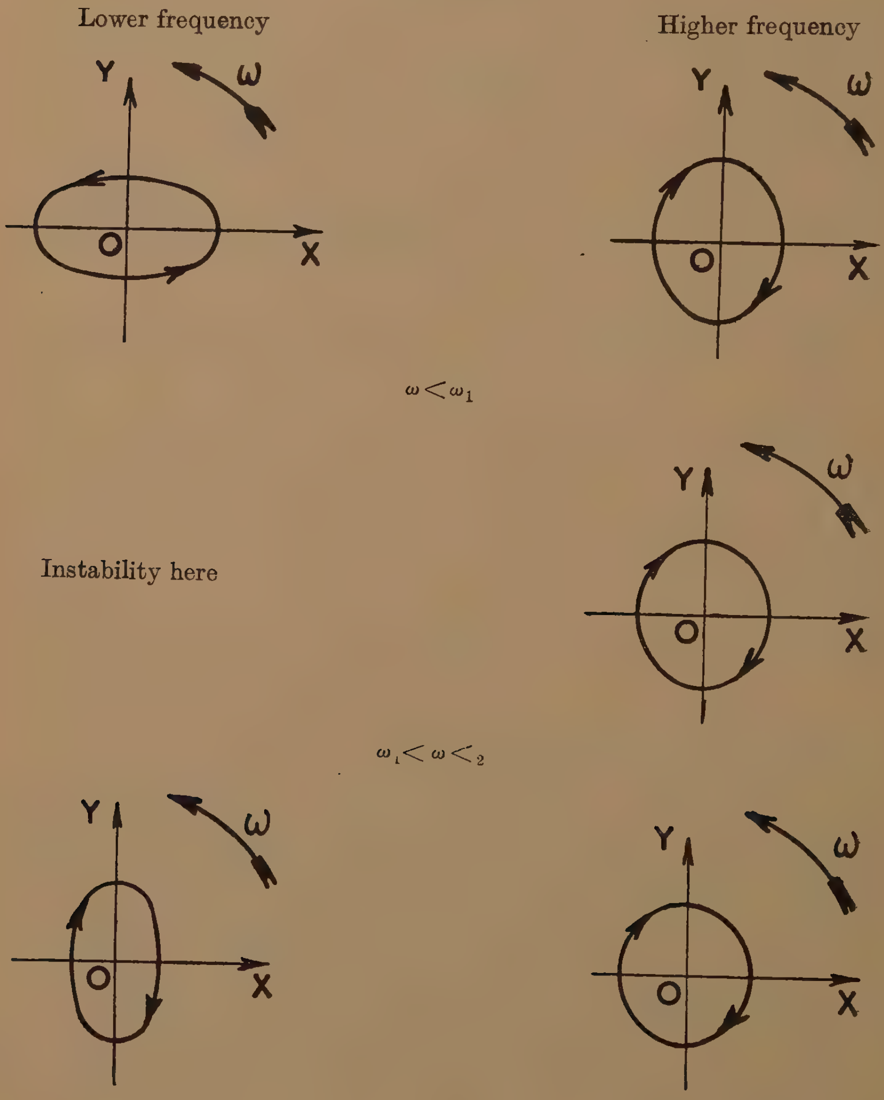
The stable orbits indicated by equations (17) and (18) are ellipses with OX and OY as axes.

Fig. 3 shows some special cases of these orbits. In the case of the orbit of higher frequency, that is, with $p^2=p_2^2$, the ellipse has its major axis along OY, but the ratio of the lengths of the axes approaches unity as ω increases. The elastic axis moves round this orbit in a direction opposite to that of the rotation of OX and OY. This orbit is stable for all values of ω, but its stability in the range $\omega_1<\omega<\omega_2$ does not affect the instability of the second locus indicated by equations (21) and (22).

In the case of the orbit of lower frequency, that is, $p^2=p_1^2$, the ellipse has its major axis along OX when $0<\omega<\omega_1$, and the ratio of the lengths of the axes varies with ω. In this range the elastic axis moves round the locus in the direction of rotation of OX and OY. The value of p_1^2 is zero when $\omega=\omega_1$ and the orbit is indeterminate. This is one of the main critical speeds. When $\omega>\omega_2$ the ellipse has its major axis along OY and the ratio of the lengths of the axes varies with ω. The elastic axis moves round the locus in a direction opposite to that of the rotation of OX and OY. When $\omega=\omega_2$, $p_1^2=0$, and the orbit for this is indeterminate. This is the second of the main critical speeds.

There are thus two obrits with the same direction of rotation giving stability when $\omega > \omega_2$, but the frequencies and the ratio of the lengths of the axis both differ.

Fig. 3.


Lower frequency
Higher frequency
Y
X
O
ω
Instability here

$\omega < \omega_1$

$\omega_1 < \omega < {}_2$

$\omega > \omega_2$

Showing orbits for free vibrations.

The terms in $e^{\beta t}$ in equations (21) and (22) show that there is instability in the range $\omega_1 < \omega < \omega_2$. The radius of rotation continues to increase unless some restricting effect is applied.

Effect of Damping Forces.

The effect of damping forces can be considered by assuming damping proportional to velocity. In this case, the equations (7) and (8) become

$$\frac{d^2x}{dt^2}+\frac{b}{\mathrm{W}/g}\frac{dx}{dt}+(\omega_1{}^2-\omega^2)x-2\omega\frac{dy}{dt}-\frac{b}{\mathrm{W}/g}\omega y=\omega^2e_1+g\cos\omega t \qquad (24)$$

and

$$\frac{d^2y}{dt^2}+\frac{b}{\mathrm{W}/g}\frac{dy}{dt}+(\omega_2{}^2-\omega^2)y+2\omega\frac{dx}{dt}+\frac{b}{\mathrm{W}/g}\omega x=\omega^2e_2-g\sin\omega t, \qquad (25)$$

where b is a damping coefficient.

The particular integrals are of the form

$$x=\mathrm{H}_0+\mathrm{H}_1\sin\omega t+\mathrm{H}_2\cos\omega t, \quad \ldots\ldots \quad (26)$$

$$y=\mathrm{K}_0+\mathrm{K}_1\sin\omega t+\mathrm{K}_2\cos\omega t, \quad \ldots\ldots \quad (27)$$

where H_0 and K_0 refer to the main whirl, and H_1, H_2, K_1, and K_2 refer to the gravitational whirl.

Main Whirl Terms with Damping.

The main whirl terms are

$$x=\mathrm{H}_0=\frac{(\omega^2e_1(\omega_2{}^2-\omega^2)+\dfrac{b\omega}{\mathrm{W}/g}\omega^2e_2}{(\omega_1{}^2-\omega^2)(\omega_2{}^2-\omega^2)+\left(\dfrac{b\omega}{\mathrm{W}/g}\right)^2} \quad \ldots\ldots \quad (28)$$

and

$$y=\mathrm{K}_0=\frac{\omega^2e^2(\omega_1{}^2-\omega^2)-\dfrac{b\omega}{\mathrm{W}/g}\omega^2e_1}{(\omega_1{}^2-\omega^2)(\omega_2{}^2-\omega^2)-\left(\dfrac{b\omega}{\mathrm{W}/g}\right)^2}. \quad \ldots\ldots \quad (29)$$

The effect of the damping is to cause a slight change in the values of ω at which the critical speeds occur, and also to set a limit to the maximum radius of whirl, compared with the case where damping is zero.

Gravitational Whirl Terms with Damping.

The terms for the whirl induced by the gravitational pull become

$$x=\mathrm{H}_1\sin\omega t+\mathrm{H}_2\cos\omega t$$

where

$$\mathrm{H}_1=\frac{g\dfrac{b\omega}{\mathrm{W}/g}\omega_2{}^2(\omega_2{}^2-\omega_1{}^2)}{\left[\{\omega_1{}^2\omega_2{}^2-2\omega^2(\omega_1{}^2+\omega_2{}^2)\}^2+\left(\dfrac{b\omega}{\mathrm{W}/g}\right)^2(\omega_1{}^2+\omega_2{}^2)^2\right]} \quad . \quad (30)$$

and

$$\mathrm{H}_2=\frac{g\left[\{\omega_1{}^2\omega_2{}^2-2\omega^2(\omega_1{}^2+\omega_2{}^2)\}(\omega_2{}^2-4\omega^2)+2\left(\dfrac{b\omega}{\mathrm{W}/g}\right)^2(\omega_1{}^2+\omega_2{}^2)\right]}{\left[\{\omega_1{}^2-2\omega^2(\omega_1{}^2+\omega_2{}^2)\}^2+\left(\dfrac{b\omega}{\mathrm{W}/g}\right)^2(\omega_1{}^2+\omega_2{}^2)^2\right]}. \quad \ldots\ldots \quad (31)$$

Also, $y=\mathrm{K}_1 \sin \omega t+\mathrm{K}_2 \cos \omega t$, where

$$\mathrm{K}_1=-\frac{g\left[\{\omega_1{}^2\omega_2{}^2-2\omega^2(\omega_1{}^2+\omega_2{}^2)\}(\omega_1{}^2-4\omega^2)+2\left(\frac{b\omega}{\mathrm{W}/g}\right)^2(\omega_1{}^2+\omega_2{}^2)\right]}{\left[\{\omega_1{}^2\omega_2{}^2-2\omega_2{}^2(\omega_1{}^2+\omega_2{}^2)\}^2+\left(\frac{b\omega}{\mathrm{W}/g}\right)^2(\omega_1{}^2+\omega_2{}^2)^2\right]}, \quad \ldots\ldots \quad (32)$$

and

$$\mathrm{K}_2=-\frac{g\frac{b\omega}{\mathrm{W}/g}\omega_1{}^2(\omega_2{}^2-\omega_1{}^2)}{\left[\{\omega_1{}^2\omega_2{}^2-2\omega^2(\omega_1{}^2+\omega_2{}^2)\}^2+\left(\frac{b\omega}{\mathrm{W}/g}\right)^2(\omega_1{}^2+\omega_2{}^2)^2\right]}. \quad . \quad (33)$$

Analysis of these shows that the effect of the gravitational forces is a whirl at twice the speed of the shaft. The effects of a normal amount of damping in this case, compared with no damping, are:

(1) there is a change in the position of the centre of the whirling circle, which is now displaced slightly from the vertical plane through the bearing axis.

(2) the radius of the whirling circle is reduced for a given value of ω, and there is now a limit to the greatest value.

(4) there is a slight change in the critical speed for this whirl.

(4) there is a change in the angular position of the section at any chosen point on the whirling circle.

Complementary Function with Damping.

With the damping terms introduced, the differential equations for the complementary functions are:

$$\frac{d^2x}{dt^2}+\frac{b}{\mathrm{W}/g}\frac{dx}{dt}+(\omega_1{}^2-\omega^2)x-2\omega\frac{dy}{dt}-\frac{b}{\mathrm{W}/g}\omega y=0. \quad . \quad . \quad (34)$$

and

$$\frac{d^2y}{dt^2}+\frac{b}{\mathrm{W}/g}\frac{dy}{dt}+(\omega_2{}^2+\omega^2)y+2\omega\frac{dx}{dt}+\frac{b}{\mathrm{W}/g}\omega x=0. \quad . \quad . \quad (35)$$

The operator equation for these is

$$\mathrm{D}^4+2\frac{b}{\mathrm{W}/g}\mathrm{D}^3+\left\{\omega_1{}^2+\omega_2{}^2+2\omega^2+\left(\frac{b}{\mathrm{W}/g}\right)^2\right\}\mathrm{D}^2+\frac{b}{\mathrm{W}/g}(\omega_1{}^2+\omega_2{}^2+2\omega^2)\mathrm{D}$$
$$+\left\{(\omega_1{}^2-\omega^2)(\omega_2{}^2-\omega^2)+\left(\frac{b\omega}{\mathrm{W}/g}\right)^2\right\}=0. \quad \ldots\ldots \quad (36)$$

If the term $(\omega_1{}^2-\omega^2)(\omega_2{}^2-\omega^2)+\left(\frac{b\omega}{\mathrm{W}/g}\right)^2$ is positive, then solutions are of the form

$$x=\mathrm{A}_5e^{-\beta_1 t}\sin(p_3t+\alpha_3)+\mathrm{A}_8e^{-\beta_2 t}\sin(p_4t+\alpha_4) \quad . \quad . \quad . \quad (37)$$

and

$$\mathrm{A}_7e^{-\beta_1 t}\cos(p_3t+\alpha_3)+\mathrm{A}_8e^{-\beta_2 t}\cos(p_4t+\alpha_4). \quad . \quad . \quad . \quad (38)$$

The movements indicated by these will decay as t increases.

The main interest lies in the effect of the damping on the instability which is indicated when $\omega_1<\omega<\omega_2$ with no damping. Instability occurs when the term $(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)+\left(\frac{b\omega}{W/g}\right)^2$ is negative. The limiting values of ω^2 are given by the equation

$$\left(\frac{b}{W/g}\right)^2\omega^2=(\omega^2-\omega_1^2)(\omega^2-\omega_2^2),$$

that is

$$\omega^4-\omega^2\left\{\omega_1^2+\omega_2^2+\left(\frac{b}{W/g}\right)^2\right\}+\omega_1^2\omega_2^2=0.$$

Each of the two solutions of this equation lies between ω_1 and ω_2 and hence the range of speed in which instability occurs is reduced by the damping. The range becomes zero when

$$\left(\frac{b}{W/g}\right)^2=\omega_1^2+\omega_2^2-2\omega_1\omega_2,$$

that is

$$\frac{b}{W/g}=\omega_2-\omega_1.$$

This is unlikely to be attained, unless $\omega_2-\omega_1$ is very small, in which case the range in which instability occurs would be small, even if there was no damping.

Conclusions.

The analysis has shown that :

(1) There are two main whirling speeds, corresponding to the frequencies of vibration along the two principal axes with the stationary shaft.

(2) With a shaft rotating about a non-vertical axis there is a whirl induced by the gravitational forces. This whirl occurs at twice the shaft speed, and the critical speed is given by

$$\omega=\frac{\omega_1\omega_2}{\sqrt{2(\omega_1^2+\omega_2^2)}}$$

where ω_1 and ω_2 are the two main critical whirling speeds.

(3) The free vibration movements can occur round orbits which are ellipses with axes lying along the principal bending axes. If the speed is less than ω_1 or greater than ω_2, there are two possible orbits, and hence two frequencies. If the speed lies between ω_1 and ω_2 there is instability and the deflection of the shaft continues to increase.

(4) The effect of damping is to cause slight changes in the values of the critical speeds, to reduce the amplitudes of the various whirls, and to reduce the range in which instability occurs.

References.

(1) W. J. McQ. Rankine, 'The Engineer' (April 9th, 1869).
(2) H. H. Jeffcott, Phil. Mag. xxxvii. pp. 304–314, (1919).
(3) D. Robertson, 'The Engineer' (August 9th to September 1st, 1933).

CXII. *The Primary and Secondary Scattering of Sunlight in a Plane-stratified Atmosphere of Uniform Composition.*—Part IV. *The Approximate Evaluation of Secondary Scattered Light.*

By A. Hammad, Ph.D., Department of Applied Mathematics, Faculty of Science, Cairo, and S. A. M. Hassanein, M.A., Meteorological Department, Cairo *.

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Introduction.

1. In an earlier paper † (here referred to as Part I.) with the same (main) title as this one, exact formulæ were given for the primary and secondary scattered sunlight of any wavelength λ, emitted and received in any direction, at any point in a plane-stratified atmosphere of uniform composition, extending to infinity in all horizontal directions. The formulæ for the reception of primary scattered light are quite simple and easy to evaluate, but those for secondary are so complicated and elaborate to evaluate that the need was urged for much simpler expressions whose numerical values agree fairly well with those of the exact formulæ. Such simpler expressions are of value in the numerical discussion of secondary scattered light itself and still more because they facilitate the calculation of the scattered light of higher orders.

In a recent paper ‡, L. Silberstein made an attempt towards that end, but considered the very special case when the sun is overhead and the radiation is weakly attenuated. His formulæ, however, are not simple enough. In this paper we have derived, for this special case, simpler approximate expressions which are more readily evaluated and which give numerical values in closer agreement with the exact values than those of Silberstein. For other wavelengths and other directions of incidence of the solar beam, simple approximate expressions are also given which are based on previously published numerical tables.

Notation.

2. As in Part I. the zenith angle of the sun is denoted by z. Relative to the vertical and the vertical half-plane through the sun, other directions are specified by their polar angles (θ, ϕ), Light travelling in the direction

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* Communicated by the Authors.

† A. Hammad and S. Chapman, Phil. Mag. ser. 7, vol. xxviii. pp. 99–110 (July, 1939).

‡ L. Silberstein, ‘Journal of the Optical Society of America,’ vol. xxxiii. No. 9, pp. 526-532 (September, 1943).

of the unit vector $\boldsymbol{k}$ and that travelling in the opposite direction $-\boldsymbol{k}$ will, however, be denoted by the same angles (θ, ϕ), so that $\sec\theta \geqslant 1$. When it is required to distinguish between upgoing and downgoing light, we attach to our symbols a suffix "*u*" or "*d*" respectively.

The fraction of the total mass of the atmosphere that lies below the horizontal plane through the point P is denoted, as in Part I., by m; the remaining fraction above that plane is $1-m$, also denoted by $\bar{m}$. We refer to m as the *level*, $\bar{m}$ as the *depth* of the point P in the atmosphere.

The absorption and scattering coefficients of the air, per unit mass, for the wavelength λ, are denoted by $\alpha(\lambda)$ and $\sigma(\lambda)$, or, more briefly, by α and σ; we define a, s, c by

$$a/\alpha = s/\sigma = \mathrm{M}, \qquad c = a + s,$$

where M denotes the total mass of air in a vertical column of unit cross-section (1 sq. cm.) extending from the ground throughout the atmosphere. We shall also write

$$\mathrm{Z} = c \sec z, \qquad \Theta = c \sec\theta.$$

Emission and Reception of Secondary Scattered Light.

3. The exact expression for the intensity of emission E_2, per unit mass of air, at any level m in any direction $\boldsymbol{k}$, is given in § 13 of Part I. This is

$$\mathrm{E}_2(\lambda, m, z, \boldsymbol{k}) = \frac{9\sigma s}{256\pi}\mathrm{I}_\infty\{\mathrm{A}_0(z, \boldsymbol{k})\mathrm{L}_1(c, z, m) + \mathrm{A}_1(z, \boldsymbol{k})\mathrm{L}_3(c, z, m)$$
$$+ \mathrm{A}_2(z, \boldsymbol{k})\mathrm{L}_5(c, z, m)\},$$

where I_∞ is the intensity of the direct sunlight (of any wavelength λ) outside the atmosphere, and the A's and the L's are functions specified and discussed in §§ 13 and 17 of Part I.

The functions L_n are given by

$$\mathrm{L}_n(c, z, m) = \int_0^1 e^{-\mathrm{Z}\bar{m}'}\mathrm{E}j_n(c\bar{m} \sim c\bar{m}')\, d\bar{m}',$$

where

$$\mathrm{E}j_n(t) = \int_1^\infty e^{-tx}x^{-n}\, dx.$$

From this definition of L_n, it may be readily seen that it decreases as either n, z, or c increases; its variation with m, however, is not monotonic, and it has maximum at a level higher than $m=\frac{1}{2}$ (*cf.* § 17, Part I.).

The exact expression for the intensity of reception R_2 of secondary scattered light is given in § 14 of Part I. For downgoing light this is

$$\mathrm{R}d_2(\lambda, m, z, \boldsymbol{k}) = \frac{9s^2}{256\pi}\mathrm{I}_\infty \sec\theta\{\mathrm{A}_0\mathrm{L}d_1(c, z, m, \theta) + \mathrm{A}_1\mathrm{L}d_3(c, z, m, \theta)$$
$$+ \mathrm{A}_2\mathrm{L}d_5(c, z, m, \theta)\},$$

where

$$\mathrm{L}d_n(c, z, m, \theta) = e^{-\Theta\bar{m}}\int_0^{\bar{m}} e^{\Theta\bar{m}'}\mathrm{L}_n(c, z, \bar{m}')\, d\bar{m}'.$$

For upgoing light the intensity is

$$\mathrm{R}u_2(\lambda, m, z, \boldsymbol{k})=\frac{9s^2}{256\pi}\mathrm{I}_\infty \sec\theta\{\mathrm{A}_0\mathrm{L}u_1(c, z, m, \theta)+\mathrm{A}_1\mathrm{L}u_3(c, z, m, \theta)$$
$$+\mathrm{A}_2\mathrm{L}u_5(c, z, m, \theta)\},$$

where

$$\mathrm{L}u_n(c, z, m, \theta)=e^{-\Theta m}\int_0^m e^{\Theta m'}\mathrm{L}_n(c, z, m')\,dm'.$$

The integrals $\mathrm{L}d_n$ and $\mathrm{L}u_n$ are evaluated in terms of known functions in § 18 of Part I., but the expressions for them are so lengthy and elaborate as practically to preclude their numerical evaluation and discussion. Our object in this paper is to give simple approximate expressions for these integrals. We shall start with the case considered by Silberstein, namely, the case when Z is small.

Approximate Forms for L_1 *for small* Z.

4.1. The integrated form of the function L_1 is given in § 17 of Part I. This is

$$\mathrm{Z}e^{\mathrm{Z}\overline{m}}\mathrm{L}_1(c, z, m)=\log\{(\mathrm{Z}+c)/(\mathrm{Z}-c)\}+\mathrm{E}i\{(\mathrm{Z}-c)\overline{m}\}-\mathrm{E}i\{-(\mathrm{Z}+c)m\}$$
$$+e^{\mathrm{Z}\overline{m}}\mathrm{E}j_1(c\overline{m})-e^{-\mathrm{Z}m}\mathrm{E}j_1(cm),$$

$\mathrm{E}i(x)$ and $\mathrm{E}i(-x)$ being the well-known exponential integrals.

Consider the right-hand side of the above expression for L_1, and let Z be so small that Z^3 (and therefore c^3) and higher powers may be neglected. Using the well-known expansions for the exponential integrals and retaining the necessary terms, we get

$$\mathrm{E}i\{(\mathrm{Z}-c)\overline{m}\}=\gamma+\log\{(\mathrm{Z}-c)\overline{m}\}+(\mathrm{Z}-c)\overline{m}+(\mathrm{Z}-c)^2\overline{m}^2/2\cdot 2\,!,$$
$$\mathrm{E}i\{-(\mathrm{Z}+c)m\}=\gamma+\log\{(\mathrm{Z}+c)m\}-(\mathrm{Z}+c)m+(\mathrm{Z}+c)^2m^2/2\cdot 2\,!,$$
$$e^{\mathrm{Z}\overline{m}}\mathrm{E}j_1(c\overline{m})=\{1+\mathrm{Z}\overline{m}+\mathrm{Z}^2\overline{m}^2/2\,!\}\{-\gamma-\log(c\overline{m})+c\overline{m}-c^2\overline{m}^2/2\cdot 2\,!\},$$
$$e^{-\mathrm{Z}m}\mathrm{E}j_1(cm)=\{1-\mathrm{Z}m+\mathrm{Z}^2m^2/2\,!\}\{-\gamma-\log(cm)+cm-c^2m^2/2\cdot 2\,!\},$$

where γ is Euler's constant $=0\cdot 57722\ldots$.

Thus

$$\log\{(\mathrm{Z}+c)/(\mathrm{Z}-c)\}+\mathrm{E}i\{(\mathrm{Z}-c)\overline{m}\}-\mathrm{E}i\{-(\mathrm{Z}+c)m\}$$
$$=\log(\overline{m}/m)+(\mathrm{Z}-c)\overline{m}+(\mathrm{Z}+c)m+(\mathrm{Z}-c)^2\overline{m}^2/4-(\mathrm{Z}+c)^2m^2/4,$$

and

$$e^{\mathrm{Z}\overline{m}}\mathrm{E}j_1(c\overline{m})-e^{-\mathrm{Z}m}\mathrm{E}j_1(cm)=-\log(\overline{m}/m)-c+c^2/4+2\overline{m}(c-c^2/4)$$
$$+\mathrm{Z}\{2c(\overline{m}-1)\overline{m}-(\overline{m}\log\overline{m}+m\log m)-\gamma-\log c+c\}$$
$$+\tfrac{1}{2}\mathrm{Z}^2\{m^2\log m-\overline{m}^2\log\overline{m}-2m(\gamma+\log c)+\gamma+\log c\}.$$

Hence

$$\begin{aligned} Ze^{Z\bar{m}}L_1 &= Z(1-\gamma-\log c)+\tfrac{1}{2}Z^2(\gamma+\log c-\tfrac{1}{2})+\tfrac{1}{2}cZ \\ &\quad +\bar{m}\{-cZ-Z^2(\gamma+\log c-\tfrac{1}{2})\}+\bar{m}^2cZ \\ &\quad -Z(\bar{m}\log\bar{m}+m\log m)-\tfrac{1}{2}Z^2(\bar{m}^2\log\bar{m}-m^2\log m). \quad . \quad (1) \end{aligned}$$

We shall now replace the function $\bar{m}\log\bar{m}+m\log m$, in the interval $0\leqslant m\leqslant 1$ by an approximate expression of the form $Cm\bar{m}$. This polynomial has the same zeros as the original function, and the constant C will be chosen so that

$$\int_0^1(\bar{m}\log\bar{m}+m\log m)\,dm=\int_0^1 Cm\bar{m}\,dm.$$

This gives $C=-3$.

Similarly we shall put

$$\bar{m}^2\log\bar{m}-m^2\log m=Dm\bar{m}(2\bar{m}-1).$$

Again the zeros of both sides are the same and the constant D is determined as before from the equation

$$\int_0^{\frac{1}{2}}(\bar{m}^2\log\bar{m}-m^2\log m)\,dm=\int_0^{\frac{1}{2}}Dm\bar{m}(2\bar{m}-1)\,dm,$$

which gives $D=-0{\cdot}818$ or $-0{\cdot}8$ approximately.

That these approximations are reasonably satisfactory is borne out by the following table :

$\bar{m}$	$\bar{m}\log\bar{m}+m\log m$	$-3m\bar{m}$	$\bar{m}^2\log\bar{m}-m^2\log m$	$-0{\cdot}8m\bar{m}(2\bar{m}-1)$
0·0	0·00	0·00	0·000	0·000
0·2	—0·50	—0·48	0·078	0·077
0·3	—0·61	—0·63	0·066	0·067
0·4	—0·67	—0·72	0·037	0·038
0·5	—0·69	—0·75	0·000	0·000

It is clear that the first two functions are symmetrical about $\bar{m}=\frac{1}{2}$, while the last two are antisymmetrical about that point, so that it is enough to stop at $\bar{m}=\frac{1}{2}$.

Introducing these approximations in (1) we get

$$L_1(c,z,\bar{m})=e^{-Z\bar{m}}\{a_0+b_0\bar{m}+c_0\bar{m}^2+d_0\bar{m}^3\}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where

$$a_0=1-g+\tfrac{1}{2}c-\tfrac{1}{2}Z(\tfrac{1}{2}-g),$$
$$b_0=3-c-Z(g-0{\cdot}1),$$
$$c_0=-3+c+1{\cdot}2Z,$$
$$d_0=-0{\cdot}8Z,$$

and

$$g=\gamma+\log c.$$

We give now the exact values * of $L_1(\overline{m})$ for $c=0{\cdot}2$, $\sec z=1$, for some values of $\overline{m}$, together with the approximate values (*in italics*) which are obtained from the last formula; and for the sake of comparison we also include, inside brackets, the values that would be obtained from Silberstein's formula †.

Values of $L_1(\overline{m})$ for $c=0{\cdot}2$, $\sec z=1$.

$\overline{m}=$		0·0	0·2	0·4	0·6	0·8	1·0
$L_1(\overline{m})=$	Exact.	1·97	2·40	2·51	2·48	2·31	1·89
	Approx.	*1·98*	*2·38*	*2·55*	*2·51*	*2·28*	*1·86*
	Approx.	(2·03)	(2·53)	(2·71)	(2·71)	(2·53)	(2·03)

The above table shows that our approximation gives numerical values in closer agreement with the exact values than those of Silberstein. Calculations carried out for other small values of c and z support the same conclusion.

4.2. We shall now compare the approximate expressions for Ld_1 obtained according to the last formula with that given by Silberstein. According to equation (2), the formula for Ld_1 will be as follows:

$$e^{\Theta\overline{m}}Ld_1(c, z, \overline{m}, \theta)=\int_0^{\overline{m}} e^{\Theta\overline{m}'}L_1(c, z, \overline{m}')\,d\overline{m}'$$

$$=\int_0^{\overline{m}} e^{(\Theta-Z)\overline{m}'}\{a_0+b_0\overline{m}'+c_0\overline{m}^{2\prime}+d_0\overline{m}'^3\}\,d\overline{m}'$$

$$=\frac{e^{Q\overline{m}}}{Q}\Big\{a_0+b_0\overline{m}+c_0\overline{m}^2+d_0\overline{m}^3-\frac{1}{Q}(b_0+2c_0\overline{m}+3d_0\overline{m}^2)$$

$$+\frac{2}{Q^2}(c_0+3d_0\overline{m})-\frac{6d_0}{Q^3}\Big\}-\frac{a_0}{Q}+\frac{b_0}{Q^2}-\frac{2c_0}{Q^3}+\frac{6d_0}{Q^4},$$

where $Q=(\Theta-Z)$.

This gives Ld_1 at any level. At the ground, *i. e.* for $\overline{m}=1$ (which is the case considered by Silberstein), this reduces to

$$e^{\Theta}Ld_1(\overline{m}=1)=\frac{Q}{Q}\Big\{a_0+b_0+c_0+d_0-\frac{1}{Q}(b_0+2c_0+3d_0)$$

$$+\frac{2}{Q^2}(c_0+3d_0)-\frac{6d_0}{Q^3}\Big\}-\frac{a_1}{Q}+\frac{b_1}{Q^2}-\frac{2c_0}{Q^3}+\frac{6d_0}{Q^4}.$$

A glance at page 531 of Silberstein's paper suffices to show that the above expression is simpler and less elaborate to evaluate than that given by Silberstein.

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* The exact values are taken from a paper by A. Hammad (Part III.), Phil. Mag. ser. 7, vol. xxxviii. pp. 515–529 (July, 1947).

† L. Silberstein, *op. cit.* p. 530.

In the following table we give the exact values of Ld_1 at ground-level, these being taken from Part III., for the case $c=0{\cdot}2$, $\sec z=1$, together with the approximate values obtained from the above formula (*shown in italics*); the values obtained from Silberstein's formula are given inside brackets.

Values of Ld_1 at ground-level for $c=0{\cdot}2$, $\sec z=1$.

$\sec\theta=$	1	2	4	6
$e^{\Theta}Ld_1(\theta)=$ Exact	2·60	2·89	3·59	4·51
Approx.	*2·60*	*2·88*	*3·58*	*4·50*
Approx.	(2·79)	(3·11)	(3·87)	(4·88)

This table shows that our approximate expression yields numerical values in closer agreement with the exact values than Silberstein's. Calculations carried out for other small values of c and z support the same conclusion.

Approximate Forms for L_n.

5.1. The results of the last article suggest that there is ground for writing

$$L_1(c, z, \overline{m})=a_1+b_1\overline{m}+c_1\overline{m}^2, \quad \ldots \ldots \quad (1)$$

where a_1, b_1 and c_1 are functions of c and z. This formula, however, cannot possibly hold for large values of Z, as can be expected from the very nature of the approximate expansions made in the previous article.

Tables for the functions L_1, L_3 and L_5 have been given in Part III. of this series of papers. By the help of these tables it is possible to determine numerically by the method of least squares * the coefficients a_1, b_1, c_1. If we take the values of $L_1(\overline{m})$ at $\overline{m}=0$, 0·2, 0·4, 0·6, 0·8, 1, we find

$$c_1=\frac{1}{0{\cdot}05973}\left\{\sum_{\overline{m}}\overline{m}^2L_1(\overline{m})-\sum_{\overline{m}}\overline{m}L_1(\overline{m})+0{\cdot}13333\sum_{\overline{m}}L_1(\overline{m})\right\},$$

$$b_1=-c_1+\frac{1}{0{\cdot}7}\left\{\sum_{\overline{m}}\overline{m}L_1(\overline{m})-\tfrac{1}{2}\Sigma L_1(\overline{m})\right\},$$

$$a_1=-\tfrac{1}{2}b_1-\frac{1{\cdot}1}{3}c_1+\frac{1}{6}\sum_{\overline{m}}L_1(\overline{m}).$$

When Z is not small, formula (1) of this article cannot be expected to hold and a correction term has to be added, which will be chosen of the exponential form $d\exp(-p\overline{m})$, as it does not give rise to difficulties in the further development of the problem. This choice is, however, justified on account of the agreement of the numerical results based on it with the exact results. Thus when Z is greater or equal to unity, we shall put

$$L_1(c, z, \overline{m})=a_1+b_1\overline{m}+c_1\overline{m}^2+de^{-p\overline{m}}. \quad \ldots \ldots \quad (2)$$

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* Whittaker and Robinson, 'Calculus of Observations,' p. 211.

For the determination of the coefficients a_1, b_1, c_1, d and p, we shall divide the range of variation of $\overline{m}$ into two parts: (i.) from 0 to $\overline{m}_1$ and (ii.) from $\overline{m}_1$ to 1, where $\overline{m}_1$ is the maximum point of $L_1(\overline{m})$. It can be proved that $\overline{m}_1$ is the solution of the equation

$$\log\{(Z+c)/(Z-c)\}+Ei\{(Z-c)\overline{m}\}+Ej_1\{(Z+c)m\}=0,$$

and that

$$ZL_1(\overline{m}_1)=Ej_1(c\overline{m}_1)-e^{-Z}Ej_1(cm_1).$$

The quantities a_1, b_1, c_1 are then determined from the range $\overline{m}_1$ to 1 by the method of least squares, given the values of L_1 for a set of values of $\overline{m}$ in that range. In other words we assume that, in this range,

$$L_1(c,z,\overline{m})=a_1+b_1\overline{m}+c_1\overline{m}^2.$$

Having thus determined a_1, b_1, c_1 for this range, *i. e.* for the range $\overline{m}_1 \ldots\ldots 1$, we extend the application of the above form to the other range $0 \ldots\ldots \overline{m}_1$, and determine the error $u(\overline{m})$ at two points, say 0 and $\overline{m}_0$, where $0<\overline{m}_0\leqslant\overline{m}_1$. The two quantities $u(0)$ and $u(\overline{m}_0)$ should be of the same sign. The function $u(\overline{m})$ is then assumed to be of the form $d\exp(-p\overline{m})$; the quantities d and p will be

$$d=u(0), \qquad p=\log\{u(0)/u(\overline{m}_0)\}/\overline{m}_0.$$

Owing to the fact that $u(\overline{m}_0)$ is much smaller than $u(0)$ and since $\overline{m}_0$ is much smaller than $\frac{1}{2}$ (*cf.* Part I. p. 108), p will be large, and the approximation will be fairly good throughout the whole range.

We have calculated the coefficients a_1. b_1, c_1, d and p for

$$c=0\cdot2,\ 0\cdot4,\ 0\cdot8,\ 1\cdot0,$$

$$\sec z=1,\ 2,\ 4,\ 6,$$

their values being given in Table I. at the end. We then calculated the approximate values of L_1 corresponding to these values of c and z, for $\overline{m}=0$, 0·2, 0·4, 0·6, 0·8 and 1, from formulæ (1) and (2) above, each in its region of application. The values thus obtained are then compared with the exact values (given in Part III.). It has been found that, in 52 cases out of the 96 cases considered, the error is less than 1 per cent of the exact value, while in 74 of the cases the errors were within 2 per cent; in only three cases has the error been as high as from 4 to 4·4 per cent.

5.2. In view of the fact that L_3 and L_5 are much smaller than L_1 and that the coefficient of L_1, in the expression of secondary scattered light, is, in most cases, much larger than the coefficients of L_3 and L_5, it is not necessary in the approximate representation of L_3 and L_5 to distinguish between the cases when Z is small and the cases when it is large. It is therefore sufficient to assume that

$$\left.\begin{aligned}L_3(c,z,\overline{m})&=a_3+b_3\overline{m}+c_3\overline{m}^2\\ L_5(c,z,\overline{m})&=a_5+b_5\overline{m}+c_5\overline{m}^2\end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

The exact values of L_3 and L_5 being given in Part III., it is easy to determine, by the method of least squares, the coefficients a_3, b_3, c_3, a_5, b_5, c_5. The equations giving these coefficients are similar to those giving a_1, b_1, c_1 in § 5.1, with the suffix 1 replaced by 3 or 5 according to the case. The values of these coefficients are given in Table I. at the end, for $c=0{\cdot}2$, $0{\cdot}4$, $0{\cdot}8$, $1{\cdot}0$, $\sec z=1, 2, 4, 6$.

Approximate Forms for Ld_n *and* Lu_n.

6. We shall now apply the approximations given in § 5 above to get approximate forms for the functions Ld_n which occur in the expression of Rd_2. A similar procedure would be done for Lu_n in Ru_2.

We have

$$e^{\Theta\bar{m}}Ld_1(c, z, \bar{m}, \theta)=\int_0^{\bar{m}} e^{\Theta\bar{m}'}L_1(c, z, \bar{m}')\, d\bar{m}'.$$

Thus when $Z<1$, we have, on using equation (1) of § 5.1,

$$e^{\Theta\bar{m}}Ld_1(c, z, \bar{m}, \theta)=\frac{e^{\Theta\bar{m}}}{\Theta}\left(a_1+b_1\bar{m}+c_1\bar{m}^2-\frac{b_1+2c_1\bar{m}}{\Theta}+\frac{2c_1}{\Theta^2}\right)$$
$$-\frac{a_1}{\Theta}+\frac{b_1}{\Theta^2}-\frac{2c_1}{\Theta^3} \quad . \quad . \quad (1)$$

For $Z\geqslant 1$, we have on using (2) of § 5.1,

$$e^{\Theta\bar{m}}Ld_1(c, z, \bar{m}, \theta)=\frac{e^{\Theta\bar{m}}}{\Theta}\left(a_1+b_1\bar{m}+c_1\bar{m}^2-\frac{b_1+2c_1\bar{m}}{\Theta}+\frac{2c_1}{\Theta^2}\right)$$
$$-\frac{a_1}{\Theta}+\frac{b_1}{\Theta^2}-\frac{2c_1}{\Theta^3}+d\,\frac{e^{(\Theta-p)\bar{m}}-1}{(\Theta-p)}.$$

These two equations give Ld_1 at any level.

In a similar way we obtain approximate forms for the functions Ld_3 and Ld_5, such forms being similar to the form (1) above, with the suffix 1 replaced by 3 or 5 according to the case.

In the following table we have given the approximate values of

$$e^{\Theta m}(Ld_1, Ld_3, Ld_5).$$

at ground-level, for the following values of c, z and θ:

$$c=0{\cdot}2,\ 0{\cdot}4,\ 0{\cdot}8,\ 1,$$

$$\sec z=1,\ 2,\ 4, \qquad \sec\theta=1,\ 2,\ 4,$$

according to the formulæ of this article. The exact values are also given (*in italics*) for comparison.

An inspection of the above table suffices to show that the approximate expressions given in this article are really adequate, the error being less than 1 per cent. in almost all the cases. Actually, in only seven cases out of the 108 cases included in the table, has the error been greater than 1 per cent, while in only one case has it exceeded 2 per cent, being nearly 2·1 per cent of the exact value.

Values of $e^{\Theta}\mathrm{L}d_n(c, z, \overline{m}=1, \theta)$ (to three significant figures).

		$e^{\Theta}\mathrm{L}d_1$.				$e^{\Theta}\mathrm{L}d_3$.				$e^{\Theta}\mathrm{L}d_5$.			
sec z	sec θ	0·2	0·4	0·8	1·0	0·2	0·4	0·8	1·0	0·2	0·4	0·8	1·0
1	1	2·59	1·97	1·42	1·26	0·445	0·404	0·347	0·325	0·230	0·214	0·189	0·180
		2·60	*1·98*	*1·42*	*1·26*	*0·445*	*0·404*	*0·346*	*0·325*	*0·230*	*0·213*	*0·189*	*0·180*
	2	2·87	2·44	2·22	2·23	0·494	0·503	0·551	0·589	0·255	0·266	0·302	0·327
		2·89	*2·45*	*2·23*	*2·24*	*0·494*	*0·503*	*0·550*	*0·589*	*0·256*	*0·266*	*0·301*	*0·326*
	4	3·57	3·88	6·20	8·51	0·615	0·806	1·59	2·34	0·318	0·427	0·873	1·31
		3·59	*3·90*	*6·26*	*8·59*	*0·616*	*0·806*	*1·58*	*2·33*	*0·318*	*0·427*	*0·870*	*1·30*
2	1	2·35	1·64	1·00	0·824	0·405	0·337	0·247	0·217	0·209	0·178	0·135	0·120
		2·36	*1·64*	*1·00*	*0·824*	*0·404*	*0·337*	*0·247*	*0·216*	*0·209*	*0·178*	*0·135*	*0·120*
	2	2·61	2·02	1·54	1·41	0·449	0·418	0·390	0·386	0·232	0·221	0·214	0·215
		2·62	*2·03*	*1·54*	*1·41*	*0·449*	*0·418*	*0·388*	*0·383*	*0·232*	*0·221*	*0·213*	*0·214*
	4	3·24	3·19	4·15	5·10	0·559	0·669	1·10	1·49	0·289	0·355	0·610	0·844
		3·25	*3·20*	*4·15*	*5·11*	*0·559*	*0·668*	*1·10*	*1·47*	*0·289*	*0·354*	*0·608*	*0·835*
4	1	1·96	1·17	0·567	0·427	0·338	0·243	0·144	0·116	0·175	0·129	0·0791	0·0651
		1·97	*1·18*	*0·568*	*0·428*	*0·338*	*0·243*	*0·143*	*0·116*	*0·175*	*0·129*	*0·0789*	*0·0648*
	2	2·17	1·43	0·839	0·692	0·375	0·301	0·222	0·201	0·194	0·159	0·123	0·114
		2·18	*1·44*	*0·839*	*0·691*	*0·375*	*0·300*	*0·221*	*0·199*	*0·194*	*0·159*	*0·123*	*0·113*
	4	2·68	2·22	2·13	2·28	0·466	0·478	0·614	0·747	0·241	0·254	0·345	0·431
		2·69	*2·23*	*2·12*	*2·27*	*0·466*	*0·477*	*0·606*	*0·732*	*0·241*	*0·254*	*0·341*	*0·424*

Conclusion.

7. In the preceding investigation a simple and satisfactory approximate expression has been derived analytically for the function L_1 when the sun is high and the attenuation coefficient is small. In this particular case, a polynomial form has been derived for the function $\exp(Z\bar{m})L_1$. In other cases, recourse is made to numerical methods making use of previously published numerical tables. It is interesting to note that the method of least squares has proved to give exceedingly good results in the problem we have dealt with.

With the simple approximate expressions we have obtained, the intensity of secondary scattered light, received at any level, can be evaluated within close limits. For the light received at the ground, the error in our estimation has been calculated for more than a hundred cases and has been found less than 1 per cent. in almost all the cases. Within such limits, an estimate of the scattered light returned to free space from planetary atmospheres can be obtained, as far as primary and secondary scattering are concerned.

It is hoped that by the use of the approximate formulæ given in this paper, it will be possible to calculate the scattered light of higher orders of scattering. Such an estimation is very desirable, particularly for low solar altitudes and large attenuation coefficients; where the contribution of tertiary and higher orders of scattering is supposed to be considerable.

Table I.

Coefficients in the Expansions of L_n.

Values of a_1, b_1, c_1, d and p.

c	sec z	a_1	b_1	c_1	d	p
0·2	1	1·9922	2·3020	−2·4060		
	2	1·8603	1·9845	−2·1732		
	4	1·6364	1·4546	−1·7672		
	6	1·7242	0·0817	−0·7084	−0·3126	13·3
0·4	1	1·3585	1·8531	−2·0313		
	2	1·2129	1·3548	−1·6501		
	4	1·2033	−0·0994	−0·4978	−0·2555	14·5
	6	1·0245	−0·4607	−0·1503	−0·2332	17·6
0·8	1	1·0111	0·5640	−0·9843	−0·2112	16·1
	2	0·9000	−0·1352	−0·4027	−0·2348	24·1
	4	0·6902	−0·6245	0·0962	−0·1900	24·6
	6	0·5470	−0·7311	0·2771	−0·1420	36·6
1·0	1	0·8780	0·2889	−0·7349	−0·2186	17·3
	2	0·7633	−0·3440	−0·1847	−0·2223	21·2
	4	0·5609	−0·6877	0·2182	−0·1592	30·4
	6	0·4293	−0·6978	0·3210	−0·1050	46·0

TABLE I. (*cont.*).

Values of a_3, b_3, c_3, a_5, b_5, c_5.

c	sec z	a_3	b_3	c_3	a_5	b_5	c_5
0·2	1	0·3827	0·1275	−0·1325	0·2001	0·0503	−0·0522
	2	0·3502	0·1111	−0·1201	0·1828	0·0439	−0·0474
	4	0·2962	0·0844	−0·0993	0·1541	0·0335	−0·0393
	6	0·2537	0·0637	−0·0824	0·1315	0·0262	−0·0335
0·4	1	0·3046	0·1791	−0·1939	0·1635	0·0765	−0·0827
	2	0·2606	0·1354	−0·1598	0·1391	0·0579	−0·0681
	4	0·1975	0·0757	−0·1100	0·1044	0·0328	−0·0472
	6	0·1559	0·0399	−0·0771	0·0818	0·0177	−0·0333
0·8	1	0·2095	0·1892	−0·2249	0·1156	0·0891	−0·1055
	2	0·1645	0·1031	−0·1530	0·0895	0·0494	−0·0723
	4	0·1121	0·0190	−0·0723	0·0597	0·0109	−0·0354
	6	0·0838	−0·0131	−0·0345	0·0440	−0·0037	−0·0181
1·0	1	0·1792	0·1743	−0·2184	0·0995	0·0850	−0·1058
	2	0·1374	0·0769	−0·1343	0·0750	0·0390	−0·0661
	4	0·0914	−0·0044	−0·0509	0·0489	0·0006	−0·0268
	6	0·0677	−0·0287	−0·0175	0·0356	−0·0109	0·0109

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CXIII. *Optical Analysis of Canal Rays.*

By Prof. Dr. B. Dasannacharya, M.A.(Madras), Dr.Phil.(Munich), F.Inst.P.(London), University Professor of Physics and Head Department of Physics, Benares Hindu University, Benares, India and Dr. Gopi Krishna Das, D.Sc.(Benares), Assistant Professor Department of Physics, Benares Hindu University, Benares, India *.

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[Plate VI.]

PREFACE.

EUGEN GOLDSTEIN (1) was led to the discovery of canal rays by the very striking colour which these rays possess. Johannes Stark (2) discovered that, on examination of this light spectroscopically, it is possible by the method of Doppler displacement (D.D.) to distinguish the light given out by the particles forming the rays from the light given out by particles which have been brought to light emission by collisions with the canal rays. Great attention has justly been paid to studies on the electromagnetic analysis but not to an analysis based on light emission. This is due mainly to the complexity of the problem. In the following, some of these problems have been subjected to detailed investigation and clarification sought (3). While in the electromagnetic analysis only charged particles can be studied and not the uncharged ones, there is no such limitation, generally, for the optical analysis.

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* Communicated by the Authors.

In the present paper, we shall give an account of our investigations made with inhomogeneous canal rays of hydrogen. We start with the idea that for every discharge voltage applied, that the Doppler displacement recorded by us for a particular potential difference shall be the maximum that can be obtained for that potential and that that value has not been vitiated by any insufficiency of photographic exposure of the spectra. Spectrograms are therefore taken with times of exposure varying in each particular potential, from one minute up to a maximum of six hours when necessary on one and the same plate. From a graph connecting the Doppler displacement against times of exposure we can read off a value for the displacement which cannot be improved by longer exposures. A saturation value which would be, therefore, independent of time, can be obtained for the maximum possible displacement. The different times of exposures enable us also to get Doppler displacements from which the intensities of the different subgroups in it could be evaluated by the application of the principle of Roscoe and Bunsen, regarding the intensities of photographic exposures, to an accuracy sufficient for our purposes.

The very striking results obtained by us enable us to explain the conflicting results obtained by previous workers who worked with inhomogeneous canal rays, from those obtained by workers who made use of homogeneous canal rays. They have led us to the discovery of the remarkable influence of ionization on light emission in atomic collisions. Critical potential, corresponding to ionization potential of an atom, has been introduced to express the effect. The prominences of the different groups composing the canal ray have been studied.

EXPERIMENTAL ARRANGEMENT.

The discharge tube of glass was of the Wien type, cylindrical tube. The cathode was made of brass 4 cm. diameter, 6 mm. thick and faced on the discharge side with a plate of aluminium 1 mm. thick in order to diminish contamination of the glass walls of the discharge tube due to the sputtering from the front face of the cathode. A bore of 2 mm. in diameter was bored right through the centre of the cathode to serve as the canal for the canal rays to pass through. The cathode fitted loosely into the glass walls and was supported by a copper rod 2 mm. in diameter screwed into it and fixed in a side tube with sealing wax coated over with picæin. The rod served the additional purpose of an electrical conductor to the cathode.

The sealing-wax joint had to be cooled by surrounding it with water in a beaker. The side tube served in addition as an inlet for fresh pure dry hydrogen gas to stream into the apparatus continuously to be able to drive away any impurities and contaminations resulting from the sealing-wax joints, etc. caused during the discharge. The walls of the discharge space in front of the cathode was slightly blown out so as to minimize the risk of its being broken. At a distance of 15 cm. from the cathode was introduced in a side tube the anode made of aluminium hollow cylinder and sealed through glass with platinum. The anode gets very much heated and to get over it the whole of the side tube containing the anode had to be kept in a beaker of water. The discharge tube beyond the anode joint has been gradually drawn out into smaller and smaller diameters so as to distribute the heat produced by the bombardment of electrons by increasing the area bombarded. The whole of the discharge space was cooled by a strong current of air from an electric fan. The tube behind the cathode was 11 cm. long and of the same diameter as the glass tube in front, being in fact the same single long tube for both. It was closed at the open back end with a window of glass fixed to the tube by means of sealing wax.

The gas outlet was at the drawn out end of the discharge tube. Hydrogen from a Kipp's apparatus produced by the action of zinc and dilute sulphuric acid passed through glass wool, granules of calcium chloride in a drying tube, then through a space filled with sticks of potassium hydroxide to remove traces of carbon dioxide and finally over P_2O_5. It was stored in a 2 litre reservoir also containing a little P_2O_5, and a manometer read the pressure of the gas before it streamed into the discharge tube through a capillary continuous leak about a metre long and 0·3 mm. diameter. A U-tube cooled in a mixture of ice and sodium nitrite and sometimes merely ice and common salt was introduced in the path of the gas after it passed through the capillary leak. This prevented to some extent mercury vapour from entering the discharge space. The pumping was effected by means of a three stage steel diffusion pump of Gaede backed by a Cenco hyvac pump. A McLeod gauge connected to the discharge space through a U-tube cooled in ice and salt mixture enables the pressure in the discharge space to be measured. This pressure is the same as that in the observation chamber behind the cathode, where the light emission from the canal rays was subjected to investigation.

The gas in the discharge space was excited by means of continuous discharge from a high tension transformer from which both the phases of the alternating supply are rectified by kenetrons and smoothed over with condensers and resistances, supplied by Etabli Gaiffe-Gallot and Pilon, Paris. The voltage was measured with a Starke and Schröder electrostatic voltmeter supplied with different scales to read off voltages from 2000 up to 50,000 volts. The same electrometer was also used for voltages below 2000 in that the deflecting plates were brought so near that a deflection for 2000 volts was read off as 20,000 on the 4000/20,000 scale.

The lowest indications on that scale were checked by means of high tension accumulator cells whose voltages for every 200 could be measured by an accurate standard electrostatic voltmeter.

The light of the canal ray was viewed longitudinally through the glass window of the observation chamber and focused by means of a $f/3$ condenser on the slit of the collimator fitted with a lens system of light gathering power $f/3$ with f equal to 19·5 cm. in a Steinheil spectrograph with three prisms with effective base of about 30 cm. The prisms were mounted each on an adjustable base fitted with a scale such that the scale readings in all would be the same for one set of adjustments. Spectrograms were obtained on the same plate on the camera side which is fitted with a lens 64 cm. in focal length. X-rays given out from the sides of the discharge tube were found to fog the plates, hence they had to be protected during the exposure by interposing a plate of iron 25 × 25 cm., square and 1·4 cm. thick. The temperature of the developing bath was not necessary to be controlled as this temperature did not vary very much and it would not affect the saturation value of the Doppler displacement at all. All the time-exposures for one particular voltage were taken on a single plate, so that the effect of a variation of temperature will be common for those exposures. Dispersions at H_β and H_γ are 16·5 and 9·5°A. respectively, per mm.

The spectrograms were examined with a travelling microscope. With this it was not always possible to determine with great accuracy the maximum Doppler displacement especially in the regions of the critical voltages (see about this later) when the edge of the maximum is exceedingly faint and not at all sharp. The determination is further rendered difficult by the prominence of the continuous and many-lined spectra. So recourse was had to getting enlarged prints of the spectrograms by using various times of printing. This method was very successful. The final measurements for the maximum D.D. have been checked by the method of enlarged prints and by the method of reading directly with a travelling microscope of suitable magnification, such that corns on the plate do not falsify the readings.

EXPERIMENTAL RESULTS.

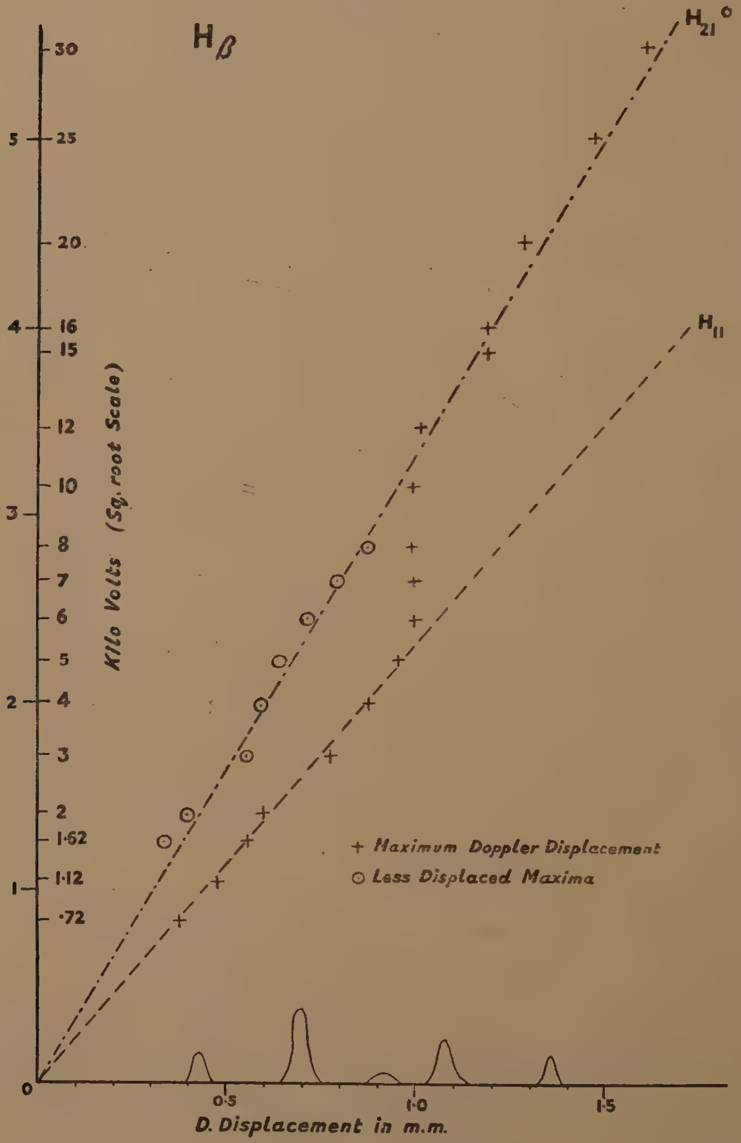
The maximum of $d\lambda$ for H_β.

The maximum attainable values come out uniquely in most cases. Some difficulty arose only when $d\lambda$ approached, or coincided with, some line of the background spectra. But it was overcome, because the Doppler strip extends on either side of the central spectral strip, whereas in most cases the lines of the background spectra do not extend similarly.

Fig. 1 gives the maximum $d\lambda$ for the H_β as abscissæ against square roots of the discharge potential as ordinates. The figure also shows lines calculated for the position of the maximum displacements for the H_{11} and H_{21} atoms against their velocities which are proportional to the square roots of the discharge potentials. It is found that the experimental points for potentials 720 up to 5500 volts fall strictly on the H_{11} line.

From 11,000 to 30,000 volts they fall on the H_{21} line, while between 5500 and 11,000 volts they lie on a line parallel to the ordinate axis. This region of potential 5500–11,000 we shall denote as *the transition region*

Fig. 1.



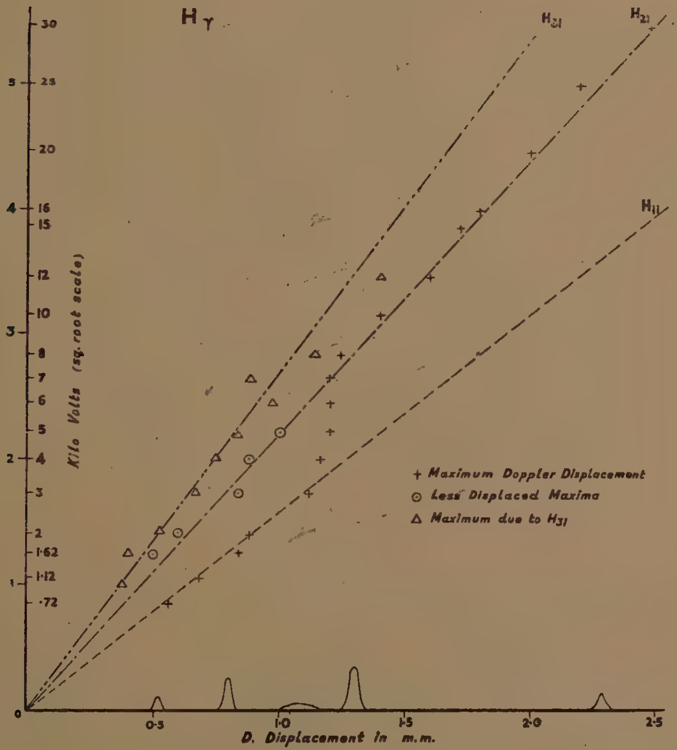
Dopper displacement H_β.

and 5500 and 11,000 as the *transition limits*, and 5500 itself as the *transition potential* because the constant maximum displacement corresponds to what is to be expected for that voltage. The significance of the transition potential we shall discuss a little later. Thus the transition potential for the H_β emission is 5500 volts. It may be remarked that the maxima corresponding to H_{21} can be noted and are plotted, down to a discharge potential of 1620 volts.

The maximum of $d\lambda$ for H_γ.

Effects similar to those observed in H_β are also found here, see fig. 2. The transition potential here is 3500 volts, the transition limits 3500 and 7000 volts. Here too the maxima due to H_{21} can be seen, down to a discharge potential of 1120 volts. An H_{31} group is to be seen and the maximum for these are plotted.

Fig. 2.



Doppler displacement Hγ.

Comparison with the Measurements of Krefft, Lion and Ney Valerius.

It can be shown that our locating of the transition potential does not go contrary to their findings. Our values bring the necessary clarification to the findings of the different workers. Where Krefft's values do not reach up to the H_{21} line they are to be attributed to insufficient exposure. The same holds good in the lower voltage region where some of the values do not reach up to the line for voltages up to 5500.

Significance of the Transition Potential.

The Balmer lines are emitted by the neutral atoms, hence we should expect that whatever cause that will enhance ionization should also show itself in the diminution of the intensity of light emitted by the neutral atoms, as the number of neutral atoms will diminish. The diminution may be maximum at a particular velocity of the neutral atoms. Our suggestion is that the bombardment by the high speed neutral atom of the canal ray against molecules of gas at rest in the chamber should lead to maximum ionization of the stream of neutral atoms, at the transition potential.

That with H_γ the transition potential of 3500 volts is smaller than that with H_β is in keeping with our idea that a neutral atom excited to emit H_γ, and thus with its electron in the energy level 5 can more easily be disturbed from emitting the excited light than with H_β for which the excited atom would have its electron in level 4 and therefore more firmly attached to the systems. In other words, it would require less energy to ionize an atom excited to H_γ than that to H_β. According to this view, the transition potential for H_δ and H_ε, etc. should progressively diminish while it should be higher for H_α. But the highest value so found cannot be greater than that which would be required to completely knock out of the atom the electron from its initial lowest level.

According to F. Goldman⁽⁴⁾ and Bartels⁽⁵⁾, from ionization experiments with homogeneous positive rays, a potential of 7000 volts gives the largest discharge cross-section. This is also the value that we should expect out of the theory of Joos and Kuhlenkampff⁽⁶⁾ for atomic collisions of positive rays, namely

$$V=\tfrac{1}{4}\frac{M}{m}V',$$

where V' is the ionization potential of an atom of mass M, and m is the mass of the electron. With V' for hydrogen as 13·5 volts, V would be 6200 volts. It is possible to take an higher value for V' by including the dissociation energy of the hydrogen molecule. Joos and Kuhlenkampff's theory postulates a high speed atom colliding against an electron of another atom and transferring the ionization energy to it. By assuming that the parent atom of the electron formed part of a molecule and that the collision results in the dissociation of the atom from the molecule and ionization in addition, we can take for V' the ionization potential of the molecule. In the case of hydrogen this value is 15·9 volts.

For our present purpose we can evidently take the electron to which energy is transferred to be that of the neutral high speed atom of the canal ray, from considerations of reciprocity. The dissociation energy will now refer to the molecule struck by the high speed atom. The critical potential could now arise to 7280 volts. A mean of the two values, namely 6740 volts would be a satisfactory value. The value for H_α may then be expected not to exceed 6740 volts. This has actually been found to be the case in the measurements to be reported separately⁽⁷⁾.

The Relative Prominences of H_{11}, H_{21} *and* H_{31}.

The relative prominences of these come out slightly differently with H_β than with H_γ. Still, the following may be taken as the general conclusions. The estimates made by visually matching intensities in strips exposed to different times are shown in Table I., see Pl. VI.

TABLE I.

Potential V in kilovolts	Intensities					
	Pressure in mm. p.	$\frac{H_{21}}{H_{11}}$	$\frac{H_{31}}{H_{11}}$	$\frac{H_{31}}{H_{21}}$	$V.\,p^2.\,10^2$	
0·720	0·17	0	0·025	—	4·92	
1·120	0·13	0	1·300	—	2·20	
1·620	0·10	1	5·000	5·0	1·62	
2·000	0·09	1	17·000	17·0	1·62	
3·000	0·072	3	20·000	6·7	1·55	
4·000	0·062	3	20·000	6·7	1·53	
5·000	0·055	12	36·000	3·0	1·51	
6·000	0·049	—	—	—	1·44	
7·000	0·042	—	—	—	1·23	= constant
8·000	0·040	—	—	—	1·28	
10·000	0·036	—	—	—	1·29	
12·000	0·032	—	—	—	1·22	
14·000	0·029	—	—	—	1·18	
16·000	0·027	—	—	1·0	1·17	
20·000	0·026	—	—	—	1·35	
24·000	0·023	—	—	—	1·27	
28·000	0·020	—	—	—	1·12	
30·000	0·019	—	—	—	1·08	

The following further results may be noted :—

(1) At 15 kv. the intensities of H_{31} become equal to that of H_{21} (pressure 0·028 mm.).

(2) Between 15 and 30 kv. (pressures between 0·028 and 0·019 mm.) H_{21} group becomes very sharp, both the upper and lower limits particulary with H_β with increase of potential.

(3) The maximum of H_{31} in contrast to what is said about H_{21} in (2), becomes indefinite at potentials above 15 kv.

(4) The portions having displacements less than corresponding to H_{31} above 15 kv. while becoming less prominent with increase of potential have nearly the same intensity as H_{21}.

(5) Immediately on starting discharge displacements due to $H_{13,1}$ $(CH^+=C^++H)$ are strong and sharply bounded on the higher potential end ; it gradually disappears after the discharge is run for about an hour or two.

(6) H_{11} strip for H_γ attains maximum intensity between 1120 and 1620 volts, say at 1370 . The corresponding value for H_β, while being below 2000, is slightly higher.

(7) H_{21} strip attains its maximum intensity at 20 kv. (0·026 mm. pressure).

(8) H_{31} strip gets its highest intensity at 5 kv. (pr.=0·055).

(9) There seems to be a strip H_{41} between 1 and 3 kv.

The finding in (2) is remarkable and the effect is certainly pronounced and unmistakable. It follows from it that, with increase of the discharge potential above 15 kv., the contribution of charged molecules from the negative glow hurling themselves down into the cathode fall discharge space and passing through without discharge becomes prominent. It is not clear why the range of velocities shown should be sharply defined. With H_γ the effect is less pronounced but the upper limit of displacement is still very sharply defined. Therefore, H_α may be expected to show this effect within still narrower range than with H_β. The narrowness of the range and the sharpness of the upper and lower limits of H_{11} both with H_β and H_γ at 720 volts seems to be a phenomenon of a similar type.

The explanation for the finding in (3) is that, due to the existence of the critical potential that has been found by us and discussed earlier, the ionization effect will have to be maximum for atoms of a particular group of velocities, the effect with lines of higher frequencies become pronounced with lower velocities and naturally the sharpness of the limits of the displacements will depend upon the effectiveness of the critical potential. The critical potential found from a study of the H_{11} group is found to be affecting the intensities of light emission of other groups. In fact we should take result (3) as a confirmation of the idea of the critical potential.

Our results may now be compared with the findings of Harnwell (8), Batho and Dempster (9) and Anna I. McPherson (10). (1) Harnwell's result is that H_{31} goes on increasing in intensity with increase of pressure in the range from zero to the highest pressure observed by him, namely 0·10 mm., is corroborated for range 0·018 to 0·055 mm. Above this pressure H_{31} will, according to us, decrease to zero by 0·18 mm. pressure. According to McPherson H_{21} was absent at 0·14 mm. while at 0·05 mm. it could be just noticed. We could find H_{21} at 1620 volts corresponding to a pressure of 0·1 mm. According to McPherson and Batho and Dempster the intensity of H_{21} increased steadily while those of H_{31} and H_{11} decreased with diminution of pressure, and that the intensities of H_{31} and H_{21} become equal at a pressure between 0·013 and 0·026 mm. Our value of pressure for equal intensities of these, as given in (1) above, is at a pressure of 0·028 mm. Below this value, H_{21} is definitely more intense than H_{31} with H_β ; with H_γ the pressure for equality of intensities is 0·026 mm. (20 kv.). Harnwell's value is 0·021 mm. For the disappearance of H_{21} our highest value is 0·1 mm., agrees with that of Harnwell, namely of 0·09 mm. According to latter, H_{11} is maximum at 0·08 mm., according to us it is at 0·12 mm. (see (6) above).

H_{41} was suggested by R. Döppel and apparently confirmed by F. Ney-Valerius was not found by Anna McPherson. It appears to be present in our photographs between discharge potentials of 1 and 3 kv. However, an investigation using apparatus giving higher dispersion than we have used will be necessary to be sure on this point.

The Doppler Effect of the 4358 Å *line of Mercury.*

Due to a slight amount of the vapour of mercury in the hydrogen discharge the 4358 line of mercury comes out in our plates near H_γ. This line comes very near the H_γ line of hydrogen. Its Doppler displacement for discharge voltages from 5 to 30 kv. as mean of measurements in four different plates show uniformly more displacements than the theoretical values calculated for a singly charged Hg atom. The displacements were measured from the centre of the undisplaced line. If we should, however, subtract the half width of the undisplaced line from these, we get displacements which correctly fall on the theoretical line.

Stark, Hermann and Kinoshita⁽¹¹⁾, who studied the Doppler displacements of 3650 A and 3654 A in the light of a mixture of hydrogen and mercury vapour, concluded that the mercury atom should have had acceleration as a doubly charged atom. This conclusion is contrary to our finding of a singly charged atom.

Undisplaced Balmer Lines and Molecular Lines under Constant Discharge Current.

The intensities of the H_β and H_γ undisplaced lines as well as of the moleculer lines have been studied in great detail by one of us (B.D.)⁽¹²⁾. The actual energy E of the canal ray beam after it has passed through the canal was measured by a thermo-junction of iron-constantan as well as the integrated intensity J of the line. It was found that the following relation:

$$\frac{J}{E\,.\,p}=V\,.\,A\,.\,e^{BV\log p} \qquad \text{(i)}$$

was true between values of pressure p varying from 0·001 mm. and 0·2 mm. and discharge potential V varying from 13 kv. to 30 kv.; A and B are constants. For molecular lines

$$\frac{J}{E\,.\,p} \qquad \text{(ii)}$$

was constant. Our present conditions of measurements of experiment are different in that we do not measure E nor do we vary p in the observation chamber arbitrarily behind the cathode. We keep p the same in the observation chamber as well as in the cathode fall space for any one value of V, and we measure J for a point almost immediately at the back of the cathode, and the current in the discharge is kept the same under all the discharge potentials.

The intensities of the undisplaced Balmer lines H_β and H_γ could be studied in our experimental plates 26, 27 and 28 and those of the molecular lines also in 29 and we find remarkably that the intensities of all these are nearly the same in all the spectra taken for the same time of exposure between discharge potentials 7 kv. to 30 kv.

If we take J to represent the intensity of a molecular line, it must naturally be proportional to the number of particles in the canal ray, *i.e.* to the pressure in the discharge space and also to the number of particles in the path of the canal ray, *i.e.* again proportional to pressure, so that J will be proportional to p^2. Secondly, the higher the discharge potential the greater will be the energy in the individual particles. So obviously

$$\frac{J}{V \cdot p^2} = \text{a constant}, \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(iii)}$$

as now J is found to be constant we should also expect $V \cdot p^2$ to be constant between 7 kv. and 30 kv. This is indeed found to be the case, as can be seen from Table I. For potentials below 7000 volts the intensities diminish with diminution of voltage although Vp^2 is found to increase. The reason for this is to be sought in the fact that in this region the cross-section of the beam in the cathode fall space is very much greater than that of the canal so that only a small portion of the total light energy streams through in canal rays, while at about 5 to 7 kv. the cross-section becomes nearly comparable to that of the canal and nearly the whole light energy can stream through.

There is no contradiction between (ii) and (iii) because E in (ii) is evidently proportional to $V \cdot p$, so that the two expressions become identical. We can therefore take that the molecular lines and undisplaced Balmer lines are having intensities independent of the potential in the discharge space between 7 kv. and 30 kv. as a fairly good approximation. From this it also follows that as far as light emission by impact is concerned both the charged as well as the uncharged particles are equally effective.

SUMMARY.

Doppler effect of H_β and H_γ lines of hydrogen canal rays has been photographed using a Steinheil three prism glass spectrograph of high light gathering power, *f*/3. Effort has been made, by giving various times of exposures, to get the maximum displacement possible and it has been found that the maximum displacements so found are proportional to the velocities of hydrogen atoms which are able to traverse the whole of the discharge space in a charged condition as a singly charged atom. This is true only provided the discharge potential is below a certain value called the critical potential, which is 5500 volts for H_β and 3500 volts for H_γ. For values of potentials between the critical potential and twice its value, the Doppler displacement is constant and independent of the potential. Beyond the upper limit of the critical region up to the highest potential of 30,000 volts, the displacements are those which would correspond to the

velocity of an atom which was accelerated in the whole of the cathode fall space as a charged molecule and which after leaving that space dissociated into a charged and an uncharged atom. The explanation of this remarkable fact of the existence of a critical potential is due to the existence of the ionization potential for atomic collisions.

The relative prominences in the Doppler effect due to H_{31}, H_{21} and H_{11} atoms have been studied. Below 1000 volts only H_{11} is present and it is maximum at 1000; at about 5000, H_{31} attains its maximum intensity and at 15,000 H_{31} and H_{21} are of equal intensities. With H_β a very interesting and remarkable sharpening of the H_{21} strip has been found, while with H_γ it is present though not so pronouncedly. A remarkable relation, namely that the intensities of the lines of the molecular spectra and of the undisplaced Balmer lines are constant between discharge potentials 7000 and 30,000 volts has been found and suitable explanation has been given for it as being due to the opposing effects of pressure and voltage balancing each other.

The Doppler displacement of the 4358 Å line of mercury has been studied and it is shown that it is due to a singly charged mercury atom that is accelerated in the cathode fall.

The continuation of this work to other lines of the Balmer series and to resonance line of H_g will be reported in a later paper.

Department of Physics;
University, Benares.
December 18th, 1946.

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CXIV. *On the Emission Spectrum of Chlorine under Conditions Productive of Joshi-Effect.*

By P. G. Deo, Ph.D., D.Sc., J.K. Fellow, College of Science, Benares Hindu University (India) *.

[Plate VII.]

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Summary.

Radiation of chlorine subjected to the silent electric discharge (which produces *Joshi-Effect* to a 70 per cent current decrease) has been studied spectroscopically in the visible region. An exposure of 300 hours was needed. The spectrum consists almost entirely of faint bands degraded towards the red. An intensity maximum is observed in the region 4900–4400 A. and 5800–5500 A. A number of new bands in the red have been observed. It is considered that the emission cannot be attributed to a neutral chlorine molecule, but is due to Cl_2^+. Absence of the atomic lines and of an emission continuum characteristic of atomic recombinations suggests that, *under the conditions appropriate to the production of a large Joshi-Effect*, reactions involving atomic chlorine occur predominantly on the container walls.

The production of a new *light-effect* observed by Prof. Joshi (1), viz. an instantaneous and reversible decrease (2) under light of the electrical conductivity of chlorine subjected to silent discharge in a Siemens' ozonizer, as also some of the chief features of this phenomenon have been reported in earlier papers (1), (2), (3), (4), (5) from these laboratories. Under optimum conditions (3), this decrease of the current on irradiation in the visible part of the spectrum is found to be as high as 93 per cent. The corresponding spectrum of the excited gas appeared to be of interest from the very beginning of work on this phenomenon (4). It is significant to observe that the intensity of the chlorine spectrum under conditions where the *Joshi-Effect* is most pronounced is exceedingly feeble; apparently, an exposure of more than a hundred hours is necessary to produce a reasonable impression on a photographic plate (1). Numerous attempts to investigate the comparative emission spectra of chlorine with and without irradiation under identical conditions and durations of exposure, and appropriate to the production of a large *Joshi-Effect* (*vide infra*) have so far failed, due mainly to the difficulties in maintaining constant experimental conditions over so long a period (1), (4). Even with a 300-hour exposure and a slit width of 0·1 mm., the spectrum (Pl. VII.) is found to be extremely faint and insufficiently sharp for an adequate spectroscopic analysis. Besides their possible significance for the mechanism of this phenomenon, the results now reported (Table I., Pl. VII.) are also of interest, as practically no information is available in the literature on the chlorine spectrum under the above type of excitation.

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* Communicated by Prof. S. S. Joshi, F.R.I.C., D.Sc. (London).

Experimental.

Chlorine prepared by the electrolysis of concentrated hydrochloric acid, dried and purified as described elsewhere ⁽⁵⁾, was admitted to a Siemens'-type glass ozonizer at a pressure of about 100 mm. Hg. At this pressure, under 13–14 kV. and 50 cycles frequency, the *Joshi-Effect* even in the visible corresponded to about 70 per cent. decrease of current observed in dark. The spectrum was excited by applying the desired potential to the ozonizer with a high-tension transformer; the potential was kept constant within ±1 per cent. by hand regulation of a resistance introduced in the primary of the transformer. Larger exciting potentials increased the intensity of emission and would have reduced sensibly the corresponding duration of exposure; this advantage was, however, outweighed by the circumstance that the life of the discharge tube was endangered; but what is more important, the corresponding percentage *Joshi-Effect* diminished at the high potential. The ozonizer was cooled constantly under fans to minimize the heat effect under so prolonged a discharge. Precautions were also taken against stray light, such as, *e.g.*, that due to any accidental sparking in the fans from falling on the slit of the spectroscope.

"End-on" position of the discharge tube was used and the spectrum (Pl. VII.) was taken with a Hilger's constant deviation glass instrument on a Kodak super-panchromatic tri-X cut film with an exposure of 300 hours. The linear dispersion of the spectrograph was 60 A./mm. at 590 mμ, and 15 A./mm. at 410 mμ. The bandheads have been measured with reference to the copper arc lines. Some of these have been identified with those reported by Ota and Uchida ⁽¹⁰⁾, or Elliott and Cameron ⁽¹³⁾ (Table I.). Due to their faint and ill-defined nature accuracy of the measurements is not considered more than 3–4 A. No analysis of the measured bands is, therefore, undertaken.

Relative variations of the spectral intensity as recorded photographically but uncorrected for any spectral sensitivity of the plate were observed (see fig. 1) with a simple photometer designed specially for the purpose. In essence, the arrangement consisted of a Kipp's 377 linear thermopile, a plate-carrying-stage movable in the horizontal and vertical directions by means of micrometer screws. A parallel beam of light passing through a fine slit was made to fall upon the plate and the intensity of the emergent beam was measured with the thermopile connected to a low-resistance Broca galvanometer. The (relative) intensity variations observed with this arrangement showed a satisfactory agreement with the microphotometric records for a number of spectra obtained with a standard instrument, viz., a Zeiss recording microphotometer.

TABLE I.

Wave-lengths of the Emission Bands of Chlorine under Conditions Productive of the *Joshi-Effect.*

Ota and Uchida A.U.	Elliott and Cameron A.U.	Present measurements			Remarks
		A.U.	Visual in-tensity	Wave numbers in vacuum cm.$^{-1}$	
	4159·2	4151	1½	24084	
		4219	1	23696	
4252·6	4253·4	4254	1½	23500	
4258·9	4258·8	4260	1	23468	
4292·2	4291·5	4289	3	23309	
	4297·1	4299	2½	23255	
4316·0	4316·0	4314	2	23174	
		4331	2½	23083	
4345·7	4348·5	4345	2½	23009	
		4395	3	22747	
	4409·6	4412	5	22659	
	4437·4	4433	4½	22552	
		4451	3½	22461	
	4462·5	4467	5	22480	
4487·5	4491·0	4490	2	22265	
4506·4	4506·0	4504	4	22196	
	4521·3	4516	6	22136	
		4537	4	22035	
	4552·6	4554	3	21952	
	4581·7	4582	8	21818	
4598·6		4600	2	21733	
4613·5	4613·7	4612	3½	21676	
	4637·1	4632	2	21583	
	4647·0	4648	8	21509	
		4673	2	21394	
	4684·6	4688	7	21325	
4715·4	4715·3	4718	8	21190	
4750·8	4751·1	4748	4½	21056	
4767·8		4769	3	20963	
	4786·7	4786	4	20888	
	4799·7	4802	2	20819	
		4818	3	20750	
	4826·9	4830	9	20698	
		4845	5	20634	
		4882	2	20478	
	4896·7	4894	4½	20427	
	4919·0	4912	4	20353	
	4942·2	4935	6½	20258	
	4969·2	4964	3	20180	
		4987	3	20047	

TABLE I. (*cont.*)

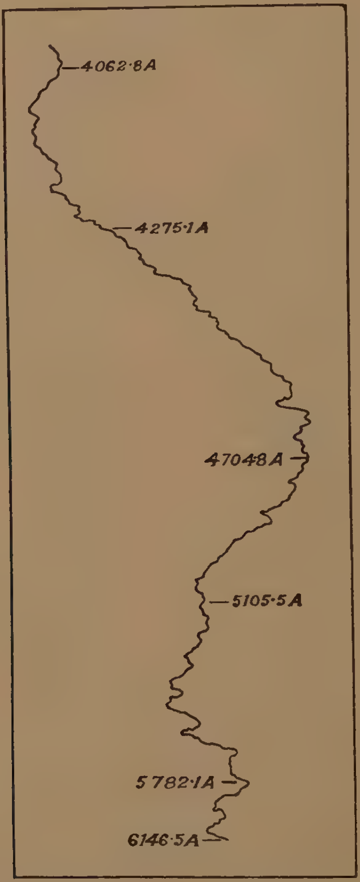
Ota and Uchida A.U.	Elliott and Cameron A.U.	Present measurements A.U.	Present measurements Visual in-tensity	Present measurements Wave numbers in vacuum cm.$^{-1}$	Remarks
	5022·9	5024	3½	19899	
	5049·0	5052	1½	19789	
		5069	2½	19722	
		5111	1	19560	
	5126·7	5125	3	19507	
	5154·1	5148	2	19420	
		5188*	2½	19270	
	5211·6	5215	2	19170	Coincides with line 5270
		5270*		18970	
		5309	2	18831	
		5357*	3	18662	
	5390·3	5385	2	18565	Coincides with line 5452
		5452*		18337	
		5466	1½	18290	
		5486	4	18223	
		5506	3	18157	
		5522	2	18104	
		5577*	4	17926	
		5600	2½	17852	
		5649*	3½	17697	
		5678	2	17607	
		5732	3	17441	
		5774*	6	17314	
		5832	1½	17142	
		5874*	1½	17019	

Results and Discussion.

It is seen that the spectrum (Pl. VII.) extends from about 6500 A. to about 4100 A. and consists almost entirely of faint bands degraded to the red. The relative intensity of the emission bands can be seen from the photometer record in fig. 1. Roughly, the spectrum has two intensity maxima : one in the region 4900–4400 A. and the other within 5800–5500 A. There are but few atomic lines in the spectrum. The bands near the red end are comparatively more diffuse and do not show marked heads. Whether this can be attributed to the comparatively high gas pressure (about 100 mm.) which is necessary for the production of the *Joshi-Effect* requires further investigation.

The absorption spectrum (6) of chlorine shows a continuum in the region 5000–2300 A. with a pronounced maximum at 3300 A.; there is also a feeble banded absorption between 6220–4785 A. In their studies of the thermo-luminescence of halogens, Kondratjew and Leipunski (7) observed that the emission from chlorine at 1000° C. corresponded with its region of continuous absorption. Kitagawa (8) noticed about 18 bands, between 6500–5590 A. in hydrogen-chlorine flame; these were identified by him with the corresponding absorption bands; it would appear that this emission of chlorine is attributable to the neutral molecule.

Fig. 1



It is significant that in the spectrum of chlorine excited under the silent electric discharge (Pl. VII.), a strong band emission occurs in the region 4800–4200 A., where there are practically no absorption bands. Kimura and Fukuda (9), working with a chlorine-filled discharge tube having internal metallic electrodes, reported that the emission and the absorption spectra were not "exactly complementary". Under essentially similar conditions of excitation, Ota and Uchida (10) classified 58 emission bands between 4870–3920 A. into three systems. No apparent connection was observed between the states determined by their analysis of the emission spectrum and those shown by Kuhn's (11) and Nakamura's (12) analysis of the absorption spectrum of chlorine. Ota and Uchida suggested, therefore, that the significant emitter cannot be a neutral chlorine molecule. Recently, Elliott and Cameron (13) measured the emission bands of chlorine under high-frequency excitation with external electrodes over a much wider spectral region. They classified the various bands mainly into two sub-systems, and attributed both to the electronic transition $^2\Pi$–$^2\Pi$. The rotational analysis of the bands showed conclusively that the rotational quantum number J is half-integral, which is a characteristic of a double state (even multiplicity); the state of a neutral chlorine molecule being of odd multiplicity. Elliott and Cameron (14) concluded, therefore, that the emitter is a Cl_2^+ molecule, a possibility also suggested by Mulliken (15).

It is interesting to compare the spectrum now obtained with that reported by Elliott and Cameron (13): "The discrete bands, degraded to the red, extend from about 3400 A. to at least 5500 A. A characteristic feature of the spectrum when viewed under low dispersion (*e. g.* a pocket spectroscope) is a series of strong bands extending from about 5100 A towards the red. These bands appear to be grouped in pairs at intervals of (*very roughly*) 300 cm.$^{-1}$, the members of a pair being about 140 cm.$^{-1}$ apart. . . . They show no marked heads, under higher dispersion become very indefinite, but may readily be seen on a microphotometer record. The region where these bands occur, particularly towards the red, is rich in atomic lines, which make their investigation difficult. While they have the appearance of diffuse bands, the evidence at present available is insufficient to give certainty on this point. . . . All attempts to obtain spectrum free from atomic lines have been unsuccessful."

In view of these remarks it is significant that the spectrum of chlorine (Pl. VII.) produced under the silent electric discharge should be practically free from the atomic lines. In the region 5900–4200 A., there are only three lines of any measurable intensity, viz., 5452 A., 5270 A. and 4356 A., the most prominent being 5270 A. None of these seems to belong to chlorine. The origin of 5270 A. is uncertain; 4356 A., and therefore perhaps 5452 A., may be identified with mercury lines 4358 A. and 5461 A. respectively. As many as 15 bands are observed in the region 6500–5900 A; their wave-lengths have not been measured. Some of these bands appear like lines, but due to their extremely faint and diffuse nature, and the insufficient dispersion of the instrument in this region, it is difficult to ascertain whether they are lines actually.

In confirmation with the observations of Elliott and Cameron, a number of new bands (shown by an asterisk in Table I.), most of them grouped in pairs, extending from 5100 A. towards the red, have been observed in the present spectrum. The frequency difference of these bands and that between the members of a pair are not, however, in good agreement with the corresponding values 300 cm.$^{-1}$ and 140 cm.$^{-1}$ respectively, mentioned by Elliott and Cameron; but it must be remembered that their observations are admittedly very rough.

Rodebush and Klingelhoefer (16) reported appreciable formation of atomic chlorine when the gas at a pressure of 0·2–0·7 mm. was subjected to an electrodeless high-frequency discharge. These conditions are essentially similar to those adopted by Elliott and Cameron. This explains partly their failure to free the spectrum from atomic lines. It is interesting to observe that there is no spectroscopic evidence, such as, *e. g.* the occurrence of atomic lines or an emission continuum characteristic of atomic recombination, for the formation of atomic chlorine under the silent discharge productive of a large *Joshi-Effect*. It is not unlikely that reactions involving atomic chlorine may occur chiefly on the walls of the reaction vessel. It is known, for instance, that the recombination of chlorine atoms is favoured by a three-body collision and occurs predominantly on the container walls. Rodebush and Klingelhoefer (16) observed that solid surfaces, *e. g.* glass, catalyze markedly the process of recombination. That the container walls play an important rôle is also suggested by the general observation (1), (17) that the magnitude of the *Joshi-Effect* is affected sensibly by the nature of the wall material.

Grateful thanks of the author are due to Mr. S. V. Raj Urs for his help in constructing the photometer, and to Prof. S. S. Joshi and Prof. R. K. Asundi for kind help and valuable guidance.

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CXV. *On the Theory of Seismometers.*—II.

By C. G. Pendse, M.A., Ph.D. *

[Received April 24, 1947.]

Introduction.

In a previous paper † the author obtained the exact equation of motion of a seismometer, taking into account the fact that the motion recorded by a seismometer is relative to a frame determined by the instrument, whereas the system of forces in question refers to axes " fixed in the earth " relative to which the frame determined by the instrument also moves ; that paper contained the discussion of an important particular case applying mainly to horizontal seismometers.

The object of this paper is to discuss another important particular case applicable to vertical seismometers. Also a correction is made here of some mechanical (copying) errors in the previous paper. It may be mentioned that these mechanical (copying) errors do not modify or affect the discussion or the conclusions in the previous paper.

This paper consists of three sections. In § 1 we state the equations (A), (B), (C) and (D) of the previous paper in the corrected form and indicate the corrections or changes. In § 2 we consider the question of the different equilibrium positions of a seismometer. In § 3 we discuss the case which applies to a vertical seismometer.

We mention below, for the sake of convenience, the notation and symbols used in the previous paper.

m is the mass of the body, G its centre of mass, O the foot of the perpendicular from G to the axis about which the body can turn and I the moment of inertia of the body about the axis, l being the distance OG. OX, OY, OZ are axes fixed in the body, OZ being along the axis of rotation, OX being along OG and OY being such that the frame OXYZ is right-handed. Oxyz is another right-handed frame through O such that Oz is along the upwards vertical, Ox, Oy being fixed relative to the earth in the horizontal plane. ϕ is the angle between the planes Ozx and OZz, θ the angle ZOz and ψ the angle between the planes ZOz and ZOX. a_x, a_y, a_z are the components of the acceleration of O relative to axes fixed in the earth. M is the moment of the forces, if any, on the system besides gravity about OZ. We assume that $M=-\lambda\dot{\psi}-\mu\psi$, where λ and μ are constants, and we put

$$\frac{\lambda}{I}=2A, \quad \frac{ml}{I}=B, \quad \frac{\mu}{I}=C, \quad n^2=Bg\sin\theta+C.$$

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* Communicated by the Author.

† C. G. Pendse, " On the Theory of Seismometers," Phil. Mag. (7) xxxv. pp. 706–713 (1944).

As a particular case we assume that

$$a_x = A_x \sin pt, \quad a_y = A_y \sin pt, \quad a_z = A_z \sin pt,$$

where A_x, A_y, A_z and p are constants ; and we put

$$H = B(A_z \sin\theta - A_x \cos\theta).$$

§ 1. *The Different Forms of the Equation of Motion of a Seismometer.*

The equations of motion of a seismometer *, in the successive stages of particularization, are :—

$$I\ddot{\psi} = M - ml \begin{vmatrix} \sin\theta\cos\phi, & \sin\theta\sin\phi, & \cos\theta \\ \cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi, & \cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi, & -\cos\psi\sin\theta \\ a_x, & a_y, & a_z + g \end{vmatrix} \quad . \quad . \quad . \quad . \quad (A)$$

$$I\ddot{\psi} = M - ml \begin{vmatrix} \sin\theta, & 0, & \cos\theta \\ \cos\theta\cos\psi, & \sin\psi, & -\cos\psi\sin\theta \\ a_x, & a_y, & a_z + g \end{vmatrix}$$

$$= M - ml[(g + a_z)\sin\theta\sin\psi + a_y\cos\psi - a_x\cos\theta\sin\psi]. \quad . \quad . \quad . \quad (B)$$

$$\ddot{\psi} + 2A\dot{\psi} + [n^2 + B(a_z\sin\theta - a_x\cos\theta)]\psi = -Ba_y. \quad . \quad . \quad . \quad (C)$$

$$\ddot{\psi} + 2A\dot{\psi} + [n^2 + H\sin pt]\psi = -BA_y\sin pt. \quad . \quad . \quad . \quad . \quad . \quad (D)$$

(A) is the most general equation. (B) is a simplified general equation obtained from (A) by putting $\phi=0$ in (A) without any real loss of generality. (C) is derived from (B) for small motions for the case in which $\psi=0$ is a position of equilibrium. (D) is derived from (C) for the particular case † in which $a_x = A_x \sin pt$, $a_y = A_y \sin pt$, $a_z = A_z \sin pt$.

§ 2. *Suitable Values of ψ for the Equilibrium Position.*

It is seen from (B) that, when the frame of the instrument is undisturbed relative to axes fixed in the earth, any value of ψ can be made an equilibrium value by applying a suitable torque about the axis of rotation.

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* The corrections which have been incorporated in (A), (B), (C) and (D) above are :—In (A), the coefficient of $\sin\psi$ in the element in the first column and the second row has to be $-\sin\phi$ instead of $-\sin\theta$. In (B), the coefficient of $\cos\psi$ in the square bracket is a_y instead of a_z. In each of (C) and (D), there is a minus sign on the right-hand side.

† In the equations (D′), (D″), (P), etc., in the previous paper there should have been $-A_y$ for A_y ; this change of sign of A_y does not modify or affect the discussion in § 4 of that paper.

But an inspection of (B) also shows that, when θ differs from 0 or $\pi/2$ and when the prescribed equilibrium value of ψ differs from zero, the equation of small motion about that equilibrium position contains all the quantities a_x, a_y, a_z in the free first-order term (corresponding to $-\mathrm{B}a_y$ on the right-hand side of (C)); and, in this case, the motion of the seismometer will not give a clear indication of only one component of the ground movement, even as far as the first-order terms are concerned.

The previous paper contains a full discussion of the case in which $\psi=0$ is a position of equilibrium, which is stable for all values of θ in the range $0\leqslant\theta\leqslant\pi/2$; when $\theta=\pi/2$, the system simply becomes a vertical pendulum. In any event the first-order term in the particular integral of (D) depends on A_y only.

The only case left out is the one in which $\theta=\pi/2$ and in which the equilibrium value of ψ differs from zero. When $\theta=\pi/2$, the equation (B) reduces to

$$\mathrm{I}\ddot{\psi}=\mathrm{M}-ml[(g+a_z)\sin\psi+a_y\cos\psi]. \quad . \quad . \quad . \quad . \quad (\mathrm{C}')$$

By applying to (C′) reasoning similar to the above, it is seen that $\psi=\pi/2$ is the only useful equilibrium value for the indication of the vertical component; for this equilibrium position $\mathrm{M}=mlg$, as is otherwise obvious.

§3. *The Equation for a Vertical Seismometer.*

In this case $\theta=\pi/2$, and $\psi=\pi/2$ is a position of equilibrium. As in the previous paper we put

$$\psi=\frac{\pi}{2}+\psi_1, \quad \mathrm{M}=mgl-\lambda\dot{\psi}_1-\mu\psi_1,$$

in order to allow for damping and elastic forces. The equation (C′) becomes

$$\mathrm{I}\ddot{\psi}_1=mgl-\lambda\dot{\psi}_1-\mu\psi_1-ml[(g+a_z)\cos\psi_1-a_y\sin\psi_1],$$

i. e. $$\mathrm{I}\ddot{\psi}_1+\lambda\dot{\psi}_1+\mu\psi_1=2mgl\sin^2\tfrac{1}{2}\psi_1-mla_z\cos\psi_1+mla_y\sin\psi_1,$$

i. e. $$\ddot{\psi}_1+2\mathrm{A}\dot{\psi}_1+\mathrm{C}\psi_1=\mathrm{B}(2g\sin^2\tfrac{1}{2}\psi_1-a_z\cos\psi_1+a_y\sin\psi_1). \quad . \quad . \quad (1)$$

For small values of ψ_1, (1) becomes

$$\ddot{\psi}_1+2\mathrm{A}\dot{\psi}_1+(\mathrm{C}-\mathrm{B}a_y)\psi_1=-\mathrm{B}a_z. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

A discussion of (2) is not possible without a knowledge of a_x, a_y, a_z as functions of t. If we assume, as in the previous paper, that

$$a_x=\mathrm{A}_x\sin pt, \quad a_y=\mathrm{A}_y\sin pt, \quad a_z=\mathrm{A}_z\sin pt,$$

the equation (2) becomes

$$\ddot{\psi}_1+2\mathrm{A}\dot{\psi}_1+(\mathrm{C}-\mathrm{B}\mathrm{A}_y\sin pt)\psi_1=-\mathrm{B}\mathrm{A}_z\sin pt. \quad . \quad . \quad . \quad . \quad (3)$$

(3) is of the same type as (D) of the previous paper, and its solution is on the same lines as that of (D). All the results of the previous paper are repeated in the case of the vertical seismometer.

There is, however, in this case a situation which does not arise in the case studied in the previous paper. In deriving (2) from (1) powers of ψ_1 higher than the first have been neglected. If, when a_x, a_y, a_z or A_x, A_y, A_z are small in comparison with g, it is not permissible to ignore $g\psi_1^2$, the equation (1) would reduce to the form

$$\ddot{\psi}_1+2A\dot{\psi}_1+C\psi_1=B(\tfrac{1}{2}g\psi_1^2-a_z+a_y\psi_1),$$

i. e.

$$\ddot{\psi}_1+2A\dot{\psi}_1+(C-Ba_y)\psi_1=B(\tfrac{1}{2}g\psi_1^2-a_z). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

(4) becomes, in the particular case of simple harmonic motion of O,

$$\ddot{\psi}_1+2A\dot{\psi}_1+(C-BA_y\sin pt)\psi_1=B(\tfrac{1}{2}g\psi_1^2-A_z\sin pt). \quad . \quad . \qquad (5)$$

(5) is far more complicated than (3) and its solution is not attempted here.

CXVI. *Note on a Paper by N. J. Durant.*

By W. M. Stone
(Iowa State College) *.

[Received January 20, 1947.]

Abstract.

A generalization of the Laplace transformation is defined by means of an infinite sum. A short list of transformation pairs is included and application is made to a finite difference equation with constant coefficients discussed by N. J. Durant (Phil. Mag. xxxvi. p. 569, Aug. 1945). Further application to finite difference equations with rational coefficients is suggested.

The Generalized Laplace transformation of Samuelson † may be applied to the solution of finite difference equations in a manner exactly analogous to the application of the regular Laplace transformation to differential equations, thereby extending to difference problems the usual advantage of modern operational methods.

The generalized Laplace transformation is defined as

$$L\{F(t)\}_E=\sum_{n=0}^{\infty}\frac{F(n)}{s^{n+1}}=f(s), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

where $F(t)$ must be defined for all positive integral values of t, is of exponential order $F(t)<M\alpha^t$, $t>T$, and $|s|>\alpha>1$. E denotes the shifting operator $E^mF(t)=F(t+m)$. Application of the formula for summation by parts *

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* Communicated by the Author.

† Bull. Amer. Math. Soc. Abstract 52-3-86 (March, 1946).

$$\sum_a^b u(n)\Delta v(n)=u(n)v(n)\Big|_a^b-\sum_a^b v(n+1)\Delta u(n), \quad . \quad . \quad . \quad (2)$$

where $\Delta=E-1$, to (1) yields

$$f(s)=\sum_{n=0}^{\infty}\frac{F(n)}{s^{n+1}}$$

$$=\frac{F(n)}{1-s}\frac{1}{s^n}\Big|_0^{\infty}-\frac{1}{1-s}\sum_{n=0}^{\infty}\frac{(E-1)F(n)}{s^{n+1}}; \quad . \quad . \quad . \quad . \quad . \quad (3)$$

therefore

$$\sum_{n=0}\frac{E\,.\,F(n)}{s^{n+1}}=sf(s)-F(0)$$

$$=L\{E\,.\,F(t)\}_E. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This may be generalized at once to

$$L\{E^m\,.\,F(t)\}_E=s^mf(s)-\sum_{j=0}^{m-1}s^{m-1-j}F(j), \quad . \quad . \quad . \quad . \quad . \quad (5)$$

or to

$$L\{E^{-m}\,.\,F(t)\}_E=s^{-m}f(s)+\sum_{j=1}^{m}s^{-m-1+j}F(-j). \quad . \quad . \quad . \quad (6)$$

Equation (5) is useful for the evaluation of transforms. Some of the more common transform pairs are :

$F(t)$	$f(s)$	
1	$\dfrac{1}{s-1}$, ...	(7)
a^t	$\dfrac{1}{s-a}$, ...	(8)
$a^tt^{(m)}$	$\dfrac{m\,!\,a^m}{(s-a)^{m+1}}$, ...	(9)
$a^t\sin wt$	$\dfrac{a\sin w}{s^2-2as\cos w+a^2}$, ...	(10)
$a^t\cos wt$	$\dfrac{s-a\cos w}{s^2-2as\cos w+a^2}$, ...	(11)
$a^t\sinh wt$	$\dfrac{a\sinh w}{s^2-2as\cosh w+a^2}$, ...	(12)
$a^t\cosh wt$	$\dfrac{s-a\cosh w}{s^2-2as\cosh w+a^2}$, ...	(13)
$\dfrac{a^t}{t+1}$	$-\dfrac{1}{a}\log\left(1-\dfrac{a}{s}\right)$, ...	(14)

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* Milne-Thomson, 'Calculus of Finite Differences,' p. 42 (1933).

$$\frac{(a^2)^t}{t} \qquad\qquad s^{-1}e^{\frac{a^2}{s}}, \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\delta(t-m) \qquad\qquad \frac{1}{s^{m+1}}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$t^{(m)}$ is the Newton polynomial $t(t-1) \ldots\ldots (t-m+1)$ and the Dirac delta function is zero for all values of the argument different from zero, unity when the argument is zero.

As an application consider the problem of the continuous beam of N equal spans discussed by N. J. Durant *. The formula for the moments at the equally-spaced supports of the beam may be written as

$$M_{n+1}+4\frac{\phi(\alpha)}{f(\alpha)}M_n+M_{n-1}=-\frac{1}{2}\frac{\psi(\alpha)}{f(\alpha)}wL^2, \quad . \quad . \quad . \quad (17)$$

where $\phi(\alpha)$, $f(\alpha)$, $\gamma(\alpha)$ are parameters involving end and loading conditions. In operational form (17) is

$$\left[E^2+4\frac{\phi(\alpha)}{f(\alpha)}E+1\right]M=-\frac{1}{2}\frac{\psi(\alpha)}{f(\alpha)}wL^2, \quad . \quad . \quad . \quad (18)$$

which may be transformed by (5) to the algebraic equation

$$s^2m(s)-sM_0-M_1+4r[sm(s)-M_0]+m(s)=-\frac{k}{s-1}, \quad . \quad . \quad (19)$$

where

$$\frac{\phi(\alpha)}{f(\alpha)}=r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$

and

$$\frac{1}{2}\frac{\psi(\alpha)}{f(\alpha)}wL^2=k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (ii)$$

By means of the equations

$$\theta=\frac{L}{24\beta}\{\psi(\alpha)wL^2+8\phi(\alpha)M_n+4f(\alpha)M_{n+1}\}, \quad . \quad . \quad . \quad . \quad (20)$$

and

$$\phi=\frac{L}{24\beta}\{\psi(\alpha)wL^2+8\phi(\alpha)M_n+4f(\alpha)M_{n+1}\}, \quad . \quad . \quad (21)$$

which give the angles the beam makes with the horizontal to right and left respectively of the nth support, the M_1 may be evaluated in terms of the M_0 for the clamaped end condition, $\phi=\theta=0$ for $n=0$:

$$\begin{aligned} M_1&=-2\frac{\phi(\alpha)}{f(\alpha)}M_0-\frac{\psi(\alpha)}{4f(\sigma)}wL^2 \\ &=-2rM_0-\frac{k}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22) \end{aligned}$$

Solving (19) for the transform,

$$m(s)=\left[M_0+\frac{k}{2(1+2r)}\right]\frac{s+2r}{s^2+4rs+1}-\frac{k}{2(1+2r)}\frac{1}{s-1}, \quad . \quad . \quad (23)$$

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* Phil. Mag. xxxvi. p. 569 (August, 1945).

and the inverse transformations (13) and (7) yield

$$M_n=\left[M_0+\frac{k}{2(1+2r)}\right](-1)^n\cosh\gamma n-\frac{k}{2(1+2r)}, \quad . \quad . \quad (24)$$

where

$$\cosh\gamma=2r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (iii)$$

and

$$\sinh\gamma=\sqrt{4r^2-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (iv)$$

By definition of *k* and *r*,

$$\frac{k}{2(1+2r)}=\frac{3\psi(\alpha)}{f(\alpha)+2\phi(\alpha)}\,\frac{wL^2}{12}=f(w), \quad . \quad . \quad . \quad . \quad . \quad . \quad (v)$$

and finally

$$M_n=(-1)^n[M_0+f(w)]\cosh\gamma n-f(w). \quad . \quad . \quad . \quad (25)$$

For $n=N$ in (25) the bending moment at the opposite end is

$$M_N=(-1)^N[M_0+f(w)]\cosh\gamma N-f(w). \quad . \quad . \quad . \quad (26)$$

Since

$$\cosh\gamma n=\frac{1}{\sinh\gamma N}[\sinh\gamma(N-n)+\cosh\gamma N\sinh\gamma n], \quad . \quad . \quad (27)$$

equation (25) may be written as

$$M_n=(-1)^n\operatorname{cosech}\gamma N[M_0+f(w)]\{\sinh\gamma(N-n)$$
$$+\cosh\gamma N\sinh\gamma n\}-f(w), \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and for $n=N$

$$\cosh\gamma N=(-1)^N\frac{M_N+f(w)}{M_0+f(w)}. \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Substitution of (29) into (28) yields

$$M_n=(-1)^n\operatorname{cosech}\gamma N\{[M_0+f(w)]\quad\sinh\gamma(N-n)$$
$$+(-1)^N[M_N+f(w)]\sinh\gamma n\}-f(w), \quad . \quad . \quad . \quad (30)$$

which reduces to the form obtained by Durant if N is even.

This method may be extended to simultaneous systems of difference equations with the advantages that no extraneous constants are introduced and that boundary conditions are folded in at the beginning of the analysis. The formula

$$L\{t^{(m)}F(t)\}_E=\left(-\frac{d}{ds}\right)^m s^m f(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

may be employed to reduce a linear difference equation with rational coefficients to a linear differential equation with rational coefficients. It is sometimes advantageous to replace a differential system by a difference system *, particularly if discontinuities are present. The Dirac delta function may be adapted to problems involving point loads on structures, impact, momentary surges of electrical voltage and other physical phenomena of similar nature.

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* Courant, Bull. Amer. Math. Soc. xlix. no. 1, p. 14 (January, 1943).

CXVII. *Direct Determination of Stresses from the Stress Equations in some Two-Dimensional Problems of Elasticity.*—Part V. *Problems of Curvilinear Boundaries.*

By Bibhutibhusan Sen *.

[Received November 12, 1945].

Introduction.

The method of solving two-dimensional problems of elasticity with the help of Airy's stress function has been discussed in almost all text-books on the subject. Problems usually treated by this method are those connected with linear, circular and elliptic boundaries. Some other elegant methods suitable for curvilinear boundaries have been suggested by different investigators. Special mention in this connection may be made of the work, of Kolossoff (1909), Muschelisvili (1932, 1933), and Stevenson (1943) †. The object of this paper is to present a different tentative method which proves to be useful in many simple problems of plates with curvilinear boundaries. The method consists of putting stress components *explicitly* in terms of certain harmonic functions in such a way that a proper choice of one of these functions leads at once to the solution of the problem. This has been illustrated in this paper by the solution of the problem of a hole in an infinite plate under all-round uniform tension, the boundary of the hole being either (1) the inverse of an ellipse, or (2) the loop of a lemniscate, or (3) an elliptic limacon, or (4) an approximate square. To these has been added the simple problem of an elliptic hole in an infinite plate under uniform normal pressures on the inner boundary. A new method of solving problems of plates with circular boundaries has been discussed by the author (Sen, 1945) in a previous paper.

§1. *Method of Solution.*

Assuming that the plate is in a state of generalized plane stress in which the average stresses are $\overset{\smile}{xx}$, $\overset{\smile}{xy}$, $\overset{\smile}{yy}$, we get the equations of equilibrium under no body forces as

$$\frac{\partial \overset{\smile}{xx}}{\partial x}+\frac{\partial \overset{\smile}{xy}}{\partial y}=0, \quad \frac{\partial \overset{\smile}{xy}}{\partial x}+\frac{\partial \overset{\smile}{yy}}{\partial y}=0, \quad . \quad . \quad . \quad . \quad . \quad (1.1)$$

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* Communicated by the Author.

† *Note added in proof*—Since the communication of this paper, similar problems have been discussed by A. C. Stevenson in the 'Proceedings of the Royal Society,' London, Ser. A, vol. 184 (1945), p. 129, and in a more general way by A. E. Green in the same journal (p. 231).

while the consistency equations are

$$\left.\begin{aligned} &\nabla_1^2\widehat{xx}+\frac{\partial^2\odot}{\partial x^2}=0, \quad \nabla_1^2\widehat{yy}+\frac{\partial^2\odot}{\partial y^2}=0,\\ &\nabla_1^2\widehat{xy}+\frac{\partial^2\odot}{\partial x\partial y}=0, \quad \nabla_1^2\odot=0,\end{aligned}\right\}\quad .\ .\ .\ .\ (1.2)$$

where

$$\odot=\widehat{xx}+\widehat{yy}, \quad\text{and}\quad \nabla_1^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

The solution of equation (1.1) satisfying the relations (1.2) can be written as

$$\left.\begin{aligned} \widehat{xx}&=\frac{y}{4}\frac{\partial\odot}{\partial y}-\frac{x}{4}\frac{\partial\odot}{\partial x}+\frac{\odot}{2}+\psi,\\ \widehat{xy}&=-\frac{x}{4}\frac{\partial\odot}{\partial y}-\frac{y}{4}\frac{\partial\odot}{\partial x}+\phi,\\ \widehat{yy}&=\frac{x}{4}\frac{\partial\odot}{\partial x}-\frac{y}{4}\frac{\partial\odot}{\partial y}+\frac{\odot}{2}-\psi,\end{aligned}\right\}\quad .\ .\ .\ .\ .\ (1.3)$$

where $\phi+i\psi$ is a function of $x+iy$.

Putting $z=f(\zeta)$ in which $\zeta=\xi+i\eta$ and $z=x+iy$, we shall now express the solution in terms of the coordinates (ξ, η).

If θ be the angle between the normal to the curve $\xi=$constant and the x-axis, we have

$$\left.\begin{aligned} &\cos\theta=h\frac{\partial y}{\partial\eta}=h\frac{\partial x}{\partial\xi}, \quad \sin\theta=h\frac{\partial y}{\partial\xi}=-h\frac{\partial x}{\partial\eta},\\ &\cos\theta\frac{\partial}{\partial y}-\sin\theta\frac{\partial}{\partial x}=h\frac{\partial}{\partial\eta}, \quad \sin\theta\frac{\partial}{\partial y}+\cos\theta\frac{\partial}{\partial x}=h\frac{\partial}{\partial\xi},\\ \text{where}\quad &\frac{1}{h^2}=\left(\frac{\partial x}{\partial\xi}\right)^2+\left(\frac{\partial y}{\partial\xi}\right)^2=\left(\frac{\partial x}{\partial\eta}\right)^2+\left(\frac{\partial y}{\partial\eta}\right)^2, \text{ so that}\\ &\nabla_1^2=h^2\left[\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right].\end{aligned}\right\}.\ (1.4)$$

Let F_1 and G_1 be two conjugate functions such that

$$F_1+iG_1=-i(\phi+i\psi)\left(\frac{\partial x}{\partial\xi}+i\frac{\partial y}{\partial\xi}\right)^2.$$

Then we have

$$F_1=\psi\left[\left(\frac{\partial x}{\partial\xi}\right)^2-\left(\frac{\partial y}{\partial\xi}\right)^2\right]+2\phi\frac{\partial x}{\partial\xi}\frac{\partial y}{\partial\xi},$$

$$G_1=-\phi\left[\left(\frac{\partial x}{\partial\xi}\right)^2-\left(\frac{\partial y}{\partial\xi}\right)^2\right]+2\psi\frac{\partial x}{\partial\xi}\frac{\partial y}{\partial\xi},$$

whence we get with the help of (1.4)

$$\left.\begin{aligned} h^2F_1&=\psi(\cos^2\theta-\sin^2\theta)+2\phi\sin\theta\cos\theta,\\ -h^2G_1&=\phi(\cos^2\theta-\sin^2\theta)-2\psi\sin\theta\cos\theta.\end{aligned}\right\}\quad .\ .\ .\ (1.5)$$

The stress components $\breve{\xi\xi}$, $\breve{\xi\eta}$ and $\breve{\eta\eta}$ are given in terms of $\breve{xx}$, $\breve{yy}$ and $\breve{xy}$ by the relations

$$\left.\begin{aligned}\breve{\xi\xi}&=\cos^2\theta\,\breve{xx}+\sin^2\theta\,\breve{yy}+2\sin\theta\cos\theta\,\breve{xy},\\ \breve{\eta\eta}&=\sin^2\theta\,\breve{xx}+\cos^2\theta\,\breve{yy}-2\sin\theta\cos\theta\,\breve{xy},\\ \breve{\xi\eta}&=-\sin\theta\cos\theta(\breve{xx}-\breve{yy})+(\cos^2\theta-\sin^2\theta)\breve{xy}.\end{aligned}\right\}\quad .\ .\ .\ . \quad (1.6)$$

Putting F/8 and G/8 for F_1 and G_1 respectively, and using the results (1.3) and (1.5), we get from (1.6)

$$\begin{aligned}\breve{\xi\xi}=&\frac{y}{4}\cos\theta\left(\cos\theta\frac{\partial\odot}{\partial y}-\sin\theta\frac{\partial\odot}{\partial x}\right)-\frac{y}{4}\sin\theta\left(\sin\theta\frac{\partial\odot}{\partial y}+\cos\theta\frac{\partial\odot}{\partial x}\right)\\ &-\frac{x}{4}\sin\theta\left(\cos\theta\frac{\partial\odot}{\partial y}-\sin\theta\frac{\partial\odot}{\partial x}\right)-\frac{x}{4}\cos\theta\left(\sin\theta\frac{\partial\odot}{\partial y}+\cos\theta\frac{\partial\odot}{\partial x}\right)\\ &+\frac{\odot}{2}+\frac{h^2F}{8},\end{aligned}$$

$$\begin{aligned}\breve{\xi\eta}=&-\frac{x}{4}\cos\theta\left(\cos\theta\frac{\partial\odot}{\partial y}-\sin\theta\frac{\partial\odot}{\partial x}\right)+\frac{x}{4}\sin\theta\left(\cos\theta\frac{\partial\odot}{\partial x}+\sin\theta\frac{\partial\odot}{\partial y}\right)\\ &-\frac{y}{4}\sin\theta\left(\cos\theta\frac{\partial\odot}{\partial y}-\sin\theta\frac{\partial\odot}{\partial x}\right)-\frac{y}{4}\cos\theta\left(\cos\theta\frac{\partial\odot}{\partial x}+\sin\theta\frac{\partial\odot}{\partial y}\right)\\ &-\frac{h^2G}{8}.\end{aligned}$$

Putting r^2 for x^2+y^2, we obtain finally

$$\frac{8\breve{\xi\xi}}{h^2}=\frac{\partial r^2}{\partial\eta}\frac{\partial\odot}{\partial\eta}-\frac{\partial r^2}{\partial\xi}\frac{\partial\odot}{\partial\xi}+\frac{4\odot}{h^2}+F,\quad .\ .\ .\ .\ . \quad (1.7\,a)$$

$$\frac{8\breve{\eta\eta}}{h^2}=\frac{\partial r^2}{\partial\xi}\frac{\partial\odot}{\partial\xi}-\frac{\partial r^2}{\partial\eta}\frac{\partial\odot}{\partial\eta}+\frac{4\odot}{h^2}-F,\quad .\ .\ .\ .\ . \quad (1.7\,b)$$

$$\frac{8\breve{\xi\eta}}{h^2}=-\frac{\partial r^2}{\partial\eta}\frac{\partial\odot}{\partial\xi}-\frac{\partial r^2}{\partial\xi}\frac{\partial\odot}{\partial\eta}-G.\quad .\ .\ .\ .\ .\ . \quad (1.7\,c)$$

Alternatively, these results can be put as

$$\begin{aligned}\frac{8\breve{\xi\xi}}{h^4}=&\frac{\partial r^2}{\partial\eta}\frac{\partial}{\partial\eta}\left(\frac{\odot}{h^2}\right)-\frac{\partial r^2}{\partial\xi}\frac{\partial}{\partial\xi}\left(\frac{\odot}{h^2}\right)+\frac{4\odot}{h^4}\\ &-\odot\left[\frac{\partial r^2}{\partial\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{h^2}\right)-\frac{\partial r^2}{\partial\xi}\frac{\partial}{\partial\xi}\left(\frac{1}{h^2}\right)\right]+\frac{F}{h^2},\quad . \quad (1.8\,a)\end{aligned}$$

$$\begin{aligned}\frac{8\breve{\eta\eta}}{h^4}=&\frac{\partial r^2}{\partial\xi}\frac{\partial}{\partial\xi}\left(\frac{\odot}{h^2}\right)-\frac{\partial r^2}{\partial\eta}\frac{\partial}{\partial\eta}\left(\frac{\odot}{h^2}\right)+\frac{4\odot}{h^4}\\ &-\odot\left[\frac{\partial r^2}{\partial\xi}\frac{\partial}{\partial\xi}\left(\frac{1}{h^2}\right)-\frac{\partial r^2}{\partial\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{h^2}\right)\right]-\frac{F}{h^2},\quad . \quad (1.8\,b)\end{aligned}$$

$$\begin{aligned}\frac{8\breve{\xi\eta}}{h^4}=&-\frac{\partial r^2}{\partial\eta}\frac{\partial}{\partial\xi}\left(\frac{\odot}{h^2}\right)-\frac{\partial r^2}{\partial\xi}\frac{\partial}{\partial\eta}\left(\frac{\odot}{h^2}\right)\\ &+\odot\left[\frac{\partial r^2}{\partial\eta}\frac{\partial}{\partial\xi}\left(\frac{1}{h^2}\right)+\frac{\partial r^2}{\partial\xi}\frac{\partial}{\partial\eta}\left(\frac{1}{h^2}\right)\right]-\frac{G}{h^2}.\quad .\ .\ . \quad (1.8\,c)\end{aligned}$$

The results (1.7) and (1.8) show that the stress components $\overset{\smile}{\xi\xi}$, $\overset{\smile}{\xi\eta}$ and $\overset{\smile}{\eta\eta}$ can be expressed explicitly in terms of the harmonic function $\odot$ and two conjugate functions F and G. If, for example, $\overset{\smile}{\xi\eta}$ is given on the boundary $\xi=$constant, we can choose a tentative harmonic function for $\odot$ involving certain unknown constants and hence determine an expression for G and thence from the known value of $\overset{\smile}{\xi\xi}$ on the boundary, the unknown constants in $\odot$ may be completely determined. An alternative method will be to assume an expression for $\odot$ and then to find the values of F and G on the boundary from the known values of the stress components on it. If the expression for $\odot$ be correctly chosen, it will be possible to find two conjugate functions F and G satisfying the required values on the boundary. Thus all the stress components will be exactly known. In choosing the expression for $\odot$, we can either take an infinite series or have it in a finite form. In any case the function must be such that in the absence of an isolated force, or a couple, it will be free from singularity in the domain considered. $\odot$ being a harmonic function, it will generally be of the form $Re\phi(z)$ or $Re\phi_0(\zeta)$ where Re denotes the real part. When such an expression is chosen the formulæ (17) may be used. But in many problems, specially when $z=\Sigma a_n e^{n\xi}$, the suitable form will be $Re\psi'(z)$ or $Re[\psi_0'(\zeta)/(dz/d\zeta)]$. In these cases the formulæ (18) may be utilized with advantage.

§ 2. *A Hole in the Form of an Inverse of an Ellipse.*

If we use the transformation

$$x+iy=c\sec(\xi+i\eta), \quad \ldots \ldots \quad (2.1)$$

we get

$$\left.\begin{aligned} x&=\frac{2c\cosh\eta\cos\xi}{\cosh 2\eta+\cos 2\xi}, \quad y=\frac{2c\sinh\eta\sin\xi}{\cosh 2\eta+\cos 2\xi}, \\ r^2&=x^2+y^2=\frac{2c^2}{\cosh 2\eta+\cos 2\xi}, \\ \frac{1}{h^2}&=\left(\frac{\partial x}{\partial\xi}\right)^2+\left(\frac{\partial y}{\partial\xi}\right)^2=\frac{2c^2(\cosh 2\eta-\cos 2\xi)}{(\cosh 2\eta+\cos 2\xi)^2}. \end{aligned}\right\} \quad \ldots \quad (2.2)$$

It can easily be seen that $\eta=\beta$ (a constant) is the inverse of an ellipse with respect to the centre. Let the hole be unstressed while T is the all-round tension at a great distance. In this case we assume

$$\odot=\mathrm{T}[k+2+k\cosh 2\eta\cos 2\xi], \quad \ldots \ldots \quad (2.3)$$

where k is a real constant to be determined. We find from the assumed value of $\odot$ that when $\eta\to 0$ and $\xi\to\pm(\pi/2)$, that is, when $r\to\infty$, $\odot\to 2\mathrm{T}$, and also that $\odot$ is a harmonic function.

Since by assumption $\overset{\smile}{\xi\eta}=0$ when $\eta=\beta$, we shall have from (1.7c)

$$\begin{aligned} [\mathrm{G}]_{\eta=\beta}&=-\left[\frac{\partial r^2}{\partial\eta}\frac{\partial\odot}{\partial\xi}+\frac{\partial r^2}{\partial\xi}\frac{\partial\odot}{\partial\eta}\right]_{\eta=\beta} \\ &=-\frac{8\mathrm{T}c^2k\sinh 2\beta\sin 2\xi}{\cosh 2\beta+\cos 2\xi}. \end{aligned} \quad \ldots \ldots \quad (2.4)$$

The above value of G on the boundary will be obtained if we take

$$F+iG=-8Tc^2k\sinh 2\beta i\frac{\sin(\xi+i\eta)}{\cos(\xi+i\eta)}, \quad . \quad . \quad . \quad . \quad (2.5)$$

which gives

$$F=\frac{8Tc^2k\sinh 2\beta\sinh 2\eta}{\cosh 2\eta+\cos 2\xi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.6)$$

Substituting this value of F into (1.7 *b*) and equating to zero the expression for $\overset{\smile}{\eta\eta}$ when $\eta=\beta$, we have

$$2(\cosh 2\beta-\cos 2\xi)-k\sinh^2 2\beta(\cosh 2\beta-\cos 2\xi)=0,$$

whence we get

$$k=2\operatorname{cosech}^2 2\beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.7)$$

Thus

$$\odot=2T[1+\operatorname{cosech}^2 2\beta(1+\cosh 2\eta\cos 2\xi)]. \quad . \quad . \quad . \quad (2.8)$$

Values of $\odot$, F and G being known, stress components are completely determined. The value of the stress componenet $\overset{\smile}{\xi\xi}$ on the boundary of the hole is given by

$$[\overset{\smile}{\xi\xi}]_{\eta=\beta}=[\odot]_{\eta=\beta}=2T[1+\operatorname{cosech}^2 2\beta(1+\cosh 2\beta\cos 2\xi)], \quad . \quad (2.9)$$

of which the maximum value is evidently

$$T[2+\operatorname{cosech}^2\beta]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.10)$$

As β increases indefinitely, that is, when the hole reduces to a very small aperture round the origin, the stress on the periphery tends to the value 2T.

§ 3. *A Hole in the Form of a Loop of Lemniscate of Bernoulli.*

Let us consider the system of orthogonal coordinates (ξ, η) defined by

$$\left.\begin{aligned}&(z+c)(z-c)=c^2e^{2\zeta}\\ &\text{where } c \text{ is a constant, and}\\ &z=x+iy, \quad \zeta=\xi+i\eta.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.1)$$

In Cartesian coordinates we obtain the relations

$$\left.\begin{aligned}&(x^2-y^2-c^2)^2+4x^2y^2=c^4e^{4\xi},\\ &(x^2-y^2-c^2)=2xy\cot 2\eta,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3.2)$$

so that the curve $\xi=0$ becomes a lemniscate of Bernoulli, of which the polar equation is

$$r^2=2c^2\cos 2\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.3)$$

We shall consider the loop of the curve that lies in the region

$$-\infty\leqslant\xi\leqslant 0, \quad -(\pi/2)\leqslant\eta\leqslant(\pi/2),$$

it being assumed that the curve forms the boundary of a hole in a plate extending indefinitely in all directions.

In the present case we have

$$\frac{1}{h^2}=\left(\frac{\partial x}{\partial\xi}\right)^2+\left(\frac{\partial y}{\partial\xi}\right)^2=\frac{c^2e^{4\xi}}{(1+e^{4\xi}+2e^{2\xi}\cos 2\eta)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad (3.4)$$

$$r^2=c^2(1+e^{4\xi}+2e^{2\xi}\cos 2\eta)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.5)$$

From these results we find that

$$\left.\begin{aligned} &\left(\frac{\partial r^2}{\partial \xi}\right)_{\xi=0}=2c^2\cos\eta, \quad \left(\frac{\partial r^2}{\partial \eta}\right)_{\xi=0}=-2c^2\sin\eta, \\ \text{and} \quad &\left(\frac{1}{h^2}\right)_{\xi=0}=\frac{c^2}{2\cos\eta}. \end{aligned}\right\} \quad . \quad . \quad (3.6)$$

For the existence of an all-round uniform tension T at a great distance, we must choose a value of $\odot$ such that $\odot\rightarrow 2T$ as $\xi\rightarrow\infty$. A harmonic function satisfying this condition will be obtained if we assume

$$\odot=2T[e^{-2\xi}\cos 2\eta+1]. \quad . \quad . \quad . \quad . \quad . \quad (3.7)$$

Since for the unstressed hole, $\overset{\smile}{\xi\xi}=0$, $\overset{\smile}{\xi\eta}=0$ when $\xi=0$, we have from (1.7 *a*) and (1.7 *b*) for the above value of $\odot$

$$\left.\begin{aligned} [F]_{\xi=0}&=-16T\,c^2\cos\eta, \\ [G]_{\xi=0}&=8Tc^2\sin\eta. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.8)$$

These relations are satisfied if we take

$$F+iG=-8Tc^2e^{-(\xi+i\eta)}-8Tc^2\cosh(\xi+i\eta). \quad . \quad . \quad . \quad (3.9)$$

It is apparent from (1.7) that the contributions of F and G to the stress componenets are through the expressions Fh^2 and Gh^2 which tend to zero as ξ tends to infinity. This shows that the expressions for F and G assumed in (3.9) are correct ones and hence that the appropriate expression for $\odot$ is given by (3.7).

From this result we have

$$[\overset{\smile}{\eta\eta}]_{\xi=0}=[\odot]_{\xi=0}=4T\cos^2\eta. \quad . \quad . \quad . \quad . \quad (3.10)$$

At the node of the loop, this stress component vanishes, while at the flat end where $\eta=0$, it has the maximum value 4T.

§ 4 *A Hole in the Form of an Elliptic Limacon.*

$$\left.\begin{aligned} \text{Let} \quad & z=a\left[e^{\zeta}+\frac{\lambda}{2}(1+e^{2\zeta})\right], \\ \text{where} \quad & \zeta=\xi+i\eta, \quad \text{and} \quad \lambda=b/a\leqslant 1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (4.1)$$

Then

$$\left.\begin{aligned} x&=a\left[e^{\xi}\cos\eta+\frac{\lambda}{2}(1+e^{2\xi}\cos 2\eta)\right], \\ y&=a\left[e^{\xi}\sin\eta+\frac{\lambda}{2}e^{2\xi}\sin 2\eta\right], \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.2)$$

and

$$r^2=a^2\left[e^{2\xi}+\frac{\lambda^2}{4}(1+e^{4\xi})+\lambda(e^{\xi}+e^{3\xi})\cos\eta+\frac{\lambda^2}{2}e^{2\xi}\cos 2\eta\right], \quad . \quad . \quad (4.3)$$

so that when $\xi=0$,

$$r^2=a^2(1+\lambda^2\cos^2\eta+2\lambda\cos\eta)=(a+b\cos\eta)^2.$$

Thus the boundary $\xi=0$ gives an elliptic limacon. From (4.2), we shall have

$$\frac{1}{h^2}=\left(\frac{\partial x}{\partial \xi}\right)^2+\left(\frac{\partial y}{\partial \xi}\right)^2=a^2[e^{2\xi}+\lambda^2e^{4\xi}+2\lambda e^{3\xi}\cos\eta]. \quad . \quad . \quad (4.4)$$

As stated in Art. 1, it will be to our advantage if we assume for boundaries of this type

$$\odot=\mathrm{Re}\left[\psi'(\zeta)\Big/\frac{dz}{d\zeta}\right]=\mathrm{Re}\left[\frac{\psi'(\zeta)}{a(\lambda e^{2\zeta}+e^{\zeta})}\right],$$

where $\psi'(\zeta)$ is a function of ζ to be determined. In order that $\odot$ may tend to $2\mathrm{T}$ when $\xi\to\infty$, we can assume

$$\odot=2\mathrm{T}\mathrm{Re}\frac{\lambda e^{2\zeta}-e^{\zeta}}{\lambda e^{2\zeta}+e^{\zeta}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.5)$$

We might of course add other terms to the numerator but ultimately they will be found to vanish. When simplified the above result becomes

$$\odot=2\mathrm{T}\frac{\lambda^2e^{4\xi}-e^{2\xi}}{e^{2\xi}+\lambda^2e^{4\xi}+2\lambda e^{3\xi}\cos\eta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.6)$$

which gives
$$\frac{\odot}{h^2}=2\mathrm{T}a^2(\lambda^2e^{4\xi}-e^{2\xi}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.7)$$

From (4.3) and (4.4) we get

$$\left.\begin{aligned}\left(\frac{\partial r^2}{\partial\xi}\right)_{\xi=0}&=2a^2(1+2\lambda\cos\eta+\lambda^2\cos^2\eta),\\ \left(\frac{\partial r^2}{\partial\eta}\right)_{\xi=0}&=-2a^2\lambda\sin\eta(1+\lambda\cos\eta),\\ \left[\frac{\partial}{\partial\xi}\left(\frac{1}{h^2}\right)\right]_{\xi=0}&=2a^2(1+3\lambda\cos\eta+2\lambda^2),\\ \left[\frac{\partial}{\partial\eta}\left(\frac{1}{h^2}\right)\right]_{\xi=0}&=-2a^2\lambda\sin\eta.\end{aligned}\right\} \quad . \quad . \quad . \quad (4.8)$$

For the unstressed hole we have $\breve{\xi\xi}=0$ and $\breve{\xi\eta}=0$ when $\xi=0$. Hence using the results (1.8 a) and (1.8 c) we obtain

$$\left[\frac{\mathrm{F}}{h^2}\right]_{\xi=0}=8\mathrm{T}a^4[(1+\lambda^2+3\lambda\cos\eta+\lambda^2\cos^2\eta)-\lambda^3(\lambda+\cos\eta)], \quad . \quad . \quad . \quad (4.9)$$

$$\left[\frac{\mathrm{G}}{h^2}\right]_{\xi=0}=8\mathrm{T}a^4\lambda\sin\eta(1+\lambda\cos\eta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.10)$$

The relations (4.9) and (4.10) are satisfied if we assume

$$\mathrm{F}+i\mathrm{G}=8\mathrm{T}a^2\frac{(1-\lambda^2/2)(2\lambda e^{2\zeta}+e^{\zeta})+\lambda^2/2\,.\,e^{3\zeta}}{\lambda e^{2\zeta}+e^{\zeta}}. \quad . \quad . \quad . \quad (4.11)$$

$\odot$, F and G being known, the stresses are completely determined. From (4.6) we have

$$[\breve{\eta\eta}]_{\xi=0}=[\odot]_{\xi=0}=-\frac{2\mathrm{T}(1-\lambda^2)}{1+\lambda^2+2\lambda\cos\eta}. \quad . \quad . \quad . \quad . \quad . \quad (4.12)$$

§ 5. *A Hole in the Form of an Approximate Square.*

Assuming we get

$$\left.\begin{aligned} z&=e^{\zeta}+ce^{-3\zeta},\\ x&=e^{\xi}\cos\eta+ce^{-3\xi}\cos 3\eta,\\ y&=e^{\xi}\sin\eta-ce^{-3\xi}\sin 3\eta, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5.1)$$

so that the boundary $\xi=0$ is an approximate square. This is the special case of the curvilinear polygon discussed by Stevenson (1943, Art. 12).

We have in the present case

$$\frac{1}{h^2}=e^{2\xi}+9c^2e^{-6\xi}-6ce^{-2\xi}\cos 4\eta, \quad . \quad . \quad . \quad . \quad (5.2)$$

$$r^2=e^{2\xi}+c^2e^{-6\xi}+2ce^{-2\xi}\cos 4\eta. \quad . \quad . \quad . \quad . \quad (5.3)$$

Let us assume

$$\begin{aligned} \odot&=\mathrm{Re}\left[\psi(\zeta)\middle/\frac{dz}{d\xi}\right]=2\mathrm{T}\mathrm{Re}\frac{e^{\zeta}+3ce^{-3\zeta}}{e^{\zeta}-3ce^{-3\zeta}}\\ &=2\mathrm{T}\frac{e^{2\xi}-9c^2e^{-6\xi}}{e^{2\xi}+9c^2e^{-6\xi}-6ce^{-2\xi}\cos 4\eta}. \quad . \quad . \quad . \quad . \quad (5.4) \end{aligned}$$

It is evident that this expression tends to 2T as ξ tends to infinity. We also have in this case

$$\frac{\odot}{h^2}\doteq 2\mathrm{T}(e^{2\xi}-9c^2e^{-6\xi}). \quad . \quad . \quad . \quad . \quad (5.5)$$

From (5.2) and (5.3) we get

$$\left.\begin{aligned} \left[\frac{\partial r^2}{\partial\xi}\right]_{\xi=0}&=2-6c^2-4c\cos 4\eta, & \left[\frac{\partial r^2}{\partial\eta}\right]_{\xi=0}&=-8c\sin 4\eta,\\ \left[\frac{\partial}{\partial\xi}\left(\frac{1}{h^2}\right)\right]_{\xi=0}&=2-54c^2+12c\cos 4\eta, & \left[\frac{\partial}{\partial\eta}\left(\frac{1}{h^2}\right)\right]_{\xi=0}&=24c\sin 4\eta. \end{aligned}\right\} \quad . \quad (5.6)$$

On the boundary of the hole $\xi=0$, we have $\overset{\smile}{\xi\xi}=\overset{\smile}{\xi\eta}=0$. Hence utilizing the above results in (1.8 *a*) and (1.8 *b*), we obtain

$$\left.\begin{aligned} \left[\frac{\mathrm{F}}{h^2}\right]_{\xi=0}&=-8\mathrm{T}(1+3c^2)[(1-27c^2)+6c\cos 4\eta],\\ \left[\frac{\mathrm{G}}{h^2}\right]_{\xi=0}&=96\mathrm{T}c(1+3c^2)\sin 4\eta. \end{aligned}\right\} \quad . \quad . \quad (5.7)$$

The above relations are satisfied if we write

$$\mathrm{F}+i\mathrm{G}=-8(1+3c^2)\mathrm{T}\frac{e^{\zeta}+9ce^{-3\xi}}{e^{\zeta}-3ce^{-3\zeta}}. \quad . \quad . \quad . \quad . \quad (5.8)$$

Expressions for $\odot$, F and G being known, the stress components are completely determined.

From the results (5.4), we find that

$$[\overset{\smile}{\eta\eta}]_{\xi=0}=[\odot]_{\xi=0}=2\mathrm{T}\frac{(1-9c^2)}{1-6c\cos 4\eta+9c^2}. \quad . \quad . \quad (5.9)$$

§ 6. *An Elliptic Hole in an Infinite Plate with Normal Pressures on the Inner Boundary.*

It is well known that the transformation

$$z=\frac{c}{2}(e^{\zeta}+e^{-\zeta})=c\cosh\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.1)$$

gives us an elliptic boundary for a constant value of ξ equal to α (say). In this case we shall have

$$\left.\begin{array}{l} x=c\cosh\xi\cos\eta, \quad y=c\sinh\xi\sin\eta, \\ r^2=\dfrac{c^2}{2}(\cosh 2\xi+\cos 2\eta), \quad \dfrac{1}{h^2}=\dfrac{c^2}{2}(\cosh 2\xi-\cos 2\eta). \end{array}\right\} \quad . \quad (6.2)$$

If the stress components vanish at a great distance, we shall have to find a form of $\odot$ which will tend to zero as $\xi\rightarrow\infty$. To satisfy this condition let us assume

$$\odot=\mathrm{B}\left[1-\mathrm{Re}\frac{\cosh(\xi+i\eta)}{\sinh(\xi+i\eta)}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.3)$$

where B is a constant to be determined. It is to be noted that all problems of elliptic holes in an infinite plate can be solved by assuming the suitable values of $\odot$ which involve

$$\mathrm{Re}\left[\phi(\zeta)\Big/\frac{dz}{d\zeta}\right]=\mathrm{Re}\frac{\cosh\zeta}{\sinh\zeta}.$$

If there be no shearing stress on the boundary of the hole we shall have $\overset{\smile}{\xi\eta}=0$ when $\xi=\alpha$. Hence from (1.7 *c*) we get

$$[\mathrm{G}]_{\xi=0}=-\frac{2B\cosh 2\alpha\sin 2\eta}{\cosh 2\alpha-\cos 2\eta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.4)$$

The boundary of value G is obtained if we take

$$\mathrm{F}+i\mathrm{G}=2\mathrm{B}\cosh 2\alpha\frac{\cosh(\xi+i\eta)}{\sinh(\xi+i\eta)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.5)$$

which also gives

$$\mathrm{F}=\frac{2\mathrm{B}\cosh 2\alpha\sinh 2\xi}{\cosh 2\xi-\cos 2\eta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.6)$$

Let us now suppose that $\overset{\smile}{\xi\xi}=-\mathrm{P}_1$ on the boundary $\xi=\alpha$. Then on using the result (6.6) in (1.7 *a*) we get

$$-4(\cosh 2\alpha-\cos 2\eta)\mathrm{P}_1=2\mathrm{B}(\cosh 2\alpha-\cos 2\eta)$$

$$\text{or } \mathrm{B}=-2\mathrm{P}_1.$$

Thus we find that

$$\odot=-2\mathrm{P}_1\left[1-\frac{\sinh 2\xi}{\cosh 2\xi-\cos 2\eta}\right]. \quad . \quad . \quad . \quad . \quad . \quad (6.7)$$

This problem was solved by Ghosh (1930) in a different manner.

References.

(1) S. Ghosh, Trans. Am. Math. Soc. xxxii. p. 47 (1930).
(2) G. V. Kolossoff, *Compte Rendus*, cxlviii. p. 1242 (1909).
(3) N. I. Muschelisvili, *Math. Annalen*, cvii. p. 282 (1932); Z.A.M.M. xiii p. 264 (1933).
(4) B. Sen, Phil. Mag. (7) xxxvi. p. 211 (1945).
(5) A. C. Stevenson, Phil. Mag. (7) xxxiv. p. 766 (1943).

B. E. College,
Sibpore,
West Bengal.

CXVIII. *Note on Dirac's Treatment of the Harmonic Oscillator.*

By Peter T. Landsberg, M.Sc.[†]

[Received October 24, 1946.]

In his treatment of the harmonic oscillator[‡], Dirac introduces two infinite matrices:

$$e^{i\omega}=\begin{array}{|ccccc}\hline 0&0&0&0&\cdot\\ 1&0&0&0&\cdot\\ 0&1&0&0&\cdot\\ 0&0&1&0&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\end{array}, \qquad e^{-i\omega}=\begin{array}{|ccccc}\hline 0&1&0&0&\cdot\\ 0&0&1&0&\cdot\\ 0&0&0&1&\cdot\\ 0&0&0&0&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\end{array}, \quad . \; . \quad (1)$$

in order to obtain a representation of the angle variable (ω). The notation suggests that $e^{-i\omega}$ is the complex conjugate matrix of $e^{i\omega}$ (*i. e.* $e^{-i\omega}=(e^{i\omega})^*$)—a condition which is evidently fulfilled. It also suggests that the products of the matrices (1) yield the unit matrix, I say. This is not strictly correct; in fact, Dirac states that the relations

$$e^{-i\omega}e^{i\omega}=\mathrm{I}, \quad e^{i\omega}e^{-i\omega}\neq\mathrm{I} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

hold, and that the second product (2) differs from I merely by having zero as its first diagonal element.

It is perhaps surprising that two symmetrical matrices of this sort should have such asymmetrical properties, and it is the purpose of this note to explain how this comes about. We shall, in the course of the discussion, throw doubt even on the first of the relations (2).

We commence by considering the corresponding finite matrices, U_n, $\mathrm{U}_n^{-1}=\mathrm{U}_n^*$, say, where the suffix n indicates the number of rows and columns. These matrices are unitary and accordingly relate two normalized bases in the corresponding vector space $\mathfrak{S}_n$. If $\{\alpha_j\}$ be a basis in $\mathfrak{S}_n$, our matrices are simply introduced by the relations

$$\alpha_{j+1}=\mathrm{U}_n\alpha_j, \quad \mathrm{U}_n^{-1}\alpha_{j+1}=\alpha_j \quad (j=1, 2, \ldots), \quad . \; . \; . \quad (3\,a)$$

[†] Communicated by the Author.

[‡] 'Quantum Mechanics,' 1st and 2nd ed., Cambridge Univ. Press.

provided the suffixes are taken module n. For $n=4$ we obtain the scheme

$$\begin{array}{cc} & \begin{array}{l} X \\ \searrow \end{array} \\ U_4= & \begin{matrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{matrix} \\ & \begin{array}{r} \nwarrow \\ Y \end{array} \end{array} , \qquad U_4^{-1}= \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{matrix} \quad . \; . \; . \quad (3\,b).$$

The matrices (3) obey the usual unitary relations

$$U_nU_n^*=U_n^*U_n=I \quad (n=1, 2, \ldots);$$

this equation corresponds to (2) except that, in the present case, the matrices are finite and the equality sign holds throughout. It is apparent, on fixing attention on the left-hand top corner of the matrices (3 *b*), that they go over into the matrices (1) as $n\rightarrow\infty$. The non-vanishing corner element goes to infinity (and is therefore ignored in (1)) ; similarly, the bottom right-hand corner of the matrices (3 *b*) is "lost." We shall see that it is this latter circumstance which gives rise to the lack of symmetry in (2).

There is no compelling reason why the usual *plane* model should be assumed for the generation of the matrices (1). The following is an alternative procedure. Consider a right-circular cylinder and let the matrix U_4 be written down on its curved surface such that the diagonal of the matrix is in a plane which is parallel to the base of the cylinder. Since $n=4$, all dimensions of the cylinder are to be finite. Somewhere on the cylinder the points X and Y (equation (3)) are arranged to be close together, and the development of the curved surface of the cylinder for a view of this area will be on the lines indicated in (4). We now fix

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & & 1 & & 0 & & \\
 & & & 0 & & & \\
 & & & \nwarrow & & & \\
 & & & & Y & & \\
 & & X & & & & \\
 & & & \searrow & & & \\
 & & & 0 & & & \\
 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & \\
0 & & 1 & & 0 & & 1
\end{array} \quad . \; . \; . \quad (4).$$

attention on this view and let $n \to \infty$. The length and diameter of the cylinder will both have to become infinite. The non-vanishing corner element is "lost," as before, but we do retain a view of both the top left-hand corner and the bottom right-hand corner of the matrix.

The above considerations show that a definite meaning can be given to the relations

$$U_\infty = \begin{array}{l}\begin{array}{|ccccc}\hline 0&0&0&0&\cdot\\ 1&0&0&0&\cdot\\ 0&1&0&0&\cdot\\ 0&0&1&0&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\end{array}\\ \qquad\qquad\begin{array}{ccccc|}\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&0&0&0&0\\ \cdot&1&0&0&0\\ \cdot&0&1&0&0\\ \cdot&0&0&1&0\\ \hline\end{array}\end{array}, \quad U_\infty^{-1} = \begin{array}{l}\begin{array}{|ccccc}\hline 0&1&0&0&\cdot\\ 0&0&1&0&\cdot\\ 0&0&0&1&\cdot\\ 0&0&0&0&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\end{array}\\ \qquad\qquad\begin{array}{ccccc|}\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&0&1&0&0\\ \cdot&0&0&1&0\\ \cdot&0&0&0&1\\ \cdot&0&0&0&0\\ \hline\end{array}\end{array}, \qquad \ldots \quad (5)$$

which are obtained by re-writing our view in the usual form *after* transition has been made to the limit. The dots indicate an infinity of elements which may be supposed to jut out of the plane of the paper. Using (5), we readily find

$$U_\infty^{-1}U_\infty = \begin{array}{l}\begin{array}{|cccc}\hline 1&0&0&\cdot\\ 0&1&0&\cdot\\ 0&0&1&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{array}\\ \qquad\qquad\begin{array}{cccc|}\cdot&\cdot&\cdot&\cdot\\ \cdot&1&0&0\\ \cdot&0&1&0\\ \cdot&0&0&0\\ \hline\end{array}\end{array}, \quad U_\infty U_\infty^{-1} = \begin{array}{l}\begin{array}{|cccc}\hline 0&0&0&\cdot\\ 0&1&0&\cdot\\ 0&0&1&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{array}\\ \qquad\qquad\begin{array}{cccc|}\cdot&\cdot&\cdot&\cdot\\ \cdot&1&0&0\\ \cdot&0&1&0\\ \cdot&0&0&1\\ \hline\end{array}\end{array}. \qquad \ldots \quad (6)$$

It is therefore evident that on the present view neither of the two products of the matrices (4) yields exactly the unit matrix: owing to the loss of the non-vanishing corner element the unitary property has definitely been forfeited. The loss of this element is required for the use to which these matrices are subsequently put.

The above results may be related to the properties of the "Auxiliary Unit Matrix" *, A_n say, and its transpose, A'_n, which are obtained if the non-vanishing corner element of respectively U_n^{-1} and U_n are replaced by zero. If one forms the products $X_n=A'_n A_n$, $Y_n=A_n A'_n$, then X_n and Y_n retain only the idempotent property $X_n^2=X_n$, $Y_n^2=Y_n$ of the unit matrix, each having one vanishing proper value. However, for both the plane and the cylindrical limiting process, the relation

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* W. H. Turnbull and A. C. Aitken, 'An Introduction to the Theory of Canonical Matrices,' Blackie, pp. 62, 143 and 151 (1932.)

$$\lim_{n\to\infty} U_n=A_\infty$$

holds.

The limiting process employed here has the advantage over the plane model that, instead of (2), the symmetrical relations (6) are obtained. It would, however, if accepted, affect some of Dirac's results.

CXIX. *A Table of Roots of* sin Z=−Z.

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

HILLMAN and SALZER * give six decimal values of the first ten non-zero roots of sin Z=Z having both real and imaginary parts positive. Here a similar table is given for the roots of sin Z=−Z. On writing $Z=x+iy$ and separating into real and imaginary parts this equation is replaced by the pair of equations

$$f(x, y)=\cosh y+\frac{x}{\sin x}=0,$$

$$g(x, y)=\cos x+\frac{y}{\sinh y}=0.$$

It is obvious that for large roots (with positive real and imaginary parts)

$$x\approx(2n+1{\cdot}5)\pi,$$

$$y\approx\log_e (4n+3)\pi.$$

The values in the table below were calculated by Newton's rule in the form

$$f(x+\delta x, y+\delta y)\approx f(x, y)+\delta x\frac{\partial f}{\partial x}+\delta y\frac{\partial f}{\partial y}=0,$$

$$g(x+\delta x, y+\delta y)\approx g(x, y)+\delta x\frac{\partial g}{\partial x}+\delta y\frac{\partial g}{\partial y}=0.$$

Starting from approximate values x, y these two equations determine $\delta x, \delta y$ and thus improved approximations $x+\delta x$, $y+\delta y$ to the roots.

---

* Hillman, A. P., and Salzer, H. E., *Phil. Mag.* (1943) (7) **34,** 575.

This process was repeated until the corrections did not affect the eighth decimal. Consequently the roots are believed reliable to the six decimals quoted.

Roots of sin Z=−Z where Z=$x+iy$.

n	x	y
0	4·212 392	2·250 729
1	10·712 537	3·103 149
2	17·073 365	3·551 087
3	23·398 355	3·858 809
4	29·708 120	4·093 705
5	36·009 866	4·283 782
6	42·306 827	4·443 446
7	48·600 684	4·581 105
8	54·892 406	4·702 096
9	61·182 590	4·810 025

Yours faithfully,
C. I. Robbins (Miss).
R. C. T. Smith, D.Phil.

Council for Scientific and
Industrial Research, Australia.

CXX. *Notices of New Books and Periodicals received.*

Velocity-modulated Thermionic Tubes. By A. H. W. Beck. [Pp. x+180. (Cambridge University Press, 1948.) Price 15*s*.

This is one of the Modern Radio Technique series, edited by J. A. Ratcliffe, which aim at giving a connected account of the very great developments in radio and radar during the war years. The writers of the books in this series have been intimately connected with the British work in the fields they discuss.

As a result of much careful development work, it has been found possible to design valves of the " conventional " type (in which a single control-grid varies the density of the electron stream), to work at frequencies at one time considered quite impossible. For the highest frequencies now in use, however, it becomes necessary to use quite different principles, making use of electron-transit time instead of trying to minimize its effects. One such principle is that of velocity modulation which is used in the klystron valve and has become of great importance. A reflex klystron is used as the local oscillator in practically all superheterodyne receivers for centimetric-wave radar or communication.

The fundamental theory of velocity modulation is clearly set out in chapter 3, which is the " core " of the book. It is preceded by an account of the design of the cavity resonators used in the klystron and is followed by a discussion of the production and focusing of electron beams in klystrons, which require to contain so many more electrons than the beam of a cathode-ray oscillograph.

The klystron amplifier is then dealt with. Its disappointing feature—a poor signal/noise ratio—is discussed, the inadequacy of present theory being pointed out. Chapters on klystron oscillators, of both double-cavity and reflex types, follow. There are valuable chapters on the limitations and defects of velocity-modulation valves and on the practical design and manufacture. There is an appendix on the travelling-wave tube, which shows promise as a wideband amplifier for centimetric waves but on which more development work is required.

The book is clearly written and well-produced. It is a very useful addition to the literature on the subject, presenting much that has only been available in scattered, recent papers.

C. R. S.

Theoretical Aerodynamics. By L. M. Milne-Thomson. [Pp. xix+363.] (Macmillan, 1948.) Price 40*s*. net.

Difficulty in teaching Aerodynamics arises chiefly from its collective and composite nature: progress often requires approach from several angles, each absorbing considerable time or space, and the range of interest is wide. The subject has grown sufficiently to warrant subdivision for educational purposes, but this is not easy to contrive.

Professor Milne-Thomson's book admirably explains, to first degree standard, certain of the mathematical approaches in use. Some two-thirds of the book is devoted to the theory of potential incompressible flow and of vortices in relation to two- and three-dimensional aerofoils, to airscrews and wind tunnel constraint. There follow introductions to subsonic and supersonic flow and to the methods employed in investigating the stability of aeroplanes in flight. As would be expected from the lucidity and care of the Author the treatment is clear and illuminating. It will be particularly helpful to students, and not only to those graduating in mathematics, since the mathematical equipment required of the reader further to elementary calculus is largely explained in place.

On the other hand, theoretical Aerodynamics is a much more comprehensive subject than is here described. For instance, practically no discussion is given of experiment, drag (other than induced drag), boundary layer theory and skin friction, or turbulence. In respect of the matters which are treated one consequently misses directives, justifications, and the like, which other parts of the theory provide at the undergraduate stage; and there arises need for parallel reading, though not in mathematics. Such remarks offer criticism as to choice of title without detracting from the value of the contribution the book makes to aerodynamical education. The production is excellent, there is a chapter on vectors and a list of 328 exercises. N. A. V. Piercy.

The Diary and Sundry Observations of Thomas Alva Edison. Edited by D. D. Runes. [Pp. xii+247.] (New York: The Philosophical Library, 15 East 40th Street, 1948.) Price $4·75.

The Diary is a fragment dealing in characteristically lively fashion with the small beer of domestic life covering the brief period July 12–21, 1885 and occupying 38 pages of the book. The rest of the book is devoted to sundry observations which are grouped under eight main headings—Autobiographical, Motion Pictures and the Arts, War and Peace, Education and Work, The Philosophy of (Tom) Paine, Man and Machine, For a Better World and The Realm Beyond.

All these topics—and they deal with what are First and Last Things for most of us—are handled in a manner which reveals a mind of great independence, whimsical and sincere and, as was to be expected from a self-made man, intensely individualistic.

In the autobiographical fragment he tells how he became a newsboy at 12 years old and how he educated himself by reading in the Detroit Public Library, where he followed the reading of the *Anatomy of Melancholy* by the study of Newton's *Principles.* The third Lord Rayleigh wrote a learned and highly mathematical article on the *Wave Theory of Light* for the Encyclopædia Britannica. The article appeared in the 10th edition which was published in the early 1900's, and on hearing that a group of American farmers had clubbed together to buy a set of the new encyclopædia and were meeting at regular intervals to read through the whole work, Rayleigh pensively asked, "I wonder how they got on with my article?" One is tempted to put a similar query concerning Edison's study of the *Principia*!

The book would have been much improved had it contained a chronological table setting out the principal events in Edison's life. But it would seem that the editor took his duties rather lightly. Spelling is a real difficulty, for Edison had a way of indulging in comic misspelling of set purpose. But (p. 7) Macauley, Sidney Smith, DeQuincey (one word) and (p. 9) Brittanica are instances of straight uncorrected mistakes. Coronor (p. 4) might be taken as an example of Americanisation, but the more usual spelling occurs on the next line.

But the reviewer does not wish to seem hypercritical. The book provides a most interesting view of a fascinating personality and does much to illustrate Edison's own dictum, "Genius is made up of one per cent inspiration and ninety-nine per cent perspiration."

The production of the volume is admirable and leaves nothing to be desired.

A. F.

Dynamics. By S. L. Green. [Pp. iv+264.] (London: University Tutorial Press, 1948.) Price 10*s.* 6*d.*

This is a workmanlike book, written on orthodox lines. It is intended for students reading for general honours, and for first year special honours students in the mathematical schools, and it admirably fulfils its intention. The opening chapter gives a useful summary of the elements of vector algebra (it is unfortunate that the term *rotor*, suggested for *localised vector*, never became popular) and succeeding chapters deal clearly and competently with the usual topics of particle dyamics and the two-dimensional motion of a rigid body. The book is well produced, and the price is moderate. It is strongly recommended. A. F.

Distillation and Rectification. By Emil Kirschbaum. Translated by M. Wulfinghoff. [Pp. xiii+426.] (Chemical Publishing Co., Inc., Brooklyn, New York, 1948.) Price 10$.

The theoretical principles underlying distillation and the separation of mixtures of liquids belong to thermodynamics and the phase rule and a knowledge of some parts of these subjects is essential for a proper understanding of the processes involved. The technical aspects of the subject, however, go far beyond what is commonly taught in these subjects and their description requires a special treatise. There have been several books of this kind, but the reviewer does not recall one which covers such a wide field so adequately as the present one. The book opens with a brief statement of the fundamental theory, based on the Duhem-Margules equation and applied to binary and ternary mixtures, and then proceeds to its main topic, the practical applications. The latter include descriptions and diagrams of apparatus and numerous graphs. The graphical method is largely used and many useful tables and curves are given. The treatment is very practical and should appeal especially to chemical engineers and those engaged in industrial distillation. At the same time the information is presented in a scientific way which makes the text suitable for students of chemical engineering. Plant design and process control receive adequate treatment, and such matters as the construction and operation of rectifying columns (including packed columns) and plate efficiency are well covered. A section on molecular distillation is rather too brief in view of the growing importance of this topic. In the translation the metric system has been retained and the numerous illustrations have descriptions in German, the translations being given below. The German subscripts D and F for vapour and liquid might well have been changed in the text. A review of recent developments is added by the translator. This is a valuable and practical book.

J. R. Partington.

Strength of Solids. [Pp. 162.] (Physical Society, 1948.) Price 25*s.* 0*d.*

In this Report are published papers which were read at the Bristol Conference in 1947, and which may be considered to indicate the state of the subject at the end of that year. The titles of the four parts into which the text is divided, namely, " Creep and Plastic Flow ", " Grain Boundaries and Recrystallization ", " Precipitation ", and " Fracture ", give some idea of the scope, but to convey properly the wide range of topics treated in the 19 papers would demand nothing less than a list of their separate titles.

It is natural to recall a former Physical Society Report on this subject, namely, that on the Solid State of Matter, published in 1935, and a comparison of the two does bring out the notable advances made in the interval between the Reports. In the earlier report, the work of Taylor, Orowan and others on the mechanism of slip, was beginning to make a distinct impression. The word " dislocation " is certainly to be found therein, but only after careful search. The 1948 Report, on the other hand, might be said to be dominated by the concept of dislocations—no longer a term demanding inverted commas. Much of the theoretical background is given in the first paper, by Mott and Nabarro, and all but a very few of the remaining papers revert at some point to one aspect or another of the dislocation idea. This gives to the later Report a unity not so obvious in the earlier. It would be a mistake, however, to leave the impression that the Report deals only with the ramifications of dislocation theory. Much remarkable new experimental work is described on such subjects as the " Thermo-electric study of cold-rolling and heat treatment of copper " (Brindley), the " Study of Slip in Aluminium Crystals by electron Diffraction, etc. " (Heidenreich and Shockley) or " the Trapping of Electrons in Silver Chloride " (Haynes and Shockley). An article on the Creep of Metals, by Prof. Andrade, recounts much of his fundamental work and gives an exposition of the physical ideas which at least one reader considers the most stimulating in the literature.

No doubt the Report is primarily for the specialist, but anyone who wishes seriously to know the present state of knowledge in this subject, could do worse than begin here.

J. P. A.

The following Journals have been received at the Office :—

Pontificia Academia Scientiarum. Commentationes, VIII., XI. (1944, 1947.)
" " " Acta, VIII., XI. (1944, 1947.)
" " " Relationes de Auctis Scientus tempore Belli, 1939–1945 (1945–1947.)
Physical Review, June–Sept. 1948.
La Houille Blanche, Jan.–Aug. 1948.
Texas Cancer Bulletin, May–June 1948.
American Journal of Science, May–Aug. 1948.
Transactions and Discussions of the Faraday Society, April–Sept. 1948.
The Advancement of Science, April–July 1948.
Hungarica Acta Mathematica, 1946–48.
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Portugaliæ Physica, 1947.
Excerpta Medica, N.D.
Half-yearly Journal of the Mysore University, Sept 1947.
Enquiries, N.D.
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Radio-Technician, Dec. 1947.
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[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

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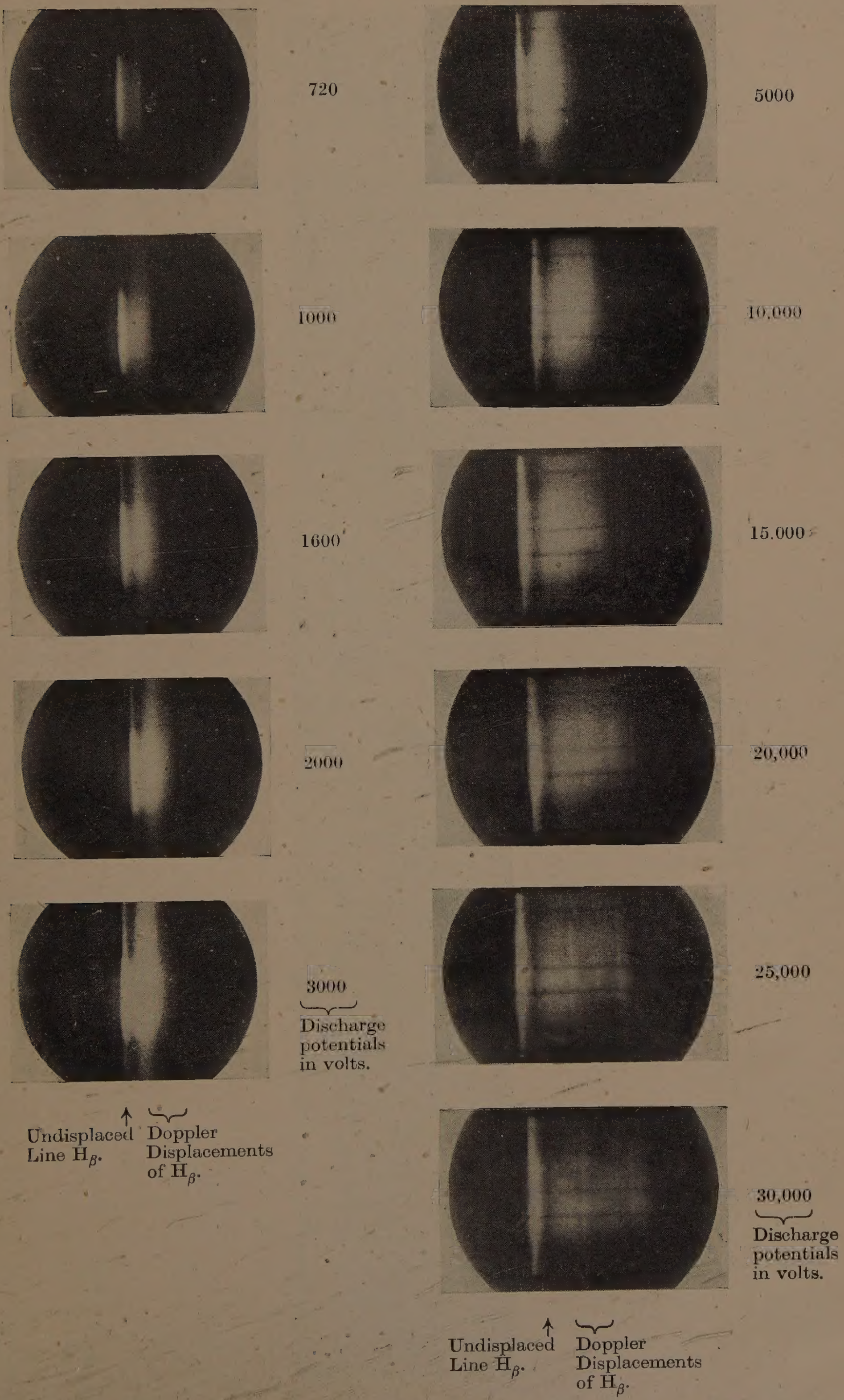
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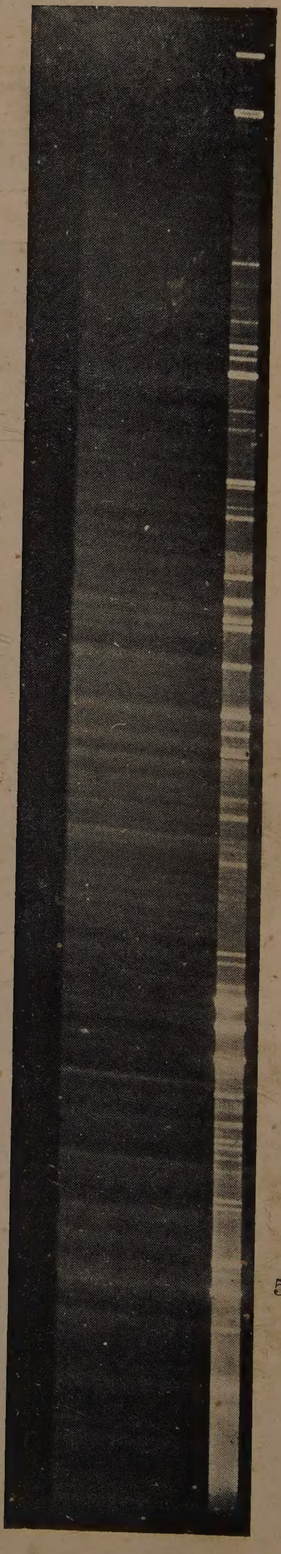
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